Online
ISSN 1346-0447

ISSN 1346-0862

Scientiae Mathematicae Japonicae

(Scientiae Mathematicae / Mathematica Japonica New Series)

Vol. 81, No. 3 Whole Number 290

Special Issue on FIM 2017

International Society for Mathematical Sciences

INTERNATIONAL SOCIETY FOR MATHEMATICAL SCIENCES
Scientiae Mathematicae Japonicae, Notices from the ISMS

The International Society for Mathematical Sciences (ISMS) is an international society consisting of mathematical scientists throughout the world.

The main activities of the ISMS are to publish (1) the (**print and online**) journal **Scientiae Mathematicae Japonicae (SCMJ)** and (2) Notices from the ISMS and to hold **assembly meeetings** in Japan and **international internet meetings** (distance symposium) of mathematical sciences (IVMS) accessible from all over the world.

SCMJ is the 21^{st} Century New Unified Series of **Mathematica Japonica (MJ)** and **Scientiae Mathematicae (SCM). MJ** was first published in 1948 and was one of the oldest mathematical journals in Japan. **SCM** was an online and print journal started in 1998 in celebration of the semi-centurial anniversary and received 26000 visits per month from 50 countries in the world. **SCMJ** contains original papers in mathematical sciences submitted from all over the world and receives **38000** visits per month now. Not only papers in pure and applied mathematics but those devoted to mathematical statistics, operations research, informatics, computer science, biomathematics, mathematical economics and other mathematical sciences are also welcome. The journal is published in January, March, May, July, September, and November in each calendar year.

The ISMS has enhanced the journal, begining from July 1995, by including excellent Research-Expository papers in the section "**International Plaza** for Mathematical Sciences " as well as original research papers. The section provides papers dealing with broad overviews of contemporary mathmatical sciences, written by experts mainly at our invitation. Papers shedding lights on open problems or new directions or new breakthroughs for future research are especially welcome.

As is shown in the **Editorial Board of SCMJ**, we have invited **many distinguished professors** of 20 countries as editors, who will receive and referee the papers of their special fields with their **high standard**.

Beginning from 2007, we make the online version of SCMJ more readable and convenient to the readers by adding the specialized contents. By this, the readers can access to the online version, in which the papers appear in the order of acceptance, from (i) the contents of the printed version, and (ii) the specialized contents of a volume. From 2007, the subscription fee of the printed version plus the online version of SCMJ becomes lower and the same of the printed version only. Therefore, the subscribers of the printed version can read the online version without no additional cost.

For benefit of the ISMS members, we publish "Notices from the ISMS" 6 times a year. We are enhancing it by adding interesting articles, including book reviewing, written by eminent professors.

The ISMS has set up a **videoconferencing system** (**IVMS**) which can connect up to twenty sites of a reserch group in the same or **different countries in the world**. Using this system, speakers of the session can write on a white board or an OHP sheet or use PowerPoint. On the other hand participants can ask questions or make comments from any connected site in the world. All these are performed similarly to the traditional meetings.

To connect with our system, you can use your own videoconferencing system only if it satisfies the International Telecommunication Union-Technical Committee Standards (ITU-T Standard).

Copyright Transfer Agreement

A copyright transfer agreement is required before a paper is published in this journal. By submitting a paper to this journal, authors are regarded to certify that the manuscript has not been submitted to nor is it under consideration for publication by another journal, conference proceedings or similar publication.

For more information, please visit http://www.jams.or.jp.

FOREWORD

The twenty-fifth annual international conference of the Forum for Interdisciplinary Mathematics (FIM2017) and the seventeenth International Symposium of Management Engineering (ISME 2017) were jointly organized at Kitakyushu International **Conference Centre, Kokura Kitakyushu, Japan during 25–28 August 2017. The Forum for Interdisciplinary Mathematics, established in India to promote interdisciplinary studies in mathematics, holds its annual international conference alternately in India and** abroad. In 2013, FIM 2013 was hosted in Kitakyushu International Conference Centre and it proved to be a resounding success. Impressed by the warm hospitality and the meticulous organization by the local organizing committee of FIM 2013 held in Kitakyushu in 2013, the Forum readily agreed to have FIM 2017 again in Kitakyushu, Japan.

Among the hundred and odd papers submitted for FIM 2017 covering topics such as disease modelling, elasticity, fuzzy systems, management engineering, pedagogy, pure mathematics and statistics, about fifty papers were selected for presentation in the conference. Of the three keynote addresses, Prof. Kato's lecture dealt with Geometry of Banach spaces while the lectures by Drs. Wan Fatima and Bo Wang were on teaching methodologies in mathematics and joint application of stochastics and fuzzy set theory for certain optimization problems respectively. The conference registration commenced in the evening of the 25th of August 2017 and the keynote addresses by Drs. Kato and Wan Fatima constituted the morning session on the 26th August. In the afternoon simultaneous paper reading sessions were held. In the morning of the 27th August apart from the keynote address by Dr. Bo Wang, a technical session of talks on fuzzy sets, information theory and Reliability analysis was held. The tour in and around Kokura that afternoon provided the participants some glimpses of the rich Japanese culture and the modern technological advances achieved in Japan. The 28th August, the final day of the conference, witnessed parallel paper reading sessions.

The members of the advisory committee comprised eminent academicians from about twenty countries, while the participants hailed mostly from China, India, Japan, Malaysia and Taiwan. The conference was marked by camaraderie and lively academic discussions. The present volume comprises refereed papers presented as well as invited articles. It is hoped that this publication will promote further research in the respective topics discussed. The conference chairs and the editors of the proceedings are beholden to the members of the local organizing committee, the international advisory committee, the government of Kitakyushu City and its Visitors Association.

P.V. SUBRAHMANYAM
JUNZO WATADA
HIROSHI SAKAI
YOSHIYUKI YABUUCHI
YOSHIYUKI MATSUMOTO

OCTAGONAL FUZZY CHOQUET INTEGRAL OPERATOR FOR MULTI-ATTRIBUTE DECISION MAKING

Felbin C Kennedy[1†] and Dhanalakshmi V[2]

[1]Research Guide & Associate Professor
[2]Research Scholar & Assistant Professor
Stella Maris College(Autonomous), Chennai, Tamil Nadu, India
† E-mail: felbinckennedy@gmail.com

Abstract. This paper introduces two types of aggregations, namely the octagonal fuzzy weighted averaging(OFWA) operator for non-interactive aggregation and octagonal fuzzy Choquet integral(OFCI) operator for interactive aggregation. The paper emphasis the use of octagonal fuzzy number as a general case of some well known linear fuzzy numbers. Procedure for solving multi-attribute decision making(MADM) problem using OFWA and OFCI operators are described and algorithms for the same are presented to handle large data. Finally, an illustrative example is provided to demonstrate the application of the OFCI operator in MADM problem.

Keyword: Octagonal fuzzy number, Choquet integral, aggregation, MADM, algorithm

1 Introduction Multi-attribute Decision Making (MADM) problems involve aggregating information from various decision makers, aggregating the interactive criteria and then the final selection through ranking the alternatives. In real situations, quantifying the quality of the alternative may not be precise[2]. Zadeh[33] suggested employing the fuzzy set theory as a modeling tool that can help overcome the situation. However, the presence of fuzziness in decision making increases the computational difficulty in aggregating and ranking the alternatives, which has been handled by various authors including us. To cite a few [1, 3, 4, 7, 8, 17, 20, 24].

The Choquet integral based aggregation finds its use in cases where individual criteria importance and group importance are required. The Choquet integral is related to a fuzzy measure which considers the interaction among the criteria to be aggregated [16, 21, 25]. For this reason, Choquet integral is more suited to deal with fuzzy MCDM problems and in recent years, many scholars have done a lot of good research in this field. Yang et. al. [31, 32] studied the real and fuzzy Choquet integrals for fuzzy integrand. Tan [23], Xu [30], Wei et.al. [28], Wu et. al. [29] used Choquet integral to propose some intuitionistic fuzzy aggregation operators. Tan [22], Qin et.al.[18], Meng et. al. [15] studied and used Choquet integral to determine attribute weight and applied it in decision making problems under interval intuitionistic environment. Rebille [19] used decision making over necessity measures through Choquet integral.

In this paper, we introduce two types of aggregations on octagonal fuzzy numbers [14], namely octagonal fuzzy weighted averaging(OFWA) operator and octagonal fuzzy Choquet integral(OFCI) operator. OFWA deals with non-interactive aggregation to aggregate the evaluations of different decision makers, OFCI operator deals with interactive aggregation that aggregates the different criteria for the same alternative.

The paper is organized as follows. Section 2 discusses some of the properties of octagonal fuzzy numbers which are used to describe the linguistic terms for expert evaluations. In the Section 3, we recall the concept of fuzzy measure, introduce octagonal fuzzy Choquet integral($OFCI$) and then investigate the aggregation properties of $OFCI$. In Section 4, we present the procedure for solving MADM problem using $OFCI$ operator, also algorithms are provided so as to apply it to the real life situations which usually comes with large number of alternatives and criteria. The application of the proposed method is given in Section 5 and conclusion is presented in Section 6.

2 Octagonal Fuzzy Numbers

Definition 2.1 [14] *A fuzzy number $\tilde{A}$ is said to be an octagonal fuzzy number denoted by $\tilde{A} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8; k, w)$ with membership function*

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1}k & \text{if } a_1 \le x \le a_2 \\ k & \text{if } a_2 \le x \le a_3 \\ \dfrac{k(a_4-x)+w(x-a_3)}{a_4-a_3} & \text{if } a_3 \le x \le a_4 \\ w & \text{if } a_4 \le x \le a_5 \\ \dfrac{k(x-a_5)+w(a_6-x)}{a_6-a_5} & \text{if } a_5 \le x \le a_6 \\ k & \text{if } a_6 \le x \le a_7 \\ \dfrac{a_8-x}{a_8-a_7}k & \text{if } a_7 \le x \le a_8 \\ 0 & otherwise \end{cases} \tag{2.1}$$

where $0 < k < w, w = height(\tilde{A}), w > k$.

Remark 2.1 *The fuzzy number defined in [14] is piecewise and made up of 8 linear curves and therefore named as 'octagonal'. Note that it satisfies the properties of fuzzy number in accordance with the definition by Klir in [13].*

Remark 2.2 *The above defined octagonal fuzzy number is a generalised form of some of the popular linear fuzzy numbers like, crisp, rectangular, triangular and trapezoidal fuzzy numbers. As all these numbers can be represented as an octagonal fuzzy number, the operations defined for octagonal fuzzy numbers will hold good for them. The equivalent forms are as follows:*

Fuzzy Numbers	*Equivalent Octagonal Fuzzy Numbers*
Crisp Numbers a	$(a, a, a, a, a, a, a, a; k, w)$
Interval Numbers $[a_1, a_2]$	$(a_1, a_1, a_1, a_1, a_2, a_2, a_2, a_2; k, w)$
Triangular Fuzzy Numbers (a_1, a_2, a_3)	$\left(a_1, \dfrac{ka_2 - ka_1 + wa_1}{w}, \dfrac{ka_2 - ka_1 + wa_1}{w}, a_2, a_2, \dfrac{-ka_3 + ka_2 + wa_3}{w}, \dfrac{-ka_3 + ka_2 + wa_3}{w}, a_3; k, w\right)$
Trapezoidal Fuzzy Number (a_1, a_2, a_3, a_4)	$\left(a_1, \dfrac{ka_2 - ka_1 + wa_1}{w}, \dfrac{ka_2 - ka_1 + wa_1}{w}, a_2, a_3 \dfrac{-ka_4 + ka_3 + wa_4}{w}, \dfrac{-ka_4 + ka_3 + wa_4}{w}, a_4; k, w\right)$

Remark 2.3 *The fuzzy numbers that are piece-wise linear and are made of less than 8 line segments can be directly expressed as octagonal fuzzy number as pointed out in Remark 2.2. Fuzzy numbers which may constitute more than 8 linear segments or those which are piece-wise non-linear are not exactly octagonal fuzzy numbers but can be approximated to octagonal fuzzy numbers in a particular sense (Theorem 2.4.1 in [5]).*

Definition 2.2 *Let $\tilde{A} = (a_1, a_2, ..., a_8; k, w)$ and $\tilde{B} = (b_1, b_2, ..., b_8, k, w)$ be two octagonal fuzzy numbers, then*
(i) $\tilde{A} + \tilde{B} = (a_1 + b_1, a_2 + b_2, ..., a_8 + b_8; k, w)$
(ii) $c\tilde{A} = \ (ca_1, ca_2, ..., ca_8; k, w), \quad for\ c \ge 0$

Remark 2.4 *In [9], it is verified that the sum and scalar multiplication obtained from definition 2.2 is as that using $\alpha-$ cut approach.*

Remark 2.5 *It is clear that $\tilde{A} + \tilde{B}$ and $c\tilde{A}$ are also octagonal fuzzy numbers.*

Proposition 2.1 *Let $\tilde{A} = (a_1, a_2, ..., a_8; k, w)$, $\tilde{B} = (b_1, b_2, ..., b_8, k, w)$ be two octagonal fuzzy numbers and let $c_1, c_2 > 0$, then we have*
(i) $\tilde{A} + \tilde{B} = \tilde{B} + \tilde{A}$
(ii) $c_1(\tilde{A} + \tilde{B}) = c_1\tilde{A} + c_1\tilde{B}$
(iii) $(c_1 + c_2)\tilde{A} = c_1\tilde{A} + c_2\tilde{A}$

Definition 2.3 *An octagonal fuzzy weighted averaging operator on a collection of n octagonal fuzzy numbers is defined as*

$$OFWA_{wv}(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) = wv_1\tilde{A}_1 + wv_2\tilde{A}_2 + ... + wv_n\tilde{A}_n \tag{2.2}$$

where $wv = (wv_1, wv_2, ..., wv_n)^T$ *is the weight vector of* $\tilde{A}_i (i = 1, 2, ..., n)$ *with* $wv_i \in [0, 1]$ *and* $\sum_{i=1}^{n} wv_i = 1$.

Definition 2.4 *Ranking using Radius of Gyration:[6] Area between the radius of gyration point* $(r_x^{\tilde{A}}, r_y^{\tilde{A}})$ *of the octagonal fuzzy number* $\tilde{A}$ *and the origin* $(0, 0)$ *is given by*

$$\mathcal{R}(\tilde{A}) = r_x^{\tilde{A}} r_y^{\tilde{A}}$$

where $r_x^{\tilde{A}} = \sqrt{\frac{I_x(\tilde{A})}{Area(\tilde{A})}}$ *and* $r_y^{\tilde{A}} = \sqrt{\frac{I_y(\tilde{A})}{Area(\tilde{A})}}$, $I_x(\tilde{A}), I_y(\tilde{A})$ *are respectively the moment of inertia with respect to the x-axis and y-axis and* $Area(\tilde{A})$ *the area of the octagonal fuzzy number* $\tilde{A}$.

Remark 2.6 *Ranking using radius of gyration is used in the procedure for defuzzification, whereas to compare the octagonal fuzzy numbers, we use the ranking algorithm introduced by us in Section 3.5 of the paper [6] . The ranking algorithm compares any two octagonal fuzzy numbers* $\tilde{A}$ *and* $\tilde{B}$ *in 10 steps and we have proved that the algorithm returns either* $\tilde{A} \prec \tilde{B}$, $\tilde{B} \prec \tilde{A}$ *or the two octagonal fuzzy numbers are equal(not just equivalent). Thus any two octagonal fuzzy numbers are comparable and the ordering is anti-symmetric.*

3 Fuzzy Measure and Choquet Integral

For the sake of completion, we recall the concept of fuzzy measure [12]. Using this, we define octagonal fuzzy Choquet integral operator which is then verified for fundamental properties of aggregation operator, like idempotency, monotonicity, boundedness and symmetry.

Definition 3.1 *[13] A fuzzy measure on X is a set function* $m : \mathcal{P}(X) \to [0, 1]$ *such that*
(i) $m(\phi) = 0,\ m(X) = 1$
(ii) $A, B \in \mathcal{P}(X),\ A \subseteq B \Rightarrow m(A) \leq m(B)$.

Considering the MADM problems, the number $m(A)$ can be interpreted as the importance of the subset A, and the monotonicity condition (ii) in Definition 3.1 of the fuzzy measure means that the importance of a subset of criteria cannot decrease when new criteria are added to it [26].
Let $E_j = \{x_j, x_{j+1}, ..., x_n\} (1 \leq j \leq n)$ be a criteria set. The interaction among the criteria in E_j can be described by employing $m(E_j)$to express the degree of importance of E_j. That is, the degree of importance of E_j is evaluated by simultaneously considering $x_j, x_{j+1}, ..., x_n$. Hence, m can be called an importance measure [27].
In order to determine such fuzzy measure, we generally need to find $2^n - 2$ values for n criteria, where $m(\phi) = 0$ and $m(X) = 1$ always. So the evaluation model obtained becomes quite complex, and the structure is difficult to grasp. To avoid the problems with computational complexity and practical estimations, $\lambda-$ fuzzy measure m, a special kind of fuzzy measure, was proposed by Sugeno, which satisfies the following additional property:

$$m(A \cup B) = m(A) + m(B) + \lambda m(A)m(B), \tag{3.1}$$

for all $A, B \in \mathcal{P}(X)$ and $A \cap B = \phi$ where $\lambda > -1$.

Definition 3.2 *[26] If X is a finite set, then* $\cup_{i=1}^{n} \{x_i\} = X$. *The* $\lambda-$ *fuzzy measure* $m : \mathcal{P}(X) \to [0, 1]$ *for every subset* $A \in \mathcal{P}(X)$, *satisfies*

$$m(A) = \begin{cases} \frac{1}{\lambda}\left(\prod_{x_i \in A} [1 + \lambda m(\{x_i\})] - 1\right) & \text{if } \lambda \neq 0 \\ \sum_{x_i \in A} m(\{x_i\}) & \text{if } \lambda = 0 \end{cases}$$

Remark 3.1 *[26] Based on the above definition of $m(A)$ and using the fact that $m(X) = 1$, we can uniquely solve λ which is equivalent to solving*

$$\lambda + 1 = \prod_{i=1}^{n}(1 + \lambda m(\{x_i\})) \tag{3.2}$$

and

$\sum_{i=1}^{n} m(\{x_i\})$	*Range of* λ	*Type of the* $\lambda-$ *fuzzy measure*
$= 1$	$\lambda = 0$	*Additive*
< 1	$\lambda > 0$	*Super-additive*
> 1	$-1 < \lambda < 0$	*Sub-additive*

Definition 3.3 *Let $\tilde{A}_i = (a_1^i, a_2^i, ..., a_8^i; k, w)(i = 1, 2, ..., n)$ be a collection of n octagonal fuzzy numbers on X and m be a $\lambda-$ fuzzy measure on X. The octagonal fuzzy Choquet integral of $\tilde{A}_i$ with respect to m is defined by*

$$OFCI(\tilde{A}_1, ..., \tilde{A}_n) = \sum_{i=1}^{n}(m(E_{(i)}) - m(E_{(i+1)})\tilde{A}_{(i)} \tag{3.3}$$

where $(\cdot)$ indicates a permutation on X such that $\tilde{A}_{(1)} \preceq \tilde{A}_{(2)} \preceq ... \preceq \tilde{A}_{(n)}$ and $E_{(i)} = \{x_i, ..., x_n\}$, $E_{(n+1)} = \phi$.

Proposition 3.1 *Let $\tilde{A}_i = (a_1^i, a_2^i, ..., a_8^i; k, w)(i = 1, 2, ..., n)$ be a collection of n octagonal fuzzy numbers on X and m be a $\lambda-$ fuzzy measure on X, then their aggregated value $OFCI(\tilde{A}_1, ..., \tilde{A}_n)$ is also an octagonal fuzzy number.*

Proof: The result follows immediately from Definition 2.2□

Proposition 3.2 *Let $\tilde{A}_i = (a_1^i, a_2^i, ..., a_8^i; k, w)(i = 1, 2, ..., n)$ be a collection of n octagonal fuzzy numbers on X, such that $\sum_{i=1}^{n} m(\{x_i\}) = 1$. Then the octagonal fuzzy choquet integral coincides with the octagonal fuzzy weighted average.*

Proof: From Remark 3.1 we see that $\lambda = 0$ here. According to Definition 3.2 the $\lambda-$fuzzy measure is given by $m(E_{(i)}) = \sum_{j=i}^{n} m(\{x_j\})$. Thus

$$\begin{aligned} OFCI(\tilde{A}_1, ..., \tilde{A}_n) &= \sum_{i=1}^{n}(m(E_{(i)}) - m(E_{(i+1)})\tilde{A}_{(i)} \\ &= \sum_{i=1}^{n}\left[\sum_{j=i}^{n} m(\{x_j\}) - \sum_{j=i+1}^{n} m(\{x_j\})\right]\tilde{A}_{(i)} \\ &= \sum_{i=1}^{n} m(\{x_i\})\tilde{A}_{(i)} \\ &= OFWA(\tilde{A}_1, ..., \tilde{A}_n) \end{aligned}$$

Here $(m(\{x_1\}), m(\{x_2\}), ..., m(\{x_n\}))^T$ is the weight vector satisfying $\sum_{i=1}^{n} m(\{x_i\}) = 1$.□

Proposition 3.3

$$OFCI(\tilde{A}, ..., \tilde{A}) = \tilde{A}$$

Proof: From equation 3.3, we have

$$\begin{aligned} OFCI(\tilde{A},...,\tilde{A}) &= \sum_{i=1}^{n}(m(E_{(i)})-m(E_{(i+1)})\tilde{A} \\ &= \tilde{A}\sum_{i=1}^{n}(m(E_{(i)})-m(E_{(i+1)})) \\ &= \tilde{A}\left(m(E_{(1)}-m(E_{(n+1)})\right) \\ &= \tilde{A}\left(m(X)-m(\phi)\right) \\ &= \tilde{A} \ \square \end{aligned}$$

Proposition 3.4 *Let $\tilde{A}_i=(a_1^i,a_2^i,...,a_8^i;k,w)$ and $\tilde{B}_i=(b_1^i,b_2^i,...,b_8^i;k,w)$ $(i=1,2,...,n)$ be a collection of $2n$ octagonal fuzzy numbers on X such that $\tilde{A}_i\preceq\tilde{B}_i$ $(i=1,2,...,n)$ but there exists no j and k such that $\tilde{A}_i\preceq\tilde{A}_j\preceq\tilde{B}_k\preceq\tilde{B}_i$ for any $j,k(\neq i)\in\{1,2,...,n\}$ and m be a $\lambda-$ fuzzy measure on X, then $OFCI(\tilde{A}_1,...,\tilde{A}_n)\leq OFCI(\tilde{B}_1,...,\tilde{B}_n)$.*

Proof: Since $E_{(i+1)}\subseteq E_{(i)}$, we have $m(E_{(i+1)})\leq m(E_{(i)})$. Thus $m(E(i))-m(E_{(i+1)})\geq 0$ for all i. Suppose after rearranging in ascending order, $\tilde{A}_i$ is moved to $\tilde{A}_{(j)}$ and $\tilde{B}_i$ is moved to $\tilde{B}_{(k)}$, then $\tilde{A}_{(j)}\preceq\tilde{B}_{(k)}$ and no $\tilde{A}_.$ or $\tilde{B}_.$ comes in between. Also, we have n such inequalities. Thus, $j=k$. i.e. $\tilde{A}_{(i)}\preceq\tilde{B}_{(i)}$ for $i=1,2,...,n$ Now,

$$\begin{aligned} OFCI(\tilde{A}_1,...,\tilde{A}_n) &= \sum_{i=1}^{n}(m(E_{(i)})-m(E_{(i+1)})\tilde{A}_{(i)} \\ &\preceq \sum_{i=1}^{n}(m(E_{(i)})-m(E_{(i+1)})\tilde{B}_{(i)} \\ &= OFCI(\tilde{B}_1,...,\tilde{B}_n) \ \square \end{aligned}$$

Proposition 3.5 *Let $\tilde{A}_i=(a_1^i,a_2^i,...,a_8^i;k,w)$ $(i=1,2,...,n)$ be a collection of n octagonal fuzzy numbers on X and m be a $\lambda-$ fuzzy measure on X, then $OFCI(\tilde{A}_1,...,\tilde{A}_n)$ is bounded.*

Proof: From the definition of $OFCI$,

$$OFCI(\tilde{A}_1,...,\tilde{A}_n)=\sum_{i=1}^{n}(m(E_{(i)})-m(E_{(i+1)})\tilde{A}_{(i)}$$

where $(\cdot)$ indicates a permutation on X such that $\tilde{A}_{(1)}\preceq\tilde{A}_{(2)}\preceq...\preceq\tilde{A}_{(n)}$. Thus

$$\begin{aligned} OFCI(\tilde{A}_1,...,\tilde{A}_n) &= \sum_{i=1}^{n}(m(E_{(i)})-m(E_{(i+1)})\tilde{A}_{(i)} \\ &\succeq \tilde{A}_{(1)}\sum_{i=1}^{n}(m(E_{(i)})-m(E_{(i+1)})) \\ &\succeq \tilde{A}_{(1)}\left(m(E_{(1)})-m(E_{(n+1)})\right) \\ &\succeq \tilde{A}_{(1)}\left(m(X)-m(\phi)\right) \\ &\succeq \tilde{A}_{(1)} \end{aligned}$$

Also

$$\begin{aligned} OFCI(\tilde{A}_1,...,\tilde{A}_n) &= \sum_{i=1}^{n}(m(E_{(i)})-m(E_{(i+1)})\tilde{A}_{(i)} \\ &\preceq \tilde{A}_{(n)}\sum_{i=1}^{n}(m(E_{(i)})-m(E_{(i+1)})) \\ &\preceq \tilde{A}_{(n)} \ \square \end{aligned}$$

From Definition 3.3, the following property can easily be obtained.

Proposition 3.6 *Let $\tilde{A}_i=(a_1^i,a_2^i,...,a_8^i;k,w)(i=1,2,...,n)$ be a collection of n octagonal fuzzy numbers on X and m be a $\lambda-$ fuzzy measure on X. If $(\tilde{A}'_1,\tilde{A}'_2,...,\tilde{A}'_n)$ is any permutation of $(\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_n)$, then $OFCI(\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_n)=OFCI(\tilde{A}'_1,\tilde{A}'_2,...,\tilde{A}'_n)$.*

Proof: The proof is obvious, as whatever the permutation the $OFCI$ first orders the given collections of octagonal fuzzy numbers and then aggregates. $\square$

4 Multi-Attribute Decision Making with OFCI Operator Consider the MADM problem handled in [7] with k decision makers $D_1, D_2, ..., D_k$. evaluating the importance of n criteria $c_1, c_2, ..., c_n$ and m alternatives $A_1, A_2, ..., A_m$ based on each of the n criteria. The problem is considered in octagonal fuzzy environment.

4.1 Abstract Algorithm for solving the MCDM problem using OFCI operator:

Step 1: Aggregate the evaluations of the decision makers:
Use $OFWA$ operator for this step, so that the problem now has a vector C of size n, which gives the importance of the n criteria and an $m \times n$ matrix, which is the evaluations of the m alternatives based on n criteria. All the entries in the vector and the matrix are octagonal fuzzy numbers

Step 2: Find the $\lambda-$fuzzy measure of the power set of the criteria set:

(i) Compute the $\lambda-$ fuzzy measure for individual criteria as

$$g_\lambda(C_i) = \frac{\mathcal{R}(C_i)}{2 \times max(\mathcal{R}(C_i))}, \ i = 1, 2, ..., n$$

where $\mathcal{R}$ is the radius of gyration as given in Definition 2.4

(ii) Solve the equation $\lambda + 1 = \prod_{i=1}^{n}(1 + \lambda g_\lambda(C_i))$ for λ and

$\lambda = 0$ if $\sum_{i=1}^{m} g_\lambda(C_i) = 1$

$\lambda < 0$ if $\sum_{i=1}^{m} g_\lambda(C_i) > 1$

$\lambda > 0$ if $\sum_{i=1}^{m} g_\lambda(C_i) < 1$

(iii) $g_\lambda(A)$ is obtained using Definition 3.2, where $A \in \mathcal{P}(\{c_1, c_2, ..., c_n\})$

Step 3: Aggregate the criterias for the alternatives:
Use the octagonal fuzzy Choquet integral operator to aggregate the n evaluations for each alternative, to obtain an octagonal fuzzy number.

Step 4: Order the alternatives:
Sort the alternatives.

4.2 Algorithms for solving the MCDM problem using OFCI operator: In the above abstract algorithm, Step 1 is direct as it is the weighted average which involves addition and scalar multiplication only. The result of this Step is the matrix DM with m rows and n columns with each entry (i, j) the aggregation of the decision makers' evaluation of i^{th} alternative versus j^{th} criteria. Also Step 2 (i) and (ii) are direct calculations. Step 3 is tricky as we have to identify the subsets of the criteria set and then the corresponding $\lambda-$measure. Hence we present an algorithm to find $g_\lambda(A)$, where A is the subset of the criteria set. In this algorithm, we will obtain matrix M with two columns and 2^n rows, the first column gives the binary equivalent of the numbers $1, 2, ..., 2^n$ and the second column gives the g_λ measure of the subset of the criteria set, which is identified using the corresponding first column entry. For example, the binary number "10110" will represent the subset $\{c_2, c_3, c_5\}$ i.e from right to left the entries denote $c_1, c_2, ..., c_n$ with each binary digit acting like a characteristic function of the subset.

Algorithm 4.1 Subset of the Criteria set and its Measure

Require: $g_\lambda(C_i), (i = 1, 2, ...n)$, n - number of criteria
  for $r \leftarrow 1$ to 2^n **do**
    $M_{r,1} = ""$
    for $i \leftarrow 1$ to n **do**
      $t_i \leftarrow floor(mod(\frac{r-1}{2^{i-1}}, 2))$
      $M_{r,1} = Concatenate(M_{r,1}, t_i)$
▷ First column of M identifies the subsets of the criteria set

```
        end for
        for i ← 1 to n do
            s_i ← floor(mod((r−1)/2^(i−1), 2)) ∗ (1 + λ g_λ(C_i))
        end for
        prod ← 1
        for j ← 1 to n do
            if s_i ≠ 0 then
                prod ← prod ∗ s_i
            end if
        end for
        M_{r,2} = (prod − 1)/λ
                ▷ Second column gives the measure of the set identified in the corresponding first column
    end for
```

Algorithm 4.2 Octagonal Fuzzy Choquet Integral to aggregate the criteria

```
Require: the order of the decision matrix
    for i ← 1, m do                                              ▷ Identifying the set E_(i)
        for l ← 1, n do
            OB_{i,l} ← ""
            t_{i,l} ← 1
        end for
        for p ← n, 1 step −1 do
            OB_{i,l} ←concatenate(OB_{i,l}, t_{i,p})
        end for
    end for
    for j ← 2, n do
        for i ← 1, m do
            for l ← 1, n do
                if s_{i,j−1} = l then
                    t_{i,l} ← 0
                end if
            end for
            for p ← n, 1 step −1 do
                OB_{i,j} ←concatenate(OB_{i,l}, t_{i,p})         ▷ OB_{i,j} denote the set E_(j)
            end for                                              ▷ for the alternative i
        end for
    end for
    for u ← 1, m do
        for r ← 1, 2^n do
            for j ← 1, n do
                if M_{r,1} = OB_{u,j} then
                    a_j ← M_{r,2}                                ▷ a_j is the measure of the set E_(j)
                end if
            end for
        end for
        a_{n+1} ← 0
        CI_u ← Σ_{s=1}^{n} DM_{u,s_{u,s}} ∗ (a_s − a_{s+1})
                ▷ CI is a vector of size n with CI_u is the aggregated evaluation for alternative u
    end for
```

To end the procedure, the vector CI is sorted using the ranking method, radius of gyration and the alternative with maximum $\mathcal{R}(CI_u)$ is the best alternative.

5 Illustration Consider an hypothetical problem of selecting a supplier among four suppliers. They determine five attributes, namely capacity, quality, cost, distance and delivery time. By the help of three experts, they evaluate all the suppliers, also the experts determine the fuzzy weights of the criteria. Assume that the experts are equally important. The evaluations are as follows:

Importance of criteria matrix

$$DC = \begin{pmatrix} VH & H & H & VH & M \\ VH & H & MH & H & MH \\ VH & H & MH & VH & M \end{pmatrix}$$

Evaluation matrix of Expert 1

$$DM1 = \begin{pmatrix} VG & VG & VG & VG & VG \\ G & VG & VG & VG & MG \\ VG & MG & G & G & G \\ G & M & M & G & MG \end{pmatrix}$$

Evaluation matrix of Expert 2

$$DM2 = \begin{pmatrix} G & MG & G & G & VG \\ G & VG & VG & VG & MG \\ G & G & MG & VG & G \\ VG & M & MG & M & G \end{pmatrix}$$

Evaluation matrix of Expert 3

$$DM3 = \begin{pmatrix} MG & MG & G & VG & VG \\ MG & MG & G & MG & G \\ VG & VG & VG & VG & MG \\ MG & VG & MG & VG & M \end{pmatrix}$$

where the corresponding octagonal fuzzy numbers for the above used linguistic term set are as given in the following table:

Linguistic term set for attributes	Linguistic term set for Weights	Corresponding octagonal fuzzy number
VP	VL	$(0, 10, 20, 30, 40, 50, 60, 70; \frac{1}{2}, 1)$
P	L	$(10, 20, 30, 40, 50, 60, 70, 80; \frac{1}{2}, 1)$
MP	ML	$(20, 30, 40, 50, 60, 70, 80, 90; \frac{1}{2}, 1)$
M	M	$(30, 40, 50, 60, 70, 80, 90, 100; \frac{1}{2}, 1)$
MG	MH	$(40, 50, 60, 70, 80, 90, 100, 100; \frac{1}{2}, 1)$
G	H	$(50, 60, 70, 80, 90, 100, 100, 100; \frac{1}{2}, 1)$
VG	VH	$(60, 70, 80, 90, 100, 100, 100, 100; \frac{1}{2}, 1)$

As the experts are considered equal, their weight vector will be $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

The first step to the problem is to aggregate the evaluations of the three experts and then to obtain the λ- fuzzy measure of the singleton sets $\{C_i\}, (i = 1, 2, ..., 5)$ which is as $0.5, 0.467, 0.43, 0.489, 0.378$ respectively.

Solving the equation

$$(1 + 0.5\lambda)(1 + 0.467\lambda)(1 + 0.43\lambda)(1 + 0.489\lambda)(1 + 0.378\lambda) - \lambda - 1 = 0$$

we get the λ- values to be $0, -0.93772, -5.19866, -2.51050 + 2.76915i, -2.51050 - 2.76915i$ and considering the cases in Remark 3.1, we let $\lambda = -0.938$

Following the algorithms, we aggregate all the information and obtain a octagonal fuzzy number for each alternative follows:

Alternative 1 $(56.206, 66.204, 76.202, 86.201, 96.199, 99.291, 99.983, 99.983; \frac{1}{2}, 1)$

Alternative 2 $(53.788, 63.786, 73.785, 83.783, 93.781, 97.962, 99.983, 99.983; \frac{1}{2}, 1)$

Alternative 3 $(54.933, 64.932, 74.93, 84.928, 94.927, 99.182, 99.983, 99.983; \frac{1}{2}, 1)$

Alternative 4 $(46.574, 56.572, 66.571, 76.569, 86.567, 93.745, 98.028, 99.983; \frac{1}{2}, 1)$

The order of the alternatives is $A_1 \succeq A_3 \succeq A_2 \succeq A_4$.

Remark 5.1 *The method proposed seems to be helpful in many cases provided the situation in any practical example can be described in terms of ideas in fuzzy sets on which the method is based.*

6 Conclusion In this paper, we introduced two aggregation operators, which are used to aggregate two types of information, namely, interactive and non-interactive. The aggregation for non-interactive information is verified to be a particular case of $OFCI$ operator. The fundamental aggregation properties are verified for $OFCI$ operator and a procedure for solving MADM problem involving the two types of aggregation is considered. An illustrative example is given to demonstrate the same. We note that algorithms are presented for complicated steps in the procedure, so that computer programs can be written to handle the real life problems which comes with large number of alternatives and criterias'(as pointed out with a concrete example in the second authors' thesis [5]). Also from Remark 2.1, we see that the problem with any other linear fuzzy numbers, like crisp, interval, triangular or trapezoidal fuzzy numbers, can be used, by considering their equivalent octagonal fuzzy numbers.

Acknowledgement The authors wish to thank Professor M.S.Rangachari, Former Director and Head, Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai for his valuable suggestions in the preparation of this paper.

References

[1] Baas, S.M., Kwakernaak, H.: Rating and ranking of multiple aspect alternatives using fuzzy sets. Automatica **13**, 47-58 (1977)

[2] Bellman, R., Zadeh, L.A.: Decision making in a fuzzy environment. Management Sciences **17B**, 141-164 (1970)

[3] Chen C.T.: Extension of the TOPSIS for group decision making under fuzzy environment. Fuzzy Sets and Systems **114**, 1–9 (2000)

[4] Chen, S.J., Hwang, C.L.: Fuzzy Multiple Attribute decision making, Methods and Applications. Lecture Notes in Economics and Mathematical Systems **375**, (1992)

[5] Dhanalakshmi V: A Study of the Structure of the Class of Octagonal Fuzzy Numbers and their Applications to Multi-Criteria Decision Making, Thesis submitted to the University of Madras, (2017)

[6] Dhanalakshmi V, Felbin C. Kennedy: Some ranking methods for Octagonal fuzzy numbers. International Journal of Mathematical Archive **5**, 177-188 (2014)

[7] Dhanalakshmi V, Felbin C. Kennedy: Some Aggregation Operations on Octagonal Fuzzy Numbers and its Application to Decision Making. International Journal of Mathematics and Scientific Computing **5**, 52-56 (2015)

[8] Deng H, Yeh CH, Willis R.J.: Inter-company comparison using modified TOPSIS with objective weights. Computers and Operations Research **27**, 963–973 (2000)

[9] Felbin C. Kennedy, Dhanalakshmi V: Cone Properties of Linear Fuzzy Numbers. Global and Stochastic Analysis **4**, 95-105 (2017)

[10] Grabisch, M., Roubens, M.: Application of the Choquet Integral in Multicriteria Decision Making. Fuzzy Measures and Integrals - Theory and Applications, Physica Verlag, Göttingen, 348-374, (2000)

[11] C.L. Hwang, K. Yoon: Multiple Attributes Decision Making Methods and Applications. Springer, Berlin Heidelberg, (1981)

[12] Guo C., Zhang D., Wu C.: Fuzzy-valued fuzzy measures and generalised fuzzy integrals. Fuzzy Sets and Systems **97**, 255-260 (1998)

[13] Klir George J, Bo Yuan: Fuzzy sets and Fuzzy logic-Theoryand Applications. Prentice Hall of India, (1997)

[14] Malini S. U., Felbin C.Kennedy: An Approach for Solving Fuzzy Transportation Problem Using Octagonal Fuzzy Numbers. Applied Mathematical Sciences **7**, 2661-2673 (2013)

[15] Meng, F., Chen, W., Zhang Qjang: Some interval-valued intuitionistic uncertain linguistic Choquet operators and their application to multi-attribute group decision making. Applied Mathematical Modeling **38**, 2543 - 2557 (2014)

[16] Murofushi, T., Sugeno, M.: A Theory of Fuzzy Measure: Representations, the Choquet Integral and null Sets. Journal of Mathematical Analysis and Applications **159**, 532 - 549 (1991)

[17] Opricovic S, Tzeng GH: Fuzzy multicriteria model for post-earthquake landuse planning. Natural Hazards Review **4**, 59–64 (2003)

[18] Qin J., Liu X.: Study on interval intuitionistic fuzzy multi-attribute group decision making method based on Choquet integral. Procedia Computer Science **17**, 465-472 (2013)

[19] Rebille Yann: Decision making over necessity measures through the Choquet integral criterion. Fuzzy Sets and Systems **157**, 3025 - 3039 (2006)

[20] Riberio, R.A.: Fuzzy multiple attribute decision making-a review and new preference elicitation techniques. Fuzzy Sets and Systems **78**, 155–181 (1996)

[21] Sugeno M., Narukawa Y., Murofushi T.: Choquet integral and fuzzy measures on locally compact space. Fuzzy Sets and Systems **99**, 205-211 (1998)

[22] Tan C: A multi-criteria interval-valued intuitionistic fuzzy group decision making with Choquet integral-based TOPSIS. Expert Systems with Applications **38(4)**, 3023-3033 (2011)

[23] Tan C., Chen X: Intuitionistic fuzzy Choquet integral operator for multi-criteria decision making. Expert Systems with Applications **37**, 149-157 (2010)

[24] Triantaphyllou E, Sanchez: A sensitivity analysis approach for some deterministic multi-criteria decision-making methods. Decision Sciences **28**, 151–194 (1997)

[25] Wang, Z., Klir, G., Wang W: Monotone set functions defined by Choquet Integral. Fuzzy Sets and Systems **81**, 241-250 (1996)

[26] Wang, Z., Klir G.J.: Fuzzy Measure Theory. Plenum Publishing Corporation, New York (1992)

[27] Wang, W., Wang, Z., Klir, G.J.: Genetic Algorithm for determining fuzzy measures from data. Journal of Intelligent and Fuzzy Systems **6**, 171-183 (1998)

[28] Wei G., Lin R., Zhoa X., Wang H.: Some Aggregation Operators based on the Choquet integral with fuzzy number intuitionistic fuzzy information and thier applications to multiple attribute decision making. Control and Cybernetics **41**, 463-480 (2012)

[29] Wu J., Chen F., Nie C., Zhang Q.: Intuitionistic fuzzy-valued Choquet integral and its application in multicriteria decision making. Information Sciences **222**, 509-527 (2013)

[30] Xu Z.S.: Choquet integrals of weighted intuitionistic fuzzy information. Information Sciences **180(5)**, 726-736 (2010)

[31] Yang R., Wang Z., Heng P., Leung K.: Fuzzy numbers and fuzzification of the Choquet integral. Fuzzy Sets and Systems **153**, 95-113 (2005)

[32] Yang R., Wang Z., Heng P., Leung K.: Real-valued Choquet integrals with fuzzy-valued integrand. Fuzzy Sets and Systems **157**, 256-269 (2006)

[33] Zadeh, L.A.: Fuzzy Sets. Information and Control **8**, 338-353 (1965)

[34] Zimmermann, H.J.: Fuzzy set Theory and its Applications. Kluwer, Nijhoff Publishing, Boston (1985)

Figure 1: MathCAD 14 programs for Algorithm 2.1

```
Order(M) := | for p ∈ 1..rows(M)
            |   B ← ((M^T)^<p>)^T
            |   for i ∈ 1..cols(M)
            |     for j ∈ i..cols(M)
            |       if Rank(B_{1,i}, B_{1,j}) = 0
            |         m ← B_{1,j}
            |         B_{1,j} ← B_{1,i}
            |         B_{1,i} ← m
            |   D ← ((M^T)^<p>)^T
            |   for i ∈ 1..cols(M)
            |     sc_{p,i} ← 0
            |     for j ∈ 1..cols(M)
            |       if (D_{1,j} = B_{1,i})
            |         n_{p,i} ← j
            |         sc_{p,i} ← sc_{p,i} + 1
            |   for i ∈ 1..cols(M)
            |     if sc_{p,i} > 1
            |       jj ← 1
            |       for k ∈ 1..cols(M)
            |         if n_{p,i} = n_{p,k}
            |           s_{jj} ← k
            |           jj ← jj + 1
            |       j ← 1
            |       for k ∈ 1..cols(M)
            |         if B_{1,i} = D_{1,k}
            |           r_j ← k
            |           j ← j + 1
            |       for j ∈ 1..sc_{p,i}
            |         n_{p,(s_j)} ← r_j
            |         sc_{p,(s_j)} ← 1
            | n
```

Figure 2: MathCAD 14 programs for Algorithm 4.1

```
r := 1..2^n

M_{r,1} := | t ← ""
           | for i ∈ 1..n
           |    t ← concat(num2str(floor(mod((r − 1)/2^(i−1), 2))), t)
           | t

M_{r,2} := | for i ∈ 1..n
           |    t_i ← floor(mod((r − 1)/2^(i−1), 2))·(1 + λ·CI_i)
           | pro ← 1
           | for i ∈ 1..n
           |    pro ← pro·t_i if t_i ≠ 0
           | (pro − 1)/λ
```

Figure 3: MathCAD 14 programs for Algorithm 4.2

```
Order_Binary(M) := | Order ← OrderManyRows(M)
                   | for i ∈ 1..rows(M)
                   |    | for l ∈ 1..cols(M)
                   |    |    t_{i,l} ← 1
                   |    | s_{i,1} ← ""
                   |    | for x ∈ 1..5
                   |    |    s_{i,1} ← concat(num2str(t_{i,x}), s_{i,1})
                   | for j ∈ 2..cols(M)
                   |    for i ∈ 1..rows(M)
                   |       | for l ∈ 1..cols(M)
                   |       |    t_{i,l} ← 0 if Order_{i,j−1} = l
                   |       | s_{i,j} ← ""
                   |       | for x ∈ 1..5
                   |       |    s_{i,j} ← concat(num2str(t_{i,x}), s_{i,j})
                   | s
```

Choquet Integral Value for the Alternatives:

```
CI_A(M) := | for u ∈ 1..rows(M)
           |    | for r ∈ 1..2^cols(M)
           |    |    for i ∈ 1..cols(M)
           |    |       a_i ← Mea_{r,3} if Mea_{r,2} = Order_Binary(M)_{u,i}
           |    | a_{cols(M)+1} ← 0
           |    | f_u ← Σ_{s=1}^{cols(M)} DM_{u, OrderManyRows(M)_{u,s}}·(a_s − a_{s+1})
           | f
```

Figure 4: Illustration

Linguistic Term Set For Attributes and Weights:

$$VP := \begin{pmatrix} 0 \\ 10 \\ 20 \\ 30 \\ 40 \\ 50 \\ 60 \\ 70 \end{pmatrix} \quad P := \begin{pmatrix} 10 \\ 20 \\ 30 \\ 40 \\ 50 \\ 60 \\ 70 \\ 80 \end{pmatrix} \quad MP := \begin{pmatrix} 20 \\ 30 \\ 40 \\ 50 \\ 60 \\ 70 \\ 80 \\ 90 \end{pmatrix} \quad M := \begin{pmatrix} 30 \\ 40 \\ 50 \\ 60 \\ 70 \\ 80 \\ 90 \\ 100 \end{pmatrix} \quad MG := \begin{pmatrix} 40 \\ 50 \\ 60 \\ 70 \\ 80 \\ 90 \\ 100 \\ 100 \end{pmatrix} \quad G := \begin{pmatrix} 50 \\ 60 \\ 70 \\ 80 \\ 90 \\ 100 \\ 100 \\ 100 \end{pmatrix} \quad VG := \begin{pmatrix} 60 \\ 70 \\ 80 \\ 90 \\ 100 \\ 100 \\ 100 \\ 100 \end{pmatrix}$$

VL := VP L := P ML := MP MH := MG H := G VH := VG

Number of Decision Makers: q := 3

Number of Alternatives: m := 4 i := 1.. m $k := \frac{1}{2}$ w := 1

Number of Attributes: n := 5 j := 1.. n

Importance of Attributes Matrix:

$$D\_C := \begin{pmatrix} VH & H & H & VH & M \\ VH & H & MH & H & MH \\ VH & H & MH & VH & M \end{pmatrix}$$

Evaluation matrix of Decision Maker 1:

$$DM1 := \begin{pmatrix} VG & VG & VG & VG & VG \\ G & VG & VG & VG & MG \\ VG & MG & G & G & G \\ G & M & M & G & MG \end{pmatrix}$$

Evaluation matrix of Decision Maker 2:

$$DM2 := \begin{pmatrix} G & MG & G & G & VG \\ G & VG & VG & VG & MG \\ G & G & MG & VG & G \\ VG & M & MG & M & G \end{pmatrix}$$

Evaluation matrix of Decision Maker 3:

$$DM3 := \begin{pmatrix} MG & MG & G & VG & VG \\ MG & MG & G & MG & G \\ VG & VG & VG & VG & MG \\ MG & VG & MG & VG & M \end{pmatrix}$$

The $\lambda-$ fuzzy measure for individual criteria is $g_\lambda(C_i) = \begin{pmatrix} 0.5 \\ 0.467 \\ 0.43 \\ 0.489 \\ 0.378 \end{pmatrix}$

Solving the equation $\lambda + 1 = (1 + 0.5\lambda)(1 + 0.467\lambda)(1 + 0.43\lambda)(1 + 0.489\lambda)(1 + 0.378\lambda)$, we get $\lambda = \begin{pmatrix} 0 \\ -5.1986 \\ -2.5105 - 2.7691i \\ -2.5105 + 2.7691i \\ -0.9377 \end{pmatrix}$ The $\lambda-$ fuzzy measure of the power set of the criteria set:

$$M=\begin{pmatrix} "00000" & 0 \\ "00001" & 0.5 \\ "00010" & 0.467 \\ "00011" & 0.748 \\ "00100" & 0.43 \\ "00101" & 0.728 \\ "00110" & 0.709 \\ "00111" & 0.876 \\ "01000" & 0.489 \\ "01001" & 0.76 \\ "01010" & 0.742 \\ "01011" & 0.894 \\ "01100" & 0.722 \\ "01101" & 0.883 \\ "01110" & 0.873 \\ "01111" & 0.963 \\ "10000" & 0.378 \\ "10001" & 0.701 \\ "10010" & 0.68 \\ "10011" & 0.861 \\ "10100" & 0.656 \\ "10101" & 0.848 \\ "10110" & 0.836 \\ "10111" & 0.944 \\ "11000" & 0.694 \\ "11001" & 0.868 \\ "11010" & 0.857 \\ "11011" & 0.955 \\ "11100" & 0.844 \\ "11101" & 0.948 \\ "11110" & 0.941 \\ "11111" & 1 \end{pmatrix}$$

$$\mathrm{Order}(DM)=\begin{pmatrix} 2 & 1 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \\ 5 & 2 & 3 & 1 & 4 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}$$

$$OB(DM)=\begin{pmatrix} "11111" & "11101" & "11100" & "11000" & "10000" \\ "11111" & "01111" & "01110" & "01100" & "00100" \\ "11111" & "01111" & "01101" & "01001" & "01000" \\ "11111" & "11011" & "11001" & "01001" & "00001" \end{pmatrix}$$

$$CI(DM)^T=\left[\begin{pmatrix} 56.206 \\ 66.204 \\ 76.202 \\ 86.201 \\ 96.199 \\ 99.291 \\ 99.983 \\ 99.983 \end{pmatrix} \begin{pmatrix} 53.788 \\ 63.786 \\ 73.785 \\ 83.783 \\ 93.781 \\ 97.962 \\ 99.983 \\ 99.983 \end{pmatrix} \begin{pmatrix} 54.933 \\ 64.932 \\ 74.93 \\ 84.928 \\ 94.927 \\ 99.182 \\ 99.983 \\ 99.983 \end{pmatrix} \begin{pmatrix} 46.574 \\ 56.572 \\ 66.571 \\ 76.569 \\ 86.567 \\ 93.745 \\ 98.028 \\ 99.983 \end{pmatrix}\right]$$

$$\mathrm{Order}\left(CI(DM)^T\right)=(4\ \ 2\ \ 3\ \ 1)$$

RULE BASED DECISION SUPPORT IN TABLE DATA SETS WITH UNCERTAINTY AND ITS EXECUTION ENVIRONMENT

H. Sakai, K.Y. Shen, G.H. Tzeng, M. Nakata

Abstract. A framework of decision support in table data sets with uncertainty is considered, and the prototype of its software tool is implemented in SQL. We follow the framework of the possible world semantics for table data sets with uncertainty, and two kinds of rules, i.e., the certain rules and the possible rules, are defined. This definition is simple and natural, but we are faced with the fact that the number of the possible worlds may exceed 10^{100}. Even in such huge number of possible worlds, the NIS-Apriori algorithm generates two kinds of rules, because this algorithm is independent from the number of the possible worlds due to the proved properties. The prototype system takes three phases for decision support, i.e.,
(i) the rule generation phase for knowing the general tendency of data sets,
(ii) the aggregation phase for decision support from the obtained rules,
(iii) the aggregation phase for decision support from data sets.
It is possible to employ (ii), if user's condition matches the condition in the obtained rules. Otherwise, it is necessary to employ (iii). The prototype system is applied to the Car Evaluation data set (a table data set without uncertainty) and the Congressional Voting data set (a table data set with uncertainty) in UCI machine learning repository. Since this prototype is implemented in SQL procedure, it will easily be applicable to any table data set on PC with SQL.

1 Introduction The data mining techniques afford to survey the instances in table data sets, and we can know the tendency and the property of data sets. Rule based decision support connected with such data mining techniques seems to be a very active research area now. Actually, we obtain more than 7700 papers for the keywords 'rule based decision support' in Scopus, whose composition ratio is 35% for computer science, 24% for engineering, 13% for medicine, 11% for mathematics, 5% for decision science, 5% for social science, 4% for business and management, 3% for biological science, etc. In these papers, fuzzy sets and rough sets seem very important. Some fuzzy frameworks are proposed in [6, 18], and the rough sets based framework named *Dominance based Rough Set Approach* (DRSA) is proposed in [4]. The authors in this paper also employ the rough sets and fuzzy sets based frameworks. The first and the fourth authors cope with rule generation, which they name *Rough Non-deterministic Information Analysis* (RNIA) [11, 12]. The second and the third authors cope with fuzzy sets and DRSA [15, 16]. This paper focuses on rule based decision support and its execution environment in SQL.

Even though there are a lot of frameworks on rule based decision support, our framework of RNIA preserves the logical aspect. Namely, the core rule generation algorithm named *NIS-Apriori* [12] is *sound* and *complete* for the rules based on the possible world semantics [13]. Therefore, the NIS-Apriori algorithm does not miss any rule for decision support. Generally, the number of the possible worlds becomes very huge, for example there are

2000 *Mathematics Subject Classification.* Primary 68T37, 03BXX; Secondary 68T15, 68P15.
Key words and phrases. decision support, association rules, NIS-Apriori algorithm, prototype in SQL, Uncertainty.

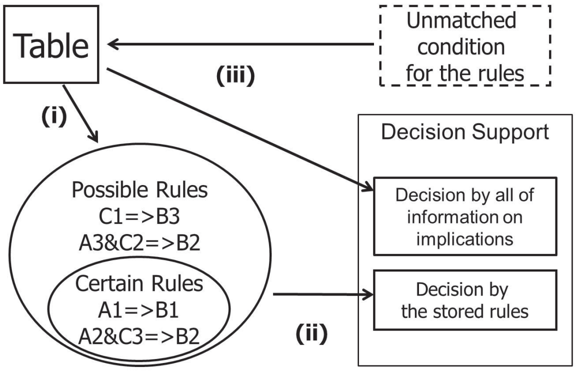


Figure 1: A chart of three phases for decision support environment in table data sets with uncertainty.

more than 10^{100} possible tables in the Mammographic data set in UCI machine learning repository [2]. Even though the definition of certain rules and possible rules is natural, it seemed hard to realize a rule generator for them. However, the NIS-Apriori algorithm affords a solution to this problem, namely this algorithm is independent from the number of the possible worlds [11, 12]. Without such property, it will be hard to address rules defined by the possible world semantics.

The main issue in this paper is to propose three phases (i), (ii), and (iii) in Figure 1.
(i) *The rule generation phase*: For two threshold values α and β, the prototype system generates rules. We will know the tendency and the character of data sets. This phase handling certain rules and possible rules based on the possible world semantics is first realized by the NIS-Apriori algorithm.
(ii) *The search phase for the obtained rules*: For the users' specified condition part $\wedge_i Con_i$, the obtained rules $\tau_k : \wedge_j Con_j \Rightarrow Dec_k$ ($\{Con_j\} \subseteq \{Con_i\}$) are examined, and triplets (Dec_k, $support(\tau_k)$, $accuracy(\tau_k)$) are generated. Users decide one decision Dec_k from the generated triplets by using $support(\tau_k)$ and $accuracy(\tau_k)$ ($support(\tau_k)$ and $accuracy(\tau_k)$ are given in the subsequent section).
(iii) *The search phase for the data set*: If there is no rule with the same condition part, all implications with the specified condition part are searched in the data set. The prototype system similarly generates triplets (Dec_k, $support(\tau_k)$, $accuracy(\tau_k)$), and users decide one decision Dec_k.

Remark 1 *In decision support, we see that the validity of the implication τ_k is measured by two values $support(\tau_k)$ and $accuracy(\tau_k)$. So, our environment tries to afford all of information about implications $\tau_k : \wedge_i Con_i \Rightarrow Dec_k$, i.e., $support(\tau_k)$ and $accuracy(\tau_k)$. We do not strongly touch about what is the final decision, which should be fixed by users.*

Remark 2 *If the phases (ii) is applicable to the specified condition part, the execution is much faster than the execution in the phase (iii). So, the application of the phase (ii) will be useful, however there may not be any rule matching the specified condition part. Thus, it is necessary to prepare the phase (iii). Even though the phase (iii) may take much execution time, this phase responds all implications with the specified condition part.*

Remark 3 *Let us consider the following three cases in Figure 1.*
(1) Let us suppose we need to have one decision under the condition $A1$*. Then, we employ the implication* $\tau : A1 \Rightarrow B1$ *(certain rule, reliable), and have the decision* $B1$*. The validity of* $B1$ *depends upon the validity of* τ*. This is an example of the phase (ii).*
(2) Let us suppose we need to have one decision under the condition $A1\&C3$*. Then, there is no rule with the condition* $A1\&C3$*. However, we have the following equation,*

$$(A1 \wedge C3 \Rightarrow Dec) = (\neg(A1 \wedge C3) \vee Dec) = (\neg A1 \vee \neg C3 \vee Dec) = ((\neg A1 \vee Dec) \vee (\neg C3 \vee Dec)) = ((A1 \Rightarrow Dec) \vee (C3 \Rightarrow Dec)).$$

Since we can conclude $A1 \wedge C3 \Rightarrow B1$ *from* $A1 \Rightarrow B1$*, we will have the decision* $B1$*. We usually say that* $A1 \wedge C3 \Rightarrow B1$ *is a redundant implication for* $A1 \Rightarrow B1$*. This is also an example of the phase (ii).*
(3) Since the phase (i) takes much execution time, we should not employ the phase (i) frequently. For the Chess data set (3196 instances, 36 attributes) in UCI machine learning repository [2], we obtained 6 rules for support ≥ 0.25 *and accuracy* ≥ 0.6 *by the implemented procedure apri, but it took more than 1 hour. So, in the phase (i), we preliminary employ the weak condition for rule generation, i.e., we employ the lower values of* α *and* β*. Even though we may have a large number of rules, the phase (ii) is effectively applied.*

This paper is organized as follows: Section 2 describes rule based decision support in table data sets without uncertainty and that in table data sets with uncertainty. Section 3 investigates some procedures in SQL, and Section 4 concludes this paper.

2 Rule Based Decision Support in Table Data Sets

This section focuses on decision support in table data sets without uncertainty and decision support in table data sets with uncertainty.

2.1 Rules from the Table Data Sets without Uncertainty

In order to consider rules from table data sets without uncertainty, we employ the Car Evaluation data set in UCI machine learning repository [2].

```
mysql> select * from `table 1` where object<5;
+--------+--------+-------+-------+---------+---------+--------+---------------+
| object | buying | maint | doors | persons | lugboot | safety | acceptability |
+--------+--------+-------+-------+---------+---------+--------+---------------+
|      1 | vhigh  | vhigh | 2     | 2       | small   | low    | unacc         |
|      2 | vhigh  | vhigh | 2     | 2       | small   | med    | unacc         |
|      3 | vhigh  | vhigh | 2     | 2       | small   | high   | unacc         |
|      4 | vhigh  | vhigh | 2     | 2       | med     | low    | unacc         |
+--------+--------+-------+-------+---------+---------+--------+---------------+
4 rows in set (0.02 sec)

mysql> select * from `table 1` where object<240 and acceptability='acc';
+--------+--------+-------+-------+---------+---------+--------+---------------+
| object | buying | maint | doors | persons | lugboot | safety | acceptability |
+--------+--------+-------+-------+---------+---------+--------+---------------+
|    228 | vhigh  | med   | 2     | 4       | small   | high   | acc           |
|    231 | vhigh  | med   | 2     | 4       | med     | high   | acc           |
|    233 | vhigh  | med   | 2     | 4       | big     | med    | acc           |
|    234 | vhigh  | med   | 2     | 4       | big     | high   | acc           |
+--------+--------+-------+-------+---------+---------+--------+---------------+
4 rows in set (0.00 sec)
```

Figure 2: Some parts of the Car Evaluation data set.

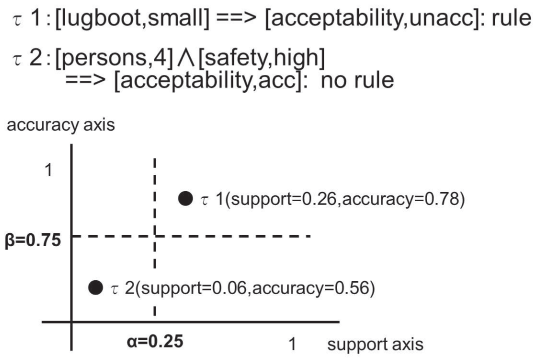


Figure 3: Rules plotted in the plane by the condition support≥0.25 and accuracy≥0.75.

This table data set consists of 1728 *objects* (instances), 6 *attributes*: *buying*, *maint(enance)*, *doors*, *persons*, *lugboot*, *safety*, 3 or 4 *attribute values* for each attribute, one decision attribute *acceptability* with 4 attribute values, *unacc*, *acc*, *good*, *vgood* in Figure 2. Each attribute value can be seen as a categorized value, and it may be hard to consider means nor variance in statistics. In such table data sets, we consider rule based decision support.

A pair $[A, val_A]$ of an attribute A and its attribute value val_A is called a *descriptor*. For a decision attribute *Dec* and a set *CON* of the attributes, we see an implication $\tau : \wedge_{A \in CON}[A, val_A] \Rightarrow [Dec, val]$ is (a candidate of) a *rule*, if τ satisfies the next two criterion values [10].

(1) For two threshold values $0 < \alpha, \beta \leq 1.0$,
$support(\tau)(= N(\wedge_{A \in CON}[A, val_A] \wedge [Dec, val])/|OB|) \geq \alpha$,
$accuracy(\tau)(= N(\wedge_{A \in CON}[A, val_A] \wedge [Dec, val])/N(\wedge_{A \in CON}[A, val_A])) \geq \beta$,
Here, $N(*)$ means the number of the objects satisfying the formula $*$, and OB means a set of all objects. We define $support(\tau) = accuracy(\tau) = 0$, if $N(\wedge_{A \in CON}[A, val_A]) = 0$.

For an implication $\tau_1 : [lugboot, small] \Rightarrow [acceptability, unacc]$ in Figure 3,

$$(2) \quad N(\tau_1) = 450,\ N([lugboot, small]) = 576,\quad support(\tau_1) = 450/1728 \fallingdotseq 0.26,\ accuracy(\tau_1) = 450/576 \fallingdotseq 0.78.$$

Similarly, for an implication $\tau_2 : [persons, 4] \wedge [safety, high] \Rightarrow [acceptability, acc]$,

$$(3) \quad N(\tau_2) = 108,\ N([persons, 4] \wedge [safety, high]) = 192,\quad support(\tau_2) = 108/1728 \fallingdotseq 0.06,\ accuracy(\tau_2) = 108/192 \fallingdotseq 0.56.$$

The $support(\tau)$ value means the occurrence ratio of the implication τ. If τ occurs much more time, this τ is much more reliable. On the other hand, the $accuracy(\tau)$ value means the consistency ratio of the implication τ. If the $accuracy(\tau)$ value is higher, this τ is more reliable.

In Figure 3, we see τ_1 and τ_2 are located in the points ($support(\tau)$,$accuracy(\tau)$) by the support and the accuracy axises. We usually fix two threshold values α and β for defining rules in each table data set. In Figure 3, we give α=0.25 and β=0.75, and we see τ_1 is a rule, and τ_2 is not a rule.

2.2 Decision Support in Table Data Sets without Uncertainty If we need to have a decision for the condition $[lugboot, small]$ in the Car Evaluation data set, we make use of the rule τ_1 and have a triplet $([acceptability, unacc], support = 0.26, accuracy = 0.78)$. Thus, we will conclude this car is unacceptable. This inference takes the phases (i) and (ii) in Figure 1.

On the other hand, we consider the condition $[lugboot, medium]$. In this case, we do not have any rule matching this condition and take the phase (iii) in Figure 1. Actually, we have Figure 4 for the condition $[lugboot, medium]$. Probably, we will conclude that this car is also *unacc(eptable)* due to the third implication in Figure 4. In Figure 4, the implemented command $srdf\_con1$ searches the Car Evaluation data set, and it took 0.33 (sec).

```
mysql> call srdf_con1('acceptability',1728,'lugboot','med');
Query OK, 0 rows affected (0.33 sec)

mysql> select * from srdf_con1;
+---------+------+---------------+------------+---------+----------+
| att1    | val1 | deci          | deci_value | support | accuracy |
+---------+------+---------------+------------+---------+----------+
| lugboot | med  | acceptability | acc        |   0.078 |    0.234 |
| lugboot | med  | acceptability | good       |   0.014 |    0.042 |
| lugboot | med  | acceptability | unacc      |   0.227 |    0.681 |
| lugboot | med  | acceptability | vgood      |   0.014 |    0.043 |
+---------+------+---------------+------------+---------+----------+
4 rows in set (0.00 sec)
```

Figure 4: All possible implications with the condition $[lugboot, med]$.

Like this, the prototype system responds all of information w.r.t. $\tau_k : \wedge_{A \in CON}[A, val_A] \Rightarrow [Dec, val_k]$.

2.3 Rules from the Table Data Sets with Uncertainty In order to consider rules from table data sets with uncertainty, we employ the Congressional Voting data set in UCI machine learning repository [2].

```
mysql> select a1,a2,a3,a4,a5,a6,a7,a12,a16,a17 from `table 1` where object < 6;
+-----+----+----+----+----+----+----+-----+-----+-----+
| a1  | a2 | a3 | a4 | a5 | a6 | a7 | a12 | a16 | a17 |
+-----+----+----+----+----+----+----+-----+-----+-----+
| rep | n  | y  | n  | y  | y  | y  | ?   | n   | y   |
| rep | n  | y  | n  | y  | y  | y  | n   | n   | ?   |
| dem | ?  | y  | y  | ?  | y  | y  | y   | n   | n   |
| dem | n  | y  | y  | n  | ?  | y  | y   | n   | y   |
| dem | y  | y  | y  | n  | y  | y  | y   | y   | y   |
+-----+----+----+----+----+----+----+-----+-----+-----+
5 rows in set (0.00 sec)
```

Figure 5: Some parts of the Congressional Voting data set.

This table data set consists of 435 objects (instances), 16 attributes: a_2, a_3, $\cdots$, a_{17}, two attribute values *y(es)* or *n(o)* for each attribute, one decision attribute a_1 with two attribute values, *rep(ublic)* or *dem(octat)* in Figure 5. In the Congressional Voting data set, there are 329 missing values expressed by the ? symbol. Of course, rules depend upon the missing values, and it is necessary for handling rules in such table data sets [7, 8, 9]. We have dealt with this problem in RNIA.

We briefly review RNIA. In a table with missing values, we usually apply the discretization procedure, and we handle a finite number of the possible values. By replacing each ? symbol with a possible value, we have a table data set without uncertainty, which we name a *derived DIS* (*DIS: Deterministic Information System*). Let $DD(\Phi)$ denote the set of all derived DISs from Φ with missing values, and we may say Φ is a *NIS: Non-deterministic Information System*. In rule generation, we employ the usual definition of a rule in DIS [10], and extend it to a certain rule and a possible rule in NIS below [11, 12]:

(A certain rule in NIS) An implication τ is a *certain rule*, if τ is a rule in each derived DIS for given α and β.
(A possible rule in NIS) An implication τ is a *possible rule*, if τ is a rule in at least one derived DIS for given α and β.

If τ is a certain rule, we can conclude τ is also a rule in the unknown actual DIS ψ^{actual}. (We see there is one derived DIS $\psi^{actual} \in DD(\Phi)$ which contains the actual values.) This property is also described in Lipski's incomplete information databases [5]. In DIS, the same set of rules are obtained by two definitions, so two definitions will be a natural extension from rules in DIS. However, the number of $DD(\Phi)$ increases exponentially, and there are more than 10^{100} derived DISs for the Congressional Voting data set. It will be hard to examine the certain rules and the possible rules by checking each derived DIS sequentially. For this problem, we afford a solution by showing some properties on rules [11, 12].

(4)

(Property 1) For NIS Φ and any implication τ, there is a derived DIS $\psi_{min} \in DD(\Phi)$ such that
$minsupp(\tau)$(defined by $support(\tau)$ in ψ_{min}) $= \min_{\psi \in DD(\Phi)}\{support(\tau) \text{ in } \psi\}$,
$minacc(\tau)$(defined by $accuracy(\tau)$ in ψ_{min}) $= \min_{\psi \in DD(\Phi)}\{accuracy(\tau) \text{ in } \psi\}$.

(Property 2) For NIS Φ and any implication τ, there is a derived DIS $\psi_{max} \in DD(\Phi)$ such that
$maxsupp(\tau)$(defined by $support(\tau)$ in ψ_{max}) $= \max_{\psi \in DD(\Phi)}\{support(\tau) \text{ in } \psi\}$,
$maxacc(\tau)$(defined by $accuracy(\tau)$ in ψ_{max}) $= \max_{\psi \in DD(\Phi)}\{accuracy(\tau) \text{ in } \psi\}$.

(Property 3) There is a calculation method of $support(\tau)$ and $accuracy(\tau)$, and this method is independent from the number of $DD(\Phi)$. The details are in [12].

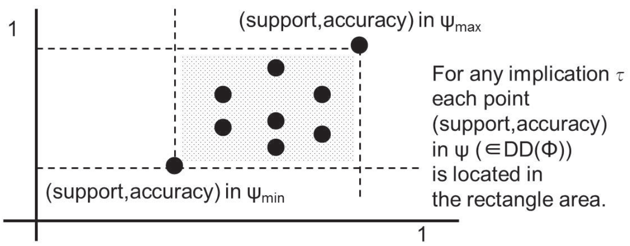


Figure 6: Each point for an implication τ is located in the rectangle area.

```
mysql> select * from c1_rule where att1>'a2' and att1<'a7';
+------+------+------+------------+---------+--------+
| att1 | val1 | deci | deci_value | minsupp | minacc |
+------+------+------+------------+---------+--------+
| a4   | n    | a1   | rep        |   0.326 |  0.798 |
| a4   | y    | a1   | dem        |   0.531 |  0.899 |
| a5   | n    | a1   | dem        |   0.563 |  0.980 |
| a5   | y    | a1   | rep        |   0.375 |  0.881 |
| a6   | n    | a1   | dem        |   0.460 |  0.948 |
| a6   | y    | a1   | rep        |   0.361 |  0.701 |
+------+------+------+------------+---------+--------+
6 rows in set (0.00 sec)
```

Figure 7: A part of the obtained certain rules satisfying $support(\tau) \geq 0.3$ and $accuracy(\tau) \geq 0.6$ in the Congressional Voting data set.

Based on the above properties, we have the chart in Figure 6. In Figure 3, the point ($support(\tau)$,$accuracy(\tau)$) in DIS is unique, but each point in $\psi \in DD(\Phi)$ is located in the rectangle area in Figure 6. There are more than 10^{100} points in the rectangle area, however we can have two points by ψ_{min} and ψ_{max} independently from the number of $DD(\Phi)$. Furthermore, we have the next properties for the certain rules and the possible rules [11, 12].

(5)

(Property 4) For NIS Φ and any implication τ, τ is a certain rule if and only if $minsupp(\tau) \geq \alpha$ and $minacc(\tau) \geq \beta$.

(Property 5) For NIS Φ and any implication τ, τ is a possible rule if and only if $maxsuppt(\tau) \geq \alpha$ and $maxacc(\tau) \geq \beta$.

We added the above two properties to the *Apriori* algorithm [1], which is the representative algorithm in data mining, and proposed the *NIS-Apriori* algorithm [11, 12]. We refer to the prototype system in SQL powered by the NIS-Apriori algorithm in the next section.

2.4 Decision Support in Table Data Sets with Uncertainty In the Congressional Voting data set, we had 22 certain rules (with one descriptor in the condition part) for α=0.3 and β=0.6 in Figure 7. They satisfy $support(\tau) \geq 0.3$ and $accuracy(\tau) \geq 0.6$ in each of more than 10^{100} derived DISs. Especially, two certain rules $[a5, n] \Rightarrow [a1, dem(ocrat)]$ and $[a5, y] \Rightarrow [a1, rep(ublic)]$ are very strong. If we have a person's answer to the attribute $a5$, we will easily conclude his supporting party. This inference takes the phases (i) and (ii) in Figure 1. We also had 26 possible rules (with one descriptor in the condition part) and one possible rule (with two descriptors in the condition part) in Figure 8. If the condition does not match any certain rule, we may apply possible rules. Furthermore, if the condition does not much any rule, we have the phase (iii) in Figure 1.

For the implications $\tau : \wedge_{A \in CON}[A, val_A] \Rightarrow [Dec, val]$ and $\tau' : \wedge_{A \in CON}[A, val_A] \Rightarrow [Dec, val']$, if $maxsupp(\tau) \leq minsupp(\tau')$ and $maxacc(\tau) \leq minsupp(\tau')$ hold, we have $support(\tau) \leq support(\tau')$ and $accuracy(\tau) \leq accuracy(\tau')$ for any DIS $\psi \in DD(\Phi)$ (Figure 9). So, we will certainly have the decision $[Dec, val']$ under the table data set with uncertainty. The concept in Figure 9 will be the extension from the concepts in Figure 3 and Figure 6.

3 Rule Based Decision Support System in SQL This section describes each phase in the prototype system. Each program is implemented in the SQL procedure.

```
mysql> select * from p2_rule;
+------------+------+------+------+------+------------+---------+--------+
| att1       | val1 | att2 | val2 | deci | deci_value | maxsupp | maxacc |
+------------+------+------+------+------+------------+---------+--------+
| a12        | n    | a7   | y    | a1   | rep        |   0.301 |  0.753 |
| end_attrib | NULL | NULL | NULL | NULL | NULL       |    NULL |   NULL |
+------------+------+------+------+------+------------+---------+--------+
2 rows in set (0.00 sec)
```

Figure 8: One possible rule with two descriptors in the condition part.

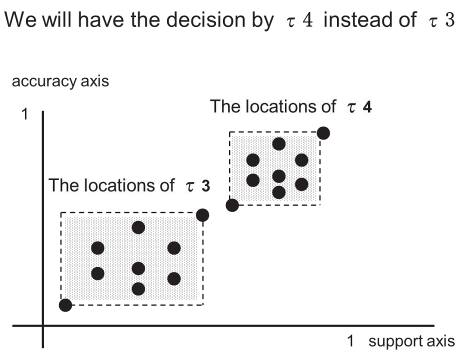


Figure 9: The locations of the implications plotted in the plane.

```
mysql> call car_rdf;
Query OK, 0 rows affected (1.58 sec)

mysql> select * from rdf where object=2;
+--------+---------------+-------+
| object | attrib        | value |
+--------+---------------+-------+
|      2 | acceptability | unacc |
|      2 | buying        | vhigh |
|      2 | doors         | 2     |
|      2 | lugboot       | small |
|      2 | maint         | vhigh |
|      2 | persons       | 2     |
|      2 | safety        | med   |
+--------+---------------+-------+
7 rows in set (0.00 sec)
```

Figure 10: The execution of car_rdf command and the generated rdf file from the Car Evaluation data set.

3.1 The Rule Generation Phase (i) in Figure 1: The Case of DISs In table data sets without uncertainty, we at first translate each csv file to the rdf format [17], and employ the Apriori algorithm for rule generation. In DISs, we implemented the following procedures in SQL.
(1) The procedure `File_name_rdf`: It translates a csv file to the rdf format file. (In Figure 10, `car_rdf` is executed.)

(2) The procedure `apri`: It generates tables rule1 (rules with one condition), rule2 (rules with two conditions), rule3 (rules with three conditions). (For the constraint $support \geq 0.25$ and $accuracy \geq 0.7$, the procedure `apri` generated three tables in 9.99 (sec) for the Car Evaluation data set, whose execution logs are in [14].)

In the rdf format, each table data is translated to a table of descriptors. In each table data set, the number of attributes and its attribute values are different, but we can uniformly handle any data set if the data set is in the rdf format. Without this property, we need to make a set of the SQL procedures for each table data set.

3.2 The Search Phase (ii) and (iii) in Figure 1: The Case of DISs Let us consider the case that we need to have a decision for a given condition. The procedures `srule_con1`, `srule_con2`, and `srule_con3` are implemented for searching lots of rules stored in tables. They are the commands for the phase (ii) in Figure 1. Figure 11 shows the execution of `srule_con1`.

```
mysql> call srule_con1('acceptability','persons','2');
Query OK, 1 row affected (0.14 sec)

mysql> select * from srule_con1;
+---------+------+---------------+-------+---------+----------+
| att1    | val1 | deci          | val   | support | accuracy |
+---------+------+---------------+-------+---------+----------+
| persons | 2    | acceptability | -     | 999.000 |  999.000 |
| persons | 2    | acceptability | unacc |   0.333 |    1.000 |
+---------+------+---------------+-------+---------+----------+
2 rows in set (0.00 sec)
```

Figure 11: The all searched rules from obtained rules for the condition $[persons, 2]$. The first line means the query and the number 999 is meaningless value. The second line is picked up from the obtained rules.

Based on Figure 11, we know all kind of information for the condition $[persons, 2]$. This search is restricted to the obtained table data, so it takes less execution time. However, if the condition does not match the obtained rules, we have no information for the condition. In order to handle such case, we consider the phase (iii) in Figure 1. Figure 4 shows the execution about the condition $[lugboot, medium]$. Even though this condition is not in the obtained rules, we will have a decision *unacc(eptable)* from Figure 4. This will be useful for decision support.

3.3 The Rule Generation Phase (i) in Figure 1: The Case of NISs In table data sets with uncertainty, we at first translate each csv file to the nrdf format [17], and employ the NIS-Apriori algorithm for rule generation. In NISs, we implemented the following procedures in SQL.

(1) The procedure `File_name_nrdf`: It translates the csv file with ? symbol and non-deterministic values to the nrdf format file.

(2) The procedure `step1`: It generates tables c1_rule (certain rules with one condition) and p1_rule (possible rules with one condition).

(3) The procedures `step2`, `step3`: They generate tables c2_rule (certain rules with two conditions), p2_rule (possible rules with two conditions), c3_rule (certain rules with three conditions), and p3_rule (possible rules with three conditions).

The execution logs of the Congressional Voting data set are in [14].

3.4 The Search Phase (ii) in Figure 1 for the Obtained Rules: The Case of NISs Let us consider the case that we need to have a decision for a given condition. The procedures `srule_con1`, `srule_con2`, and `srule_con3` are implemented for searching lots of rules stored in tables. Figure 12 shows the execution of `srule_con2`.

```
mysql> call snrule_con2('a1','a5','y','a9','n');
Query OK, 0 rows affected (0.25 sec)

mysql> select * from snrule_con2;
+-----------+------+------+------+------+------+------+---------+---------+---------+---------+
| type      | att1 | val1 | att2 | val2 | deci | val  | minsupp | minacc  | maxsupp | maxacc  |
+-----------+------+------+------+------+------+------+---------+---------+---------+---------+
| Condition | a5   | y    | a9   | n    | a1   | -    | 999.000 | 999.000 | 999.000 | 999.000 |
| Certain   | a5   | y    | NULL | NULL | a1   | rep  |   0.375 |   0.881 | 999.000 | 999.000 |
| Certain   | a9   | n    | NULL | NULL | a1   | rep  |   0.306 |   0.731 | 999.000 | 999.000 |
| Possible  | a5   | y    | NULL | NULL | a1   | rep  | 999.000 | 999.000 |   0.382 |   0.922 |
| Possible  | a9   | n    | NULL | NULL | a1   | rep  | 999.000 | 999.000 |   0.331 |   0.762 |
+-----------+------+------+------+------+------+------+---------+---------+---------+---------+
5 rows in set (0.00 sec)
```

Figure 12: The all searched rules from obtained rules for the condition $[a5, y] \wedge [a9, n]$. The number 999 is meaningless value.

Based on Figure 12, we know all of information for the condition $[a5, y] \wedge [a9, n]$. The implication $[a5, y] \wedge [a9, n] \Rightarrow [a1, rep]$ is redundant for two certain rules $[a5, y] \Rightarrow [a1, rep]$ and $[a9, n] \Rightarrow [a1, rep]$. In both cases, $[a5, y]$ and $[a9, n]$ conclude $[a1, rep]$. We will probably have the decision value *rep(ublic)* in Figure 12. This search is restricted to the obtained table data, so it takes less execution time. However, if the condition does not match the obtained rules, we have no information for the condition.

3.5 The Search Phase (iii) in Figure 1 for Data Sets: The Case of NISs Let us consider the case that we need to have a decision for a given condition. The procedures `snrdf_con1`, `snrdf_con2`, and `snrdf_con3` are implemented for searching tables with uncertainty. In this case, we employ the same condition $[a5, y] \wedge [a9, n]$ in Figure 12. Figure 13 shows the execution of `snrdf_con2`.

```
mysql> call snrdf_con2('a1',435,'a5','y','a9','n');
Query OK, 0 rows affected (4.94 sec)

mysql> select * from snrdf_con2;
+------+------+------+------+------+------+------+---------+---------+--------+--------+
| pkey | att1 | val1 | att2 | val2 | deci | val  | minsupp | maxsupp | minacc | maxacc |
+------+------+------+------+------+------+------+---------+---------+--------+--------+
|    1 | a5   | y    | a9   | n    | a1   | dem  |   0.025 |   0.032 |  0.071 |  0.096 |
|    2 | a5   | y    | a9   | n    | a1   | rep  |   0.303 |   0.329 |  0.904 |  0.929 |
+------+------+------+------+------+------+------+---------+---------+--------+--------+
2 rows in set (0.00 sec)
```

Figure 13: The all searched rules with the condition part $[a5, y] \wedge [a9, n]$ for the Congressional Voting data set.

Based on Figure 13, we know all of information for the condition $[a5, y] \wedge [a9, n]$. In this case, the procedure `snrdf_con2` searches the table nrdf, and it took 4.94 (sec). The execution time is about 20 times longer than that of `snrule_con2`. For two implications τ : $[a5, y] \wedge [a9, n] \Rightarrow [a1, dem]$ and τ' : $[a5, y] \wedge [a9, n] \Rightarrow [a1, rep]$, $maxsupp(\tau) \leq minsupp(\tau')$

and $maxacc(\tau) \leq minacc(\tau')$ hold. This is corresponding to the case in Figure 9, and we will easily have the decision value *rep(ublic)*.

3.6 The Validity of the Implementation We have previously implemented the NIS-Apriori algorithm in C and Prolog. This time, we employed SQL, because it will be difficult to use Prolog for the large size data sets. So, we had two independent systems, and we had the same results by the two systems. The execution logs are in [14].

4 Concluding Remarks and Discussion This paper clarified rule based decision support on RNIA, and reported its prototype system. The definition of the certain rules and the possible rules seems natural, however there is less software tool for handling them, because the rules are defined by all derived DISs whose number may exceed 10^{100}. Without effective property, it will be hard to obtain rules. The NIS-Apriori algorithm affords a solution to this problem, and we implemented the prototype by NIS-Apriori in SQL. This algorithm takes the core part for handling the uncertainty, and we applied it to decision support environment.

Now, let us consider each phase of (i), (ii), and (iii). The phase (i) generates all certain rules and possible rules, which have the characteristic properties. However, it is time-consuming, so the frequent usage of the phase (i) will not be appropriate, and we need to employ the lower values of α and β. In this situation, we need the phase (ii) much more. If we have the large number of rules, the method to find the rules matching the condition may not be easy, and we realized some procedures in the phase (ii). The phase (iii) will be necessary to cope with the case that any rule does not match the condition. In table data sets, the implications are located in the plane like Figure 3. On the other hand in the tables with uncertainty, the implications are located in the plane like Figure 6 and Figure 9. The extension from Figure 3 to Figure 6 and Figure 9 is the key concept for considering decision support for the tables with uncertainty.

However, there may be the cases like Figure 14 and Figure 15, where it is difficult to have a decision even by using the phase (iii). In such cases, we will need other criteria like the type I error and the type II error in the statistical hypothesis tests instead of the support and accuracy values. Furthermore, it is important to have the theoretical property of the distribution of points (implications) with the same conditions and the different decision. Even though we consider that Figure 14 and Figure 15 express the rare cases, the next new challenges are open for them.

Acknowledgment: The authors would be grateful to the anonymous referees and Professor Junzo Watada (Waseda University and Universiti Teknologi PETRONAS, Malaysia) for their useful comments. This work is supported by JSPS (Japan Society for the Promotion of Science) KAKENHI Grant Number 26330277.

References

[1] Agrawal, R., Srikant, R.: Fast algorithms for mining association rules in large databases. Proc. VLDB'94, Morgan Kaufmann, 487–499 (1994)

[2] Frank, A., Asuncion, A.: UCI machine learning repository. Irvine, CA: University of California, School of Information and Computer Science (2010) `http://mlearn.ics.uci.edu/MLRepository.html`

[3] Grzymała-Busse, J.: Data with missing attribute values: Generalization of indiscernibility relation and rule induction. Transactions on Rough Sets 1, 78–95 (2004)

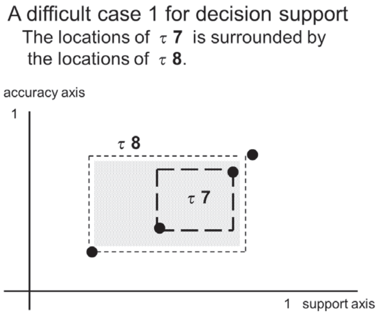


Figure 14: A difficult case 1 for having one decision from the implications with the same conditions and the different decision.

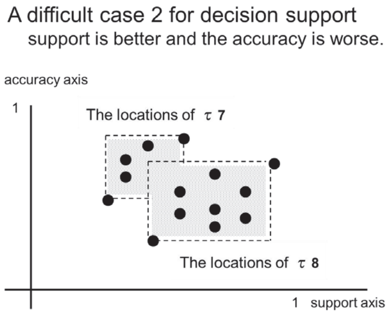


Figure 15: A difficult case 2 for having one decision from the implications with the same conditions and the different decision.

[4] Kadziński, M., Słowiński, R., Szeląg, M.: Dominance-based rough set approach to multiple criteria ranking with sorting-specific preference information. Studies in Computational Intelligence 606, 155-171 (2016)

[5] Lipski, W.: On semantic issues connected with incomplete information databases. ACM Transactions on Database Systems 4(3), 262–296 (1979)

[6] Minutolo, A., Esposito, M., De Pietro, G.: A fuzzy framework for encoding uncertainty in clinical decision-making. Knowledge-Based Systems 98, 95–116 (2016)

[7] Nakata, M., Sakai, H.: Twofold rough approximations under incomplete information. Int'l. J. General Systems 42(6), 546–571 (2013)

[8] Orłowska, E., Pawlak, Z.: Representation of nondeterministic information. Theoretical Computer Science 29(1-2), 27–39 (1984)

[9] Pawlak, Z.: Systemy Informacyjne: Podstawy Teoretyczne (in Polish) WNT (1983)

[10] Pawlak, Z.: Rough Sets: Theoretical aspects of reasoning about data. Kluwer Academic Publishers (1991)

[11] Sakai, H., et al.: Rules and apriori algorithm in non-deterministic information systems. Transactions on Rough Sets 9, 328–350 (2008)

[12] Sakai, H., Wu, M., Nakata, M.: Apriori-based rule generation in incomplete information databases and non-deterministic information systems. Fundamenta Informaticae 130(3), 343–376 (2014)

[13] Sakai, H., Wu, M.: The completeness of NIS-Apriori algorithm and a software tool getRNIA. In: Proc. Int'l. Conf. on AAI2014, IEEE, 115–121 (2014).

[14] Sakai, H.: Software Tools for RNIA (Rough Non-deterministic Information Analysis) Web Page (2016) http://www.mns.kyutech.ac.jp/~sakai/RNIA/

[15] Shen, K.Y., Tzeng, G.H.: Contextual improvement planning by fuzzy-rough machine learning: A novel bipolar approach for business analytics. International Journal of Fuzzy Systems 18(6), 940–955 (2016)

[16] Shen, K.Y., Tzeng, G.H.: A novel bipolar MCDM model using rough sets and three-way decisions for decision aids. In: Proc. SCIS-ISIS, IEEE, 53–58 (2016)

[17] Ślęzak, D., Sakai, H.: Automatic extraction of decision rules from non-deterministic data systems: Theoretical foundations and SQL-based implementation. DTA2009 Springer CCIS Vol.64, 151–162 (2009)

[18] Zarikas, V., Papageorgiou, E., Regner, P.: Bayesian network construction using a fuzzy rule based approach for medical decision support. Expert Systems 32(3), 344-369 (2016)

STRUCTURE STUDY OF SYMMETRIC FUZZY NUMBERS

Rosy Joseph[1] and Dhanalakshmi V[2]

[1]Associate Professor, Stella Maris College(Autonomous), Chennai
rosyjosef@gmail.com
[2]Assistant Professor, Stella Maris College(Autonomous), Chennai

Received September 27, 2017

Abstract. In many practical situations, intervals or fuzzy numbers are used to model imprecise observations derived from uncertain measurements or linguistic assessments. When using fuzzy numbers the shape of the membership function is important in modelling. In this paper, we consider the fuzzy numbers whose membership function is symmetric with respect to a vertical axis. For $\alpha \in (0,1]$ the $\alpha-$ cuts of such fuzzy numbers will have a constant mid-point and the upper end of the interval will be a non-increasing function of α, the lower end will be the image of this function. Hence these symmetric fuzzy numbers can be fully described by a constant and a non-increasing function. Based on this description, we define the arithmetic operations and a ranking technique to order the symmetric fuzzy numbers. We also discuss various properties of interest. Using Radstorm embedding theorem[5], we conduct a structure study on symmetric fuzzy numbers.

1 Introduction The operations on the set of fuzzy numbers are usually obtained by the Zadeh extension principle [7], [8], [6]. These definitions can have some disadvantages for the applications, both by an algebraic point of view and by logical and practical aspects. In particular, the shape of fuzzy numbers is not preserved by multiplication, the indeterminateness of the sum and product is often too increasing.

Dong Qiu et.al. [1] studied the algebraic properties of fuzzy numbers using equivalence classes on fuzzy numbers and identified the group structure for addition. In this paper, we are studying a special class of fuzzy numbers, namely the symmetric fuzzy numbers, whose membership function is symmetric with respect to a vertical axis, define various arithmetic operations anew to suit our need. Also applying Radstorm embedding theorem[5] we are identifying the vector space structure. We define the arithmetic operations, such as addition, subtraction, scalar multiplication, product, inverse on symmetric fuzzy numbers in a way that the resultants are also symmetric fuzzy numbers.

Section 2 introduces symmetric fuzzy numbers, the arithmetic operations and the ranking technique on them. We also verify various properties of the arithmetic operations in this section. Based on the properties verified, section 3 gives an embedding of the class of symmetric fuzzy numbers into a collection of equivalence classes of symmetric fuzzy numbers which forms a group and a vector space.

2 Symmetric Fuzzy Numbers

Definition 2.1 The characteristic function χ_A of a crisp set $A \subseteq X$ assigns a value either 0 or 1 to each member in X. This function can be generalized to a function

$\mu_{\tilde{A}}$ such that the value assigned to the element of the universal set X fall within a specified range i.e. $\mu_{\tilde{A}} : X \to [0,1]$. The assigned value indicates the membership grade of the element in the set A. The function $\mu_{\tilde{A}}$ is called the membership function and the set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ is called a fuzzy set.

Definition 2.2 A fuzzy set $\tilde{A}$, defined on the universal set of real numbers $\mathbb{R}$, is said to be a fuzzy number if its membership function has the following characteristics:

i. $\tilde{A}$ is convex i.e. $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)) \ \forall \ x_1, x_2 \in \mathbb{R}, \ \forall \ \lambda \in [0,1]$

ii. $\tilde{A}$ is normal i.e. $\exists x_0 \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x_0) = 1$

iii. $\mu_{\tilde{A}}$ is piecewise continuous

The height of a fuzzy set $A \in \mathcal{F}(X)$, is the value $hgt(A) = \sup_{x \in X} \mu_A(x)$. From the definition of a fuzzy set it is immediate that $hgt(A) \leq 1$. If there exists $x_0 \in X$ such that $hgt(A) = \mu_A(x_0) = 1$, then the fuzzy set A is called normal.
The core of a fuzzy set $A \in \mathcal{F}(X)$ is denoted with $core(A)$ and it is given by $core(A) = \{x \in X \mid \mu_A(x) = 1\}$. The support of a fuzzy set $A \in \mathcal{F}(X)$ is denoted with $supp(A)$ and represents the set of all elements of X with a nonzero degree of membership, that is $supp(A) = \{x \in X \mid \mu_A(x) > 0\}$
For $\alpha \in [0,1]$, the α-cut of a fuzzy set $A \in \mathcal{F}(X)$ denoted by $[A]_\alpha$ and is given by $[A]_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$. It is clear that $[A]_0 = X$ and $[A]_1 = core(A)$.

Remark 2.1. For a fuzzy number, the $\alpha-$cut will be a closed interval.

Definition 2.3 Let $[\tilde{A}]_\alpha = [a_1^\alpha, a_2^\alpha]$ be the $\alpha-$cut of the fuzzy number $\tilde{A}$, then $\tilde{A}$ is said to be symmetric if the mid-point $m_\alpha(\tilde{A}) = \dfrac{a_1^\alpha + a_2^\alpha}{2}$ is constant $\forall \alpha \in [0,1]$.

Remark 2.2. The spread $S_\alpha(\tilde{A}) = \dfrac{a_2^\alpha - a_1^\alpha}{2}$ is non-negative and a non-increasing function of α. It is the factor that determines the fuzziness of the quantity measured. As a particular case, when the spread is zero, the quantity reduces to a crisp quantity.

Remark 2.3. The $\alpha-$cut $[\tilde{A}]_\alpha = [a_1^\alpha, a_2^\alpha]$ of the symmetric fuzzy numbers $\tilde{A}$ can also be represented as $[\tilde{A}]_\alpha = \dfrac{a_1^\alpha + a_2^\alpha}{2} + \dfrac{a_2^\alpha - a_1^\alpha}{2}[-1,1]$.

2.1 Ranking Technique For two symmetric fuzzy numbers $\tilde{A}$ and $\tilde{B}$ with $\alpha-$cuts $[\tilde{A}]_\alpha = [a_1^\alpha, a_2^\alpha]$ and $[\tilde{B}]_\alpha = [b_1^\alpha, b_2^\alpha]$, define $\tilde{A} \prec \tilde{B}$ if either $\dfrac{a_1^\alpha + a_2^\alpha}{2} < \dfrac{b_1^\alpha + b_2^\alpha}{2}$ or $\dfrac{a_1^\alpha + a_2^\alpha}{2} = \dfrac{b_1^\alpha + b_2^\alpha}{2}$ and $\dfrac{a_2^\alpha - a_1^\alpha}{2} < \dfrac{b_2^\alpha - b_1^\alpha}{2}$. If $\dfrac{a_1^\alpha + a_2^\alpha}{2} = \dfrac{b_1^\alpha + b_2^\alpha}{2}$ and $\dfrac{a_2^\alpha - a_1^\alpha}{2} = \dfrac{b_2^\alpha - b_1^\alpha}{2}$, then $\tilde{A} = \tilde{B}$

2.2 Arithmetic Operations

Definition 2.4 Let $\tilde{A}$ and $\tilde{B}$ be two symmetric fuzzy numbers with $\alpha-$cuts $[\tilde{A}]_\alpha = [a_1^\alpha, a_2^\alpha]$, $[\tilde{B}]_\alpha = [b_1^\alpha, b_2^\alpha]$ and $\lambda \in \mathbb{R}$, then the $\alpha-$cut of the arithmetic operations are defined as follows:

Sum

$$[\tilde{A} + \tilde{B}]_\alpha = \frac{a_1^\alpha + a_2^\alpha}{2} + \frac{b_1^\alpha + b_2^\alpha}{2} + \left(\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1] \tag{2.1}$$

Difference

$$[\tilde{A}-\tilde{B}]_\alpha = \frac{a_1^\alpha + a_2^\alpha}{2} - \frac{b_1^\alpha + b_2^\alpha}{2} + \left(\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1] \tag{2.2}$$

Scalar Multiplication

STRUCTURE STUDY OF SYMMETRIC FUZZY NUMBERS

$$[\lambda\tilde{A}]_\alpha = \lambda\frac{a_1^\alpha + a_2^\alpha}{2} + \left(|\lambda|\frac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1] \tag{2.3}$$

Product

$$\begin{aligned}[\tilde{A}.\tilde{B}]_\alpha &= \frac{a_1^\alpha + a_2^\alpha}{2}.\frac{b_1^\alpha + b_2^\alpha}{2} + \\ &\left(\frac{a_2^\alpha - a_1^\alpha}{2}\left|\frac{b_1^\alpha + b_2^\alpha}{2}\right| + \left|\frac{a_1^\alpha + a_2^\alpha}{2}\right|\frac{b_2^\alpha - b_1^\alpha}{2} + \frac{a_2^\alpha - a_1^\alpha}{2}\frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1]\end{aligned} \tag{2.4}$$

Inverse

$$\left[\frac{1}{\tilde{A}}\right]_\alpha = \frac{2}{a_1^\alpha + a_2^\alpha} + \frac{1}{2}\left(\frac{1}{a_1^\alpha} - \frac{1}{a_2^\alpha}\right)[-1,1] \tag{2.5}$$

here either $a_1^0 > 0$ or $a_2^0 < 0$

Proposition 2.1. *Let $\tilde{A}$ be a symmteric fuzzy number with $\alpha-$cut $[\tilde{A}]_\alpha = [a_1^\alpha, a_2^\alpha]$, then for $\alpha < \beta$, $[a_1^\alpha, a_2^\alpha] \supseteq [a_1^\beta, a_2^\beta]$.*

Proof. $\tilde{A}$ is a symmetric fuzzy number $\implies$ the mid-point $m_\alpha(\tilde{A})$ is constant and the spread $S_\alpha(\tilde{A})$ is a non-increasing function of α.

Thus $\alpha < \beta \implies \frac{a_1^\alpha + a_2^\alpha}{2} = \frac{a_1^\beta + a_2^\beta}{2}$ and $\frac{a_2^\alpha - a_1^\alpha}{2} \geq \frac{a_2^\beta - a_1^\beta}{2}$

$\implies [a_1^\alpha, a_2^\alpha] \supseteq [a_1^\beta, a_2^\beta]$ □

Remark 2.4. Proposition 2.1 proves that the symmetric fuzzy number is convex.

Proposition 2.2. *If $\tilde{A}$ and $\tilde{B}$ are symmetric fuzzy numbers, then so are $\tilde{A}+\tilde{B}, \tilde{A}-\tilde{B}, \lambda\tilde{A}(\lambda \in \mathbb{R}), \tilde{A}.\tilde{B}, \frac{1}{\tilde{A}}$.*

Proof. Let the $\alpha-$cuts of $\tilde{A}$ and $\tilde{B}$ be $[\tilde{A}]_\alpha = [a_1^\alpha, a_2^\alpha]$ and $[\tilde{B}]_\alpha = [b_1^\alpha, b_2^\alpha]$ respectively, then we know that for $\alpha < \alpha'$,
the mid-points

$$\frac{a_1^\alpha + a_2^\alpha}{2} = \frac{a_1^{\alpha'} + a_2^{\alpha'}}{2} \tag{2.6}$$

$$\frac{b_1^\alpha + b_2^\alpha}{2} = \frac{b_1^{\alpha'} + b_2^{\alpha'}}{2} \tag{2.7}$$

the spreads

$$\frac{a_2^\alpha - a_1^\alpha}{2} \geq \frac{a_2^{\alpha'} - a_1^{\alpha'}}{2} \tag{2.8}$$

$$\frac{b_2^\alpha - b_1^\alpha}{2} \geq \frac{b_2^{\alpha'} - b_1^{\alpha'}}{2} \tag{2.9}$$

Sum $[\tilde{A}+\tilde{B}]_\alpha = \dfrac{a_1^\alpha + a_2^\alpha}{2} + \dfrac{b_1^\alpha + b_2^\alpha}{2} + \left(\dfrac{a_2^\alpha - a_1^\alpha}{2} + \dfrac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1]$

Adding equations 2.6 and 2.7, we get the mid-point of $\tilde{A}+\tilde{B}$ to be constant and adding 2.8 and 2.9 we see the spread of $\tilde{A}+\tilde{B}$ to be non-increasing. Thus $\tilde{A}+\tilde{B}$ is symmetric.

Difference $[\tilde{A}-\tilde{B}]_\alpha = \dfrac{a_1^\alpha + a_2^\alpha}{2} - \dfrac{b_1^\alpha + b_2^\alpha}{2} + \left(\dfrac{a_2^\alpha - a_1^\alpha}{2} + \dfrac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1]$

Subtracting equations 2.6 and 2.7, we get the mid-point of $\tilde{A}-\tilde{B}$ to be constant and adding 2.8 and 2.9 we see the spread of $\tilde{A}-\tilde{B}$ to be non-increasing. Thus $\tilde{A}-\tilde{B}$ is symmetric.

Scalar Multiplication $[\lambda\tilde{A}]_\alpha = \lambda\dfrac{a_1^\alpha + a_2^\alpha}{2} + \left(|\lambda|\dfrac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1]$

By the definition, it is clear that $\lambda\tilde{A}$ is a symmetric fuzzy number.

Product $[\tilde{A}.\tilde{B}]_\alpha = \dfrac{a_1^\alpha + a_2^\alpha}{2}.\dfrac{b_1^\alpha + b_2^\alpha}{2} +$
$\left(\dfrac{a_2^\alpha - a_1^\alpha}{2}\left|\dfrac{b_1^\alpha + b_2^\alpha}{2}\right| + \left|\dfrac{a_1^\alpha + a_2^\alpha}{2}\right|\dfrac{b_2^\alpha - b_1^\alpha}{2} + \dfrac{a_2^\alpha - a_1^\alpha}{2}\dfrac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1]$

Product of two constants is also a constant $\implies$ the mid-point of $\tilde{A}.\tilde{B} = \dfrac{a_1^\alpha + a_2^\alpha}{2}.\dfrac{b_1^\alpha + b_2^\alpha}{2}$ is constant.

For $\alpha < \alpha'$,

$$\begin{aligned}
S_\alpha(\tilde{A}.\tilde{B}) &= \frac{a_2^\alpha - a_1^\alpha}{2}\left|\frac{b_1^\alpha + b_2^\alpha}{2}\right| + \left|\frac{a_1^\alpha + a_2^\alpha}{2}\right|\frac{b_2^\alpha - b_1^\alpha}{2} + \frac{a_2^\alpha - a_1^\alpha}{2}\frac{b_2^\alpha - b_1^\alpha}{2} \\
&= \frac{a_2^\alpha - a_1^\alpha}{2}\left|\frac{b_1^{\alpha'} + b_2^{\alpha'}}{2}\right| + \left|\frac{a_1^{\alpha'} + a_2^{\alpha'}}{2}\right|\frac{b_2^\alpha - b_1^\alpha}{2} + \frac{a_2^\alpha - a_1^\alpha}{2}\frac{b_2^\alpha - b_1^\alpha}{2} \\
&\qquad \text{using 2.6 and 2.7} \\
&\geq \frac{a_2^{\alpha'} - a_1^{\alpha'}}{2}\left|\frac{b_1^{\alpha'} + b_2^{\alpha'}}{2}\right| + \left|\frac{a_1^{\alpha'} + a_2^{\alpha'}}{2}\right|\frac{b_2^{\alpha'} - b_1^{\alpha'}}{2} + \frac{a_2^{\alpha'} - a_1^{\alpha'}}{2}\frac{b_2^{\alpha'} - b_1^{\alpha'}}{2} \\
&\qquad \text{using 2.8 and 2.9 and the fact that } \frac{a_2^{\alpha'} - a_1^{\alpha'}}{2}, \frac{b_2^{\alpha'} - b_1^{\alpha'}}{2} \\
&\qquad \text{are non-negative} \\
&= S_{\alpha'}(\tilde{A}.\tilde{B})
\end{aligned}$$

Thus $\tilde{A}.\tilde{B}$ is symmetric.

Inverse $\left[\dfrac{1}{\tilde{A}}\right]_\alpha = \dfrac{2}{a_1^\alpha + a_2^\alpha} + \dfrac{1}{2}\left(\dfrac{1}{a_1^\alpha} - \dfrac{1}{a_2^\alpha}\right)[-1,1]$

It is clear that $m_\alpha\left(\dfrac{1}{\tilde{A}}\right)$ is constant. To prove $S_\alpha\left(\dfrac{1}{\tilde{A}}\right)$ is non-increasing.

Let $\alpha < \alpha'$, then

$$\begin{aligned}
S_\alpha\left(\frac{1}{\tilde{A}}\right) &= \frac{1}{2}\left(\frac{1}{a_1^\alpha} - \frac{1}{a_2^\alpha}\right) \\
&= \frac{1}{2}\left(\frac{a_2^\alpha - a_1^\alpha}{a_1^\alpha a_2^\alpha}\right)
\end{aligned}$$

$$= \frac{S_\alpha(\tilde{A})}{a_1^\alpha a_2^\alpha} \tag{2.10}$$

We have $a_1^\alpha \leq a_1^{\alpha'} \leq a_2^{\alpha'} \leq a_2^\alpha$. Since $\tilde{A}$ is symmetric, $a_1^{\alpha'} - a_1^\alpha = a_2^\alpha - a_2^{\alpha'} = k(say)$, let $a_2^{\alpha'} - a_1^{\alpha'} = c$, i.e

$$\begin{aligned}
a_1^{\alpha'} &= a_1^\alpha + k \\
a_2^{\alpha'} &= a_1^\alpha + k + c \\
a_2^\alpha &= a_1^\alpha + k + c + k \\
\text{Thus } a_1^\alpha a_2^\alpha - a_1^{\alpha'} a_2^{\alpha'} &= a_1^\alpha(a_1^\alpha + k + c + k) - (a_1^\alpha + k)(a_1^\alpha + k + c) \\
&= -k^2 - kc \\
&\leq 0 \text{ as } k \geq 0, c \geq 0 \\
\implies \frac{1}{a_1^\alpha a_2^\alpha} &\geq \frac{1}{a_1^{\alpha'} a_2^{\alpha'}} \qquad (2.11) \\
\tilde{A} \text{ is symmetric} \implies S_\alpha(\tilde{A}) &\geq S_{\alpha'}(\tilde{A}) \qquad (2.12)
\end{aligned}$$

From the definition of the inverse all the terms appearing in equations 2.11 and 2.12 are positive, hence multiplying equations 2.11 and 2.12 and applying it in equation 2.10, we get $S_\alpha\left(\frac{1}{\tilde{A}}\right) \geq S_{\alpha'}\left(\frac{1}{\tilde{A}}\right)$.

Thus $\frac{1}{\tilde{A}}$ is symmetric.

□

Theorem 2.1. *[Properties of Arithmetic Operators] Let $\tilde{A}, \tilde{B}, \tilde{C}$ be symmetric fuzzy numbers, then the following properties hold:*

1. $\tilde{A} + \tilde{B} = \tilde{B} + \tilde{A}$ *(commutative)*
2. $\tilde{A}.\tilde{B} = \tilde{B}.\tilde{A}$ *(commutative)*
3. $(\tilde{A} + \tilde{B}) + \tilde{C} = \tilde{A} + (\tilde{B} + \tilde{C})$ *(associative)*
4. $(\tilde{A}.\tilde{B}).\tilde{C} = \tilde{A}.(\tilde{B}.\tilde{C})$ *(associative)*
5. $\tilde{A} + \tilde{0} = \tilde{0} + \tilde{A} = \tilde{A}$ *(identity)*
6. $\tilde{A}.\tilde{1} = \tilde{1}.\tilde{A} = \tilde{A}$ *(identity)*
7. $\tilde{A} + \tilde{B} = \tilde{A} + \tilde{C} \implies \tilde{B} = \tilde{C}$ *(cancellation)*
8. $\tilde{A}.\tilde{B} = \tilde{A}.\tilde{C} \implies \tilde{B} = \tilde{C}$ *(cancellation)*
9. *Scalar multiplication by non-negative real scalars satisfies:*
   (a) $\lambda(A + B) = \lambda A + \lambda B$
   (b) $(\lambda + \mu)A = \lambda A + \mu A$
   (c) $(\lambda\mu)A = \lambda(\mu A)$
10. $A.\tilde{}\ (B\tilde{} + C\tilde{}) \preceq A.\tilde{}\ B\tilde{} + A.\tilde{}\ C\tilde{}$ *(sub-distributive)*

11. $(\tilde{A}+\tilde{B})-\tilde{C}=\tilde{A}+(\tilde{B}-\tilde{C})$

12. $(\tilde{A}+\tilde{B})-\tilde{B}\neq\tilde{A}$

13. $\tilde{A}\preceq\tilde{C}$ *and* $\tilde{B}\preceq\tilde{D} \implies \tilde{A}+\tilde{B}\preceq\tilde{C}+\tilde{D}$ *and* $\tilde{A}-\tilde{B}\preceq\tilde{C}-\tilde{D}$ *(inclusion monotonicity)*

Proof. Let the $\alpha-$cut of the given symmetric fuzzy numbers be $[\tilde{A}]_\alpha=[a_1^\alpha,a_2^\alpha]$, $[\tilde{B}]_\alpha=[b_1^\alpha,b_2^\alpha]$, $[\tilde{C}]_\alpha=[c_1^\alpha,c_2^\alpha]$ and $[\tilde{0}]_\alpha=[0,0]$, $[\tilde{1}]_\alpha=[1,1]$ and thus in the mid-point and spread notation

$$[\tilde{A}]_\alpha=\frac{a_1^\alpha+a_2^\alpha}{2}+\frac{a_2^\alpha-a_1^\alpha}{2}[-1,1]$$
$$[\tilde{B}]_\alpha=\frac{b_1^\alpha+b_2^\alpha}{2}+\frac{b_2^\alpha-b_1^\alpha}{2}[-1,1]$$
$$[\tilde{C}]_\alpha=\frac{c_1^\alpha+c_2^\alpha}{2}+\frac{c_2^\alpha-c_1^\alpha}{2}[-1,1]$$

1. To prove $\tilde{A}+\tilde{B}=\tilde{B}+\tilde{A}$

$$\begin{aligned}[\tilde{A}+\tilde{B}]_\alpha &= \frac{a_1^\alpha+a_2^\alpha}{2}+\frac{b_1^\alpha+b_2^\alpha}{2}+\left(\frac{a_2^\alpha-a_1^\alpha}{2}+\frac{b_2^\alpha-b_1^\alpha}{2}\right)[-1,1]\\ &= \frac{b_1^\alpha+b_2^\alpha}{2}+\frac{a_1^\alpha+a_2^\alpha}{2}+\left(\frac{b_2^\alpha-b_1^\alpha}{2}+\frac{a_2^\alpha-a_1^\alpha}{2}\right)[-1,1]\\ &= [\tilde{B}+\tilde{A}]_\alpha\end{aligned}$$

2. To prove $\tilde{A}.\tilde{B}=\tilde{B}.\tilde{A}$

$$\begin{aligned}[\tilde{A}.\tilde{B}]_\alpha &= \frac{a_1^\alpha+a_2^\alpha}{2}.\frac{b_1^\alpha+b_2^\alpha}{2}+\\ &\quad\left(\frac{a_2^\alpha-a_1^\alpha}{2}\left|\frac{b_1^\alpha+b_2^\alpha}{2}\right|+\left|\frac{a_1^\alpha+a_2^\alpha}{2}\right|\frac{b_2^\alpha-b_1^\alpha}{2}+\frac{a_2^\alpha-a_1^\alpha}{2}\frac{b_2^\alpha-b_1^\alpha}{2}\right)[-1,1]\\ &= \frac{b_1^\alpha+b_2^\alpha}{2}.\frac{a_1^\alpha+a_2^\alpha}{2}+\\ &\quad\left(\frac{b_2^\alpha-b_1^\alpha}{2}\left|\frac{a_1^\alpha+a_2^\alpha}{2}\right|+\left|\frac{b_1^\alpha+b_2^\alpha}{2}\right|\frac{a_2^\alpha-a_1^\alpha}{2}+\frac{b_2^\alpha-b_1^\alpha}{2}\frac{a_2^\alpha-a_1^\alpha}{2}\right)[-1,1]\\ &= [\tilde{B}.\tilde{A}]_\alpha\end{aligned}$$

3. To prove $(\tilde{A}+\tilde{B})+\tilde{C}=\tilde{A}+(\tilde{B}+\tilde{C})$

$$\begin{aligned}[(\tilde{A}+\tilde{B})+\tilde{C}]_\alpha &= \left(\frac{a_1^\alpha+a_2^\alpha}{2}+\frac{b_1^\alpha+b_2^\alpha}{2}\right)+\frac{c_1^\alpha+c_2^\alpha}{2}\\ &\quad+\left\{\left(\frac{a_2^\alpha-a_1^\alpha}{2}+\frac{b_2^\alpha-b_1^\alpha}{2}\right)+\frac{c_2^\alpha-c_1^\alpha}{2}\right\}[-1,1]\\ &= \frac{a_1^\alpha+a_2^\alpha}{2}+\left(\frac{b_1^\alpha+b_2^\alpha}{2}+\frac{c_1^\alpha+c_2^\alpha}{2}\right)\\ &\quad+\left\{\frac{a_2^\alpha-a_1^\alpha}{2}+\left(\frac{b_2^\alpha-b_1^\alpha}{2}+\frac{c_2^\alpha-c_1^\alpha}{2}\right)\right\}[-1,1]\\ &= [\tilde{A}+(\tilde{B}+\tilde{C})]_\alpha\end{aligned}$$

4. To prove $(\tilde{A}.\tilde{B}).\tilde{C} = \tilde{A}.(\tilde{B}.\tilde{C})$

$$
\begin{aligned}
&[(\tilde{A}.\tilde{B}).\tilde{C}]_\alpha \\
&= \left\{\frac{a_1^\alpha + a_2^\alpha}{2}.\frac{b_1^\alpha + b_2^\alpha}{2} + \right. \\
&\quad \left. \left(\frac{a_2^\alpha - a_1^\alpha}{2}\left|\frac{b_1^\alpha + b_2^\alpha}{2}\right| + \left|\frac{a_1^\alpha + a_2^\alpha}{2}\right|\frac{b_2^\alpha - b_1^\alpha}{2} + \frac{a_2^\alpha - a_1^\alpha}{2}\frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1]\right\} \\
&\qquad .\left\{\frac{c_1^\alpha + c_2^\alpha}{2} + \frac{c_2^\alpha - c_1^\alpha}{2}[-1,1]\right\} \\
&= \left(\frac{a_1^\alpha + a_2^\alpha}{2}.\frac{b_1^\alpha + b_2^\alpha}{2}\right)\left(\frac{c_1^\alpha + c_2^\alpha}{2}\right) \\
&\quad + \left\{\left(\frac{a_2^\alpha - a_1^\alpha}{2}\left|\frac{b_1^\alpha + b_2^\alpha}{2}\right| + \left|\frac{a_1^\alpha + a_2^\alpha}{2}\right|\frac{b_2^\alpha - b_1^\alpha}{2} + \frac{a_2^\alpha - a_1^\alpha}{2}\frac{b_2^\alpha - b_1^\alpha}{2}\right)\left|\frac{c_1^\alpha + c_2^\alpha}{2}\right|\right. \\
&\qquad + \left|\frac{a_1^\alpha + a_2^\alpha}{2}.\frac{b_1^\alpha + b_2^\alpha}{2}\right|\frac{c_2^\alpha - c_1^\alpha}{2} \\
&\qquad + \left(\frac{a_2^\alpha - a_1^\alpha}{2}\left|\frac{b_1^\alpha + b_2^\alpha}{2}\right| + \left|\frac{a_1^\alpha + a_2^\alpha}{2}\right|\frac{b_2^\alpha - b_1^\alpha}{2}\right. \\
&\qquad\qquad \left.\left. + \frac{a_2^\alpha - a_1^\alpha}{2}\frac{b_2^\alpha - b_1^\alpha}{2}\right)\frac{c_2^\alpha - c_1^\alpha}{2}\right\}[-1,1] \\
&= \frac{a_1^\alpha + a_2^\alpha}{2}\left(\frac{b_1^\alpha + b_2^\alpha}{2}\frac{c_1^\alpha + c_2^\alpha}{2}\right) \\
&\quad + \left\{\frac{a_2^\alpha - a_1^\alpha}{2}\left|\frac{b_1^\alpha + b_2^\alpha}{2}\right|\left|\frac{c_1^\alpha + c_2^\alpha}{2}\right| + \left|\frac{a_1^\alpha + a_2^\alpha}{2}\right|\frac{b_2^\alpha - b_1^\alpha}{2}\frac{c_1^\alpha + c_2^\alpha}{2}\right. \\
&\qquad + \frac{a_2^\alpha - a_1^\alpha}{2}\frac{b_2^\alpha - b_1^\alpha}{2}\left|\frac{c_1^\alpha + c_2^\alpha}{2}\right| + \left|\frac{a_1^\alpha + a_2^\alpha}{2}\right|\left|\frac{b_1^\alpha + b_2^\alpha}{2}\right|\frac{c_2^\alpha - c_1^\alpha}{2} \\
&\qquad + \frac{a_2^\alpha - a_1^\alpha}{2}\left|\frac{b_1^\alpha + b_2^\alpha}{2}\right|\frac{c_2^\alpha - c_1^\alpha}{2} + \left|\frac{a_1^\alpha + a_2^\alpha}{2}\right|\frac{b_2^\alpha - b_1^\alpha}{2}\frac{c_2^\alpha - c_1^\alpha}{2} \\
&\qquad \left. + \frac{a_2^\alpha - a_1^\alpha}{2}\frac{b_2^\alpha - b_1^\alpha}{2}\frac{c_2^\alpha - c_1^\alpha}{2}\right\}[-1,1] \\
&= \frac{a_1^\alpha + a_2^\alpha}{2}\left(\frac{b_1^\alpha + b_2^\alpha}{2}\frac{c_1^\alpha + c_2^\alpha}{2}\right) \\
&\quad + \left\{\frac{a_2^\alpha - a_1^\alpha}{2}\left|\frac{b_1^\alpha + b_2^\alpha}{2}\frac{c_1^\alpha + c_2^\alpha}{2}\right|\right. \\
&\qquad + \left|\frac{a_1^\alpha + a_2^\alpha}{2}\right|\left(\frac{b_2^\alpha - b_1^\alpha}{2}\left|\frac{c_1^\alpha + c_2^\alpha}{2}\right| + \left|\frac{b_1^\alpha + b_2^\alpha}{2}\right|\frac{c_2^\alpha - c_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\frac{c_2^\alpha - c_1^\alpha}{2}\right) \\
&\qquad \left. + \frac{a_2^\alpha - a_1^\alpha}{2}\left(\frac{b_2^\alpha - b_1^\alpha}{2}\left|\frac{c_1^\alpha + c_2^\alpha}{2}\right| + \left|\frac{b_1^\alpha + b_2^\alpha}{2}\right|\frac{c_2^\alpha - c_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\frac{c_2^\alpha - c_1^\alpha}{2}\right)\right\} \\
&\qquad\qquad [-1,1] \\
&= [\tilde{A}.(\tilde{B}.\tilde{C})]_\alpha
\end{aligned}
$$

5. To prove $\tilde{A} + \tilde{0} = \tilde{0} + \tilde{A} = \tilde{A}$

$$
\begin{aligned}
[\tilde{A} + \tilde{0}]_\alpha &= \frac{a_1^\alpha + a_2^\alpha}{2} + \frac{0+0}{2} + \left(\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{0-0}{2}\right)[-1,1] \\
&= \frac{a_1^\alpha + a_2^\alpha}{2} + \frac{a_2^\alpha - a_1^\alpha}{2}[-1,1] \\
&= [\tilde{A}]_\alpha
\end{aligned}
$$

Similarly, $\tilde{0}+\tilde{A}=\tilde{A}$.

6. To prove $\tilde{A}.\tilde{1}=\tilde{1}.\tilde{A}=\tilde{A}$

$$\begin{aligned}
[\tilde{A}.\tilde{1}]_\alpha &= \frac{a_1^\alpha+a_2^\alpha}{2}.\frac{1+1}{2} \\
&\quad + \left(\frac{a_2^\alpha-a_1^\alpha}{2}\left|\frac{1+1}{2}\right| + \left|\frac{a_1^\alpha+a_2^\alpha}{2}\right|\frac{1-1}{2} + \frac{a_2^\alpha-a_1^\alpha}{2}\frac{1-1}{2}\right)[-1,1] \\
&= \frac{a_1^\alpha+a_2^\alpha}{2} + \frac{a_2^\alpha-a_1^\alpha}{2}[-1,1] \\
&= [\tilde{A}]_\alpha
\end{aligned}$$

Similarly, $\tilde{1}.\tilde{A}=\tilde{A}$.

7. $\tilde{A}+\tilde{B}=\tilde{A}+\tilde{C} \implies \tilde{B}=\tilde{C}$ (cancellation)

$$\begin{aligned}
& [\tilde{A}+\tilde{B}]_\alpha = [\tilde{A}+\tilde{C}]_\alpha \\
\implies\; & \frac{a_1^\alpha+a_2^\alpha}{2} + \frac{b_1^\alpha+b_2^\alpha}{2} + \left(\frac{a_2^\alpha-a_1^\alpha}{2} + \frac{b_2^\alpha-b_1^\alpha}{2}\right)[-1,1] \\
& \quad = \frac{a_1^\alpha+a_2^\alpha}{2} + \frac{c_1^\alpha+c_2^\alpha}{2} + \left(\frac{a_2^\alpha-a_1^\alpha}{2} + \frac{c_2^\alpha-c_1^\alpha}{2}\right)[-1,1] \\
\implies\; & \frac{a_1^\alpha+a_2^\alpha}{2} + \frac{b_1^\alpha+b_2^\alpha}{2} = \frac{a_1^\alpha+a_2^\alpha}{2} + \frac{c_1^\alpha+c_2^\alpha}{2} \\
& \text{and } \frac{a_2^\alpha-a_1^\alpha}{2} + \frac{b_2^\alpha-b_1^\alpha}{2} = \frac{a_2^\alpha-a_1^\alpha}{2} + \frac{c_2^\alpha-c_1^\alpha}{2} \\
\implies\; & \frac{b_1^\alpha+b_2^\alpha}{2} = \frac{c_1^\alpha+c_2^\alpha}{2} \\
& \text{and } \frac{b_2^\alpha-b_1^\alpha}{2} = \frac{c_2^\alpha-c_1^\alpha}{2} \\
\implies\; & [\tilde{B}]_\alpha = [\tilde{C}]_\alpha \\
\implies\; & \tilde{B} = \tilde{C}
\end{aligned}$$

8. $\tilde{A}.\tilde{B}=\tilde{A}.\tilde{C} \implies \tilde{B}=\tilde{C}$ if $\tilde{A}\neq 0$ (cancellation)

$$\begin{aligned}
& \tilde{A}.\tilde{B} = \tilde{A}.\tilde{C} \\
\implies\; & [\tilde{A}.\tilde{B}]_\alpha = [\tilde{A}.\tilde{B}]_\alpha \\
\implies\; & \frac{a_1^\alpha+a_2^\alpha}{2}.\frac{b_1^\alpha+b_2^\alpha}{2} = \frac{a_1^\alpha+a_2^\alpha}{2}.\frac{c_1^\alpha+c_2^\alpha}{2} \\
& \text{and } \frac{a_2^\alpha-a_1^\alpha}{2}\left|\frac{b_1^\alpha+b_2^\alpha}{2}\right| + \left|\frac{a_1^\alpha+a_2^\alpha}{2}\right|\frac{b_2^\alpha-b_1^\alpha}{2} + \frac{a_2^\alpha-a_1^\alpha}{2}\frac{b_2^\alpha-b_1^\alpha}{2} \\
& \quad = \frac{a_2^\alpha-a_1^\alpha}{2}\left|\frac{c_1^\alpha+c_2^\alpha}{2}\right| + \left|\frac{a_1^\alpha+a_2^\alpha}{2}\right|\frac{c_2^\alpha-c_1^\alpha}{2} + \frac{a_2^\alpha-a_1^\alpha}{2}\frac{c_2^\alpha-c_1^\alpha}{2} \\
\implies\; & \frac{b_1^\alpha+b_2^\alpha}{2} = \frac{c_1^\alpha+c_2^\alpha}{2} \\
& \text{and } \left|\frac{a_1^\alpha+a_2^\alpha}{2}\right|\frac{b_2^\alpha-b_1^\alpha}{2} + \frac{a_2^\alpha-a_1^\alpha}{2}\frac{b_2^\alpha-b_1^\alpha}{2} \\
& \quad = \left|\frac{a_1^\alpha+a_2^\alpha}{2}\right|\frac{c_2^\alpha-c_1^\alpha}{2} + \frac{a_2^\alpha-a_1^\alpha}{2}\frac{c_2^\alpha-c_1^\alpha}{2}
\end{aligned}$$

$$\implies \left(\left|\frac{a_1^\alpha + a_2^\alpha}{2}\right| + \frac{a_2^\alpha - a_1^\alpha}{2}\right)\left(\frac{b_2^\alpha - b_1^\alpha}{2} - \frac{c_2^\alpha - c_1^\alpha}{2}\right) = 0$$
$$\implies \frac{b_2^\alpha - b_1^\alpha}{2} = \frac{c_2^\alpha - c_1^\alpha}{2}$$

as $\left|\frac{a_1^\alpha + a_2^\alpha}{2}\right| + \frac{a_2^\alpha - a_1^\alpha}{2} = 0$ would mean $\left|\frac{a_1^\alpha + a_2^\alpha}{2}\right| = 0$ and $\frac{a_2^\alpha - a_1^\alpha}{2} = 0$ $\implies \tilde{A} = 0$

9. Scalar multiplication by non-negative real scalars satisfies:

   (a) $\lambda(A+B) = \lambda A + \lambda B$ for $\lambda \geq 0$

$$\begin{aligned}
[\lambda(A+B)]_\alpha &= \lambda\left\{\frac{a_1^\alpha + a_2^\alpha}{2} + \frac{b_1^\alpha + b_2^\alpha}{2} + \left(\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1]\right\} \\
&= \lambda\left\{\frac{a_1^\alpha + a_2^\alpha}{2} + \frac{b_1^\alpha + b_2^\alpha}{2}\right\} \\
&\quad + \left\{|\lambda|\left(\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\right)\right\}[-1,1] \\
&= \left\{\lambda\frac{a_1^\alpha + a_2^\alpha}{2} + \lambda\frac{b_1^\alpha + b_2^\alpha}{2}\right\} \\
&\quad + \left(|\lambda|\frac{a_2^\alpha - a_1^\alpha}{2} + |\lambda|\frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1] \\
&= \lambda\frac{a_1^\alpha + a_2^\alpha}{2} + \left(|\lambda|\frac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1] \\
&\quad + \lambda\frac{b_1^\alpha + b_2^\alpha}{2} + \left(|\lambda|\frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1] \\
&= [\lambda A]_\alpha + [\lambda B]_\alpha
\end{aligned}$$

   (b) $(\lambda + \mu)A = \lambda A + \mu A$ for $\lambda, \mu \geq 0$

$$\begin{aligned}
[(\lambda+\mu)A]_\alpha &= (\lambda+\mu)\left\{\frac{a_1^\alpha + a_2^\alpha}{2} + \left(\frac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1]\right\} \\
&= (\lambda+\mu)\frac{a_1^\alpha + a_2^\alpha}{2} + \left(|\lambda+\mu|\frac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1] \\
&= \lambda\frac{a_1^\alpha + a_2^\alpha}{2} + \mu\frac{a_1^\alpha + a_2^\alpha}{2} \\
&\quad + \left(|\lambda|\frac{a_2^\alpha - a_1^\alpha}{2} + |\mu|\frac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1] \text{ as } \lambda, \mu \geq 0 \\
&= \lambda\frac{a_1^\alpha + a_2^\alpha}{2} + \left(|\lambda|\frac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1] \\
&\quad + \mu\frac{a_1^\alpha + a_2^\alpha}{2} + \left(|\mu|\frac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1] \\
&= [\lambda A]_\alpha + [\mu A]_\alpha
\end{aligned}$$

   (c) $(\lambda\mu)A = \lambda(\mu A)$ for $\lambda, \mu \geq 0$

$$[(\lambda\mu)A]_\alpha = (\lambda\mu)\frac{a_1^\alpha + a_2^\alpha}{2} + \left(|\lambda\mu|\frac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1]$$

$$= \lambda\left\{\mu\frac{a_1^\alpha+a_2^\alpha}{2}+\left(|\mu|\frac{a_2^\alpha-a_1^\alpha}{2}\right)[-1,1]\right\} \text{ as } \lambda\geq 0$$
$$= [\lambda(\mu A)]_\alpha$$

10. $\tilde{A}.(\tilde{B}+\tilde{C}) \preceq \tilde{A}.\tilde{B}+\tilde{A}.\tilde{C}$ (sub-distributive)

$$\begin{aligned}
[\tilde{A}.(\tilde{B}+\tilde{C})]_\alpha &= \left\{\frac{a_1^\alpha+a_2^\alpha}{2}+\frac{a_2^\alpha-a_1^\alpha}{2}[-1,1]\right\}.\left\{\frac{b_1^\alpha+b_2^\alpha}{2}+\frac{c_1^\alpha+c_2^\alpha}{2}\right.\\
&\qquad \left.+\left(\frac{b_2^\alpha-b_1^\alpha}{2}+\frac{c_2^\alpha-c_1^\alpha}{2}\right)[-1,1]\right\}\\
&= \frac{a_1^\alpha+a_2^\alpha}{2}.\left(\frac{b_1^\alpha+b_2^\alpha}{2}+\frac{c_1^\alpha+c_2^\alpha}{2}\right)\\
&\quad +\left\{\frac{a_2^\alpha-a_1^\alpha}{2}\left|\frac{b_1^\alpha+b_2^\alpha}{2}+\frac{c_1^\alpha+c_2^\alpha}{2}\right|\right.\\
&\qquad +\left|\frac{a_1^\alpha+a_2^\alpha}{2}\right|\left(\frac{b_2^\alpha-b_1^\alpha}{2}+\frac{c_2^\alpha-c_1^\alpha}{2}\right)\\
&\qquad \left.+\frac{a_2^\alpha-a_1^\alpha}{2}\left(\frac{b_2^\alpha-b_1^\alpha}{2}+\frac{c_2^\alpha-c_1^\alpha}{2}\right)\right\}[-1,1]\\
&\preceq \frac{a_1^\alpha+a_2^\alpha}{2}\frac{b_1^\alpha+b_2^\alpha}{2}+\frac{a_1^\alpha+a_2^\alpha}{2}\frac{c_1^\alpha+c_2^\alpha}{2}+\\
&\quad \left\{\frac{a_2^\alpha-a_1^\alpha}{2}\left|\frac{b_1^\alpha+b_2^\alpha}{2}\right|+\frac{a_2^\alpha-a_1^\alpha}{2}\left|\frac{c_1^\alpha+c_2^\alpha}{2}\right|\right.\\
&\qquad +\left|\frac{a_1^\alpha+a_2^\alpha}{2}\right|\frac{b_2^\alpha-b_1^\alpha}{2}+\left|\frac{a_1^\alpha+a_2^\alpha}{2}\right|\frac{c_2^\alpha-c_1^\alpha}{2}\\
&\qquad \left.+\frac{a_2^\alpha-a_1^\alpha}{2}\frac{b_2^\alpha-b_1^\alpha}{2}+\frac{a_2^\alpha-a_1^\alpha}{2}\frac{c_2^\alpha-c_1^\alpha}{2}\right\}[-1,1]\\
&= \frac{a_1^\alpha+a_2^\alpha}{2}\frac{b_1^\alpha+b_2^\alpha}{2}+\left\{\frac{a_2^\alpha-a_1^\alpha}{2}\left|\frac{b_1^\alpha+b_2^\alpha}{2}\right|+\left|\frac{a_1^\alpha+a_2^\alpha}{2}\right|\frac{b_2^\alpha-b_1^\alpha}{2}\right.\\
&\qquad \left.+\frac{a_2^\alpha-a_1^\alpha}{2}\frac{b_2^\alpha-b_1^\alpha}{2}\right\}[-1,1]\\
&\quad +\frac{a_1^\alpha+a_2^\alpha}{2}\frac{c_1^\alpha+c_2^\alpha}{2}+\left\{\frac{a_2^\alpha-a_1^\alpha}{2}\left|\frac{c_1^\alpha+c_2^\alpha}{2}\right|+\left|\frac{a_1^\alpha+a_2^\alpha}{2}\right|\frac{c_2^\alpha-c_1^\alpha}{2}\right.\\
&\qquad \left.+\frac{a_2^\alpha-a_1^\alpha}{2}\frac{c_2^\alpha-c_1^\alpha}{2}\right\}[-1,1]\\
&= [\tilde{A}.\tilde{B}]_\alpha+[\tilde{A}.\tilde{C}]_\alpha
\end{aligned}$$

11. $(\tilde{A}+\tilde{B})-\tilde{C}=\tilde{A}+(\tilde{B}-\tilde{C})$

$$\begin{aligned}
[(\tilde{A}+\tilde{B})-\tilde{C}]_\alpha &= \left\{\frac{a_1^\alpha+a_2^\alpha}{2}+\frac{b_1^\alpha+b_2^\alpha}{2}+\left(\frac{a_2^\alpha-a_1^\alpha}{2}+\frac{b_2^\alpha-b_1^\alpha}{2}\right)[-1,1]\right\}\\
&\qquad -\left\{\frac{c_1^\alpha+c_2^\alpha}{2}+\left(\frac{c_2^\alpha-c_1^\alpha}{2}\right)[-1,1]\right\}\\
&= \left(\frac{a_1^\alpha+a_2^\alpha}{2}+\frac{b_1^\alpha+b_2^\alpha}{2}\right)-\frac{c_1^\alpha+c_2^\alpha}{2}\\
&\quad +\left\{\left(\frac{a_2^\alpha-a_1^\alpha}{2}+\frac{b_2^\alpha-b_1^\alpha}{2}\right)+\frac{c_2^\alpha-c_1^\alpha}{2}\right\}[-1,1]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{a_1^\alpha + a_2^\alpha}{2} + \left(\frac{b_1^\alpha + b_2^\alpha}{2} - \frac{c_1^\alpha + c_2^\alpha}{2}\right) \\
&\quad + \left\{\frac{a_2^\alpha - a_1^\alpha}{2} + \left(\frac{b_2^\alpha - b_1^\alpha}{2} + \frac{c_2^\alpha - c_1^\alpha}{2}\right)\right\}[-1,1] \\
&= [\tilde{A} + (\tilde{B} - \tilde{C})]_\alpha
\end{aligned}
$$

12. $(\tilde{A} + \tilde{B}) - \tilde{B} \neq \tilde{A}$

$$
\begin{aligned}
[(\tilde{A} + \tilde{B}) - \tilde{B}]_\alpha &= \left\{\frac{a_1^\alpha + a_2^\alpha}{2} + \frac{b_1^\alpha + b_2^\alpha}{2} + \left(\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1]\right\} \\
&\quad - \left\{\frac{b_1^\alpha + b_2^\alpha}{2} + \left(\frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1]\right\} \\
&= \left(\frac{a_1^\alpha + a_2^\alpha}{2} + \frac{b_1^\alpha + b_2^\alpha}{2}\right) - \frac{b_1^\alpha + b_2^\alpha}{2} \\
&\quad + \left\{\left(\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\right) + \frac{b_2^\alpha - b_1^\alpha}{2}\right\}[-1,1] \\
&= \frac{a_1^\alpha + a_2^\alpha}{2} + \left\{\frac{a_2^\alpha - a_1^\alpha}{2} + b_2^\alpha - b_1^\alpha\right\}[-1,1] \\
&\neq [\tilde{A}]_\alpha
\end{aligned}
$$

13. $\tilde{A} \preceq \tilde{C}$ and $\tilde{B} \preceq \tilde{D} \implies \tilde{A} + \tilde{B} \preceq \tilde{C} + \tilde{D}$ (inclusion monotonicity)
$\tilde{A} \preceq \tilde{C} \implies m_\alpha(\tilde{A}) < m_\alpha(\tilde{C})$ or $\left[m_\alpha(\tilde{A}) = m_\alpha(\tilde{C}) \text{ and } S_\alpha(A) \leq S_\alpha(\tilde{C})\right]$
Similarly,
$\tilde{B} \preceq \tilde{D} \implies m_\alpha(\tilde{B}) < m_\alpha(\tilde{D})$ or $\left[m_\alpha(\tilde{B}) = m_\alpha(\tilde{D}) \text{ and } S_\alpha(B) \leq S_\alpha(\tilde{D})\right]$

Case i: $m_\alpha(\tilde{A}) < m_\alpha(\tilde{C})$ and $m_\alpha(\tilde{B}) < m_\alpha(\tilde{D})$
$m_\alpha(\tilde{A} + \tilde{B}) = m_\alpha(\tilde{A}) + m_\alpha(\tilde{B}) < m_\alpha(\tilde{C}) + m_\alpha(\tilde{D}) = m_\alpha(\tilde{C} + \tilde{D})$
$\implies \tilde{A} + \tilde{B} \preceq \tilde{C} + \tilde{D}$

Case ii: $m_\alpha(\tilde{A}) < m_\alpha(\tilde{C})$ and $\left[m_\alpha(\tilde{B}) = m_\alpha(\tilde{D}) \text{ and } S_\alpha(B) \leq S_\alpha(\tilde{D})\right]$
$m_\alpha(\tilde{A} + \tilde{B}) = m_\alpha(\tilde{A}) + m_\alpha(\tilde{B}) < m_\alpha(\tilde{C}) + m_\alpha(\tilde{D}) = m_\alpha(\tilde{C} + \tilde{D})$
$\implies \tilde{A} + \tilde{B} \preceq \tilde{C} + \tilde{D}$

Case iii: $\left[m_\alpha(\tilde{A}) = m_\alpha(\tilde{C}) \text{ and } S_\alpha(A) \leq S_\alpha(\tilde{C})\right]$ and $m_\alpha(\tilde{B}) < m_\alpha(\tilde{D})$
Similar to Case ii

Case iv: $\left[m_\alpha(\tilde{A}) = m_\alpha(\tilde{C}) \text{ and } S_\alpha(A) \leq S_\alpha(\tilde{C})\right]$ and $\left[m_\alpha(\tilde{B}) = m_\alpha(\tilde{D}) \text{ and } S_\alpha(B) \leq S_\alpha(\tilde{D})\right]$
$m_\alpha(\tilde{A} + \tilde{B}) = m_\alpha(\tilde{A}) + m_\alpha(\tilde{B}) = m_\alpha(\tilde{C}) + m_\alpha(\tilde{D}) = m_\alpha(\tilde{C} + \tilde{D})$
$S_\alpha(\tilde{A} + \tilde{B}) = S_\alpha(\tilde{A}) + S_\alpha(\tilde{B}) \leq S_\alpha(\tilde{C}) + S_\alpha(\tilde{D}) = S_\alpha(\tilde{C} + \tilde{D})$
$\implies \tilde{A} + \tilde{B} \preceq \tilde{C} + \tilde{D}$

□

3 Embedding To extend the concepts of coherent prevision and probability in a fuzzy ambit, it is necessary to obtain a structure of vector space based on fuzzy numbers. But, whatever definition of sum is utilized, the sum of two fuzzy numbers has left and right spreads greater than the spreads of the individual fuzzy numbers. Then we cannot have the additive inverse of a non degenerate fuzzy number and fuzzy numbers are neither a group nor a vector space.

In this section we prove that we can overcome this obstacle by introducing a suitable equivalence relation $\sim$ on the set $\mathcal{SF}$ of fuzzy numbers and by considering the quotient set $\mathcal{SF}/\sim$ and the induced structures. In fact, in this case we obtain a vector space.

Theorem 3.1. *[5]*

A. *Let M be a commutative semigroup in which the law of cancellation holds. That is, For $A, B, C \in M$, if*

   1. $(A + B) + C = A + (B + C)$
   2. $A + B = B + A$
   3. $A + C = B + C \implies A = B$

   then M can be embedded in a group N. Furthermore N can be chosen so as to be minimal in the following sense: If G is any group in which M is embedded, then N is isomorphic to a subgroup of G containing M.

B. *If a multiplication by non-negative real scalars satisfying:*

   4. $\lambda(A + B) = \lambda A + \lambda B$
   5. $(\lambda_1 + \lambda_2)A = \lambda_1 A + \lambda_2 A$
   6. $\lambda_1(\lambda_2)A = \lambda_1\lambda_2 A$
   7. $1A = A$

   is defined on M, then a multiplication be real scalars can be defined on N so as to make N a vector space and so that for $\lambda \geq 0$ and $A \in M$ the product λA coincides with the one given on M.

Theorem 2.1 shows that the collection of symmetric fuzzy numbers $\mathcal{SF}$ satisfy conditions 1 to 7 of theorem 3.1. Hence $\mathcal{SF}$ can be embedded into $\mathcal{SFN}$ which will be a group and a vector space. According to the proof of theorem 3.1 in [5], the class $\mathcal{SFN}$ consists of equivalence classes of pairs $(\tilde{A}, \tilde{B})$ of elements of $\mathcal{SF}$. The equivalence relation, $\sim$ is defined by $(\tilde{A}, \tilde{B}) \sim (\tilde{C}, \tilde{D})$ if and only if $\tilde{A} + \tilde{D} = \tilde{B} + \tilde{C}$ i.e. $m_\alpha(\tilde{A} + \tilde{D}) = m_\alpha(\tilde{B} + \tilde{C})$ and $S_\alpha(\tilde{A} + \tilde{D}) = S_\alpha(\tilde{B} + \tilde{C})$. The equivalence class containing the pair (A, B) is denoted by $[A, B]$.
Define addition on $\mathcal{SFN}$ as

$$[\tilde{A}, \tilde{B}] + [\tilde{C}, \tilde{D}] = [\tilde{A} + \tilde{C}, \tilde{B} + \tilde{D}]$$

and scalar multiplication as

$$c[\tilde{A}, \tilde{B}] = \begin{cases} [c\tilde{A}, c\tilde{B}] & \text{if } c \in \mathbb{R}_+ \\ [-c\tilde{B}, -c\tilde{A}] & \text{otherwise} \end{cases}$$

and the order relation may be defined on $\mathcal{SFN}$ as $[\tilde{A}, \tilde{B}] \preceq [\tilde{C}, \tilde{D}]$ if $\tilde{A} + \tilde{D} \preceq \tilde{B} + \tilde{C}$ holds.
The zero element in $\mathcal{SFN}$ will be $[\tilde{0}, \tilde{0}]$ and the inverse of $[A, B]$ will be $[B, A]$.
The element $\tilde{A} \in \mathcal{SF}$ will be identified with the class $[\tilde{A}, \tilde{0}] \in \mathcal{SFN}$, where $\tilde{0}$ is the zero element in $\mathcal{SF}$.

4 Conclusion In this paper, a special class of fuzzy numbers is considered, the symmetric fuzzy numbers whose shape is symmetric with respect to a vertical line. We introduced the necessary arithmetic operations on these numbers and also verified that they belong to the same class. When studying the structure of the class, we see that it forms a commutative semi-group with the cancellation property. Also it satisfies certain other properties that are required in the Radstorm embedding theorem. Hence using Radstorm embedding theorem, the class of symmetric fuzzy numbers are embedded into a class of equivalent pairs of symmetric fuzzy numbers which form a group and a vector space.

References

[1] Dong Qiu, Chongxia Lu, Wei Zhang, Yaoyao Lan, Algebraic Properties and topological properties of the quotient space of fuzzy numbers based on Mares equivalence relation, Fuzzy Sets and Systems, 245,(16), 63-82 (2014).

[2] Klaus D Schmidt, Embedding theorems for classes of convex sets in a hypernormed vector space, Analysis 6, 57-96 (1986).

[3] Mares M, Addition of fuzzy quantities: disjunction-conjunctionapproach, Kybernetika 25, 104-116 (1989).

[4] Maturo A, On some structures of fuzzy numbers, Iranian Journal of Fuzzy Systems, 6 49-59 (2009).

[5] Radstrom H, An embedding theorem for spaces of convex sets, Proc. Amer. Math. Soc., 3, 165-169 (1952).

[6] R. Yager, A characterization of the extension principle, Fuzzy Sets and Systems, 18(3), 205-217 (1986).

[7] L. Zadeh, Fuzzy sets, Inf. Control, 8, 338-353 (1965).

[8] L. Zadeh, Probability measures of fuzzy events, J. Math. Anal. Appl., 23, 421-427 (1968).

A VANISHING THEOREM OF ADDITIVE HIGHER CHOW GROUPS

Toshiro Hiranouchi

Abstract. We show that the additive higher Chow group of the form $\mathrm{TCH}^{\dim(X)+q}(X, q; m)$ becomes 0 for some scheme X over a perfect field of positive characteristic and for $q \geqslant 2$. This is an analogy of Akhtar's theorem on the higher Chow groups: $\mathrm{CH}^{\dim(X)+q}(X, q) = 0$ for $q \geqslant 2$.

1 Introduction As a continuation[1] of [5], we study an analogy between

$$\mathrm{CH}^a(X, b) \quad \longleftrightarrow \quad \mathrm{TCH}^a(X, b; m).$$

Here, $\mathrm{CH}^a(X, b)$ is the higher Chow group of an appropriate scheme X over a field k and $\mathrm{TCH}^a(X, b; m)$ is the additive higher Chow group of X (see Sect. 2 for the definitions). An objective of this note is to show the following theorem:

Theorem 1.1 (Thm. 3.5). *Let X be a projective smooth variety over a perfect field k with positive characteristic. Then, for $q \geqslant 2$,*

$$\mathrm{TCH}^{d+q}(X, q; m) = 0,$$

where $d = \dim(X)$ is the dimension of X.

This is an additive version of Akhtar's theorem ([1], Cor. 7.1) on the higher Chow group: For $q \geqslant 2$,

$$\mathrm{CH}^{d+q}(X, q) = 0,$$

when X is a smooth quasi-projective variety of $d = \dim(X)$ over a *finite field*.

Our motivation is to define an additive variant of Somekawa type K-groups. Recall that a Mackey functor over a field k is a contravariant functor from the category of étale schemes over k to that of abelian groups equipped with a covariant structure for finite morphisms satisfying some conditions (for the precise definition, see Def. 3.1). The higher Chow group $\mathrm{CH}^a(X, b)$ defines a Mackey functor

$$\mathscr{C}\mathrm{H}^a(X, b) : k'/k \mapsto \mathrm{CH}^a(X_{k'}, b),$$

where k' is a finite field extension of k and $X_{k'} = X \otimes_k k'$. For some schemes X, X' over k with $d = \dim(X)$ and $d' = \dim(X')$, the Milnor type K-group

$$K(k; \mathscr{C}\mathrm{H}^{d+a}(X, a), \mathscr{C}\mathrm{H}^{d'+a'}(X', a'))$$

introduced by Raskind and Spiess ([12], Def. 2.1.1, see also Rem. 2.4.2) is defined by the quotient

$$\left(\bigoplus_{k'/k:\ \text{finite}} \mathscr{C}\mathrm{H}^{d+a}(X, a)(k') \otimes_{\mathbb{Z}} \mathscr{C}\mathrm{H}^{d'+a'}(X', a')(k') \right) \Big/ \ \textbf{(PF)}\ \&\ \textbf{(Rec)}, \tag{1}$$

where "**(PF)** & **(Rec)**" stands for the subgroup generated by elements of the following form: Put $\mathscr{M} := \mathscr{C}\mathrm{H}^{d+a}(X, a)$ and $\mathscr{M}' := \mathscr{C}\mathrm{H}^{d'+a'}(X', a')$.

2010 *Mathematics Subject Classification.* 19D45, 14C25.
Key words and phrases. additive higher Chow groups, Milnor K-groups.

[1] This short note is taken from the preprint [4], Sect. 5 which has been deleted before publication ([5]).

(PF) Let $k \subset k_1 \subset k_2$ be finite field extensions and $j = j_{k_2/k_1} : \mathrm{Spec}(k_2) \to \mathrm{Spec}(k_1)$ the canonical map. The elements are of the form

$$\begin{aligned} &j^*(x) \otimes x' - x \otimes j_*(x') \quad \text{for } x \in \mathscr{M}(k_2) \text{ and } x' \in \mathscr{M}'(k_1), \text{ and} \\ &x \otimes j^*(x') - j_*(x) \otimes x' \quad \text{for } x \in \mathscr{M}(k_1) \text{ and } x' \in \mathscr{M}'(k_2). \end{aligned}$$

(Rec) Let F be a function field in one variable over k and $f \in F^\times, g \in \mathscr{M}(F), g' \in \mathscr{M}'(F)$. The required elements are of the form

$$\sum_v \partial_v(f \otimes g \otimes g'),$$

where the sum is taken over all places v of F/k, and

$$\partial_v : F^\times \otimes_{\mathbb{Z}} \mathscr{M}(F) \otimes_{\mathbb{Z}} \mathscr{M}'(F) \to \mathscr{M}(k(v)) \otimes_{\mathbb{Z}} \mathscr{M}'(k(v))$$

is the *local symbol*. This is given by using the connecting map in the localization sequence of higher Chow groups[2].

Using this, it is known the following expressions:

- $K(k; \mathscr{C}\mathrm{H}^1(k,1), \mathscr{C}\mathrm{H}^1(k,1)) \simeq K(k; \mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$, where the right side is Somekawa's K-group associated to the multiplicative groups $\mathbb{G}_{\mathrm{m}}$ [14], and
- $K(k; \mathscr{C}\mathrm{H}^{d+a}(X,a), \mathscr{C}\mathrm{H}^{d'+a'}(X',a')) \simeq \mathrm{CH}^{d+d'+a+a'}(X \times X', a+a')$ (*cf.* Thm. 3.3).

In our previous work [4], we introduced an additive variant of Somekawa's K-group of the form

$$K(k; \mathbb{W}_m, \mathbb{G}_{\mathrm{m}}),$$

where $\mathbb{W}_m$ is the Witt group scheme of lenth $m \in \mathbb{Z}_{>0}$. We *expect* to define the group of the form

$$K(k; \mathscr{T}\mathrm{CH}^{d+a}(X,a;m), \mathscr{C}\mathrm{H}^{d'+a'}(X',a'))$$

which gives

- $K(k; \mathscr{T}\mathrm{CH}^1(k,1;m), \mathscr{C}\mathrm{H}^1(k,1)) \simeq K(k; \mathbb{W}_m, \mathbb{G}_{\mathrm{m}})$, and
- $K(k; \mathscr{T}\mathrm{CH}^{d+a}(X,a;m), \mathscr{C}\mathrm{H}^{d'+a'}(X',a') \simeq \mathrm{TCH}^{d+d'+a+a'}(X \times X', a+a';m)$.

However, the localization property to define the condition corresponding to **(Rec)** above is not known on the additive higher Chow groups (due to lack of homotopy invariance). Instead of Somekawa type K-group, we consider the Mackey product

$$\left(\mathscr{T}\mathrm{CH}^{d+a}(X,a;m) \overset{M}{\otimes} \mathscr{C}\mathrm{H}^{d'+a'}(X',a')\right)(k)$$

which is defined using the "projection formula" only as follows:

$$\left(\bigoplus_{k'/k} \mathscr{T}\mathrm{CH}^{d+a}(X,a;m)(k') \otimes_{\mathbb{Z}} \mathscr{C}\mathrm{H}^{d'+a'}(X',a')(k')\right) \Big/ \ \mathbf{(PF)},$$

where **(PF)** is the subgroup defined similarly as in (1) (for the precise definition, see Def. 3.2). In this note, we present the following surjective homomorphism on 0-cycles (Thm. 3.4):

$$\left(\mathscr{T}\mathrm{CH}^{d+a}(X,a;m) \overset{M}{\otimes} \mathscr{C}\mathrm{H}^{d'+a'}(X',a')\right)(k) \twoheadrightarrow \mathrm{TCH}^{d+d'+a+a'}(X \times X', a+a';m).$$

It is easy to show that the Mackey product on the left hand side becomes trivial when k has positive characteristic so that we obtain the main theorem noted above (Thm. 3.5).

[2] Although the precise definition of **(Rec)** is not given in [12], but we do not mention about the local symbol more on this. About this topic, see [6] and [1].

Notation In this note, a **variety** over a field k we mean an integral and separated scheme of finite type over $\mathrm{Spec}(k)$. For a field k, we use

- $\mathrm{char}(k)$: the characteristic of k.

For varieties X and Y over a field k, we denote by

- $\dim(X)$: the dimension of X,
- $X_{k'} := X \otimes_k k' := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$: the base change of X for an extension field k'/k, and
- $X \times Y := X \times_{\mathrm{Spec}(k)} Y$.

Acknowledgments This work was supported by KAKENHI 17K05174.

2 (Additive) higher Chow groups of schemes In this section, we recall the definitions of (additive) higher Chow groups following [2],[10], and [3]. Throughout this section, we use the following notation:

- k : a field as a base field,
- $\square^q := (\mathbb{P}^1 \setminus \{1\})^q$ and we use the coordinates $(y_1, \dots, y_q)$ on $\square^q$, and
- X : a scheme of finite type over k.

Higher Chow groups The subscheme of $\square^q$ defined by equations $y_{i_1} = \varepsilon_1, \dots, y_{i_s} = \varepsilon_s$ for $\varepsilon_j \in \{0, \infty\}$ is called a **face** of $\square^q$. For $\varepsilon \in \{0, \infty\}$ and $i = 1, \dots, q-1$, let $\iota_{q,i,\varepsilon} : \square^{q-1} \to \square^q$ be the inclusion defined by $(y_1, \dots, y_{q-1}) \mapsto (y_1, \dots, y_{i-1}, \varepsilon, y_i, \dots, y_{q-1})$.

Definition 2.1. Let $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{\geqslant 0}$.

(i) We denote by $\underline{\mathrm{z}}_p(X, q)$ the free abelian group on integral closed subschemes Z of $X \times \square^q$ of dimension $p + q$ that intersect all faces of $\square^q$ properly.

(ii) For each $1 \leqslant i \leqslant q$ and $\varepsilon \in \{0, \infty\}$, let $\partial_i^\varepsilon := \mathrm{Id}_X \times \iota_{q,i,\varepsilon}^*$, where $\mathrm{Id}_X : X \to X$ is the identity morphism. The abelian groups $\underline{\mathrm{z}}_p(X, \bullet) = \{\underline{\mathrm{z}}_p(X, q)\}_{q \geqslant 0}$ form a complex with boundary map

$$\sum_{i=1}^{q} (-1)^i (\partial_i^\infty - \partial_i^0) : \underline{\mathrm{z}}_p(X, q) \to \underline{\mathrm{z}}_p(X, q-1).$$

The **higher Chow complex** $\mathrm{z}_p(X, \bullet)$ is $\underline{\mathrm{z}}_p(X, \bullet)$ modulo the complex consists of the degenerate cycles, that is, the cycles on $X \times \square^q$ pulled back from cycles on $X \times \square^{q-1}$ by a projection $X \times \square^q \to X \times \square^{q-1}$ of the form $(x, y_1, \dots, y_q) \mapsto (x, y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_q)$ for some index j. The homology group

$$\mathrm{CH}_p(X, q) := H_q(\mathrm{z}_p(X, \bullet))$$

is called the **higher Chow group** of X.

If the scheme X is equidimensional of $d = \dim(X)$ over k, we write

$$\mathrm{z}^p(X, q) := \mathrm{z}_{d-p}(X, q), \quad \text{and} \quad \mathrm{CH}^p(X, q) := H_q(\mathrm{z}^p(X, \bullet)) = \mathrm{CH}_{d-p}(X, q).$$

The higher Chow groups have functorial properties induced from the proper push-forward, and the flat pull-back of cycles. In particular, for a finite field extension k'/k, the projection $j = j_{k'/k} : X_{k'} = X \otimes_k k' \to X$ induces

$$N_{k'/k} := j_* : \mathrm{CH}^p(X_{k'}, q) \to \mathrm{CH}^p(X, q). \tag{2}$$

These functorial properties enable us to give the structure of $\mathrm{CH}^p(X,q)$ a Mackey functor as follows:

(3) $$\mathscr{C}\mathrm{H}^p(X,q) : k' \mapsto \mathrm{CH}^p(X_{k'},q),$$

where $X_{k'} = X \otimes_k k'$, for a finite field extension k' of k. For two schemes X, Y of finite type over k, one can construct

$$\boxtimes : \mathrm{z}_p(X,\bullet) \otimes_{\mathbb{Z}} \mathrm{z}_r(Y,\bullet) \to \mathrm{z}_{p+r}(X \times Y, \bullet).$$

On integral cycles, it is defined by $Z \boxtimes W := \tau_*(Z \times W)$, where $\tau : X \times \square^p \times Y \times \square^r \to X \times Y \times \square^{p+r}$ is the exchange of factors (*cf.* [8], Sect. 1.3). On homology groups, $\boxtimes$ induces the **external product**

(4) $$\boxtimes : \mathrm{CH}_p(X,q) \otimes_{\mathbb{Z}} \mathrm{CH}_r(Y,s) \to \mathrm{CH}_{p+r}(X \times Y, q+s).$$

If X is smooth over k, then pulling back of $\boxtimes$ along the diagonal $\Delta : X \to X \times X$, we have the **intersection product**

(5) $$\cap : \mathrm{CH}^p(X,q) \otimes_{\mathbb{Z}} \mathrm{CH}^r(X,s) \to \mathrm{CH}^{p+r}(X, q+s).$$

We list some relevant calculations of higher Chow groups: There is a natural isomorphism $\mathrm{CH}^p(X,0) \simeq \mathrm{CH}^p(X)$, where the latter is the ordinary Chow group. In the case of $p = q$, we have the following theorem:

Theorem 2.2 ([11], [15])**.** *There is a canonical isomorphism*

$$\phi : \mathrm{CH}^q(k,q) \xrightarrow{\simeq} K_q^M(k),$$

where the latter group is the Milnor K-group of the field k.

In particular, in the case of $q = 1$, we have

$$\mathrm{CH}^1(k,1) \simeq k^\times = \mathbb{G}_{\mathrm{m}}(k),$$

where $\mathbb{G}_{\mathrm{m}}$ is the multiplicative group scheme. This extends to an isomorphism

(6) $$\mathscr{C}\mathrm{H}^1(k,1) \simeq \mathbb{G}_{\mathrm{m}}$$

of Mackey functors. Here, we refer the construction of the map ϕ in Thm. 2.2. By the very definition, $\mathrm{CH}^q(k,q)$ is generated by classes $[P]$ represented by a closed point $P : \operatorname{Spec} k(P) \to \square^q$. It is determined by the maps $y_i(P) : \operatorname{Spec} k(P) \to \square^q \xrightarrow{y_i} \square$ for $i = 1, \dots, q$ and they give $y_i(P) \in k(P)^\times$ for each i. The map ϕ is defined by

$$\phi([P]) := N_{k(P)/k}\{\, y_1(P), \dots, y_q(P)\,\},$$

where $N_{k(P)/k} : K_q^M(k(P)) \to K_q^M(k)$ is the norm map of the Milnor K-groups and $\{\, y_1(P), \dots, y_q(P)\,\}$ is the element in $K_q^M(k(P))$ represented by $y_1(P) \otimes \cdots \otimes y_q(P) \in k(P)^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} k(P)^\times$.

Additive higher Chow groups The additive higher Chow groups are defined similarly to the higher Chow groups using B_q below instead of cubes $\square^q$. Let

- $B_q := \mathbb{A}^1 \times \square^{q-1}$, and
- $\overline{B}_q := \mathbb{A}^1 \times (\mathbb{P}^1)^{q-1} \supset B_q$. We use the coordinates $(t, y_1, \dots, y_{q-1})$ on $\overline{B}_q$.

The subscheme of B_q defined by equations $y_{i_1} = \varepsilon_1, \dots, y_{i_s} = \varepsilon_s$ for $\varepsilon_j \in \{0, \infty\}$ is called a **face** of B_q. For $\varepsilon \in \{0, \infty\}$ and $i = 1, \dots, q-1$, let $\iota_{q,i,\varepsilon} : B_{q-1} \to B_q$ be the inclusion defined by $(t, y_1, \dots, y_{q-2}) \mapsto (t, y_1, \dots, y_{i-1}, \varepsilon, y_i, \dots, y_{q-2})$. On $\overline{B}_q$, let $F^1_{q,i}$ be the Cartier divisor defined by $y_i = 1$ and $F_{q,0}$ the Cartier divisor defined by $t = 0$.

Definition 2.3. Let $p \in \mathbb{Z}$, and $q, m \in \mathbb{Z}_{>0}$.

(i) Define $\underline{\mathrm{Tz}}_p(X, 1; m)$ to be the free abelian group on integral closed subschemes Z of $X \times \mathbb{A}^1$ of dimension p satisfying $Z \cap (X \times \{0\}) = \varnothing$ and the modulus condition defined below. For the integer $q > 1$, $\underline{\mathrm{Tz}}_p(X, q; m)$ is the free abelian group on integral closed subschemes Z of $X \times B_q$ of dimension $p + q - 1$ satisfying the following two conditions:

(Good position) For each face F of B_q, Z intersects $X \times F$ properly.

(Modulus condition) Let $\pi : \overline{Z}^N \to Z \subset X \times \overline{B}_q$ be the normalization of the closure $\overline{Z}$ of Z in $X \times \overline{B}_q$. Then

$$(m+1)\pi^*(X \times F_{q,0}) \leqslant \pi^*(X \times F^1_q)$$

as Weil divisors, where $F^1_q := \sum_{i=1}^{q-1} F_{q,i}$.

(Here, we adapt the modulus condition M_{sum} in Def. 2.1 in [9]. For the other similar conditions on modulus and their relations, see [9], Sect. 2).

(ii) For each $1 \leqslant i \leqslant q-1$ and $\varepsilon \in \{0, \infty\}$, let $\partial^\varepsilon_i := \mathrm{Id}_X \times \iota^*_{q,i,\varepsilon}$. The boundary map of $\underline{\mathrm{Tz}}_p(X, \bullet; m)$ is given by

$$\sum_{i=1}^{q-1} (-1)^i (\partial^\infty_i - \partial^0_i) : \underline{\mathrm{Tz}}_p(X, q; m) \to \underline{\mathrm{Tz}}_p(X, q-1; m).$$

The **additive cycle complex** $\mathrm{Tz}_p(X, \bullet; m)$ is the nondegenerate complex associated to $\underline{\mathrm{Tz}}_p(X, \bullet; m)$. Its homology group

$$\mathrm{TCH}_p(X, q; m) := H_q(\mathrm{Tz}_p(X, \bullet; m))$$

is called the **additive higher Chow group** of X with modulus m.

If the scheme X is equidimensional of $d = \dim(X)$ over k, we write

$$\mathrm{Tz}^p(X, q; m) := \mathrm{Tz}_{d+1-p}(X, q; m), \quad \text{and} \quad \mathrm{TCH}^p(X, q; m) := H_q(\mathrm{Tz}^p(X, \bullet; m)).$$

The additive higher Chow groups have also functorial properties as projective push-forward, and the flat pull-back. For a finite field extension k'/k with the projection $j = j_{k'/k} : X_{k'} := X \otimes_k k' \to X$, we have

$$\mathrm{Tr}_{k'/k} := j_* : \mathrm{TCH}^p(X_{k'}, q; m) \to \mathrm{TCH}^p(X, q; m). \tag{7}$$

The assignment

$$\mathscr{T}\mathrm{CH}^p(X, q; m) : k' \mapsto \mathrm{TCH}^p(X \otimes_k k', q; m) \tag{8}$$

gives a structure of Mackey functors.

For two equidimensional schemes X, Y of finite type over k, one can construct the product

$$\boxtimes : \mathrm{z}_p(X, \bullet) \otimes_{\mathbb{Z}} \mathrm{Tz}_r(Y, \bullet; m) \to \mathrm{Tz}_{p+r}(X \times Y, \bullet; m).$$

On integral cycles it is defined by $Z \boxtimes W := \tau_*(Z \times W)$ where $\tau : X \times \square^p \times Y \times B_r \to X \times Y \times B_{p+r}$ is the exchange of factors (*cf.* [8], Sect. 4.1). On homology groups, $\boxtimes$ induces the **external product**

$$\boxtimes : \mathrm{CH}_p(X, q) \otimes_{\mathbb{Z}} \mathrm{TCH}_r(Y, s; m) \to \mathrm{TCH}_{p+r}(X \times Y, q + s; m).$$

If we assume that X is a *smooth and projective* variety over k, we obtain the **intersection product**

$$\cap : \mathrm{CH}_p(X, q) \otimes_{\mathbb{Z}} \mathrm{TCH}_r(X, s; m) \to \mathrm{TCH}_{p+r}(X, q + s; m). \tag{9}$$

Essentially, this product is defined by the pullback of $\boxtimes$ along the diagonal map $\Delta : X \to X \times X$ (see [8], Thm. 4.10 for the precise construction). The intersection product is natural with flat pull-back, and satisfying the projection formula:

$$f_*(f^*(x) \cap y) = x \cap f_*(y) \tag{10}$$

for a morphism $f : X \to Y$ of smooth projective varieties over k. If f is flat, we also have

$$f_*(x \cap f^*(y)) = f_*(x) \cap y. \tag{11}$$

Putting $\mathrm{TCH}^p(k, q; m) := \mathrm{TCH}^p(\mathrm{Spec}(k), q; m)$ we also have the following theorem:

Theorem 2.4 ([13], Thm. 3.20)**.** *For a field k with characteristic $\neq 2$, there is a canonical isomorphism*

$$\phi : \mathrm{TCH}^q(k, q; m) \xrightarrow{\simeq} \mathbb{W}_m \Omega_k^{q-1},$$

where the latter group is the generalized de Rham-Witt group.

In particular, in the case of $q = 1$, we have

$$\mathrm{TCH}^1(k, 1; m) \simeq \mathbb{W}_m(k), \quad \text{and hence} \quad \mathscr{T}\mathrm{CH}^1(k, 1; m) \simeq \mathbb{W}_m, \tag{12}$$

where $\mathbb{W}_m$ is the Witt group scheme. Recall the construction of the map ϕ in Thm. 2.4. The additive higher Chow group $\mathrm{TCH}^q(k, q; m)$ is generated by classes $[P]$ represented by a closed point $P : \mathrm{Spec}\, k(P) \to B_q$. It is determined by the maps $t(P) : \mathrm{Spec}\, k(P) \to B_q \xrightarrow{t} \mathbb{A}^1$ and $y_i(P) : \mathrm{Spec}\, k(P) \to \square^q \xrightarrow{y_i} \square$. They give $t(P) \in k(P), y_i(P) \in k(P)^\times$. The map ϕ is defined by

$$\phi([P]) := \mathrm{Tr}_{k(P)/k} \left([t(P)^{-1}]\, \mathrm{dlog}[y_1(P)] \cdots \mathrm{dlog}[y_{q-1}(P)] \right),$$

where $[-]$ is the Teichmüller lift. Note that the modulus condition assures $t(P) \neq 0$.

3 Mackey product and additive higher Chow groups In this section, we assume

- k : a *perfect* field.

Mackey product We recall the definition of the Mackey functor.

Definition 3.1 (*cf.* [12], Sect. 3)**.** A **Mackey functor** $\mathscr{A}$ (over k) is a contravariant functor from the category of étale schemes over k to the category of abelian groups equipped with a covariant structure for finite morphisms such that $\mathscr{A}(X_1 \sqcup X_2) = \mathscr{A}(X_1) \oplus \mathscr{A}(X_2)$ and if

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a Cartesian diagram, then the induced diagram

$$\begin{array}{ccc} \mathscr{A}(X') & \xrightarrow{g'_*} & \mathscr{A}(X) \\ {\scriptstyle f'^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathscr{A}(Y') & \xrightarrow{g_*} & \mathscr{A}(Y) \end{array}$$

commutes.

For a Mackey functor $\mathscr{A}$, we denote by $\mathscr{A}(k')$ its value $\mathscr{A}(\mathrm{Spec}(k'))$ for a field extension k' of k.

Definition 3.2 (*cf.* [7])**.** For Mackey functors $\mathscr{A}_1, \dots, \mathscr{A}_q$, their **Mackey product** $\mathscr{A}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{A}_q$ is defined as follows: For any finite field extension k'/k,

$$\left(\mathscr{A}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{A}_q\right)(k') := \left(\bigoplus_{k''/k' \colon \text{finite}} \mathscr{A}_1(k'') \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathscr{A}_q(k'')\right)/R, \tag{13}$$

where R is the subgroup generated by elements of the following form:

(PF) For finite field extensions $k' \subset k'_1 \subset k'_2$, and if $x_{i_0} \in \mathscr{A}_{i_0}(k'_2)$ and $x_i \in \mathscr{A}_i(k'_1)$ for all $i \neq i_0$, then

$$j^*(x_1) \otimes \cdots \otimes x_{i_0} \otimes \cdots \otimes j^*(x_q) - x_1 \otimes \cdots \otimes j_*(x_{i_0}) \otimes \cdots \otimes x_q,$$

where $j = j_{k'_2/k'_1} : \mathrm{Spec}(k'_2) \to \mathrm{Spec}(k'_1)$ is the canonical map.

For the Mackey product $\mathscr{A}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{A}_q$, we write $\{ x_1, \dots, x_q \}_{k'/k}$ for the image of $x_1 \otimes \cdots \otimes x_q \in \mathscr{A}_1(k') \otimes \cdots \otimes \mathscr{A}_q(k')$ in the product $\left(\mathscr{A}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{A}_q\right)(k)$. For any field extension k'/k, the canonical map $j = j_{k'/k} : \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ induces the pull-back

$$\mathrm{Res}_{k'/k} := j^* : \left(\mathscr{A}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{A}_q\right)(k) \longrightarrow \left(\mathscr{A}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{A}_q\right)(k').$$

If the extension k'/k is finite, then the push-forward

$$N_{k'/k} := j_* : \left(\mathscr{A}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{A}_q\right)(k') \longrightarrow \left(\mathscr{A}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{A}_q\right)(k) \tag{14}$$

is given by $N_{k'/k}(\{ x_1, \dots, x_q \}_{k''/k'}) = \{ x_1, \dots, x_q \}_{k''/k}$.

Main theorem In the rest of this section, we use

- X, X' : smooth projective varieties over k, and
- $d = \dim(X), d' = \dim(X')$.

Theorem 3.3 ([12], Thm. 2.2, (2.4.4)). *For $a, a' \in \mathbb{Z}_{\geqslant 0}$, we have*

$$\psi : K(k; \mathscr{C}\mathrm{H}^{d+a}(X,a), \mathscr{C}\mathrm{H}^{d'+a'}(X',a')) \xrightarrow{\simeq} \mathrm{CH}^{d+d'+a+a'}(X \times X', a+a').$$

Recall the definition of ψ. We denote by $\{ x, x' \}_{k'/k}$ the image of $x \otimes x' \in \mathrm{CH}^{d+a}(X_{k'}, a) \otimes_{\mathbb{Z}} \mathrm{CH}^{d'+a'}(X_{k'}, a)$ in $K(k; \mathscr{C}\mathrm{H}^{d+a}(X_{k'}, a), \mathscr{C}\mathrm{H}^{d'+a'}(X_{k'}, a))$ (*cf.* (1)). Define

$$\psi(\{ x, x' \}_{k'/k}) := N_{k'/k}(p^*(x) \cap (p')^*(x')),$$

where $\cap$ is the intersection product (5), $N_{k'/k} = j_*$ is the push-forward along $X_{k'} \to X$ (*cf.* (2)), and $p : (X \times X')_{k'} \to X_{k'}$ and $p' : (X \times X')_{k'} \to (X')_{k'}$ are the projections.

As we explained in Introduction, for $a, m \in \mathbb{Z}_{>0}, a' \in \mathbb{Z}_{\geqslant 0}$, we consider the Mackey product (*cf.* Def. 3.2)

$$\left(\mathscr{T}\mathrm{CH}^{d+a}(X,a;m) \overset{M}{\otimes} \mathscr{C}\mathrm{H}^{d'+a'}(X',a';m) \right)(k).$$

Define a homomorphism

$$\psi : \left(\mathscr{T}\mathrm{CH}^{d+a}(X,a;m) \overset{M}{\otimes} \mathscr{C}\mathrm{H}^{d'+a'}(X',a';m) \right)(k) \to \mathrm{TCH}^{d+d'+a+a'}(X \times X', a+a'; m)$$

by the intersection product (9) (*cf.* [12], Proof of Thm. 2.2) as

$$\psi(\{ x, x' \}_{k'/k}) := \mathrm{Tr}_{k'/k}((p')^*(x') \cap p^*(x)),$$

for any finite extension field k'/k, where $\mathrm{Tr}_{k'/k} = j_*$ is the push-forward along $j : \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ and $p : (X \times X')_{k'} \to X_{k'}$ and $p' : (X \times X')_{k'} \to (X')_{k'}$ are the projections. From the projection formula of the intersection product ((10) and (11)), the map ψ is well-defined.

Theorem 3.4. *For $a, m \in \mathbb{Z}_{>0}, a' \in \mathbb{Z}_{\geqslant 0}$, the map*

$$\psi : \left(\mathscr{T}\mathrm{CH}^{d+a}(X,a;m) \overset{M}{\otimes} \mathscr{C}\mathrm{H}^{d'+a'}(X',a') \right)(k) \to \mathrm{TCH}^{d+d'+a+a'}(X \times X', a+a'; m)$$

is surjective.

Proof. Put

- $\mathscr{X} := X \times X'$,
- $\alpha = a + a'$, and
- $\delta = d + d'$.

By the very definition (Def. 2.3), the group $\mathrm{TCH}^{\delta+\alpha}(\mathscr{X}, \alpha; m)$ consists of 0-cycles on $\mathscr{X} \times B_\alpha$. Take a closed point $P : \mathrm{Spec}(k(P)) \to \mathscr{X} \times B_\alpha$ as a generator and it is enough to show the cycle $[P]$ associated to P is in the image of ψ. By the definition of ψ, the trace map on the additive Chow groups and the norm map on the Mackey products are compatible as in the following commutative diagram:

$$\begin{array}{ccc}
\left(\mathscr{T}\mathrm{CH}^{d+a}(X_{k(P)},a;m) \overset{M}{\otimes} \mathscr{C}\mathrm{H}^{d'+a'}({X'}_{k(P)},a') \right)(k(P)) & \xrightarrow{\psi} & \mathrm{TCH}^{\delta+\alpha}(\mathscr{X}_{k(P)}, \alpha; m) \\
\downarrow N_{k(P)/k} & & \downarrow \mathrm{Tr}_{k(P)/k} \\
\left(\mathscr{T}\mathrm{CH}^{d+a}(X,a;m) \overset{M}{\otimes} \mathscr{C}\mathrm{H}^{d'+a'}(X',a') \right)(k) & \xrightarrow{\psi} & \mathrm{TCH}^{\delta+\alpha}(\mathscr{X}, \alpha; m).
\end{array}$$

Thus, to show the assertion that $[P]$ is in the image of ψ we may assume that P is a k-rational point, that is, $k(P) = k$. The point P is determined by the maps $P_X : \mathrm{Spec}(k) \to X \times B_a$ and $P_{X'} : \mathrm{Spec}(k) \to X' \times \square^{a'}$ satisfying $\tau_*(P_X \times P_{X'}) = P$, where $\tau : (X \times B_a) \times (X' \times \square^{a'}) \to \mathscr{X} \times B_\alpha$ is the exchange of factors. This gives cycles $[P_X]$ on $\mathrm{TCH}^{d+a}(X, a; m)$ and $[P_{X'}]$ on $\mathrm{CH}^{d'+a'}(X', a')$. Therefore, denoting by $p : \mathscr{X} \to X$ and $p' : \mathscr{X} \to X'$ the projection maps, we have

$$\psi(\{\,[P_X], [P_{X'}]\,\}_{k/k}) = (p')^*([P_{X'}]) \cap p^*([P_X]) = [P],$$

where the last equality follows from the very definition of the intersection product. The assertion follows from this. □

Theorem 3.5. *Let X be a projective smooth variety of dimension d over a perfect field k with* $\mathrm{char}(k) > 0$. *Then,*

$$\mathrm{TCH}^{d+q}(X, q; m) = 0, \quad \text{for } q \geqslant 2.$$

Proof. There are isomorphisms $\mathscr{C}\mathrm{H}^1(k, 1) \simeq \mathbb{G}_\mathrm{m}$ (from (6)) and $\mathscr{T}\mathrm{CH}^1(k, 1; m) \simeq \mathbb{W}_m$ (from (12)) as Mackey functors. By Theorem 3.3 and Theorem 3.4, we have surjective homomorphisms

$$\Big(\mathbb{W}_m \overset{M}{\otimes} \overbrace{\mathbb{G}_\mathrm{m} \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathbb{G}_\mathrm{m}}^{q-1} \overset{M}{\otimes} \mathscr{C}\mathrm{H}^d(X)\Big)(k) \twoheadrightarrow \Big(\mathscr{T}\mathrm{CH}^1(k, 1; m) \overset{M}{\otimes} \mathscr{C}\mathrm{H}^{d+q-1}(X, q-1)\Big)(k) \overset{\psi}{\twoheadrightarrow} \mathrm{TCH}^{d+q}(X, q; m)$$

for $q \geqslant 2$. The far left vanishes from the lemma below and the assertion follows. □

Lemma 3.6 ([5], Lem. 2.2)**.** *Let G be a unipotent smooth and commutative algebraic group over a field F and A a semi-abelian variety over F. If F is a perfect field of* $\mathrm{char}(F) > 0$, *we have $G \overset{M}{\otimes} A = 0$.*

References

[1] R. Akhtar, *Milnor K-theory of smooth varieties*, K-Theory **32** (2004), no. 3, 269–291. 1, 2

[2] S. Bloch, *Algebraic cycles and the Lie algebra of mixed Tate motives*, J. Amer. Math. Soc. **4** (1991), no. 4, 771–791. 3

[3] S. Bloch and H. Esnault, *An additive version of higher Chow groups*, Ann. Sci. École Norm. Sup. (4) **36** (2003), no. 3, 463–477. 3

[4] T. Hiranouchi, *Somekawa's K-groups and additive higher Chow groups*, arXiv:1208.6455v2. 1, 2

[5] ———, *An additive variant of Somekawa's K-groups and Kähler differentials*, J. K-Theory **13** (2014), no. 3, 481–516. 1, 9

[6] F. Ivorra and K. Rülling, *K-groups of reciprocity functors*, J. Algebraic Geom. **26** (2017), no. 2, 199–278. 2

[7] B. Kahn, *The decomposable part of motivic cohomology and bijectivity of the norm residue homomorphism*, Algebraic K-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989), Contemp. Math., vol. 126, Amer. Math. Soc., Providence, RI, 1992, pp. 79–87. 7

[8] A. Krishna and M. Levine, *Additive higher Chow groups of schemes*, J. Reine Angew. Math. **619** (2008), 75–140. 4, 6

[9] A. Krishna and J. Park, *DGA-structure on additive higher Chow groups*, Int. Math. Res. Not. IMRN (2015), no. 1, 1–54. 5

[10] M. Levine, *Bloch's higher Chow groups revisited*, Astérisque (1994), no. 226, 10, 235–320, K-theory (Strasbourg, 1992). 3

[11] Y. P. Nesterenko and A. A. Suslin, *Homology of the general linear group over a local ring, and Milnor's K-theory*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, 121–146. 4

[12] W. Raskind and M. Spiess, *Milnor K-groups and zero-cycles on products of curves over p-adic fields*, Compositio Math. **121** (2000), 1–33. 1, 2, 7, 8

[13] K. Rülling, *The generalized de Rham-Witt complex over a field is a complex of zero-cycles*, J. Algebraic Geom. **16** (2007), no. 1, 109–169. 6

[14] M. Somekawa, *On Milnor K-groups attached to semi-abelian varieties*, K-Theory **4** (1990), 105–119. 2

[15] B. Totaro, *Milnor K-theory is the simplest part of algebraic K-theory*, K-Theory **6** (1992), no. 2, 177–189. 4

Department of Basic Sciences
Graduate School of Engineering
Kyushu Institute of Technology
1-1 Sensui-cho, Tobata-ku, Kitakyushu-shi,
Fukuoka 804-8550 JAPAN
email: `hira@mns.kyutech.ac.jp`

CONSTRUCTION OF A POSSIBILISTIC REGRESSION MODEL BASED ON POSSIBILITY GRADES WITH VAGUENESS AND RELATIONSHIP WITH PARAMETERS

Yoshiyuki Yabuuchi

Abstract. A possibilistic regression model is an interval-type model. An interval-type model intuitively helps us to understand the possibilities of the target system. The data distribution defines the possibility interval of the system, which may hinder our understanding of the analysis results. Improved models have reported using outlier problem approaches. We propose models to deal with the vagueness included in a possibility grade derived from a possibilistic regression model and samples. Unfortunately, the results obtained by the proposed models were not as expected. Then, the improved model was proposed to handle the vagueness included in possibility grades. The numerical example confirmed that the proposed model could eliminate the influence of unusual samples and describe the possibilities of a focal system. The paper reports the improved model and the results by using a numerical example.

1 Introduction The interval-type possibilistic regression model proposed by Tanaka and Watada [16], as used in this paper, includes all samples. An interval output illustrates the possibility distribution of a focal system. This interval type is rewritten in linear programming (LP), and can be obtained easily. Furthermore, there are various models [1, 4, 7, 12] using possibilistic regression in addition to the least-squares model proposed by Diamond [2, 3]. Fuzzy least squares based on a fuzzy random variable [9, 10] provides a lot of information. However, we use an interval type from the viewpoint of soft computing, because an interval model helps us to understand the analysis object intuitively.

An interval type illustrates the possibilities of an analyzed system by including all samples. The shape of a model is defined by that of the data distribution. For this reason, an interval type is susceptible to the shape of the data distribution. Therefore, processing of outliers for an interval type [14, 15], in which a model coincides with a focal system [5, 8, 11, 18, 19, 20, 21, 22, 23], a linguistic regression model [17], and so forth, are reported. We have proposed a model to deal with the vagueness included in a possibility grade derived from a possibilistic regression model and samples [24, 25]. The objectives of the proposed model are to remove the influence of unusual samples and describe the possibilities of a focal system so that it can be understood subjectively. Unfortunately, the results obtained by the proposed method were not as expected. That model is sometimes unable to remove the influence of unusual samples and distortion of the model. Therefore, a model dealing with the vagueness included in the possibility grade has been built [24, 25]. The proposed model made it possible to eliminate the influence of unusual samples and describe the possibilities of a focal system [26].

This paper is organized as follows. Section 2 briefly explains the interval type of the possibility regression model dealt with in this paper. Section 3 explains the proposed model to process vagueness included in possibility grades. In Section 4, we confirm the usefulness of the proposed model using a simple numerical example. Section 5 concludes this paper.

Key words and phrases. Fuzzy Regression Model, Fuzzy Number, Possibility Grade, Vagueness, Error.

2 Possibilistic Regression Model Consider a possibilistic regression equation using triangular fuzzy regression coefficients:

$$\boldsymbol{Y}_i = (a_0, c_0) + (a_1, c_1)x_{i1} + \cdots + (a_p, c_p)x_{ip} = (\boldsymbol{a}\boldsymbol{x}_i, \boldsymbol{c}|\boldsymbol{x}_i|). \tag{1}$$

The independent and dependent variables are $\boldsymbol{x}_i = (1, x_{i1}, \ldots, x_{ip})$ and y_i in samples $(\boldsymbol{x}_i, y_i)(i = 1, 2, \ldots, n)$. The center and width of the coefficient shown in equation (1) are $\boldsymbol{a} = (a_1, a_2, \ldots, a_p)$ and $\boldsymbol{c} = (c_1, c_2, \ldots, c_p)$, respectively. An output of equation (1) contains this dependent variable. In addition, the vagueness of this model, that is the widths, should be small. Therefore, a possibilistic regression model can be rewritten in the following LP:

$$\begin{array}{ll} \text{min.} & F \\ \text{s.t.} & \boldsymbol{a}\boldsymbol{x}_i - \boldsymbol{c}|\boldsymbol{x}_i| \le y_i \le \boldsymbol{a}\boldsymbol{x}_i + \boldsymbol{c}|\boldsymbol{x}_i|, i = 1, 2, \ldots, n. \end{array} \tag{2}$$

In equation (2), F employs various functions such as widths of coefficients, $F = \sum_j^p c_j$, and widths of forecasted values, $F = \sum_i^n \boldsymbol{c}|\boldsymbol{x}_i|$.

The regression coefficients are a symmetrical triangular fuzzy number, and the model describes the possibility distribution of the target system. The predicted value $\boldsymbol{Y}$=(Y_i^C, Y_i^W) in the independent variable x_i is the interval value with the center $Y_i^C = \boldsymbol{a}\boldsymbol{x}_i$ and the width $Y^W = \boldsymbol{c}|\boldsymbol{x}_i|$. The possibility grade $\mu(y_i, \boldsymbol{x}_i)$ is written as follows:

$$\mu(y_i, \boldsymbol{x}_i) = \max\left(0, 1 - \frac{|y_i - Y_i^C|}{Y_i^W}\right). \tag{3}$$

As shown by equation (3), the range of possibility grades is $[0, 1]$. When the regression coefficients are symmetric triangular fuzzy regression coefficients, their outputs are also symmetric triangles. The possibility grade is the maximum value 1 at the center of the distribution, and becomes the minimum value 0 when leaving the center. The conventional possibility regression model does not consider the possibility grade because it is a model with the least vagueness. On the other hand, the models we propose maximize the possibility grade. The model proposed in this paper deals with vagueness included in the possibility grade. For this reason, the proposed model can eliminate the influence of unusual samples and illustrate the possibility of the focal system. The next section describes the proposed model.

3 Possibilistic Regression Model with Vagueness in Possibility Grades Observed variables include various errors. Errors included in sample attribute values are discussed in statistics and probability, and many research results have been reported. For a possibility grade [6], research results dealing with grade fluctuations are reported using Type-2 fuzzy sets. However, the method using Type-2 fuzzy sets is more complicated than handling using Type-1 fuzzy sets. Therefore, we do not use Type-2 fuzzy sets in this work, and consider a method to easily handle the vagueness included in possibility grades.

Here, because attribute values contain an error, it is natural to think that possibility grades obtained from attribute values also contain an error. Therefore, although possibility grades can be obtained depending on a relationship between membership functions and samples, we assume that a grade has flexibility [24, 25].

In this paper, the proposed regression model handles samples with vagueness in the possibility grade to illustrate the possibility of the focal system. For that purpose, this section explains handling with samples and LP problems to obtain the proposed model.

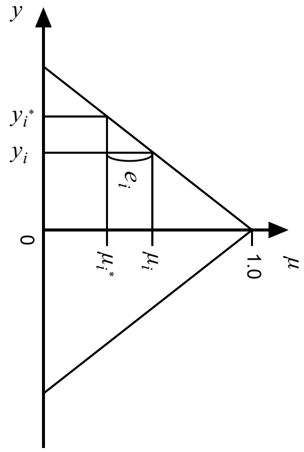


Figure 1: Vagueness included in a possibility grade

3.1 Dealing with Vagueness Including Possibility Grades The possibility grade of attribute value y_i is assumed as μ_i. That is, let us consider that possibility grades, μ_i, contain an error, e_i. At this time, as shown in Fig. 1, let the true possibility grade be μ_i^*. Then, the attribute value corresponding to the true possibility grade μ_i^* will be the value corresponding to y_i^* in Fig. 1. Let Y^C be the center of the membership function and Y^W be the width, then we can obtain the following:

$$y_i^* = y_i + e_i Y^W. \tag{4}$$

Then we replace y_i^* and y_i to find a possibility regression model.

A possibilistic regression model as shown by equation (2) explains the proposed method. A possibility grade μ_i of the ith sample contains an error e_i, and the following relationship holds with the true possibility grade μ_i^* that contains none of error:

$$\mu_i = \mu_i^* + e_i. \tag{5}$$

Here, because a possibility grade takes values of $[0, 1]$, e_i also takes values of $[-1, 1]$.

3.2 Formulation of Model Handling Vagueness Included in Possibility Grades From the above, the inclusion relation between y_i and a model output $\boldsymbol{Y}_i = (\boldsymbol{a}\boldsymbol{x}_i, \boldsymbol{c}|\boldsymbol{x}_i|)$ are as follows:

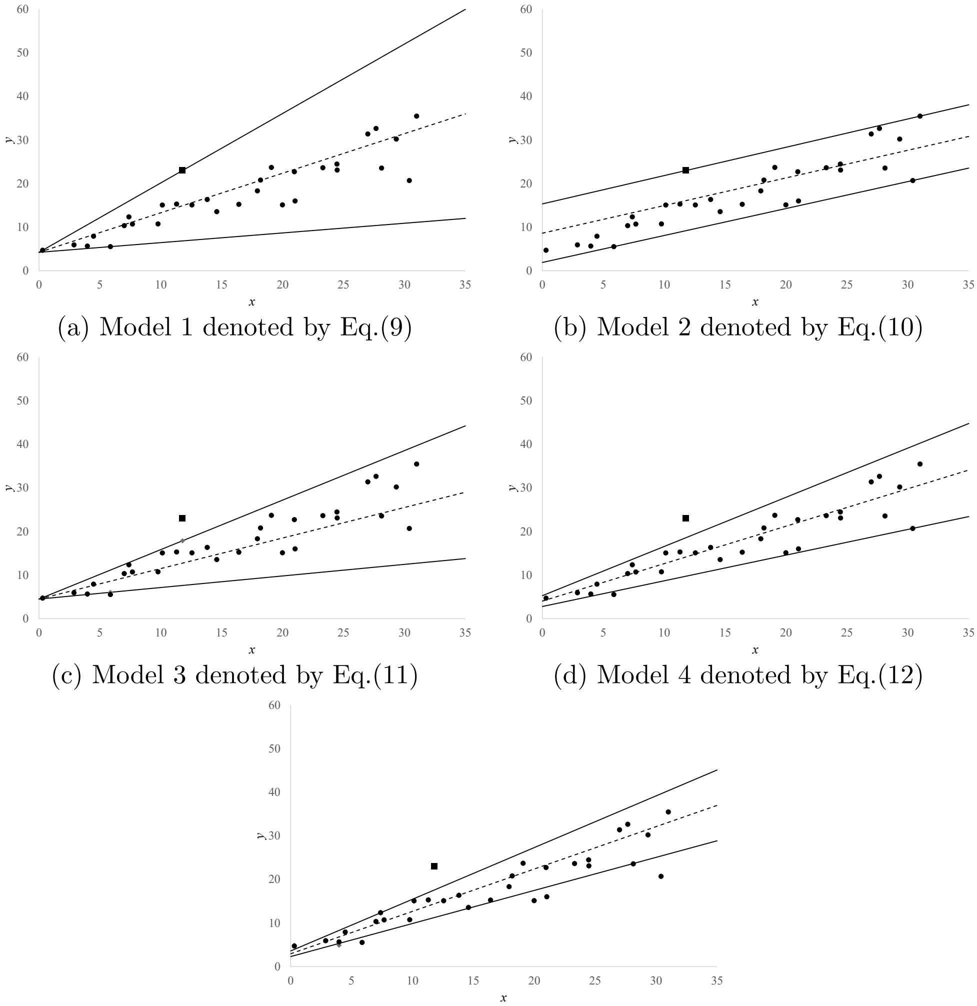
$$\boldsymbol{a}\boldsymbol{x}_i - \boldsymbol{c}|\boldsymbol{x}_i| \leq y_i + e_i\boldsymbol{c}|\boldsymbol{x}_i| \leq \boldsymbol{a}\boldsymbol{x}_i + \boldsymbol{c}|\boldsymbol{x}_i|, i = 1, 2, \ldots, n. \tag{6}$$

As a result, equation (2) can be rewritten as follows:

$$\begin{aligned} \text{min.} \quad & F \\ \text{s.t.} \quad & \boldsymbol{a}\boldsymbol{x}_i - \boldsymbol{c}|\boldsymbol{x}_i| \leq y_i + e_i\boldsymbol{c}|\boldsymbol{x}_i| \leq \boldsymbol{a}\boldsymbol{x}_i + \boldsymbol{c}|\boldsymbol{x}_i|, \\ & |e_i| \leq \varepsilon, i = 1, 2, \ldots, n. \end{aligned} \tag{7}$$

Here, ε is a parameter that specifies the range of vagueness included in the possibility grade. As possibility grades are real numbers, ε is also a real number. Furthermore, the objective function F uses an appropriate function according to the data, similar to the conventional possibilistic regression model.

Using only this, the influence of unusual samples can be removed. We confirm this concretely using a numerical example.



(a) Model 1 denoted by Eq.(9)
(b) Model 2 denoted by Eq.(10)
(c) Model 3 denoted by Eq.(11)
(d) Model 4 denoted by Eq.(12)
(e) Model 5 denoted by Eq.(13)

Figure 2: Obtained models in the numerical example

4 Numerical Example In this section, the same numerical example as in [25] is used. The numerical example adds errors with probability to the two variables, x and y, in the relationship of $y = x$. In addition, samples contain one unusual sample, and the model parameter constraint is set to $|e_i| \leq \varepsilon = 1$. In the numerical example, the following possibilistic regression equation will be found:

$$\boldsymbol{Y} = (a_0, c_0) + (a_1, c_1)x. \tag{8}$$

We obtain model 1 with $F = \sum_j^p c_j$ as the objective function of the interval-type possibilistic regression model shown by equation (2), and model 2 with the objective function $F = \sum_i^n \boldsymbol{c}|\boldsymbol{x}_i|$. In addition to models 3 and 4, which add the vagueness of grades to models 1 and 2, we also obtain model 5 that considers the vagueness of possibility grades to the model proposed by Yabuuchi [24].

The outputs of models 1 to 5 are denoted as $\boldsymbol{Y}_1$ to $\boldsymbol{Y}_5$, respectively. The five models

Table 1: Features of obtained models in the numerical examples

	Model 1	Model 2	Model 3	Model 4	Model 5
Sum of widths of regression coefficients	0.685	6.733	0.435	1.513	0.857
Sum of widths of forecasted values	344.521	432.172	437.480	348.199	254.751
Sum of possibility grades derived from the model and samples	21.473	15.927	16.420	17.003	14.018
Sum of possibility grades for widths of forecasted values	2.542	2.287	3.077	3.569	4.244
Outside samples of intervals			3	3	7

are as follows:

(9) $$\boldsymbol{Y}_1 = (4.232, 0) + (0.908, 0.685)x,$$

(10) $$\boldsymbol{Y}_2 = (8.627, 6.717) + (0.634, 0.016)x,$$

(11) $$\boldsymbol{Y}_3 = (4.495, 0) + (0.701, 0.435)x,$$

(12) $$\boldsymbol{Y}_4 = (4.039, 1.244) + (0.859, 0.270)x,$$

(13) $$\boldsymbol{Y}_5 = (2.944, 0.643) + (0.973, 0.214)x.$$

The least squares is as follows:

(14) $$\boldsymbol{Y}_S = 4.316 + 0.827x.$$

In Fig. 2, the original sample is rounded, and the values converted by equation (4) are indicated by a rhombus. Fig. 2 shows that the models handling vagueness included in possibility grades are not distorted. However, the value of the constant term seems to be large owing to the influence of a specific sample. For this reason, the center of model 3 has a small inclination. The center of model 3 is similar to model 1, the constant term is slightly larger, and the inclination seems to be smaller. On the other hand, in model 5, the centers of the model and the data distribution almost coincide, the width of the forecasted value becomes small, and the possibility of the system can be understood intuitively.

The information obtained from these models is listed in Table 1. The possibility grade is large when the sample is close to the center, so the model with the small width of the interval has the small sum of possibility grade. For this reason, the sum of the possibility grades of model (9) has the maximum value, and that of model (13) has the minimum value. However, in Table 1, the sum of the possibility grade for the width of the forecasted value is opposite to the sum of the possibility grades. This is, the sum of the possibility grade for the width of the forecasted value of the model (9) has the second smallest value, and that of the model (13) is the maximum value.

From the above, we can summarize the features of the proposed model that consider the vagueness included in the possibility grade. First, it was subjectively perceptible that the model describes the data distribution. Second, the influence of the outlier was eliminated, and a mode without distortion in shape was obtained.

In addition, its effect was improved by using the model in conjunction with the model proposed by Yabuuchi [24] that maximizes the sum of the possibility grade for the width of the forecasted value.

In the above, the parameter ε of models 3–5 has been set to 1 because the range of possibility grades is $[0, 1]$. On the other hand, because models 1 and 2 are conventional models, this parameter was not used. Here, the models are obtained by using 0.5, 1.0,

Table 2: The coefficients of the three models using $\varepsilon = \{0.5, 1.0, 1.5, 2.0\}$

		$\varepsilon = 0.5$	$\varepsilon = 1.0$	$\varepsilon = 1.5$	$\varepsilon = 2.0$
Model 3	$\boldsymbol{A}_0$	(4.232, 0)	(4.495, 0)	(4.232, 0)	(4.363, 0)
	$\boldsymbol{A}_1$	(0.908, 0.457)	(0.701, 0.435)	(0.908, 0.274)	(0.891, 0.230)
Model 4	$\boldsymbol{A}_0$	(3.697, 1.137)	(4.039, 1.244)	(3.990, 2.169)	(4.132, 1.495)
	$\boldsymbol{A}_1$	(0.956, 0.359)	(0.859, 0.270)	(0.808, 0.138)	(0.786, 0.191)
Model 5	$\boldsymbol{A}_0$	(2.866, 0.709)	(2.944, 0.643)	(2.944, 0.643)	(2.944, 0.643)
	$\boldsymbol{A}_1$	(1.056, 0.475)	(0.973, 0.214)	(0.973, 0.214)	(0.973, 0.214)

Table 3: Features of the three models using $\varepsilon = \{0.5, 1.0, 1.5, 2.0\}$

		$\varepsilon = 0.5$	$\varepsilon = 1.0$	$\varepsilon = 1.5$	$\varepsilon = 2.0$
Model 3	Index 1	459.361	437.480	275.617	231.563
	Index 2	18.403	16.420	13.721	12.253
	Index 3	3.151	3.077	3.731	3.912
Model 4	Index 1	431.065	348.199	273.368	284.759
	Index 2	18.044	17.003	13.359	13.483
	Index 3	3.175	3.569	3.326	3.393
Model 5	Index 1	521.995	254.751	254.751	254.751
	Index 2	18.663	14.018	14.018	14.018
	Index 3	2.928	4.244	4.244	4.244

Index 1: Sum of widths of forecasted values

Index 2: Sum of possibility grades derived from the model and samples

Index 3: Sum of possibility grades to widths of forecasted values

1.5, and 2.0 as the parameter ε, and the characteristics are confirmed. Table 2 lists the coefficients obtained by the models. Even if the parameter is changed, the center of the models does not change significantly. In addition, the width of the model decreased by increasing the value of the parameter. Furthermore, in model 5, the same model was obtained when $\varepsilon \geq 1.0$.

Table 3 lists the features of the model obtained by changing the parameter ε. When ε was changed from 1.5 to 2.0, the possibility grade of models 3 and 4 did not change significantly. In particular, when ε was increased, the width of the predicted value and the value of the possibility grade became smaller. However, index 3, which divided the possibility grade by the width of the predicted value, increased. In general, if the width of the predicted value is small, the sum of the possibility grade is also small. Then, the relationship between indices 1 and 2 can understand. Index 3 has a large value when the samples are near the center of the possibility interval. Therefore, increasing the value of the parameter ε gathers samples near the center of the possibility interval.

As described above, the width of the solution search space is increased by increasing the value of the parameter, and an unexpected solution is obtained from LP. Although the upper limit of the number of samples processed with fuzziness possibility grade was limited, index 1 of model 4 is larger for $\varepsilon = 2.0$ than for $\varepsilon e = 1.5$. In addition, model 5 uses possibility grades for the objective function. For this reason, model 5 might not be influenced by the parameter ε more than necessary.

To confirm these results, the models are shown in Figs. 3–5. In Figs. 3–5, the boundaries of the model when ε is changed to 0.5, 1.0, 1.5, and 2.0 are shown by a dashed-dotted line, dashed-two dotted line, dashed line, and dotted line, respectively. The features listed in Tables 2 and 3 are confirmed by the results in Figs. 3–5.

The statistical model emphasizes samples away from the center of gravity of the

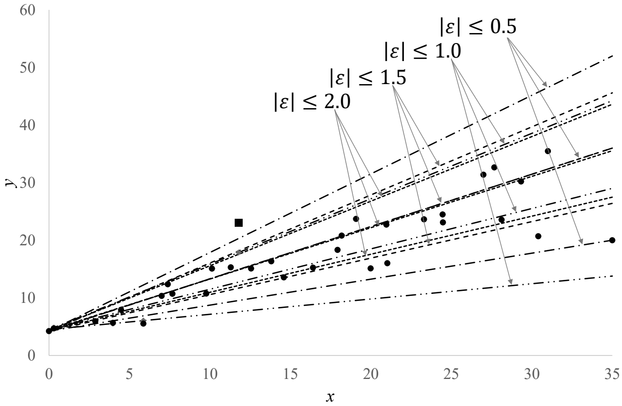


Figure 3: Model 3 using $\varepsilon = \{0.5, 1.0, 1.5, 2.0\}$

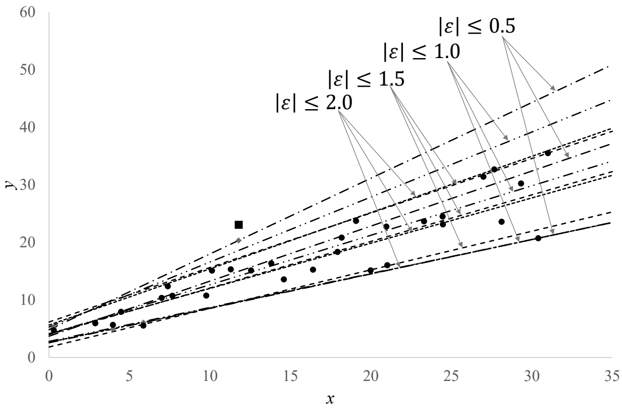


Figure 4: Model 4 using $\varepsilon = \{0.5, 1.0, 1.5, 2.0\}$

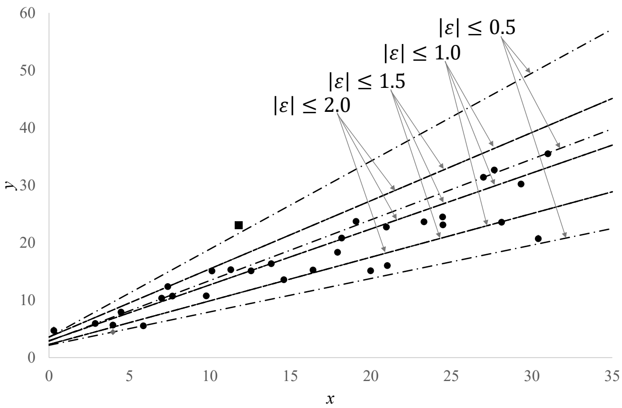


Figure 5: Model 5 using $\varepsilon = \{0.5, 1.0, 1.5, 2.0\}$

data distribution. On the other hand, samples away from the center of the interval model distort the model. We also found that the parameter of the proposed model adjusts the influence of samples away from the center of this model.

5 Conclusion In this paper, we have proposed a possibility regression model considering the vagueness included in possibility grades. Then, the usefulness of the proposed model was confirmed by using the numerical example with outliers. The proposed technique improved the forecast accuracy of models and eliminated the influence of unusual samples. In addition, by adjusting the parameter ε, it is possible to adjust the influence of samples away from the center of the model.

Furthermore, it has been improved by using it in conjunction with the model proposed by Yabuuchi to maximize the sum of the possibility grade for the width of the forecasted value. Finally, the proposed model only arranges the constraints as shown in equation (6), and sufficient results have been obtained.

REFERENCES

[1] R. Coppi, P. D'Urso, P. Giordani, A. Santoro: Least squares estimation of a linear regression model with LR Fuzzy response. *Comput. Statist. Data Anal.* 51(1), 267–286 (2006)

[2] P. Diamond: Fuzzy Least Squares. *Inform. Sci.*, 46(3), 141–157 (1988)

[3] P. Diamond: Least squares and maximum likelihood regression for fuzzy linear models. In J. Kacprzyk and M. Fedrizzi (Eds.), *Fuzzy Regression Analysis*, Omnitech Press, 137–151 (1992)

[4] P. D'Urso and T. Gastaldi: A least-squares approach to fuzzy linear regression analysis. *Comput. Statist. Data Anal.*, 34(4), 427–440 (2000)

[5] P. Guo and H. Tanaka: Fuzzy DEA: a perceptual evaluation method. *Fuzzy Sets and Systems*, 119(1), 149–160 (2001)

[6] A. Honda and Y. Okazaki: Identification of Fuzzy Measures with Distorted Probability Measures. *J. Advanced Computational Intelligence and Intell. Informatics*, 9(5), 467–76 (2005)

[7] D.H. Hong, C. Hwang, C. Ahn: Ridge estimation for regression models with crisp inputs and Gaussian fuzzy output. *Fuzzy Sets and Systems*, 142(2), 307–319 (2004)

[8] M. Inuiguchi and T. Tanino: Interval Linear Regression Methods Based on Minkowski Difference: A Bridge between Traditional and Interval Linear Regression Models. *Kybernetika*, 42(4), 423–440 (2006)

[9] H. Kwakernaak: Fuzzy random variables-I. definitions and theorems, *Inform. Sci.*, 15(1), 1–29 (1978)

[10] H. Kwakernaak: Fuzzy random variables-II. Algorithms and examples for the discrete case, *Inform. Sci.*, 17(3), 253–278 (1979)

[11] H. Lee and H. Tanaka: Upper and lower approximation models in interval regression using regression quantile techniques. *Eur. J. Oper. Res.*, 116(3), 653–666 (1999)

[12] M. Modarres, E. Nasrabadi and M.M. Nasrabadi: Fuzzy linear regression model with least square errors. *Appl. Mathematics and Computation*, 163(2), 977–989 (2005)

[13] M. L. Puri and D. A. Ralescu: The Concept of Normality for Fuzzy Random Variables, *Ann. Probab.*, 13(4), 1373–1379 (1985)

[14] A.A. Ramli, J. Watada and W. Pedrycz: Real-time fuzzy regression analysis: A convex hull approach. *Eur. J. Oper. Res.*, 210(3), 606–617 (2011)

[15] A.A. Ramli, J. Watada and W. Pedrycz: A Combination of Genetic Algorithm-based Fuzzy C-Means with a Convex Hull-based Regression for Real-Time Fuzzy Switching Regression Analysis: Application to Industrial Intelligent Data Analysis. *IEEJ Trans. Elect. Electron. Eng.*, 9(1), 71–82 (2014)

[16] H. Tanaka and J. Watada: Possibilistic Linear Systems and Their Application to The Linear Regression Model. *Fuzzy Sets and Systems*, 27(3), 275–289 (1988)

[17] J. Watada and W. Pedrycz: A Fuzzy Regression Approach to Acquisition of Linguistic Rules. In W. Pedrycz, A. Skowron and V. Kreinovich (Eds.), *Handbook of Granular Computing*, Wiley, 719–740 (2008)

[18] Y. Yabuuchi and J. Watada: Model Building Based on Central Position for a Fuzzy Regression Model. *Proc. Czech-Japan Seminar 2006*, 114–119 (2006)

[19] Y. Yabuuchi and J. Watada: Fuzzy Regression Model Building through Possibility Maximization and Its Application. *Innovative Computing, Inform. Control Express Lett.*, 4(2), 505–510 (2010)

[20] Y. Yabuuchi and J. Watada: Fuzzy Robust Regression Model by Possibility Maximization. *J. Advanced Computational Intelligence and Intell. Informatics*, 15(4), 479–484 (2011)

[21] Y. Yabuuchi: Japanese Economic Analysis by a Fuzzy Regression Model building through Possibility Maximization. *Proc. 6th Int. Conf. Soft Computing and Intelligent Syst., and 13th Int. Symp. Advanced Intelligent Syst.*, 1772–1777 (2012)

[22] Y. Yabuuchi and J. Watada: Fuzzy Robust Regression Model building through Possibility Maximization and Analysis of Japanese Major Rivers. *Innovative Computing, Inform. Control Express Lett.*, 9(4), 1033–1041 (2015)

[23] Y. Yabuuchi: Centroid-Based Fuzzy Robust Regression Model. *Innovative Computing, Inform. Control Express Lett.*, 9(12), 3299–3306 (2015)

[24] Y. Yabuuchi: The Difference between the Formulations of Possibilistic Robust Regression Model. *Proc. Joint 17th World Congr. Int. Fuzzy Syst. Assoc. and 9th Int. Conf. Soft Computing Intelligent Syst.*, N.P. (6 pages) (2017)

[25] Y. Yabuuchi: Possibility Grades with Vagueness in Fuzzy Regression Models. *Proc. KES 2017*, 1470–1478 (2017)

[26] Y. Yabuuchi: Construction of a Possibilistic Regression Model based on Possibility Grades with Vagueness. *Proc. Joint Conf. 26th Forum of Interdisciplinary Mathematics and 14th Int. Symp. Manage. Eng.*, 32–41 (2017)

Shimonoseki City Univerisity

2-1-1 Daigaku-cho, Shimonoseki, Yamaguchi 751-8510, Japan

Email: yabuuchi@shimonoseki-cu.ac.jp

AN INTELLIGENT MESSAGE AUTHENTICATION SCHEME FOR EMERGENCY AND SAFETY RELATED MESSAGES IN A VEHICULAR NETWORK

Prakash Veeraraghavan and Dalal Hanna
Department of Computer Science and Information Technology
La Trobe University, Victoria 3086, Australia

Abstract. Road safety and traffic efficiency are two important applications of a Vehicular Ad-hoc Network (VANET). In VANET, safety and emergency messages are broadcasted to all vehicles in a risk zone before the validity of the message expires. Emergency and safety-related communications have a very strict real-time requirement of 100ms latency from an originating host's application layer to destination host's application layer and a Packet Delivery Ratio (PDR) of 90% and above. Due to one-to-many nature of these emergency messages, public-key encryptions may not be employed. Furthermore, vehicles on the road have no constant access to the Roadside infrastructure. Thus, access to a Public-key Infrastructure or a Certificate Authority is not always guaranteed. Exploiting this weakness, any attacker with malicious intention can broadcast falsified emergency messages with spoofed identity to disrupt the normal operation. They may also do in order to launch a terror-like attack. Since the identity of the originating malicious vehicle cannot be established, it is not possible to take any legal action against the owner of these vehicles.

In this paper, we propose a smart digital certificate mechanism using a modified threshold cryptography scheme, that we call it as a pseudo-identity based encryption to identify the origin of every emergency message. Since the keys are not forgeable, any such malicious activities are immediately known to the receiving host vehicles and vehicle registration authorities, thus facilitating legal action. The main advantage of our proposed scheme is that it can work without constant access to a Public-key Infrastructure or a Certificate Authority. Our scheme satisfies the identical security requirements as that of the underlying public-key cryptography and incurs the same memory and run-time complexity.

The proposed scheme can also be implemented in a Mobile Ad hoc environment or a distributed environment, where source authentication is an important factor, and there is no constant access to the backbone of the network.

1 Introduction In this paper, we demonstrate a class of attack on the emergency and safety message transmission in a *Vehicular Ad hoc network* (*VANET*), by exploiting the integrity and the authentication of the message transmission. Due to real-time requirements, the current state-of-art in a vehicular network does not offer any solution to this problem. In this paper, we introduce a mathematical framework that we call as a *Pseudo-identity based encryption* that can potentially offer an efficient solution to the demonstrated attack without incurring many overheads.

An ad hoc network is a new paradigm of wireless communication for mobile nodes. It has two special characteristics, which make it different from the conventional wired network. First, there is no fixed infrastructure like a wired or a cellular phone network. There are no base stations, switching centers or routers to route packets to destinations. Secondly, in an ad hoc network, the network topology is not fixed due to the mobility of nodes. In ad hoc networks, nodes that are in the same radio range of each other communicates directly in a peer-to-peer fashion. However, nodes that are not in the same radio range may still communicate through the help of intermediate nodes. In this case, intermediate nodes act as routers to establish a multihop communication. Thus in a mobile ad hoc network (*MANET*), a node may act as a router as well as an end-node. Depending on the application environment a mobile node may have more than one role to play apart from acting as a router and an end node. Even though these networks were originally developed for military tactical applications, due to the reduction in the cost of wireless transceivers, hardware and the increase in the popularity of ubiquitous applications, ad hoc networks are deployed everywhere from a small home, video games to a battle field.

Vehicular Ad hoc Network (*VANET*) is a special type of a Mobile Ad Hoc Network (MANET), where the mobile hosts are the vehicles on the road. They communicate with each other wirelessly to establish a network. Although, passenger (and driver) safety technologies such as airbags, seat belts and anti-skid brakes are available, the deaths due to road accidents have not come down. At this moment, road traffic fatalities are the 8th leading cause of death globally, and the leading cause of death for young people aged between 15 and 29 [1]. If no action is taken to address the current crisis, global road traffic fatalities are forecasted to rise to more than 2.4 million deaths per annum by 2030 [7]. In order to reduce the number of fatalities and serious injuries, expensive sensors, radars, cameras and other state-of-art technologies are currently integrated into vehicles. These devices communicate with neighbouring vehicles in an ad hoc fashion when it detects an abnormal situation like an accident, slippery road conditions or any other noticeable hazard.

Dedicated Short-Range Communication (*DSRC*) refers to the use of *Vehicle-to-Vehicle* (*V2V*) and *Vehicle-to-Infrastructure* (*V2I*) communication that was designed to improve road safety and transportation efficiency. DSRC supports several applications. Among them, *Cooperative Collision Avoidance* (*CCA*) is the most important one. In DSRC, V2V communications are established through the use of VANET. VANETs use on-road vehicles as nodes to create an ad hoc network. DSRC supported applications can be classified into three major classes. They are: Safety-related applications, Non-safety-related applications and Infotainment. Speed management and Cooperative navigation are two examples of non-safety applications. Tourist and Traveller Information Support, Streaming music are two examples of Infotainment. These two classes of applications require communication infrastructure such as Roadside Unit (RSU). The motivation for allowing non-safety applications over DSRC is to create commercial opportunities, thereby, making the DSRC technology more cost-effective.

Road safety is not the only road issue. Traffic efficiency is another major issue, especially in metropolitan areas all around the world. The cost of the time spent sitting in traffic has been estimated at $11.1 billion, annually [6]. This figure does not include the cost of the fuel burned waiting for traffic to move, the cost to the environment or the flow-on costs to the nation's health system. Particularly in Australia, statistics show that the cost of congestion was $9.4 billion in 2005, and the social costs of congestion are forecasted to reach $20.4 billion by 2020 [3]. These figures and statistics indicate that the need for a significant reduction in both traffic congestion and vehicle crashes is a serious challenge throughout the world.

In VANET, safety messages are broadcasted to all vehicles in a risk zone before the validity of the message expires. A risk zone is the area in which the content of a specific safety message is relevant to all vehicles. The size of the risk zones varies depending upon the requirements of different safety applications. The risk zone of a particular application might be much larger than the one-hop transmission range of the source node. As a result, multi防op broadcasting is required for vehicles in the risk zone which are not in the one-hop transmission range of the source node. Thus a vehicle receiving a multi-hop safety message needs to rebroadcast before the expiry of its lifetime. The Time-to-live (TTL) value is the number of hops the emergency message is valid before it is discarded. The source or the originating vehicle of an emergency message sets the TTL value. Every vehicle that receives an emergency message reduces this value by one before the message is rebroadcasted. A vehicle that receives an emergency message with a TTL-value 1 will not rebroadcast the message.

Vehicle-to-Vehicle (V2V) safety communication has a very strict real-time requirement of 100ms latency from source host's application-layer to a destination host's application-layer, and a Packet Delivery Ratio (PDR) of at least 90% [12]. Most of the safety messages in a vehicular network are applicable to a region (or a smaller neighbourhood like accident zone), rather than to another individual vehicle. Thus, broadcasting is the most efficient way of disseminating emergency messages. Due to this real-time requirement, heavy cryptography mechanisms are not employed. Furthermore, due to one-to-many nature of the safety messages, the use of any encryption is not preferred. Whenever a vehicle received a safety message from another vehicle, it is impossible to identify the source and the authenticity of the message. To verify the identity of the source vehicle, digital signature may be used. However, without access to a Public-Key Infrastructure (PKI) or a local trusted Certificate Authority (CA); it is impossible to verify a digital signature. In a vehicular communication, we cannot always assume that a vehicle has access to a Roadside Unit (RSU) to obtain the public-key information. We exploit this weakness in this paper to demonstrate a message falsification attack. This is presented in Section 2. In Section 3, we define a pseudo-identity based encryption scheme based on Shamir's threshold cryptography [10]. Based on the proposed scheme, the identity of the transmitting vehicle can immediately be identified even if there is no access to a RSU or CA or PKI. In Section 4, we present our solution architecture to solve the message falsification attack presented in Section 2. In Section 5, we present the concluding remarks and future direction.

The readers are referred to Hartenstein and Laberteaux [5] for more fundamental details on VANET.

2 Message Falsification attack In this section, we explain how a malicious vehicle can exploit the absence of a safety message authentication to launch a message falsification attack. Traditional security threats in wireless communication, such as eavesdropping, forgery, and modification, could be easily taken advantage of in VANETs [13]. In order for the CCA to work effectively, all vehicles in the road network must trust each other and are able to trust the alerts and warnings issued by V2V devices working with messages from other V2V devices. This is a major assumption and can be exploited to launch an attack.

Any vehicle that detects a road-safety concern will immediately broadcast an emergency message. The message will contain information about the specific condition. All vehicles that receive this safety message must process them and take appropriate action. If the message is applicable to a multi-hop environment, the receiving vehicle must rebroadcast the safety message.

Let T be a vehicle with a modified DSRC protocol stack. It is capable of sending forged

Figure 1: [12]

emergency messages using falsified vehicle identification. Since DSRC use the traditional 802.11 wireless spectrum, the vehicle may also be equipped with one or more mobile wireless devices capable of sending forged emergency messages. Even though the DSRC standard dictate the amount of transmission power to be used in broadcasting emergency messages, T may violate this standard and transmit these forged messages at a much higher power level to reach a larger region. T may transmit a variety of safety-critical messages such as accident, road closure, severe congestion-ahead etc. to divert the vehicles behind through an alternative congested route. Even though T may not gain any financial advantage through this attack, he may disrupt the legitimate DSRC services to launch a *Denial of Service* (*DoS*) attack. In a worst case scenario, the aides of T may launch a terror like attack on the congested road.

In the following subsection, we simulate an emergency communication in a vehicular network and demonstrate how important is the central region surrounding an emergency zone (like an accident, or the eye of a congestion).

2.1 Communication overhead in a Central region In our simulation, we follow the DSRC standard that every vehicle's transmitter has the same transmission range as that of other vehicles in the network (typically 300m). In the literature, vehicles on the roads are modeled as an *Interval graph* [5]. However, we note that this modeling is valid only for single lane traffic. In typical multilane freeway traffic, vehicles are located in an $n \times m$ rectangle. Since each vehicle has the same transmission distance, without loss of generality, we assume that all the vehicles have the unit transmission distance. If the distance between two vehicles is less than one, we join them by an edge. Thus, it is easy to see that the network topology in this case is a *unit-disc* graph. For each vehicle T, we define $r(T)$ as the number of emergency messages the vehicle T has rebroadcasted. A realistic vehicular scenario is presented in Figure 1.

We use the *Simulation of Urban Mobility* (*SUMO*) traffic simulator to place k number of vehicles (k range from 5 to 100 in step of 5) in a $1km \times 8$-lane road grid (4 lanes on either direction). We create a random emergency zone within the first 50m of this grid, as in Figure 1. Any vehicle that approaches this emergency zone will trigger an emergency broadcast message. We set a *Time-to-live* (*TTL*) value of 3 for each triggered message. Thus every vehicle that receives an emergency message with a TTL value of 2 or 3 will rebroadcast the emergency message after reducing the TTL value by one. For a given experiment, we find the maximum, minimum, mean and median number of emergency messages rebroadcasted by a vehicle. For each k, we generate 50 different topologies. We take the average across all the 50 different topologies to remove any random simulation artifacts. We present our findings in Figure 2. As we can see min $\{r(T)\}$ is almost zero for the all the topology we generated. This is because that, there are always vehicles in the edge of the emergency zone that do not rebroadcast any message. On the otherhand, the max $\{r(T)\}$ grows rapidly with respect to the number of vehicles in the topology. On a topology with 100 vehicless, the max $\{r(T)\}$ is 1500. At all the stages the mean and median are close-by (this property also proves that our simulation results are unbiased and the traffic generation is symmetric). We have the mean and median values close to 33% of the maximum $\{r(T)\}$.

Figure 2: Communication overload

Based on the above experiment, a vehicle (or a group of vehicles) with malicious intention will transmit as many as thousands of falsified emergency messages using different forged vehicle identifications to launch a DoS attack. They may also use the TTL-value other than 3 to pretent that they are not the originating source of an emergency message. In the absence of any digital certificate, it is hard for a law enforcing agencies to take legal action against these vehicles.

3 Smart message authentication scheme for safety and emergency messages in a Vehicular network In this section, we propose a smart message authentication scheme to protect vehicular communication from the message falsification attack mentioned in Section 2. Even though, there are a number of proposals available in the literature, our proposed scheme will work in the absence of any PKI or CA. Thus, our proposal is more suitable for a VANET and MANET environments, where there is no guarantee to have access to a central infrastructure.

In VANET, message falsification attack is possible due to the lack of a message authentication feature. However, traditionally, digital certificate is used to solve the message authentication problem. In order for the message authentication system to work effectively, the public key of the transmitting vehicle must be available with all the receiving vehicles. This can be done in two ways; the local registration authority may load securely the public key of all registered vehicles to every vehicle in the country. A list of revoked keys is transmitted to vehicles whenever, they have access to RSU. Thus every vehicle 痴 public-key database is up to date to a certain degree. However, the database will be large due to the number of registered vehicles in every city (or state or country). This not only requires more storage, but increases the latency due to database search. In the worst case scenario, database search consumes $O(n)$-time complexity for a linear search or $O(log(n))$ time, in case the data is organized in a binary tree. Thus, the real-time requirement of 100ms for emergency and safety messages in a vehicular network may not be achievable. An alternative to the offline storage of public key is to have a constant access to a trusted certificate authority to obtain the public key of the transmitting vehicle. This requirement may be achievable in a wired network; however, in a VANET or in a MANET environment, there is no guarantee to have a constant access to a trusted CA.

These requirements force us to look for a new paradigm to provide solution to the message falisification attack.

In our earlier paper [9], we introduced the notion of pseudo-identity based encryption. In this paper, we extend our original idea and provide a solution to the message falsification attack in a vehicular network.

In a vehicular network, more than one vehicle may detect the same road emergency condition. If the condition is severe like road accident, road closure or severe congestion, several vehicles may detect it simultaneously. Thus, if the same emergency condition is transmitted by several vehicles as the originating source (with the maximum TTL value), then the transmitted message may be trustable. We also incorporate this observation in our framework. For this purpose, we use the threshold cryptography introduced by Shamir [10].

In a (k, n)-threshold cryptographic scheme, the secret key d is divided among n-shareholders such that

- The knowledge of k or more shares make it possible to compute the global secret key d.
- The knowledge of k-1 or fewer share make it impossible to compute the global secret key d.

In threshold cryptographic systems, k is chosen in such a way that any adversary can break (k-1) or less shareholders. Thus the system may have less than k malicious shareholders.

Since our proposed algorithm is based on threshold cryptographic scheme, we describe briefly here for completeness.

The threshold cryptography is based on *polynomial interpolation.* Given k distinct points in the two dimensional plane $(x_1, y_1), (x_2, y_2), \ldots (x_k, y_k)$, (with distinct $x_i's$), there is one and only polynomial $P(x)$ of degree k-1 passing through all the k-points.

Let P_{pub}, P_{pri} be the public and the private key for an underlying public key cryptography (like RSA). k-threshold cryptography is used to share the private key P_{pri} to all legitimate nodes (called shareholders), through a random polynomial $f(x)$ of degree k-1.

Even though, polynomial interpolations are defined over $\mathbb{R}$ or over any general ring, threshold crypto systems use polynomial interpolation over $\mathbb{Z}_n$. The choice of n will be decided by the underlying public key crypto system.

Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots a_{k-1}x^{k-1}$ be a polynomial of degree k-1 such that $f(0) = P_{pri} mod\, n$ and $a_1, a_2, \ldots, a_{k-1}$ belong to some arbitrary ring $\mathcal{Q}$. For each legitimate node with node identity v_i, its secret share is $SK_i = (f(v_i) mod\, n)$. For any coalition of k-nodes $v_1, v_2, \ldots v_k$, Lagrange interpolation states that $f(0) = P_{pri} = \sum_{j=1}^{k} SK_j * l_{V_j}(0)\ mod(n)$, where $l_{V_j}(x) = \frac{(x-v_1)(x-v_2)\ldots(x-v_{j-1})(x-v_{j+1})\ldots(x-v_k)}{(v_j-v_1)(v_j-v_2)\ldots(v_j-v_{j-1})(v_j-v_{j+1})\ldots(v_j-v_k)}$ is the Lagrange coefficient. Let $P_{V_j} = SK_j * l_{V_j}(0)$. The knowledge of P_{V_j} can expose SK_j. Thus, they cannot be revealed to any one.

Let X be any arbitrary message for which we wish to compute the digital signature $X^{P_{Pri}}$. Since, none of the shareholders have the knowledge of P_{pri}, we have to contact k shareholders, say, (v_1, v_2, $\cdots$, v_k) to obtain their partial digital signatures X^{SK_i}. Since the discrete log problem is computationally hard, from the partial signature, no adversary can compute SK_i. We can then compute the digital signature of X using the formula: $\prod_{j=1}^{k} (X^{SK_j})^{l_{V_j}(0)} = (X)^{\sum_{j=1}^{k} SK_j * l_{V_j}(0)} = X^{P_{pri}}$.

Thus by the coalition of k-shareholders, any message can be digitally signed by the global secret key without the presence of an CA. Therefore, there is no need for the CA after the bootstrap process.

The threshold cryptography in its original form has the following disadvantages:

1. From partial digital signatures X^{SK_i}, it is impossible to obtain the identity of the signed shareholder.

2. It is not possible to verify whether a shareholder whose identity is v_i has signed properly or not.

Since the above two properties are important for providing solution to message falsification attack, we cannot use the threshold cryptography.

Shamir [11], proposed the concept of *identity based cryptography.* In this scheme, a user's public identity like the email address is used as his public key. As a result identity-based cryptography eliminate the need for a PKI or a CA. Although Shamir proposed the concept, he was unable to construct any identity-based cryptographic scheme and conjectured the existence of such a scheme. His conjecture was independently solved by Boneh and Franklin [2] and Cocks [4].

Boneh and Franklin's solution is based on the *Weil Paring.* Their algorithm is called as *BasicIdent.* Elliptical curve variant of the *bilinear Diffie-Hellman* (BDH) problem is considered as the underlying hard problem in their scheme. It has been proved that in a random-oracle model, the protocol is semantically secure under the BDH assumption. Though their algorithm is computationally secure, it is hard to implement on a MANET/VANET environment due to its processor and memory requirements. In a VANET environment, BasicIdent may not satisfy the real-time requirement due to its run-time complexity.BasicIdent is not chosen ciphertext secure. However, Fujisaki-Okamoto transformation allows for conversion to a scheme having this property called *FullIdent.*

Cocks model uses *quadratic residues modulo* over a large composite integer as their underlying hard-problem. Though his solution is much simpler compared with [2], it is not practical as it uses bit-by-bit encryption, which is not economical.This scheme also does not preserve key-privacy, i.e. a passive adversary can recover meaningful information about the identity of the recipient observing the ciphertext.

3.1 Pseudo-Identity based threshold cryptography It is important to note that in a threshold cryptographic scheme, the private share of each shareholder may not have a public key component. Even, if some private share has a public key component, the public key may not reflect the identity of the node.

In this section, we modify the threshold cryptographic scheme in such a way that every secret share has a corresponding public key component and the public key component will be related to the identity of the node. We call it as pseudo-identity based threshold cryptography. We outline the importance of the threshold parameter and its releveance to our work in the following subsection. In vehicular communication, the CA may be the local registration authority or someone designated by the local registration authority.

As like the Shamir's threshold cryptography, the CA must construct the global private and public key pair for any underlying public key cryptography (We assume it to be RSA here). We outline the process as follows:

Let P, Q be two *safe-primes.* That is $P = 2P_1 + 1$ and $Q = 2Q_1 + 1$, where P_1 and Q_1 are prime numbers. Let $N = P * Q$. N is used as the modulus for both the public and private keys. The RSA, being a block cipher, both the plain text and the cipher text are integers between 0 and N-1. Then the Euler's tortient function $\phi(N) = (P-1)(Q-1) = 4P_1Q_1$. The CA then choose a non-trivial number d as its *global secret key* in such a way that d has no common factor with N and $\phi(N)$. Since $\phi(N)$ is an even number, it follows that d must be an odd number. The CA then choose the global public key e in such a way that $d * e \equiv 1(mod\, \phi(N))$. It is easy to see that d and e will have the following properties:

1. They are odd numbers
2. e and d are not equal to P, P_1, Q, Q_1 and their multiples.

We now outline the modification that leads to our proposed design:

Let $k \neq 1$ be the threshold system parameter. Let $f(x) = d + R(x)$; where $R(x) = a_1x + a_2x^2 + \ldots + a_{k-1}x^{k-1}$ and $a_i's$ belong to some ring of integers $\mathcal{Q}$ be the threshold system polynomial. Except k, all the other system parameter are kept secret and not known to anyone except the CA.

Let X_i be the identity of the i-th node in the network. Then according to the traditional threshold cryptography, its secret share is $SK_i = f(X_i) = d + R(X_i)\ (mod\, N)$, where N is the integer modulo defined above. In order for SK_i to have a public key component, it must satisfy the above two properties.

We first derive a condition to ensure that SK_i is odd for every integer i. Since d is an odd number, SK_i is an odd number if and only if $R(X_i)$ is an even number.

Theorem 1. *Let $R(x) = a_1x + a_2x^2 + \ldots + a_{k-1}x^{k-1}$, where $a_i's$ belong to some ring of integers $\mathcal{Q}$. $R(i)$ is an even number for every integer i if and only if the number of odd $a_i's$ are even.*

Let $R(x)$ be an even integer for every integer $i \in \mathcal{Q}$. In particular, $R(x)$ is an even integer for $x = 1$. That is $R(1) = a_1 + a_2 + \ldots + a_{k-1}$ is an even integer. By grouping odd and even $a_i's$, we have $R(1) =$ (sum of odd $a_i's$) + (sum of even $a_i's$). This implies that (sum of odd $a_i's$) is an even number; and hence the number of odd $a_i's$ are even.

Conversely, let the number of odd $a_i's$ are even. Let x be an even integer. Then a_ix^i is always an even integer. Since $R(x)$ is a sum of even integers, it is an even integer. Now let y be an odd integer. Then a_iy^i is an even number whenever a_i is an even number and odd number if a_i is an odd number. Since the number of odd $a_i's$ are even, it follows that in this case also $R(y)$ is an even integer.

We now present an algorithm in which the keys are computed in such a way that every secret share has a corresponding public key component.

Step 1: Let X be the i-th shareholder whose non-forgeable identity (similar to the MAC address; in case of VANET, it is the vehicle's registration number (REGO)) is X_i. Let $f(x) = d + R(x)$ be the secret polynomial, where $R(x)$ is an even number for every integer x. Choose the smallest integer r_i such that $SK_i = f(X_i + r_i) = d + R(X_i + r_i)(modN)$ has a public key component and SK_i is not distributed to any shareholder before. Since the modulo N is large, such r_i will always exist.

Now SK_i is the secret share for the node X with the non-forgeable ID X_i and PK_i is its public key component for this corresponding SK_i. $< SK_i, PK_i, N >$ is loaded on to X_i's secure module during the registration or bootstrap process.

Step 2: After computing the public key for n-shareholders whose non-forgeable IDs are $X_1, X_2, \ldots, X_n$, the CA then computes the *public key polynomial* $P(x) = b_0 + b_1x + b_2x^2 + \ldots + b_{n-1}x^{n-1}$ of degree $n - 1$ as whose $b_i's$ are given as follows:

$$\begin{bmatrix} 1 & X_1 & X_1^2 & \dots & X_1^{n-1} \\ 1 & X_2 & X_2^2 & \dots & X_2^{n-1} \\ 1 & X_3 & X_3^2 & \dots & X_3^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & X_n & X_n^2 & \dots & X_n^{n-1} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} PK_1 \\ PK_2 \\ PK_3 \\ \vdots \\ PK_{n-1} \end{bmatrix}$$

Since $X_1, X_2, \cdots, X_{n-1}$ are different, the above matrix equation has a unique solution. Thus there is a unique polynomial $P(x)$ of degree $n-1$ such that $P(X_i) = PK_i$. We call this polynomial as a *hash polynomial* for our threshold crypto system. The CA will load this hash-polynomial in the tamper-resistant available in every vehicle. This polynomial is used to generate the public key of any shareholder, provide a vehicle know the REGO of a shareholder. If, all the coefficient of this hash-polynomial is known to an adversary, he will not able to compromise the system. The main advantage in distributing this polynomial is that if a node knows the identity of any other node, it can easily compute its public key without the presence of an CA.

After this step, there is no need for the existence of an CA.

4 The Solution Architecture In this section, we provide an elegant solution to the message falsification attack in a vehicular network. We make the following natural assumptions about the system.

1. Each vehicle on the road has a unique registration number provided by the registration authority. This registration number is used as a public identity of the vehicle.
2. We assume that vehicles are equipped with an on-board camera that can recognize the registration number of vehicles in front and behind. Even, some of the current budget model cars are equipped with an onboard camera that can recognize street signs, speed limits and traffic signals.
3. Vehicles are equipped with tamper-resistant storage and processor modules like (Trusted Platform Module (TPM) [8]), where the hash function and crypto schemes are securely stored by the registration authority. Since the current and the future generation cars are controlled by real-time onboard computers, their kernel needs to be protected from malware. Thus, a tamper-resistant module is necessary.

   For every vehicle with REGO X_i its shared private key component SK_i, the hash-polynomial are securely loaded to the tamper-resistant module by the certificate authority.
4. Whenever a vehicle has access to the roadside infrastructure (RSU), vehicles will sync their key revocation information with the certificate authority.

The threshold value k is chosen by the CA in such a way that some severe road conditions (like major accident, road closure, etc.) can be detected by k vehicles independently within a reasonable amount of time.

We now present the black box model of our proposed crypto scheme. We call it as a black box model the entire architecture is implemented in a tamper-resistant hardware like Trusted Platform Module [8].

Figure 3: Singature generation module

4.1 Emergency and Safety Message Transmission Whenever a vehicle's hardware detects a safety and emergency condition, it will pass on the message to the secure module for digitally signing the message. For our discussion, we assume that the underlying crypto system is RSA. This may be replaced with any public-key cryptography. The RSA engine will securely retrieve the private key SK_i from the secure storage space. Then <Plain text emergency message, signed emergency message, REGO >is broadcasted to every vehicle with the appropriate TTL value.

The block diagram is presented in Figure 3

4.2 Signature verification Let a vehicle T receive an emergency message. The following steps are taken for verifying the signature:

1. The onboard camera reads the REGO of vehicles around to see if the REGO in the message can be recognized. If the REGO is recognized, then the *onboard_camera_check* flag is set to 1; otherwise, it is set to 0. In several cases, due to obstructions, it may not be possible to verify the REGO by the onboard camera. This flag is not going to influence the action to be taken for this emergency message. It will only serve as an extra layer of security.

   If the same emergency message from the same REGO is seen before either with a same TTL value or lower, the message is discarded.

2. The REGO is passed on to the hash polynomial module as an input. The hash polynomial will output PK_i, the public-key component for this REGO.

3. PK_i along with the signed message is passed on to the RSA module. The RSA module will decrypt the signed message using PK_i and outputs the plain text.

4. The received plain text message along with the decrypted plain text message is passed on to the comparison module. If both the messages are the same, the signature is verified; else the signature verification failed. The received message is discarded if the signature verification failed.

Once the signature is verified, the following actions are taken:

a. Appropriate response to this emergency message is taken.

b. If the message is applicable to a multi hop region, and the TTL value is greater than one, T will rebroadcast the message after reducing the TTL value by one and append its REGO along with the originally received REGO.

c. If the message is critical and needs to obtain the global signature the threshold cryptography, the message is passed on to the temporary storage area, until k similar messages from different originating vehicles are obtained.

The process block diagram for this module is given in Figure 4

Figure 4: Singature Verification module

Figure 5: Global Singature generation module

4.3 Global signature generation If a critical safety-related message independently originates from k or more vehicles, any vehicle can combine all the partial signatures to a globally signed message. Once k similar messages from different originating vehicles are available in the temporary storage area, it is then passed on to the threshold signature generation module. This module will combine all partial signatures and generate the globally signed message as per the threshold cryptographic algorithm outlined in Section 2.

The process block diagram for this module is given in Figure 5

5 Conclusion and Future direction In this paper, we proposed an elegant source authentication scheme based on the modified threshold cryptography. The proposed scheme can be modified effectively to identify vehicles that transmit false safety and emergency messages with fictitious vehicle identity. Our proposed scheme is also used by the law enforcing agencies to precisely to identify the owner of the malicious vehicles. They may also able to revoke their registration dynamically. The key revocation information is transmitted to every vehicle whenever they have access to RSU.

We present below the security analysis of our proposed scheme.

5.1 Security Analysis of the proposed solution

1. Since SK_i and the hash polynomial are loaded onto a tamper-resistant module securely by the registration authority, no user has access to them.
2. The secure module will not sign any non-standard emergency messages. This is to ensure that no user (including the owner of the vehicle) launch a chosen plaintext attack to guess the secret key. The crypt-analysis of our proposed scheme is equivalent to the crypt-analysis of the underlying RSA and the threshold system. Since the underlying RSA and the threshold cryptography are secure, it follows that our proposed model is secure.
3. Since the private key share and the hash polynomial are not disclosed to the owner of a vehicle, even change of ownership of a vehicle does not affect the security of the key.
4. In the event that a key is revoked (incase the associated vehicle registration is suspended), the CA will communicate with every vehicle, whenever they have access to a RSU. During this time, vehicles will sync their key revocation database. This database is stored in the secure storage area.

Our proposed architecture can also be implemented in a MANET or in a distributed environment where the source authentication is an important factor, and there is no constant access to the backbone network. Our future work involves implementing the proposed scheme in VANET hardware to obtain the real-time performance measures, especially, to evaluate the introduced latency of our scheme in a sparse, average and dense vehicular network.

References

[1] K.Bilstrup, A survey regarding wireless communication standards intended for a high-speed vehicle environment, Technical Report; IDE0712, Halmstad University, Halmstad, 2007.

[2] D.Boneh and M.Franklin. Identity-based encryption from the weil pairing. Proceedings of CRYPTO'01, LNCS, pages 213-229, 2001.

[3] Bureau of Transport and Regional Economics (BTRE). Estimating urban traffic and congestion cost trends for Australian cities, working paper 71. Australian Government, Department of Transport and Regional Services, BTRE, Canberra ACT, 2007.

[4] C.Cocks. An identity-based encryption scheme based on quadratic residues. Proceedings of IMA'01, LNCS, 2260:360-363, 2001.

[5] H.Hartenstein and K.P.Laberteaux, VANET: Vehicular Applications and Inter-Networking Technologies, John Wiley & Sons, NY, USA (2010)

[6] IBM. The roads to a smarter planet from the ibm executive series: Smarter thinking for a smarter planet @ONLINE. https://www.ibm.com/smarterplanet/global/files /nz_en_nz_health_ibmlbn0041_transtasman_book.pdf, May 2015.

[7] X.Ma and X.Chen. Saturation performance of IEEE 802.11 broadcast networks. Communications Letters, IEEE, 11(8): 686-6898, 2007.

[8] TPM: The Trusted Platform Module, @ONLINE. https://en.wikipedia.org/wiki/ Trusted_Platform_Module, Dec.2017

[9] P.Veeraraghavan, Pseudo-identity based encryption and its application in mobile ad hoc networks, IEEE 10th Malaysia International Conference on Communications, (2011), Malaysia.

[10] A.Shamir. How to share a secret. Communications of the ACM, 22(11):612-613, 1979.

[11] A.Shamir. Identity-based cryptosystems and signature schemes. Proceedings of CRYPTO84, LNCS, pages 47-53, 1984.

[12] H. Trivedi, P. Veeraraghavan and et.al., Routing mechanisms and cross-layer design for Vehicular Ad Hoc Networks: A survey, 2011 IEEE Symposium on Computers Informatics (ISCI), 2011.

[13] F.Wang et.al., 2FLIP: A Two-Factor Lightweight Privacy-Preserving Authentication Scheme for VANET, IEEE Transactions On Vehicular Technology, VOL. 65, NO. 2, February 2016

SOME RESULTS ON DIRECT SUMS OF BANACH SPACES — A SURVEY

MIKIO KATO*, TAKUYA SOBUKAWA† AND TAKAYUKI TAMURA‡

ABSTRACT. We shall discuss three notions of direct sums of Banach spaces, Z-, ψ-, and A-direct sums, which are in fact all isometric. Weak nearly uniform smoothness, uniform non-squareness and uniform non-ℓ_1^n-ness etc. will be discussed, especially in the general A-direct sum setting. As applications some examples of Banach spaces will be presented concerning FPP as well as super-reflexivity.

1 Introduction

Direct sums of Banach spaces have been often treated in the context of geometry of Banach spaces and the fixed point property (e.g. [2, 3, 6, 7, 8, 9, 10, 11, 14, 15, 16, 21, 22, 23, 25, 26, 27, 28, 29, 30, 32, 33, 36, 40, 41, 42, 43]). We shall discuss three notions of direct sums of Banach spaces.

It is known that every absolute normalized norm $\|\cdot\|_{AN}$ on $\mathbb{R}^N$ corresponds to a unique convex function ψ on the standard simplex in $\mathbb{R}^{N-1}$ (we shall mention it precisely in Section 2). So we shall write $\|\cdot\|_\psi$ for $\|\cdot\|_{AN}$ and refer to as a ψ-norm. Let $\|\cdot\|_Z$ and $\|\cdot\|_A$ be an absolute and an arbitrary norm on $\mathbb{R}^N$ respectively, which we shall call a Z-norm and an A-norm.

A Z-direct sum $(X_1 \oplus \cdots \oplus X_N)_Z$ of Banach spaces $X_1, \ldots, X_N$ is their direct sum equipped with the norm

$$\|(x_1, \ldots, x_N)\|_Z = \|(\|x_1\|, \ldots, \|x_N\|)\|_Z, \quad (x_1, \ldots, x_N) \in X_1 \oplus \cdots \oplus X_N,$$

where the norm $\|\cdot\|_Z$ in the right side is an absolute norm on $\mathbb{R}^N$. A ψ-direct sum $(X_1 \oplus \cdots \oplus X_N)_\psi$ and an A-direct sum $(X_1 \oplus \cdots \oplus X_N)_A$ are defined in the same way by means of a ψ-norm $\|\cdot\|_\psi$ and an A-norm $\|\cdot\|_A$.

In Section 2 the correspondence will be mentioned between the set AN_N of all absolute normalized norms on $\mathbb{R}^N$ and the collection Ψ_N of all convex functions satisfying certain conditions on the standard simplex Δ_N in $\mathbb{R}^{N-1}$. A couple of subclasses $\Psi_N^{(1)}$ and $\Psi_N^{(\infty)}$ of Ψ_N will be discussed, which were introduced in Kato and Tamura [29, 30] to discuss weak nearly uniform smoothness and uniform non-squareness for direct sums. These classes can be described in terms of Properties T_1^N and T_∞^N, which Dowling and Saejung [10] introduced to discuss uniform non-squareness for Z-direct sums.

In Section 3 it will be seen that any A-direct sum is isometrically isomorphic to a ψ-direct sum with some $\psi \in \Psi_N$ ([8]); therefore the direct sums stated above are all isometrically isomorphic and ψ-direct sums are general enough. In Sections 4, 5, and 6 we shall obtain A-direct sum versions of previous results.

Section 4 will deal with weak nearly uniform smoothness (WNUS-ness in short). *Every WNUS space has FPP, the fixed point property* (for nonexpansive mappings; [15, 14]). A characterization of WNUS-ness for $(X_1 \oplus \cdots \oplus X_N)_\psi$ will be presented by means of the class $\Psi_N^{(1)}$.

2010 *Mathematics Subject Classification.* 52A51, 26A51, 47H10, 46B10, 46B20.

Key words and phrases. absolute norm, convex function, Properties T_1^N and T_∞^N, direct sum of Banach spaces, weak nearly uniform smoothness, uniform non-squareness, uniform non-ℓ_1^N-ness, fixed point property.

*, †, ‡ The first and third authors, resp. the second author were supported in part by JSPS KAKENHI (C) Grant number JP26400131, resp. JSPS KAKENHI (B) Grant number JP15H03621.

In Section 5 we shall discuss uniform non-squareness (UNSQ-ness) which has been playing an important roll in geometry of Banach spaces. The starting point of our discussion is the following result in Kato-Saito-Tamura [22]: *A ψ-direct sum $X \oplus_\psi Y$ is UNSQ if and only if X and Y are UNSQ and $\psi \neq \psi_1, \psi_\infty$, where ψ_1 and ψ_∞ are the corresponding convex functions to the ℓ_1- and ℓ_∞-norms, respectively.* They [22] asked for a characterization for N Banach spaces. We shall present a sequence of partial results by Dowling-Saejung [10], Betiuk-Pilarska and Prus [2], and Dhompongsa-Kato-Tamura [8]. In [10] the following was shown: *Under the assumption $\|\cdot\|_Z$ is strictly monotone, $(X_1 \oplus \cdots \oplus X_N)_Z$ is UNSQ if and only if $X_1, \ldots, X_N$ are UNSQ and $\|\cdot\|_Z$ has Properties T_1^N and T_∞^N.* In the case $N = 3$ this assumption was dropped. More precise results are shown in [8] for ψ-direct sums in terms of $\Psi_N^{(1)}$, from which the A-direct sum versions are derived. In [2] it was shown that *$(X_1 \oplus \cdots \oplus X_N)_Z$ is UNSQ if and only if $X_1, \ldots, X_N$ and $(\mathbb{R}^N, \|\cdot\|)$ are UNSQ*, where it remains unknown when $(\mathbb{R}^N, \|\cdot\|)$ is UNSQ.

Recently Kato-Tamura [30, in preparation] obtained a characterization of UNSQ-ness for $(X_1 \oplus \cdots \oplus X_N)_\psi$ as well as A-direct sum without any additional assumption, which covers all the above-mentioned results and explains why the case $N = 3$ is successful in [10].

In Section 6 uniform non-ℓ_1^n-ness will be discussed. When $n = 2$, uniform non-ℓ_1^2-ness coincides with UNSQ-ness. Every uniformly non-ℓ_1^n space is uniformly non-ℓ_1^{n+1}. The above result for UNSQ-ness of $X \oplus_\psi Y$ ([22]) is extended to uniform non-ℓ_1^n-ness ([23]). The spaces $X \oplus_1 Y$ and $X \oplus_\infty Y$ cannot be UNSQ, while they can be uniformly non-ℓ_1^n, $n \geq 3$. We shall discuss when they are uniformly non-ℓ_1^n.

In the last Section 7 applications to FPP will be discussed. As UNSQ spaces have FPP ([16]), it is natural to ask whether every uniformly non-ℓ_1^3-space has FPP. We shall see a plenty of Banach spaces (direct sums) with FPP which is not UNSQ can be constructed. Super-reflexivity will be treated as well.

In the following $X, X_1, \ldots, X_N$ will stand for Banach spaces. Let S_X and B_X denote the unit sphere and the closed unit ball of X. Let $\mathbb{R}_+^N$ denote the set of all points in $\mathbb{R}^N$ with nonegative entries.

2 Absolute norms on $\mathbb{R}^N$ and convex functions A norm $\|\cdot\|$ on $\mathbb{R}^N$ is called *absolute* if $\|(a_1, \cdots, a_N)\| = \|(|a_1|, \cdots, |a_N|)\|$ for all $(a_1, \cdots, a_N) \in \mathbb{R}^N$, and *normalized* if $\|(1, 0, \cdots, 0)\| = \cdots = \|(0, \cdots, 0, 1)\| = 1$. A norm $\|\cdot\|$ on $\mathbb{R}^N$ is called *monotone* provided that, if $|a_j| \leq |b_j|$ for $1 \leq j \leq N$, $\|(a_1, \ldots, a_N)\| \leq \|(b_1, \ldots, b_N)\|$. $\|\cdot\|$ is called *strictly monotone* provided it is monotone and, if $|a_j| < |b_j|$ for some $1 \leq j \leq N$, $\|(a_1, \ldots, a_N)\| < \|(b_1, \ldots, b_N)\|$. The following is known.

Lemma 2.1 (Bhatia [4], see also [30]) *A norm $\|\cdot\|$ on $\mathbb{R}^N$ is absolute if and only if it is monotone.*

We shall see that for every absolute normalized norm on $\mathbb{R}^N$ there corresponds a unique convex function ψ on a certain convex set in $\mathbb{R}^{N-1}$ ([38, 5]).

Lemma 2.2 *Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^N$ and define*

$$\psi(s) = \left\|(1 - \sum_{i=1}^{N-1} s_i, s_1, \ldots, s_{N-1})\right\|, \quad s = (s_1, \cdots, s_{N-1}) \in \Delta_N, \tag{2.1}$$

where $\Delta_N = \{s = (s_1, \cdots, s_{N-1}) \in \mathbb{R}^{N-1} : \sum_{i=1}^{N-1} s_i \leq 1,\ s_i \geq 0\}$. Then:

(i) *The norm $\|\cdot\|$ is normalized if and only if*

$$\psi(0, \cdots, 0) = \psi(1, 0, \cdots, 0) = \cdots = \psi(0, \cdots, 0, 1) = 1. \tag{A_0}$$

(ii) For each $1 \le k \le N$ the following (a) and (b) are equivalent.
(a) The norm $\|\cdot\|$ is monotone in the k-th entry, that is,

$$|x_k| \ge |y_k| \;\Rightarrow\; \|(x_1, ..., \underbrace{x_k}^{k}, ..., x_N)\| \ge \|(x_1, ..., \underbrace{y_k}^{k}, ..., x_N)\|$$

(b) The convex function ψ satisfies

$$(A_k) \qquad \psi(s_1, \ldots, s_{N-1}) \ge (1 - s_k)\psi\left(\frac{s_1}{1 - s_k}, \ldots, \overbrace{0}^{k-1}, \ldots, \frac{s_{N-1}}{1 - s_k}\right)$$

In the case $k = 1$, (A_1) should be understood as

$$(A_1) \qquad \psi(s_1, \cdots, s_{N-1}) \ge (1 - s_0)\psi\left(\frac{s_1}{1 - s_0}, \ldots, \frac{s_{N-1}}{1 - s_0}\right),$$

where $s_0 = 1 - \sum_{i=1}^{N-1} s_i$.

Let

$$AN_N = \{\text{all absolute normalized norms on } \mathbb{R}^N\},$$
$$\Psi_N = \{\text{all convex functions } \psi \text{ satisfying } (A_k), 0 \le k \le N\}.$$

Theorem 2.3 (Saito-Kato-Takahashi [38]) (i) *For any* $\|\cdot\| \in AN_N$ *let*

$$(2.1) \qquad \psi(s) = \left\|(1 - \sum_{i=1}^{N-1} s_i, s_1, \ldots, s_{N-1})\right\|, \quad s = (s_1, \cdots, s_{N-1}) \in \Delta_N.$$

Then $\psi \in \Psi_N$, *that is,*

$$(A_0) \qquad \psi(0, \cdots, 0) = \psi(1, 0, \cdots, 0) = \cdots = \psi(0, \cdots, 0, 1) = 1;$$

and for each $1 \le k \le N$

$$(A_k) \qquad \psi(s_1, \ldots, s_{N-1}) \ge (1 - s_k)\psi\left(\frac{s_1}{1 - s_k}, \ldots, \overbrace{0}^{k-1}, \ldots, \frac{s_{N-1}}{1 - s_k}\right).$$

Conversely
(ii) *For any* $\psi \in \Psi_N$ *define*

$$(*) \quad \|(a_1, \cdots, a_N)\|_\psi = \begin{cases} \left(\sum_{j=1}^N |a_j|\right)\psi\left(\dfrac{|a_2|}{\sum_{j=1}^N |a_j|}, \cdots, \dfrac{|a_N|}{\sum_{j=1}^N |a_j|}\right) & \\ \qquad\qquad \textit{if } (a_1, \cdots, a_N) \ne (0, \cdots, 0), \\ 0 \qquad\qquad \textit{if } (a_1, \cdots, a_N) = (0, \cdots, 0). \end{cases}$$

Then $\|\cdot\|_\psi \in AN_N$ *and* $\|\cdot\|_\psi$ *satisfies* (2.1).

In fact, since an absolute normalized norm is monotone by Lemma 2.1, the statement (i) is a consequence of Lemma 2.2. For the assertion (ii) we refer the reader to [38]. Thus AN_N and Ψ_N correspond in a one-to-one way.

Remark 2.4 *(i) Let us see why we defined the norm* $\|\cdot\|_\psi$ *by the formula* $(*)$ *from* $\psi \in \Psi_N$. *For an arbitrary norm* $\|\cdot\|$ *on* $\mathbb{R}^N$ *let* ψ *be a convex function given by (2.1). Then the norm* $\|\cdot\|$ *is represented by means of* ψ *as follows. Let* $M = \sum_{j=1}^N |a_j|$ *for nonzero* $(a_1, \dots, a_N) \in \mathbb{R}^N$. *Then*

$$\|(a_1, \cdots, a_N)\| = M\|(a_1/M, \cdots, a_N/M)\| = M\psi\left(\frac{|a_2|}{M}, \cdots, \frac{|a_N|}{M}\right).$$

(ii) In the case $N = 2$ *a convex funtion* ψ *on* $\Delta_2 = [0, 1]$ *belongs to* Ψ_2 *if and only if* $\max\{1-t, t\} \le \psi(t) \le 1$ *for* $0 \le t \le 1$, *from which* $\psi(0) = \psi(1) = 1$ *is derived. Thus if we draw the graph of a convex function* $\psi \in \Psi_2$ *in this triangle area we shall obtain an absolute normalized norm* $\|\cdot\|_\psi$ *on* $\mathbb{R}^2$.

Example 2.5 *The* ℓ_p*-norm on* $\mathbb{R}^N$,

$$\|(a_1, \dots, a_N)\|_p = \begin{cases} \{\sum_{j=1}^N |a_j|^p\}^{1/p} & 1 \le p < \infty, \\ \max_{1\le j\le N} |a_j| & p = \infty \end{cases}$$

is absolute normalized and the corresponding convex function ψ_p *is given by*

$$\begin{aligned}\psi_p(s_1, \dots, s_{N-1}) \quad &:= \quad \|(1 - \sum_{i=1}^{N-1} s_i,\ s_1, \dots, s_{N-1})\|_p \\ &= \begin{cases} \left\{\left(1 - \sum_{i=1}^{N-1} s_i\right)^p + s_1^p + \cdots + s_{n-1}^p\right\}^{1/p} & \text{if } p < \infty, \\ \max\left\{1 - \sum_{i=1}^{N-1} s_i, s_1, \dots, s_{n-1}\right\} & \text{if } p = \infty. \end{cases}\end{aligned}$$

In particular $\psi_1(s_1, \dots, s_{N-1}) = 1$.

Now, the following subclasses $\Psi_N^{(1)}$ and $\Psi_N^{(\infty)}$ of Ψ_N will play an important role in our later discussion. In the following let T be a nonempty subset of $\{1, \dots, N\}$, χ_T the characteristic function of T. For $\boldsymbol{a} = (a_1, \dots, a_N) \in \mathbb{R}_+^N$ let

$$\boldsymbol{a}_T = \sum_{j\in T} a_j \mathbf{e}_j = (\chi_T(1)a_1, \dots, \chi_T(N)a_N),$$

where $\boldsymbol{e}_j = (0, \dots, \overset{j}{\smile}{1}, \dots, 0)$.

Definition 2.6 (Kato-Tamura [27, 30]) *(i) Let* $\psi \in \Psi_N$. *We say* $\psi \in \Psi_N^{(1)}$ *if there exists* $\boldsymbol{a} \in \mathbb{R}_+^N$ *and* $T \subsetneq \{1, \dots, N\}$ $(T \ne \emptyset)$ *such that*

$$\|\mathbf{a}\|_\psi = \|\mathbf{a}_T\|_\psi + \|\mathbf{a}_{T^c}\|_\psi, \quad \textit{where } \|\mathbf{a}_T\|_\psi,\ \|\mathbf{a}_{T^c}\|_\psi > 0.$$

(ii) We say $\psi \in \Psi_N^{(\infty)}$ *if there exists* $\boldsymbol{a} \in \mathbb{R}_+^N$ *and* $T \subsetneq \{1, \dots, N\}$ $(T \ne \emptyset)$ *such that*

$$\|\mathbf{a}\|_\psi = \|\mathbf{a}_T\|_\psi = \|\mathbf{a}_{T^c}\|_\psi > 0.$$

The ℓ_1-norm $\|\cdot\|_1$ has the above property (i), and the ℓ_∞-norm $\|\cdot\|_\infty$ has the property (ii) (see the example below). These properties (i) and (ii) are much weaker than ℓ_1-norm's and ℓ_∞-norm's, respectively. We call, in general, a norm $\|\cdot\|$ on $\mathbb{R}^N$ with the properties (i) and (ii) a *partial ℓ_1-norm* and a *partial ℓ_∞-norm*, respectively.

Example 2.7 *$\psi_1 \in \Psi_N^{(1)}$ and $\psi_\infty \in \Psi_N^{(\infty)}$ since*

$$\left\|(1, \frac{1}{N-1}, \dots, \frac{1}{N-1})\right\|_1 = \|(1,0,\dots,0)\|_1 + \left\|(0, \frac{1}{N-1}, \dots, \frac{1}{N-1})\right\|_1,$$
$$\|(1,1,\dots,1)\|_\infty = \|(1,0,\dots,0)\|_\infty = \|(0,1,\dots,1)\|_\infty,$$

where $T = \{1\}$ in both cases.

On the other hand Dowling-Saejung [10] introduced the following notions.

Definition 2.8 *For $\boldsymbol{a} = (a_j) \in \mathbb{R}^N$ let* supp $\boldsymbol{a} = \{j : a_j \neq 0\}$.
(i) A norm $\|\cdot\|$ on $\mathbb{R}^N$ is said to have Property T_1^N if

$$\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = \frac{1}{2}\|\boldsymbol{a}+\boldsymbol{b}\| = 1,\ \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^N \implies \text{supp}\ \boldsymbol{a} \cap \text{supp}\ \boldsymbol{b} \neq \emptyset.$$

(ii) A norm $\|\cdot\|$ on $\mathbb{R}^N$ is said to have Property T_∞^N if

$$\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = \|\boldsymbol{a}+\boldsymbol{b}\| = 1 \implies \text{supp}\ \boldsymbol{a} \cap \text{supp}\ \boldsymbol{b} \neq \emptyset.$$

Note that ℓ_1-norm $\|\cdot\|_1$ and ℓ_∞-norm $\|\cdot\|_\infty$ do not have Property T_1^N and Property T_∞^N, respectively. We have the following.

Theorem 2.9 (Dhompomgsa-Kato-Tamura [8]) *Let $\psi \in \Psi_N$. Then*
(i) $\|\cdot\|_\psi$ *has Property T_1^N if and only if $\psi \notin \Psi_N^{(1)}$.*
(ii) $\|\cdot\|_\psi$ *has Property T_∞^N if and only if $\psi \notin \Psi_N^{(\infty)}$.*

3 Direct sums Let $\|\cdot\|_Z$ be an *absolute* norm on $\mathbb{R}^N$. The *Z-direct sum* $(X_1 \oplus \cdots \oplus X_N)_Z$ of Banach spaces $X_1, \dots, X_N$ is their direct sum equipped with the norm

$$\|(x_1, \cdots, x_N)\|_Z := \|(\|x_1\|, \cdots, \|x_N\|)\|_Z, \quad (x_1, \dots, x_N) \in X_1 \oplus \cdots \oplus X_N$$

(cf. Dowling-Saejung [10]).

Remark 3.1 *In the above definition the Z-norm $\|\cdot\|_Z$ on $\mathbb{R}^N$ is sometimes assumed to be absolute and monotone in $\mathbb{R}_+^N$ in [10]. But the latter condition can be dropped because of Lemma 2.1.*

A direct sum constructed in the same way as above from an *absolute normalized* norm $\|\cdot\|_{AN} = \|\cdot\|_\psi$ on $\mathbb{R}^N$ is called a *ψ-direct sum* and denoted by

$$(X_1 \oplus \cdots \oplus X_N)_\psi,$$

where ψ is the convex function corresponding to the norm $\|\cdot\|_{AN}$ (Kato-Saito-Tamura [21]; cf. [40]).

Let $\|\cdot\|_A$ be an *arbitrary* norm on $\mathbb{R}^N$. The *A-direct sum* $(X_1 \oplus \cdots \oplus X_N)_A$ is the direct sum of $X_1, \dots, X_N$ equipped with the norm

$$\|(x_1, \dots, x_N)\|_A = \|(\|x_1\|, \dots, \|x_N\|)\|_A, \quad (x_1, \dots, x_N) \in X_1 \oplus \cdots \oplus X_N$$

(Dhompongsa-Kato-Tamura [8]). Clearly, a ψ-direct sum is a Z-direct sum, which is an A-direct sum. These notions of direct sums are in fact all isometric.

Theorem 3.2 (Kato-Tamura [30]) *Let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. Then there exists $\psi \in \Psi_N$ such that $(X_1 \oplus \cdots \oplus X_N)_A$ is isometrically isomorphic to $(X_1 \oplus \cdots \oplus X_N)_\psi$. More precisely*

$$\|(x_1, \ldots, x_N)\|_A = \|(c_1 x_1, \ldots, c_N x_N)\|_\psi, \quad (x_1, \ldots, x_N) \in X_1 \oplus \cdots \oplus X_N,$$

where $c_k = \|(0, \ldots, 0, \overset{k}{\breve{1}}, 0, \ldots, 0)\|_A$ $(1 \le k \le N)$.

Sketch of proof Take $e_j \in X_j$ with $\|e_j\| = 1$ $(1 \le j \le N)$ and define a norm $\|\cdot\|_B$ on $\mathbb{R}^N$ by

$$\|(a_1, \ldots, a_N)\|_B = \|(a_1 e_1, \ldots, a_N e_N)\|_A.$$

Then $\|\cdot\|_B$ is absolute. Let

$$\|(x_1, \cdots, x_N)\|_B = \|(\|x_1\|, \cdots, \|x_N\|)\|_B$$

for $(x_1, \ldots, x_N) \in X_1 \oplus \cdots \oplus X_N$. Then

$$\|(x_1, \ldots, x_N)\|_A = \|(x_1, \ldots, x_N)\|_B,$$

Thus we may assume that, without loss of generality, the original norm $\|\cdot\|_A$ on $\mathbb{R}^N$ is absolute to construct the A-direct sum $(X_1 \oplus \cdots \oplus X_N)_A$. Next let $c_k = \|(0, \ldots, 0, \overset{k}{\breve{1}}, 0, \ldots, 0)\|_B$ and define a norm $\|\cdot\|_C$ on $\mathbb{R}^N$ by

$$\|(a_1, \ldots, a_N)\|_C = \|(a_1/c_1, \ldots, a_N/c_N)\|_B.$$

Then $\|\cdot\|_C$ is absolute and normalized, and

$$\|(a_1, \ldots, a_N)\|_B = \|(c_1 a_1, \ldots, c_N a_N)\|_C$$

Consequently we have

$$\|(x_1, \ldots, x_N)\|_A = \|(c_1 x_1, \ldots, c_N x_N)\|_C$$

for $(x_1, \ldots, x_N) \in X_1 \oplus \cdots \oplus X_N$. Thus $(X_1 \oplus \cdots \oplus X_N)_A$ is isometric to $(X_1 \oplus \cdots \oplus X_N)_C = (X_1 \oplus \cdots \oplus X_N)_\psi$ with some function $\psi \in \Psi_N$.

In particular any Z-direct sum is isometrically isomorphic to a ψ-direct sum. The advantage of the latter is to allow us to use a convex function $\psi \in \Psi_N$ in our discussion, especially to construct examples.

We shall see some basic properties for direct sums. A Banach space X is called *strictly convex* if

$$x, y \in S_X, x \ne y \implies \left\| \frac{x+y}{2} \right\| < 1.$$

X is called *uniformly convex* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x - y\| \ge \varepsilon, x, y \in S_X \implies \left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

Theorem 3.3 ([21, 40]) *A ψ-direct sum $(X_1 \oplus \cdots \oplus X_N)_\psi$ is strictly (uniformly) convex if and only if $X_1, \ldots, X_N$ are strictly (uniformly) convex and ψ is strictly convex.*

Now, ψ is strictly convex if and only if $\|\cdot\|_\psi$ is strictly convex ([38]), we have the general A-direct sum version of this theorem by Theorem 3.2.

Theorem 3.4 *An A-direct sum $(X_1 \oplus \cdots \oplus X_N)_A$ is strictly (uniformly) convex if and only if $X_1, \ldots, X_N$ are strictly (uniformly) convex and $\|\cdot\|_A$ is strictly convex.*

For similar results for the dual notions, smoothness and uniform smoothness we refer the reader to Mitani-Oshiro-Saito [33].

4 Weak nearly uniform smoothness A Banach space X is called *weakly nearly uniformly smooth* (WNUS in short) if X is reflexive and $R(X) < 2$, $R(X)$ is the *García-Falset coefficient*:

$$R(X) = \sup\{\liminf_{n\to\infty} \|x_n + x\|\},$$

where the supremum is taken over all weakly null sequences $\{x_n\}$ in B_X and all $x \in B_X$. (cf. García-Falset [14]; we refer the reader to Kutzarova et al. [31] for the original definition; cf. [27]). A Banach space X is said to have the *fixed point property* for nonexpansive mappings (FPP in short) provided that for any bounded closed convex subset C of X every nonexpansive self-mapping T on C has a fixed point, where T is called nonexpansive if

$$\|Tx - Ty\| \le \|x - y\| \quad \forall x, y \in C.$$

Uniformly convex resp., uniformly smooth spaces are WNUS ([35]). We also have

Theorem 4.1 (García-Falset [15, 14]) *Every weakly nearly uniformly smooth space has FPP.*

For WNUS-ness of direct sums we have the following.

Theorem 4.2 (Kato-Tamura [27]) *Let $X_1, \ldots, X_N$ be of infinite dimension. Let $\psi \in \Psi_N$. Then, the following are equivalent.*
(i) *$(X_1 \oplus \cdots \oplus X_N)_\psi$ is WNUS.*
(ii) *$X_1, \ldots, X_N$ are WNUS and $\psi \notin \Psi_N^{(1)}$.*

Remark 4.3 *(i) The implication (ii) $\Rightarrow$ (i) is valid without the assumption on the dimension of X_j's.*
(ii) For the case some of X_j's are of finite dimension we refer the reader to [30].

If $\psi \in \Psi_N$ is strictly convex, $\psi \notin \Psi_N^{(1)}$ ([27]). Therefore, taking Remark 4.3(i) into account, the next previous result is derived from Theorem 4.2.

Corollary 4.4 (Dhompongsa et al. [6]) *Let $X_1, \ldots, X_N$ be arbitrary Banach spaces. Let $\psi \in \Psi_N$ be strictly convex. Then the following are equivalent.*
(i) *$(X_1 \oplus \cdots \oplus X_N)_\psi$ is WNUS.*
(ii) *$X_1, \ldots, X_N$ are WNUS.*

Recall that $\psi \notin \Psi_N^{(1)}$ if and only if $\|\cdot\|_\psi$ has Property T_1^N (Theorem 2.9). Owing to Theorem 3.2, Theorem 4.2 is reformulated in the general A-direct sum setting as follows.

Theorem 4.5 *Let $X_1, \ldots, X_N$ be infinite dimensional. Let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. Then the following are equivalent.*
(i) *$(X_1 \oplus \cdots \oplus X_N)_A$ is WNUS.*
(ii) *$X_1, \ldots, X_N$ are WNUS and $\|\cdot\|_A$ has Property T_1^N.*

5 Uniform non-squareness A Banach space X is called *uniformly non-square* (R. C. James [17]) (UNSQ in short) if there exists a constant $\varepsilon > 0$ such that

$$\min\{\|x+y\|, \|x-y\|\} \leq 2(1-\varepsilon) \quad \text{for all } x, y \in S_X.$$

It is immediate to see that uniformly convex spaces are strictly convex and UNSQ, while there is no implication between the latter two notions. The UNSQ-ness has been playing an improtant role in the geometry of Banach spaces and FPP. One of the most remarkable recent results is the following.

Theorem 5.1 (García-Falset et al. [16]) *UNSQ spaces have FPP.*

An important classical result says that UNSQ spaces are reflexive ([17]); in fact, super-reflexive ([18]); we shall mention about super-reflexivity again in Section 7. Thus UNSQ-ness lies between uniform convexity and super-reflexivity, as well as FPP. It is worth stating that some geometric constants have close connections with these notions. In fact UNSQ-ness are characterized by $C_{NJ}(X) < 2$ and also by $J(X) < 2$, where $C_{NJ}(X)$ and $J(X)$ are the von Neumann-Jordan and the James constants of a Banach space X ([39, 13]; cf. [24, 20]). Therefore, if $C_{NJ}(X) < 2$ or $J(X) < 2$, X is reflexive and has FPP. These constants have been calculated for many concrete Banach spaces (we omit precise descriptions).

Theorem 5.2 (Kato-Saito-Tamura [22]) *A ψ-direct sum $X \oplus_\psi Y$ is uniformly non-square if and only if X, Y are uniformly non-square and $\psi \neq \psi_1, \psi_\infty$, that is, $\|\cdot\|_\psi \neq \|\cdot\|_1, \|\cdot\|_\infty$.*

In this paper they posed the following problem:

Problem 1. Characterize the uniform non-squareness for $(X_1 \oplus \cdots \oplus X_N)_\psi$.

This problem is quite complicated since in the case $N \geq 3$ we need to remove much more convex functions in Ψ_N (norms in AN_N). Dowling and Saejung [10] presented a partial answer for Z-direct sums, a fortiori, for ψ-direct sums.

Theorem 5.3 (Dowling-Saejung [10]) *Assume that Z-norm $\|\cdot\|_Z$ or the dual norm $\|\cdot\|_Z^*$ on $\mathbb{R}^N$ is strictly monotone. Then the following are equivalent.*

(i) *$(X_1 \oplus \cdots \oplus X_N)_Z$ is UNSQ.*

(ii) *$X_1, \ldots, X_N$ are UNSQ and $\|\cdot\|_Z$ has Properties T_1^N and T_∞^N.*

For the case $N = 3$ they dropped the assumption on strict monotonicity, which answers Problem 1 for $N = 3$:

Theorem 5.4 (Dowling-Saejung [10]) *The following are equivalent.*

(i) *$(X_1 \oplus X_2 \oplus X_3)_Z$ is UNSQ.*

(ii) *X_1, X_2, X_3 are UNSQ and $\|\cdot\|_Z$ has Properties T_1^3 and T_∞^3.*

Any A-direct sum is isometric to a Z-direct sum, whence we have the following.

Theorem 5.5 *Let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. Then the following are equivalent.*

(i) *$(X_1 \oplus X_2 \oplus X_3)_A$ is UNSQ.*

(ii) *X_1, X_2, X_3 are UNSQ and $\|\cdot\|_A$ has Properties T_1^3 and T_∞^3.*

Remark 5.6 *Why did they succeed in the case $N = 3$? Later we shall see the reason, which is a quite natural consequence of a recent result by Kato and Tamura [30].*

In 2015 Dhompongsa, Kato and Tamura [8] gave more precise results for Theorem 5.3 in the A-direct sum setting.

Theorem 5.7 *Let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. Assume that $\|\cdot\|_A$ is strictly monotone. Then the following are equivalent.*
(i) $(X_1 \oplus \cdots \oplus X_N)_A$ *is UNSQ.*
(ii) $X_1, \ldots, X_N$ *are UNSQ and the norm $\|\cdot\|_A$ has Property T_1^N.*

Theorem 5.8 *Let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. Assume that the dual norm $\|\cdot\|_A^*$ is strictly monotone. Then the following are equivalent.*
(i) $(X_1 \oplus \cdots \oplus X_N)_A$ *is UNSQ.*
(ii) $X_1, \ldots, X_N$ *are UNSQ and the norm $\|\cdot\|_A$ has Property T_∞^N.*

If $(X_1 \oplus \cdots \oplus X_N)_A$ is UNSQ, the norm $\|\cdot\|_A$ has Properties T_1^N and T_∞^N. (This is a corresponding result to Theorem 5.2; cf. [8, Cororally 4.5] and [30]). Therefore from Theorems 5.7 and 5.8 the following A-direct sum version of Theorem 5.3 is derived.

Corollary 5.9 *Let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. Assume that $\|\cdot\|_A$ or $\|\cdot\|_A^*$ is strictly monotone. Then the following are equivalent.*
(i) $(X_1 \oplus \cdots \oplus X_N)_A$ *is UNSQ.*
(ii) $X_1, \ldots, X_N$ *are UNSQ and $\|\cdot\|_A$ has Properties T_1^N and T_∞^N.*

Remark 5.10 *Dhompongsa-Kato-Tamura [8] first proved Theorems 5.7, 5.8, and Corollary 5.9 for ψ-direct sums by means of $\Psi_N^{(1)}$ and $\Psi_N^{(\infty)}$, and then derived these results for A-direct sums by Theorems 3.2 and 2.9. We shall see below the ψ-direct sum version of Theorem 5.5.*

Theorem 5.5′ ([8]) *Let $\psi \in \Psi_N$. Assume that $\|\cdot\|_\psi$ is strictly monotone. Then the following are equivalent.*
(i) $(X_1 \oplus \cdots \oplus X_N)_\psi$ *is UNSQ.*
(ii) $X_1, \ldots, X_N$ *are UNSQ and $\psi \notin \Psi_N^{(1)}$.*

Now we are in a position to explain why Dowling-Saejung [10] succeeded for the case $N = 3$. Very recently by introducing the class $\Psi_N^{(mix)}$ as the class which should be excluded, Kato-Tamura [30, in preparation] answered Problem 1 without the assumption on strict monotonicity:

A ψ-direct sum $(X_1 \oplus \cdots \oplus X_N)_\psi$ is UNSQ if and only if $X_1, \ldots, X_N$ are UNSQ and $\psi \notin \Psi_N^{(mix)}$.

(This will appear elsewhere.) They showed $\Psi_3^{(mix)} = \Psi_3^{(1)} \cup \Psi_3^{(\infty)}$ for $N = 3$ and obtained the following as a corollary.

Theorem 5.11 *Let $\psi \in \Psi_N$. Then the following are equivalent.*
(i) $(X_1 \oplus X_2 \oplus X_3)_\psi$ *is UNSQ.*
(ii) X_1, X_2, X_3 *are UNSQ and $\psi \notin \Psi_3^{(1)} \cup \Psi_3^{(\infty)}$.*

According to Theorem 2.9, $\psi \notin \Psi_3^{(1)} \cup \Psi_3^{(\infty)}$ if and only if $\|\cdot\|_\psi$ has Properties T_1^3 and T_∞^3. As any Z-direct sum is isometric to a ψ-direct sum, we have Dowling-Saejung's result for $(X_1 \oplus X_2 \oplus X_3)_Z$. Theorem 5.3 is also derived from the above-announced result by Kato-Tamura [30].

We shall conclude this section with the following result.

Theorem 5.12 (Betiuk-Pilarska and Prus [2]) *The following are equivalent.*
(i) $(X_1 \oplus \cdots \oplus X_N)_Z$ *is UNSQ.*
(ii) $X_1, \ldots, X_N$ *are UNSQ and* $(\mathbb{R}^N, \|\cdot\|_Z)$ *is UNSQ.*

Here it remains unknown when the space $(\mathbb{R}^N, \|\cdot\|_Z)$ is UNSQ. Kato-Tamura [30] answered to this question by introducing Property T^N_{mix} in the A-direct sum setting.

6 Uniform non-ℓ_1^n-ness A Banach space X is called *uniformly non-ℓ_1^n* if there exists ε $(0 < \varepsilon < 1)$ for which

$$\forall x_1, \cdots, x_n \in S_X,\ \exists \theta = (\theta_j)\ (\theta_j = \pm 1)\ s.t.\ \left\| \sum_{j=1}^{n} \theta_j x_j \right\| \le n(1-\epsilon). \tag{6.1}$$

When $n = 2$ the uniform non-ℓ_1^2-ness coincides with the uniform non-squareness. For $n = 3$ uniformly non-ℓ_1^3 spaces are called *uniformly non-octahedral.* In the case $n = 1$ no Banach space is uniformly non-ℓ_1^1.

Proposition 6.1 *Uniformly non-ℓ_1^n spaces are uniformly non-ℓ_1^{n+1}.*

Theorem 5.2 for UNSQ-ness of $X \oplus_\psi Y$ is extended to uniform non-ℓ_1^n-ness.

Theorem 6.2 (Kato-Saito-Tamura [23]) *Assume that neither X nor Y is uniformly non-ℓ_1^{n-1}. Then the following are equivalent.*
(i) $X \oplus_\psi Y$ *is uniformly non-ℓ_1^n.*
(ii) X *and* Y *are uniformly non-ℓ_1^n and* $\psi \neq \psi_1, \psi_\infty$.

Remark 6.3 *(i) Theorem 6.2 covers Theorem 5.2 as the case $n = 2$, since no Banach space is uniformly non-ℓ_1^1.*
(ii) We cannot drop the assumption "neither X nor Y is uniformly non-ℓ_1^{n-1}".
(iii) For the N Banach spaces case some results were obtained under the assumption that the ψ-norm $\|\cdot\|_\psi$ is strictly monotone in Kato and Tamura [29].

As before we obtain the A-direct sum version of Theorem 6.2.

Theorem 6.4 *Let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. Assume that neither X nor Y is uniformly non-ℓ_1^{n-1}. Then the following are equivalent.*
(i) $X \oplus_A Y$ *is uniformly non-ℓ_1^n.*
(ii) X *and* Y *are uniformly non-ℓ_1^n and* $\|\cdot\|_A \neq \|\cdot\|_1, \|\cdot\|_\infty$.

Now, we shall look into the extreme cases, ℓ_1- and ℓ_∞-sums, which were excluded in Theorems 6.2 (and 6.4). According to this theorem, $X \oplus_1 Y$ and $X \oplus_\infty Y$ cannot be UNSQ for all X and Y, while $X \oplus_1 Y$ and $X \oplus_\infty Y$ can be uniformly non-ℓ_1^n $(n \ge 3)$ if either X or Y is uniformly non-ℓ_1^{n-1}. In fact the following are true.

Theorem 6.5 (Kato-Saito-Tamura [23]) *The following are equivalent.*
(i) $X \oplus_1 Y$ *is uniformly non-ℓ_1^n,* $n \ge 3$.
(ii) *There exist $n_1, n_2 \in \mathbb{N}$ with $n_1 + n_2 = n - 1$ such that X is uniformly non-$\ell_1^{n_1+1}$ and Y is uniformly non-$\ell_1^{n_2+1}$.*

As corollaries the following are obtained.

Corollary 6.6 *The following are equivalent.*
(i) $X \oplus_1 Y$ *is uniformly non-ℓ_1^3.*
(ii) X *and* Y *are UNSQ.*

Corollary 6.7 *The following are equivalent.*
(i) $X \oplus_1 Y$ *is uniformly non-*ℓ_1^4.
(ii) X *is UNSQ and* Y *is uniformly non-octahedral.*

Theorem 6.5 is extended as follows.

Theorem 6.8 *The following are equivalent.*
(i) $(X_1 \oplus \cdots \oplus X_N)_1$ *is uniformly non-*ℓ_1^{N+1}.
(ii) $X_1, \ldots, X_N$ *are UNSQ.*

This implies especially that the space ℓ_1^n is uniformly non-ℓ_1^{n+1}. For ℓ_∞-sums we have the following.

Theorem 6.9 (Kato-Tamura [26]) *Let* $n \geq 2$*. The following are equivalent.*
(i) $(X_1 \oplus \cdots \oplus X_{2^n-1})_\infty$ *is uniformly non-*ℓ_1^{n+1}.
(ii) $X_1, \ldots, X_{2^n-1}$ *are UNSQ.*

Corollary 6.10 *The following are equivalent.*
(1) $(X \oplus Y \oplus Z)_\infty$ *is uniformly non-*ℓ_1^3.
(2) X, Y *and* Z *are UNSQ.*

Remark 6.11 *Recall that the* ℓ_1*-sum* $X \oplus_1 Y$ *is uniformly non-*ℓ_1^3 *if and only if* X *and* Y *are UNSQ. Contrary to this, if* X *and* Y *are UNSQ, the* ℓ_∞*-sum* $X \oplus_\infty Y$ *is uniformly non-*ℓ_1^3 *([23, Corollary 5.3bis]), but the converse is not true ([23, Remark 5.5]). We added one more space* Z *to obtain the equivalence in Corollary 6.9. Compare also Theorems 6.7 and 6.8. These observations show one aspect of the defference between* ℓ_1*- and* ℓ_∞*-sums.*

7 Applications All UNSQ spaces have FPP. Thus it is natural to ask whether all uniformly non-octahedral spaces have FPP. We have the following.

Theorem 7.1 (Kato-Tamura [26]) *Let* X *be uniformly non-octahedral. If* X *is isometric to an* ℓ_∞*-sum of 3 Banach spaces,* X *has FPP, while* X *is not UNSQ.*

More generally we have

Theorem 7.2 *Let* X *be uniformly non-*ℓ_1^{n+1}*. If* X *is isometric to an* ℓ_∞*-sum of* $2^n - 1$ *Banach spaces,* X *has FPP, while* X *is not UNSQ.*

Example 7.3 *Let* $1 < p < \infty$*. Since* L_p *is uniformly convex, it is UNSQ. Therefore the space* $X = (L_p \oplus L_p \oplus L_p)_\infty$ *is uniformly non-octahedral by Theorem 6.9, and hence* X *has FPP by Theorem 7.1, while it is not UNSQ since* X *contains* ℓ_∞^3 *as a subspace.*

In the same way, the ℓ_∞*-sum of* $2^n - 1$ L_p*'s is uniformly non-*ℓ_1^{n+1}*, which is weaker than uniform non-octahedralness, has FPP but is not UNSQ.*

By using Theorem 4.2 a plenty of Banach spaces with FPP which fail to be UNSQ is constructed.

Example 7.4 (Kato-Tamura [27]) *Let* $N \geq 3$ *and let* $\varphi, \psi_1 \in \Psi_2$, $\varphi \neq \psi_1$*. Let*

$$\psi(s_1, \ldots, s_{N-1}) = \max\left\{ \|(1 - \sum_{i=1}^{N-1} s_i, s_1)\|_\varphi,\ \|(s_1, s_2)\|_\varphi,\ \|(s_2, s_3)\|_\varphi, \cdots, \|(s_{N-2}, s_{N-1})\|_\varphi \right\}$$

for $(s_1, \ldots, s_{N-1}) \in \Delta_N$*.*

Then $\psi \in \Psi_N$ *and*

$$\|(a_1, a_2 \dots, a_N)\|_\psi = \max\{\|(a_1,a_2)\|_\varphi, \|(a_2,a_3)\|_\varphi, \dots, \|(a_{N-1},a_N)\|_\varphi\} \quad \text{for } (a_1,\dots,a_N) \in \mathbb{R}^N.$$

Further, $\psi \notin \Psi_N^{(1)}$ *and* $\|\cdot\|_\psi$ *is not UNSQ.*

Since WNUS spaces have FPP, we have the following.

Theorem 7.5 (Kato-Tamura [27]) *Let* $X_1, \dots, X_N$ *be WNUS* ($N \geq 3$)*. Let* $\psi \in \Psi_N$ *be as in Example 7.4. Then* $(X_1 \oplus \cdots \oplus X_N)_\psi$ *has FPP, whereas it is not UNSQ.*

Indeed, since $\psi \notin \Psi_N^{(1)}$, $X = (X_1 \oplus \cdots \oplus X_N)_\psi$ is WNUS by Theorem 4.2 (with Remark 4.3 (i)) and hence X has FPP. On the other hand, X is not UNSQ as $(\mathbb{R}^N, \|\cdot\|_\psi)$ is not UNSQ.

Next, as $\psi_\infty \notin \Psi_N^{(1)}$, we have

Theorem 7.6 *Let* $X_1, \dots, X_N$ *be WNUS. Then* $(X_1 \oplus \cdots \oplus X_N)_\infty$ *has FPP, whereas it is not UNSQ.*

Example 7.7 *Let* $1 < p_k < \infty,\ 1 \leq k \leq N$.

(i) Let $\psi \in \Psi_N$ *be as in Example 7.4. Since* L_{p_k} *are uniformly convex and hence WNUS, the space* $X = (L_{p_1} \oplus \cdots \oplus L_{p_N})_\psi$ *has FPP, while* X *is not UNSQ by Theorem 7.6.*

(ii) The ℓ_∞*-sum* $X = (L_{p_1} \oplus \cdots \oplus L_{p_N})_\infty$ *is WNUS and hence has FPP. On the other hand, the space* X *is not UNSQ.*

Finally we shall discuss super-reflexivity. A Banach space Y is said to be *finitely representable in* X provided for any $\epsilon > 0$ and for any finite dimensional subspace F of Y there is a finite dimensional subspace E of X with $\dim F = \dim E$ such that $d(F,E) := \inf\{\|T\|\|T^{-1}\| : T \text{ is an isomorphism of } F \text{ onto } E\} < 1 + \epsilon$. A Banach space X is called *super-reflexive* if every Banach space Y which is finitely representable in X is reflexive ([18]; cf. [1]). The next celebrated result states the connection between super-reflexivity and uniform convexity as well as UNSQ-ness.

Theorem 7.8 (cf. [12, 34, 18]) *The following are equivalent.*

(i) *X is super-reflexive.*

(ii) *X admits an equivalent uniformly convex norm.*

(iii) *X admits an equivalent uniformly non-square norm.*

UNSQ spaces are super-reflexive ([17]), whereas uniformly non-octahedral spaces are not always reflexive ([19]). For ℓ_1-sum spaces we have the following ([26]).

Theorem 7.9 *Let* X *be a uniformly non-octahedral Banach space which is isometric to the* ℓ_1*-sum of 2 Banach spaces. Then* X *is super-reflexive.*

Indeed, if X is isometric to $X_1 \oplus_1 X_2$, $X_1 \oplus_1 X_2$ is uniformly non-octahedral, from which it follows that X_1 and X_2 are UNSQ by Corollary 6.5, hence super-reflexive. Consequently the ℓ_1-sum, and hence X is super-reflexive.

By Theorem 6.9 we have the similar result for ℓ_∞-sum spaces.

Theorem 7.10 *Let* X *be a uniformly non-octahedral Banach space which is isometric to the* ℓ_∞*-sum of 3 Banach spaces. Then* X *is super-reflexive.*

Acknowledgement. The first author would like to thank the organizers of the conference FIM&ISME2017, especially, Professors J. Watada, Y. Yabuuchi and H. Sakai for their kind invitation and warm hopsitality.

References

[1] B. Beauzamy, Introduction to Banach spaces and their geometry, 2nd ed., North-Holland, 1985.

[2] A. Betiuk-Pilarska and S. Prus, Uniform nonsquareness of direct sums Banach spaces, Topol. Methods Nonlinear Anal. **34** (2009), 181–186.

[3] A. Betiuk-Pilarska and S. Prus, Moduli $RW(a, X)$ and $MW(X)$ of direct sums of Banach spaces, J. Nonlinear Convex Anal. **18** (2017), 309–315.

[4] R. Bhatia, Matrix Analysis, Springer-Verlag, New York, 1997.

[5] F. F. Bonsall and J. Duncan, *Numerical Ranges II*, London Math. Soc. Lecture Note Ser. **10** (1973).

[6] S. Dhompongsa, A. Kaewcharoen and A. Kaewkhao, Fixed point property of direct sums, Nonlinear Anal. **63**(2005), e2177–e2188.

[7] S. Dhompongsa, A. Kaewkhao and S. Saejung, Uniform smoothness and U-convexity of ψ-direct sums, J. Nonlinear Convex Anal. **6** (2005), 327–338.

[8] S. Dhompongsa, M. Kato and T. Tamura, Uniform non-squareness for A-direct sums of Banach spaces with a strictly monotone norm, Linear Nonlinear Anal. **1** (2015), 247–260.

[9] P. N. Dowling, On convexity properties of ψ-direct sums of Banach spaces, J. Math. Anal. Appl. **288** (2003), 540–543.

[10] P. N. Dowling and S. Saejung, Non-squareness and uniform non-squareness of Z-direct sums, J. Math. Anal. Appl. **369** (2010), 53–59.

[11] P. N. Dowling and B. Turett, Complex strict convexity of absolute norms on $\mathbb{C}^n$ and direct sums of Banach spaces, J. Math. Anal. Appl. **323** (2006), 930–937.

[12] P. Enflo, Banach spaces which can be given an equivalent uniformly convex norm, Israel J. Math. **13** (1972), 281–288.

[13] J. Gao and K. S. Lau, On two classes of Banach spaces with uniform normal structure, Studia Math. **99** (1991), 41–56.

[14] J. García-Falset, Stability and fixed points for nonexpansive mappings, Houston J. Math. **20** (1994), 495–506.

[15] J. García-Falset, The fixed point property in Banach spaces with the NUS property, J. Math. Anal. Appl. **215** (1997), 532–542.

[16] J. García-Falset, E. Llorens-Fuster and E. M. Mazcuñan-Navarro, Uniformly nonsquare Banach spaces have the fixed point property for nonexpansive mappings, J. Funct. Anal. **233** (2006), 494–514.

[17] R. C. James, Uniformly non-square Banach spaces, Ann. of Math. **80** (1964), 542–550.

[18] R. C. James, Super-reflexive Banach spaces, Canad. J. Math. **24** (1972), 896–904.

[19] R. C. James, A nonreflexive Banach space that is uniformly non-octahedral, Israel J. Math. **18** (1974), 145–155.

[20] M. Kato, L. Maligranda and Y. Takahashi, On James, Jordan-von Neumann constants and the normal structure coefficient of Banach spaces, Studia Math. **144** (2001), 275–295.

[21] M. Kato, K.-S. Saito and T. Tamura, On the ψ-direct sums of Banach spaces and convexity, J. Aust. Math. Soc. **75** (2003), 413–422.

[22] M. Kato, K.-S. Saito and T. Tamura, Uniform non-squareness of ψ-direct sums of Banach spaces $X \oplus_\psi Y$, Math. Inequal. Appl. **7** (2004), 429–437.

[23] M. Kato, K.-S. Saito and T. Tamura, Uniform non-ℓ_1^n-ness of ψ-direct sums of Banach spaces, J. Nonlinear Convex Anal. **11** (2010), 113–133.

[24] M. Kato and Y. Takahashi, On the von Neumann-Jordan constant for Banach spaces, Proc. Amer. Math. Soc. **125** (1997), 1055–1062.

[25] M. Kato and T. Tamura, Uniform non-ℓ_1^n-ness of ℓ_1-sums of Banach spaces, Comment. Math. Prace Mat. **47** (2007), 161–169.

[26] M. Kato and T. Tamura, Uniform non-ℓ_1^n-ness of ℓ_∞-sums of Banach spaces, Comment. Math. **49** (2009), 179–187.

[27] M. Kato and T. Tamura, Weak nearly uniform smoothness of the ψ-direct sums $(X_1 \oplus \cdots \oplus X_N)_\psi$, Comment. Math. **52** (2012), 171–198.

[28] M. Kato and T. Tamura, Direct sums of Banach spaces with FPP which fail to be uniformly non-square, J. Nonlinear Convex Anal. **16** (2015), 231–241.

[29] M. Kato and T. Tamura, On the uniform non-ℓ_1^n-ness and new classes of convex functions, J. Nonlinear Convex Anal. **16** (2015), 2225–2241.

[30] M. Kato and T. Tamura, On the uniform non-squareness of direct sums of Banach spaces, in preparation.

[31] D. Kutzarova, S. Prus and B. Sims, Remarks on orthogonal convexity of Banach spaces, Houston J. Math. **19** (1993), 603–614.

[32] J. Markowicz and S. Prus, James constant, García-Falset coefficient and uniform Opial property in direct sums of Banach spaces, J. Nonlinear Convex Anal. **17** (2016), 2237–2253.

[33] K-I. Mitani, S. Oshiro and K.-S. Saito, Smoothness of ψ-direct sums of Banach spaces, Math. Inequal. Appl. **8** (2005), 147–157.

[34] G. Pisier, Martingales with values in uniformly convex spaces, Israel J. Math. **20** (1975), 326–350.

[35] S. Prus, Nearly uniformly smooth Banach spaces, Boll. U. M. I.**(7)3-B** (1989), 507–521.

[36] K.-S. Saito and M. Kato, Uniform convexity of ψ-direct sums of Banach spaces, J. Math. Anal. Appl. **277** (2003), 1–11.

[37] K.-S. Saito, M. Kato and Y. Takahashi, Von Neumann-Jordan constant of absolute normalized norms on $\mathbb{C}^2$, J. Math. Anal. Appl. **244** (2000), 515–532.

[38] K.-S. Saito, M. Kato and Y. Takahashi, On absolute norms on $\mathbb{C}^n$, J. Math. Anal. Appl. **252** (2000), 879–905.

[39] Y. Takahashi and M. Kato, Von Neumann-Jordan constant and uniformly non-square Banach spaces, Nihonkai Math. J. **9** (1998), 155–169.

[40] Y. Takahashi, M. Kato and K.-S. Saito, Strict convexity of absolute norms on $\mathbb{C}^2$ and direct sums of Banach spaces, J. Inequal. Appl. **7** (2002), 179–186.

[41] T. Tamura, On Dominguez-Benavides coefficient of ψ-direct sums $(X_1 \oplus \cdots \oplus X_N)_\psi$ of Banach spaces, Linear Nonlinear Anal. **3** (2017), 87–99.

[42] T. Zachariades, On ℓ_ψ spaces and infinite ψ-direct sums of Banach spaces, Rocky Mount. J. Math. **41** (2011), 971-997.

[43] A. Wiśnicki, On the fixed points of nonexpansive mappings in direct sums of Banach spaces, Studia Math., **207** (2011), 75–84.

(M. KATO) FACULTY OF ENGINEERING, KYUSHU INSTITUTE OF TECHNOLOGY, KITAKYUSHU 804-8550, JAPAN
Email: katom@mns.kyutech.ac.jp

(T. SOBUKAWA) GLOBAL EDUCATION CENTER, WASEDA UNIVERSITY, TOKYO 169-8050, JAPAN
Email: sobu@waseda.jp

(T. TAMURA) GRADUATE SCHOOL OF SOCIAL SCIENCES, CHIBA UNIVERSITY, CHIBA 263-8522, JAPAN
Email: ttakayuki@faculty.chiba-u.jp

A THEOREM ON SUMMABILITY FACTORS FOR THE WEIGHTED MEAN METHOD FOR DOUBLE SERIES IN ULTRAMETRIC FIELDS

P N Natarajan
Old No. 2/3, New No. 3/3, Second Main Road, R.A. Puram,
Chennai 600 028, India
e-mail: pinnangudinatarajan@gmail.com

Abstract. Throughout this paper, K denotes a complete, non-trivially valued, ultrametric (or non-archimedean) field. 4-dimensional infinite matrices, double sequences and double series have entries in K. In the present paper, we prove a theorem on summability factors for the Weighted mean method for double series in K.

1 Introduction and Preliminaries Throughout the present paper, K denotes a complete, non-trivially valued, ultrametric (or non-archimedean) field. 4-dimensional infinite matrices, double sequences and double series have entries in K. We recall the following definitions and results briefly (for details, see [2]) for the sake of completeness.

Definition 1.1. *For a double sequence $\{x_{m,n}\}$ in K and $x \in K$, we write*

$$^{1}\lim_{m+n\to\infty} x_{m,n} = x,$$

if for every $\epsilon > 0$, the set $\{(m,n) \in \mathbb{N}^2 : |x_{m,n} - x| > \epsilon\}$ is finite, $\mathbb{N}$ being the set of non-negative integers. In such a case, x is unique and x is called the limit of the double sequence $\{x_{m,n}\}$. We also say that $\{x_{m,n}\}$ converges to x.

Definition 1.2. *Let $\{x_{m,n}\}$ be a double sequence in K and $s \in K$. We write*

$$\sum_{m,n=0}^{\infty,\infty} x_{m,n} = s,$$

if

$$\lim_{m+n\to\infty} s_{m,n} = s,$$

where

$$s_{m,n} = \sum_{k,\ell=0}^{m,n} x_{k,\ell}, \quad m,n = 0,1,2,\ldots.$$

In such a case, we say that the double series $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ converges to s.

Remark 1.3. *If $\{x_{m,n}\}$ converges, then $\{x_{m,n}\}$ is bounded.*

Theorem 1.4. *[2, Lemma 1]* $\lim_{m+n\to\infty} x_{m,n} = x$ *if and only if*

2010 *Mathematics Subject Classification.* 40.
Key words and phrases. Ultrametric (or non-archimedean) field, summability factor, double sequence, double series, 4-dimensional infinite matrix, regular method, Weighted mean method.

[1]This notation was suggested by Late Prof. Wim Schikhof in a private communication to the author

(i) $\lim_{m\to\infty} x_{m,n} = x$, *$n = 0, 1, 2, \ldots$;*

(ii) $\lim_{n\to\infty} x_{m,n} = x$, *$m = 0, 1, 2, \ldots$;* *and*

(iii) for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_{m,n} - x| < \epsilon$, $m, n \geq N$, which is written as

$$\lim_{m,n\to\infty} x_{m,n} = x,$$

noting that this is Pringsheim's definition of convergence of a double sequence.

Proof. We can suppose that $x = 0$. Leaving out the trivial part of the theorem, let (i), (ii), (iii) hold. Using (iii), we can choose a positive integer N_1 such that

$$|x_{m,n}| \leq \epsilon, \quad m, n > N_1.$$

In view of (i), there exists a positive integer N_2 such that

$$|x_{m,n}| \leq \epsilon, \quad m > N_2, n = 0, 1, 2, \ldots, N_1.$$

In view of (ii), there exists a positive integer N_3 such that

$$|x_{m,n}| \leq \epsilon, \quad n > N_3, m = 0, 1, 2, \ldots, N_1.$$

Let $N = \max\{N_1, N_2, N_3\}$. Then it is possible that in the square $0 \leq m, n \leq N$,

$$|x_{m,n}| > \epsilon.$$

Note that outside this square,

$$|x_{m,n}| \leq \epsilon.$$

Thus,

$$\{(m,n) \in \mathbb{N}^2 : |x_{m,n}| > \epsilon\} \text{ is finite,}$$

$$\text{i.e.,} \lim_{m+n\to\infty} x_{m,n} = 0,$$

completing the proof. □

Theorem 1.5. *[2, Lemma 2]* $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ *converges if and only if*

$$\lim_{m+n\to\infty} x_{m,n} = 0.$$

Remark 1.6. *Let $K = \mathbb{Q}_2$, the 2-adic field. Consider the series* $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$, *where,* $x_{m,n} = 3^m 2^n$, *$m, n = 0, 1, 2, \ldots$.*

$$|x_{m,n}|_2 = |2|_2^n \to 0, \quad n \to \infty,$$

since $|3|_2 = 1$, from which we have,

$$\lim_{m,n\to\infty} x_{m,n} = 0.$$

Easy calculation shows that

$$S_{m,n} = \sum_{p,q=0}^{m,n} 3^p 2^q = \frac{(3^{m+1}-1)}{2}(2^{n+1}-1)$$

and $S_{m+1,n} - S_{m,n} = (2^{n+1}-1)3^{m+1}$,
so that

$$\begin{aligned} |S_{m+1,n} - S_{m,n}|_2 &= |2^{n+1}-1|_2 |3^{m+1}|_2 \\ &= 1.1 \\ &= 1 \not\to 0, \ m,n \to \infty, \end{aligned}$$

using the fact that

$$|a+b|_2 = \max(|a|_2, |b|_2),$$

if $|a|_2 \neq |b|_2$, $|\cdot|_2$ *being an ultrametric valuation. Consequently,* $\{S_{m,n}\}$ *is not Cauchy and so* $\sum_{m,n=0}^{\infty,\infty} 3^m 2^n$ *does not converge in the Pringsheim's sense. Thus* $\sum_{m,n=0}^{\infty,\infty} 3^m 2^n$ *diverges in the sense of Definition 1.1. Thus,* $\lim_{m,n\to\infty} x_{m,n} = 0$ *does not ensure convergence of* $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ *in the sense of Definition 1.1.*

Remark 1.7. *In the case of simple series, it is well-known that* $\sum_{n=0}^{\infty} x_n$ *converges if and only if*

$$\lim_{n\to\infty} x_n = 0.$$

(see [1], p. 25, Theorem 1.1). Theorem 1.5 shows that Definition 1.1 is more suited in the ultrametric case than Pringsheim's definition of convergence of a double sequence.

Definition 1.8. *Given a 4-dimensional infinite matrix* $A = (a_{m,n,k,\ell})$, $a_{m,n,k,\ell} \in K$, $m,n,k,\ell = 0,1,2,\dots$ *and a double sequence* $\{x_{k,\ell}\}$, $x_{k,\ell} \in K$, $k,\ell = 0,1,2,\dots$, *by the A-transform of* $x = \{x_{k,\ell}\}$, *we mean the double sequence* $A(x) = \{(Ax)_{m,n}\}$,

$$(Ax)_{m,n} = \sum_{k,\ell=0}^{\infty,\infty} a_{m,n,k,\ell} x_{k,\ell}, \quad m,n = 0,1,2,\dots,$$

where it is supposed that the double series on the right converge. If $\lim_{m+n\to\infty} (Ax)_{m,n} = s$, *we say that the double sequence* $x = \{x_{k,\ell}\}$ *is A-summable or summable A to s, written as*

$$x_{k,\ell} \to s(A).$$

If $\lim_{m+n\to\infty} (Ax)_{m,n} = s$, *whenever* $\lim_{k+\ell\to\infty} x_{k,\ell} = s$, *we say that A is regular. A double series* $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ *is said to be A-summable to s, if* $\{s_{m,n}\}$ *is A-summable to s, where* $s_{m,n} = \sum_{k,\ell=0}^{m,n} x_{k,\ell}$, $m,n = 0,1,2,\dots$.

The following important result, due to Natarajan and Srinivasan [2], gives a criterion for a 4-dimensional infinite matrix to be regular in terms of its entries.

Theorem 1.9 (Silverman-Toeplitz). *The 4-dimensional infinite matrix* $A = (a_{m,n,k,\ell})$ *is regular if and only if*

(1) $$\sup_{m,n,k,\ell} |a_{m,n,k,\ell}| < \infty;$$

(2) $$\lim_{m+n\to\infty} a_{m,n,k,\ell} = 0, \quad k,\ell = 0,1,2,\ldots;$$

(3) $$\lim_{m+n\to\infty} \sum_{k,\ell=0}^{\infty,\infty} a_{m,n,k,\ell} = 1;$$

(4) $$\lim_{m+n\to\infty} \sup_{k\geq 0} |a_{m,n,k,\ell}| = 0, \quad \ell = 0,1,2,\ldots;$$

and

(5) $$\lim_{m+n\to\infty} \sup_{\ell\geq 0} |a_{m,n,k,\ell}| = 0, \quad k = 0,1,2,\ldots.$$

2 Weighted Mean Method for Double Sequences in K The Weighted mean method $(\overline{N}, p_{m,n})$ for double sequences and double series in K was introduced earlier by Natarajan and Sakthivel in [5].

Definition 2.1. *Given* $p_{m,n} \in K$, $m,n = 0,1,2,\ldots$, *the Weighted mean method, denoted by* $(\overline{N}, p_{m,n})$, *is defined by the 4-dimensional infinite matrix* $(a_{m,n,k,\ell})$, $m,n,k,\ell = 0,1,2,\ldots$, *where*

$$a_{m,n,k,\ell} = \begin{cases} \frac{p_{k,\ell}}{P_{m,n}}, & \text{if } k \leq m \text{ and } \ell \leq n; \\ 0, & \text{otherwise,} \end{cases}$$

$P_{m,n} = \sum_{k,\ell=0}^{m,n} p_{k,\ell}$, $m,n = 0,1,2,\ldots$ *with the double sequence* $\{p_{m,n}\}$ *of weights satisfying the conditions:*

$$p_{m,n} \neq 0, \ m,n = 0,1,2,\ldots;$$

and for every fixed pair (i,j),

$$|p_{k,\ell}| \leq |P_{i,j}|, \quad \begin{array}{l} k = 0,1,2,\ldots,i; i = 0,1,2,\ldots; \\ \ell = 0,1,2,\ldots,j; j = 0,1,2,\ldots. \end{array}$$

Natarajan and Sakthivel [5] proved the following result.

Theorem 2.2. *[5, Theorem 3.1]* $(\overline{N}, p_{m,n})$ *is regular if and only if*

$$\lim_{m+n\to\infty} |P_{m,n}| = \infty;$$

$$\lim_{m+n\to\infty} \frac{\max_{0\leq k\leq m} |p_{k,\ell}|}{P_{m,n}} = 0, \ \ell = 0,1,2,\ldots;$$

and

$$\lim_{m+n\to\infty} \frac{\max_{0\leq \ell\leq n} |p_{k,\ell}|}{P_{m,n}} = 0, \ k = 0,1,2,\ldots.$$

3 Main Theorem Some properties of the Weighted mean method for double sequences in K were studied in [5].

A theorem on summability factors for the Weighted mean method for simple series in K was proved in [4] and more generally, a theorem on summability factors for any regular method for simple series in K was proved in [3]. For the definition of summability factors for simple series in the classical case, the reader can refer to [6], pp. 38-39. We retain the same definition for double series in the ultrametric set up with suitable changes.

We now prove the main result of the paper, which deals with summability factors for the Weighted mean method for double series in K.

Theorem 3.1. *If* $\sum_{m,n=0}^{\infty,\infty} a_{m,n}$ *is* $(\overline{N}, p_{m,n})$ *summable,* $(\overline{N}, p_{m,n})$ *being regular and if* $\{b_{m,n}\}$ *converges, then* $\sum_{m,n=0}^{\infty,\infty} a_{m,n}b_{m,n}$ *is* $(\overline{N}, p_{m,n})$ *summable too.*

Proof. Let $s_{m,n} = \sum_{k,\ell=0}^{m,n} a_{k,\ell}$, $t_{m,n} = \sum_{k,\ell=0}^{m,n} a_{k,\ell}b_{k,\ell}$, $m, n = 0, 1, 2, \ldots$. Let $\{\alpha_{m,n}\}$, $\{\beta_{m,n}\}$ be the $(\overline{N}, p_{m,n})$-transforms of $\{s_{m,n}\}$, $\{t_{m,n}\}$ respectively so that

$$\alpha_{m,n} = \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} p_{k,\ell}s_{k,\ell},$$

$$\beta_{m,n} = \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} p_{k,\ell}t_{k,\ell},$$

$m, n = 0, 1, 2, \ldots$. Let $\lim_{m+n\to\infty} \alpha_{m,n} = s$ and $\lim_{m+n\to\infty} b_{m,n} = m$. Let

$$b_{m,n} = m + \varepsilon_{m,n},\ m, n = 0, 1, 2, \ldots$$

so that

$$\lim_{m+n\to\infty} \varepsilon_{m,n} = 0.$$

Now,

$$\begin{aligned}\alpha_{m,n} &= \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} p_{k,\ell}s_{k,\ell} \\ &= \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} p_{k,\ell} \left(\sum_{i,j=0}^{k,\ell} a_{i,j} \right) \\ &= \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} a_{k,\ell} \left(\sum_{i,j=k,\ell}^{k,\ell} p_{i,j} \right).\end{aligned}$$

Similarly,

$$\beta_{m,n} = \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} a_{k,\ell}b_{k,\ell} \left(\sum_{i,j=k,\ell}^{m,n} p_{i,j} \right).$$

Thus,

$$\begin{aligned}\beta_{m,n} &= \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} a_{k,\ell}(m+\varepsilon_{k,\ell}) \left(\sum_{i,j=k,\ell}^{m,n} p_{i,j} \right) \\ &= m \left[\frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} a_{k,\ell} \left(\sum_{i,j=k,\ell}^{m,n} p_{i,j} \right) \right] \\ &\quad + \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} a_{k,\ell}\varepsilon_{k,\ell} \left(\sum_{i,j=k,\ell}^{m,n} p_{i,j} \right) \\ &= m\alpha_{m,n} + \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} a_{k,\ell}\varepsilon_{k,\ell} \left(\sum_{i,j=k,\ell}^{m,n} p_{i,j} \right) \\ &= m\alpha_{m,n} + \sum_{k,\ell=0}^{\infty,\infty} a_{m,n,k,\ell}\varepsilon_{k,\ell},\end{aligned}$$

where the 4-dimensional infinite matrix $(a_{m,n,k,\ell})$, $m,n,k,\ell = 0,1,2,\dots$ is defined by

$$a_{m,n,k,\ell} = \begin{cases} \dfrac{a_{k,\ell}\left(\sum_{i,j=k,\ell}^{m,n} p_{i,j} \right)}{P_{m,n}}, & \text{if } k \le m \text{ and } \ell \le n; \\ 0, & \text{otherwise.} \end{cases}$$

Using the fact that $(\overline{N}, p_{m,n})$ is regular, one can prove that A transforms all null double sequences, i.e., all double sequences converging to zero into convergent double sequences.

Since $\lim_{k+\ell\to\infty} \varepsilon_{k,\ell} = 0$,

$$\lim_{m+n\to\infty} \left(\sum_{k,\ell=0}^{\infty,\infty} a_{m,n,k,\ell}\varepsilon_{k,\ell} \right) \text{ exists.}$$

Thus,

$$\lim_{m+n\to\infty} \beta_{m,n} = ms + \lim_{m+n\to\infty} \left(\sum_{k,\ell=0}^{\infty,\infty} a_{m,n,k,\ell}\varepsilon_{k,\ell} \right).$$

In other words, $\sum_{m,n=0}^{\infty,\infty} a_{m,n}b_{m,n}$ is $(\overline{N}, p_{m,n})$ summable to $ms + \lim_{m+n\to\infty} \left(\sum_{k,\ell=0}^{\infty,\infty} a_{m,n,k,\ell}\varepsilon_{k,\ell} \right)$. This completes the proof of the theorem. □

I thank the referee for his constructive suggestions.

References

[1] G. Bachman, Introduction to p-adic numbers and valuation theory, Academic Press, 1964.

[2] P.N. Natarajan and V. Srinivasan, Silverman-Toeplitz theorem for double sequences and series and its application to Nörlund means in non-archimedean fields, Ann. Math. Blaise Pascal, 9 (2002), 85–100.

[3] P.N. Natarajan, A theorem on summability factors for regular methods in complete ultrametric fields, Contemporary Mathematics, Amer. Math. Soc., 319 (2003), 223–225.

[4] P.N. Natarajan, More about $(\bar{N}, p_n)$ methods in non-archimedean fields, Indian J. Math., 46 (2004), 87–100.

[5] P.N. Natarajan and S. Sakthivel, Weighted means for double sequences in non-archimedean fields, Indian J. Math., 48 (2006), 201–220.

[6] A. Peyerimhoff, Lectures on summability, Lecture Notes in Mathematics, 107, Springer, 1969.

ON APPROXIMATE SOLUTION TO THE INVERSE QUASI-VARIATIONAL INEQUALITY PROBLEM

Soumitra Dey and V. Vetrivel

Abstract. In the recent past, several existence theorems for the solution of inverse variational problem which is a special case of variational inequality problems have been established by several authors. In this paper, we have define an approximate solution to inverse quasi-variational inequality problem in a locally convex Hausdorff topological vector space.

1 Introduction The theory of variational inequality problems (VIP) and its applications are well known in the last five decades. The notion of inverse variational inequality problem (IVIP) has received the attention of researchers recently due to its applications in various fields, such as traffic network problems, economic equilibrium problems (see, for example [1]). Though, the inverse variational inequality problem is a special case (see [13]) of variational inequality problems, various authors [1, 11] have explored new sufficient conditions for the existence of solution to inverse variational inequality problem, because of the fact that the existence theorems for inverse variational inequality problem are stronger than those for variational inequality problems.

He et. al. [6] introduced the inverse variational inequality problem to study the bipartite market equilibrium problem. Zou et. al. [13] gave a novel method to solve inverse variational inequality problems based on neural networks.

Recently, Aussel et. al. [2] have studied the inverse quasi-variational inequality problem (IQVIP) with an application to road pricing problem and Han et. al. [1] have established the existence of solution to the inverse quasi-variational inequality problem using fixed point theorem and Fan-Knaster-Kuratowski-Mazurkiewicz (KKM) Lemma.

Let K be a non-empty subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. Let $\Phi : \mathbb{R}^n \to 2^K$ be a set-valued mapping. The inverse quasi-variational inequality problem is to find a vector $x \in \mathbb{R}^n$ such that

$$(1) \qquad f(x) \in \Phi(x), \qquad \langle x, y - f(x)\rangle \geq 0, \forall y \in \Phi(x).$$

When $\Phi(x) = \Omega$ for all $x \in \mathbb{R}^n$, where Ω is a non-empty subset of $\mathbb{R}^n$, the inverse quasi-variational inequality problem reduces to the inverse variational inequality problem, that is, to find an $x \in \mathbb{R}^n$ such that

$$f(x) \in \Omega, \qquad \langle x, y - f(x)\rangle \geq 0, \forall y \in \Omega.$$

For more details, one can also refer to [3, 4, 5, 7, 8, 9, 10].

Han et. al. [1] proved the following existence theorem.

2010 *Mathematics Subject Classification.* 49J40, 47J20, 47H04, 47H10.

Key words and phrases. Inverse variational inequality, inverse quasi-variational inequality, best approximation, upper semi-continuity, lower semi-continuity, Kakutani factorization, Lassonde fixed point theorem.

Theorem 1.1 *Let $f^{-1}(K)$ be bounded convex and $K \subseteq f(\mathbb{R}^n)$ be compact. Assume that*
(i) f is continuous on $\mathbb{R}^n$ and natural quasi $\mathbb{R}^n_+$ -convex on $f^{-1}(K)$,
(ii) f is monotone on $f^{-1}(K)$, and $f^{-1}(.)$ is l.s.c. on K,
(iii) Φ is continuous on $\mathbb{R}^n$ and for each $u \in \mathbb{R}^n$, $\Phi(u)$ is convex closed, and $f^{-1}(\Phi(u))$ is bounded and convex with $f^{-1}(\Phi(u)) \subseteq \mathbb{R}^n_+$.
Then, the inverse quasi-variational inequality has a solution.

When there is no solution to the inverse quasi-variational inequality problem especially when the ranges of f and Φ do not intersect, one can look for an approximate solution, as there is no possibility of the existence of solution. In this paper we give sufficient conditions for the existence of an approximate solution to the inverse quasi-variational inequality problem in infinite dimensional setting.

2 Basic definitions and results

Let K be a non-empty subset of a locally convex Hausdorff topological vector space X, p be a continuous semi-norm on X and $\langle \cdot, \cdot \rangle$ be a continuous bilinear functional on $X \times X$. Then, for any $x \in X$ define $d_p(x, K) = \inf\{p(x-y), \forall y \in K\}$. A point $z \in K$ is said to be a best approximation to x with respect to p from K if $p(z-x) = d_p(x, K)$. It is well known that[12] if K is a non-empty compact convex subset, then such a best approximation from K exists to given any x in X. We say that K is relatively compact if it's closure is compact.

Definition 2.1 *[15] Let X and Y be two Hausdorff topological spaces. A set-valued mapping $\Phi : X \to 2^Y$ is said to be*
(i) upper semi-continuous (in short u.s.c) at $x_0 \in X$ if for any neighbourhood $\mathcal{N}_0$ of $\Phi(x_0)$, there exists a neighbourhood $\mathcal{N}(x_0)$ of x_0 such that

$$\Phi(x) \subseteq \mathcal{N}_0, \text{ for all } x \in \mathcal{N}(x_0).$$

(ii) lower semi-continuous (in short l.s.c) at $x_0 \in X$ if for any $y_0 \in \Phi(x_0)$ and any neighbourhood $\mathcal{N}(y_0)$ of y_0, there exists a neighbourhood $\mathcal{N}(x_0)$ of x_0 such that

$$\Phi(x) \cap \mathcal{N}(y_0) \neq \emptyset, \text{ for all } x \in \mathcal{N}(x_0).$$

A set-valued mapping $\Phi : X \to 2^Y$ is said to be continuous at a point $x_0 \in X$ if it is both u.s.c and l.s.c at $x_0 \in X$. It is said to be continuous on X, if it is continuous at every point $x \in X$.

Definition 2.2 *[17] Let X and Y be two topological vector spaces. A set-valued mapping $\Phi : X \to 2^Y$ is said to be concave if*

$$\Phi(\lambda x + (1-\lambda)y) \subseteq \lambda\Phi(x) + (1-\lambda)\Phi(y), \text{ for all } x, y \in X \text{ and } \lambda \in [0,1].$$

Lemma 2.3 *[15] Assume that X and Y are any two topological spaces and $\Phi : X \to 2^Y$ is a set-valued mapping. Then Φ is lower semi-continuous at $x_0 \in X$ if and only if for any net $\{x_\alpha\} \subseteq X$ with $x_\alpha \longrightarrow x_0$ and for any $y_0 \in \Phi(x_0)$, there exists a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ and a net $y_\beta \in \Phi(x_\beta)$ such that $y_\beta \longrightarrow y_0$.*

Let K be a non-empty subset of X. We call a set-valued mapping $\Phi : K \to 2^K$ Kakutani factorizable [14] if $\Phi = \Phi_n \circ \Phi_{n-1} \circ ..... \circ \Phi_0$, that is, if there is a diagram

$$\Phi : K \xrightarrow{\Phi_0} K_1 \xrightarrow{\Phi_1} K_2 \to .... \xrightarrow{\Phi_n} K_{n+1} = K,$$

where for each Φ_i is a non-empty set-valued mapping and K_i is a convex subset of X. For such Kakutani factorizable mappings, Lassonde [14] proved the following fixed point theorem.

Theorem 2.4 *[14] Let K be non-empty convex subset of a locally convex Hausdorff topological vector space X and a set-valued mapping $\Phi : K \to 2^K$ be Kakutani factorizable, that is $\Phi = \Phi_n \circ \Phi_{n-1} \circ ..... \circ \Phi_0$, where each Φ_i is non-empty compact convex valued upper semi-continuous set-valued mapping. If $\Phi(K)$ is relatively compact, then Φ has a fixed point, that is, there exists $x_0 \in K$ such that $x_0 \in \Phi(x_0)$.*

We end this section with the following theorem which we will need in the proof of our main theorem.

Theorem 2.5 *[12, Theorem B] Let E and F be two locally convex topological vector spaces, X be a non-empty compact and convex subset of E, Y be a non-empty subset of F, and $f, g : X \times Y \to \mathbb{R}$. If*
(i) $f(x, y) \leq g(x, y)$,
(ii) for each $x \in X$, $\{y \in Y : f(x, y) > 0\}$ is convex,
(iii) for each $y \in Y$, $x \to f(x, y)$ is lower semi-continuous on X,
(iv) for each $y \in Y$, $\{x \in X : g(x, y) \leq 0\}$ be non-empty and convex,
(v) g is lower semi-continuous on $X \times Y$,
then there exists $x_0 \in X$ such that $f(x_0, y) \leq 0$ for all $y \in Y$.

3 Existence of approximate solution to IQVIP We now prove our main theorem.

Theorem 3.1 *Let K be a non-empty compact and convex subset of a locally convex Hausdorff topological vector space X. Let $f : K \to X$ be a continuous mapping and $\Phi : K \to 2^K$ be a continuous set-valued mapping with non-empty compact convex values, satisfying the following conditions:*
(i) Φ is concave and $\Phi(K)$ is convex
(ii) for x_1, $x_2 \in K$ and $u_1 \in \Phi(x_1)$, $u_2 \in \Phi(x_2)$, we have

$$\langle x_1, u_2 - z\rangle + \langle x_2, u_1 - z\rangle \geq 0, \text{ for all } z \in \mathbb{A}_p$$

(iii) for each $y \in \Phi(K)$, $\{x \in K : \langle x, z - y\rangle \leq 0\}$ is non-empty and convex for all $z \in \mathbb{A}_p$, where $\mathbb{A}_p = \bigcup_{x \in K} \{z \in \Phi(x) : p(z - f(x)) = d_p(f(x), \Phi(x))\}$.
Then, the inverse quasi-variational inequality problem (1) admits an approximate solution, that is, there exist $x_0 \in K$ and $z_0 \in \Phi(x_0)$ such that

$$p(z_0 - f(x_0)) = d_p(f(x_0), \Phi(x_0)) \text{ and } \langle x_0, y - z_0\rangle \geq 0, \text{ for all } y \in \Phi(x_0).$$

Proof. Define a set-valued mapping $S : K \to 2^K$ by $S = S_1 \circ S_0$, where $S_0 : K \to 2^{\mathbb{A}_p}$ and $S_1 : \mathbb{A}_p \to 2^K$ with

$$S_0(x) = \{z \in \Phi(x) : p(z - f(x)) = d_p(f(x), \Phi(x))\} \text{ and}$$
$$S_1(z) = \{\omega \in K : \langle \omega, y - z\rangle \geq 0, \forall y \in \Phi(\omega)\}.$$

We first claim that S is a Kakutani factorizable set-valued mapping. By our assumption, $\Phi(x)$ is compact, convex for each $x \in K$, and thus for every $x \in K$, $f(x)$ has a best approximation from $\Phi(x)$. Hence $S_0(x)$ is non-empty.

To show that $S_0(x)$ is closed, let $\{z_\alpha\}$ be any net in $S_0(x)$ which converges to z. We show that $z \in S_0(x)$. Since $\{z_\alpha\}$ belongs to $S_0(x)$,

$$p(z_\alpha - f(x)) = d_p(f(x), \Phi(x)).$$

As $\Phi(x)$ is compact, $z \in \Phi(x)$. Letting $\alpha \longrightarrow \infty$, we see that $d_p(f(x), \Phi(x)) = p(z - f(x))$, that is, $z \in S_0(x)$ and hence $S_0(x)$ is closed. For each $x \in K$, $\Phi(x)$ is compact, hence $S_0(x)$

is compact.

Also, $S_0(x)$ is convex for each $x \in K$. Indeed, let z_1 and z_2 belong to $S_0(x)$ for fixed $x \in K$. This implies that

$$p(z_1 - f(x)) = d_p(f(x), \Phi(x)) \text{ and } p(z_2 - f(x)) = d_p(f(x), \Phi(x)).$$

We show that for any $\lambda \in [0,1]$, $\lambda z_1 + (1-\lambda)z_2 \in S_0(x)$. Since $\Phi(x)$ is convex, $\lambda z_1 + (1-\lambda)z_2 \in \Phi(x)$ for any $\lambda \in [0,1]$. For $\lambda \in [0,1]$

$$\begin{aligned} p(\lambda z_1 + (1-\lambda)z_2 - f(x)) &\leq \lambda p(z_1 - f(x)) + (1-\lambda)p(z_2 - f(x)) \\ &= \lambda d_p(f(x), \Phi(x)) + (1-\lambda)d_p(f(x), \Phi(x)) \\ &= d_p(f(x), \Phi(x)) \\ &\leq p(\lambda z_1 + (1-\lambda)z_2 - f(x)), \end{aligned}$$

which implies that $\lambda z_1 + (1-\lambda)z_2 \in S_0(x)$. Hence $S_0(x)$ is convex.

We now show that S_0 is upper semi-continuous. Let B be any non-empty closed subset of $\Phi(K)$. To show that $S_0^{-1}(B)$ is closed, it is enough to show that if $\omega_\alpha \in S_0^{-1}(B)$ and $\omega_\alpha \longrightarrow \omega$, then $\omega \in S_0^{-1}(B)$. Let $\omega_\alpha \in S_0^{-1}(B)$ and $\omega_\alpha \longrightarrow \omega$. This implies that $S_0(\omega_\alpha) \cap B \neq \emptyset$. Let $\zeta_\alpha \in S_0(\omega_\alpha) \cap B$. Since $\Phi(K)$ is compact, without loss of generality, we can assume that $\zeta_\alpha \longrightarrow \zeta$. This implies that $\zeta \in B$ as B is closed. Now we show that $\zeta \in S_0(\omega)$. Indeed, since $\zeta_\alpha \in S_0(\omega_\alpha)$, $p(\zeta_\alpha - f(\omega_\alpha)) = d_p(f(\omega_\alpha), \Phi(\omega_\alpha))$. Now, as $\alpha \longrightarrow \infty$, we get $p(\zeta - f(\omega)) = d_p(f(\omega), \Phi(\omega))$, that is, $\zeta \in S_0(\omega)$ and hence $\zeta \in S_0(\omega) \cap B$. Thus $S_0^{-1}(B)$ is closed and S_0 is upper semi-continuous.

We next show that $S_1(z)$ is non-empty. Fix $z \in \mathbb{A}_p$ and define $f_z : K \times \Phi(K) \to \mathbb{R}$ by $f_z(x, y) = \langle x, z - y\rangle$. By assumption (iii), the continuity of $\langle \cdot, \cdot \rangle$, it is easy to see that all the conditions of Theorem 2.5 are satisfied by taking $f_z(,) = g_z(,)$. Therefore there exists $x_0 \in K$ such that $\langle x_0, z - y\rangle \leq 0$ for all $y \in \Phi(K)$. In particular there exists $x_0 \in K$ such that $\langle x_0, z - y\rangle \leq 0$ for all $y \in \Phi(x_0)$, that is, there exists $x_0 \in K$ such that $\langle x_0, y - z\rangle \geq 0$ for all $y \in \Phi(x_0)$. Hence $S_1(z)$ is non-empty.

To show the compactness of $S_1(z)$, it is enough to show that it is closed. Let $\{x_\alpha\}$ be a net in $S_1(z)$ such that $x_\alpha \longrightarrow x$. Since $x_\alpha \in S_1(z)$, $\langle x_\alpha, y - z\rangle \geq 0$, for all $y \in \Phi(x_\alpha)$, for each α. Let us show that $\langle x, y - z\rangle \geq 0$, for all $y \in \Phi(x)$. Let $y \in \Phi(x)$. Since $x_\alpha \longrightarrow x$ and Φ is lower semi continuous, by Lemma 2.3, there exists a net $y'_\alpha \in \Phi(x_\alpha)$ such that $y'_\alpha \longrightarrow y$. This implies that $\langle x_\alpha, y'_\alpha - z\rangle \geq 0$, as $x_\alpha \in S_1(z)$. Since $x_\alpha \longrightarrow x$, $\langle \cdot, \cdot \rangle$ is continuous and $y'_\alpha \longrightarrow y$, as $\alpha \longrightarrow \infty$, we see that $\langle x, y - z\rangle \geq 0$. Since y is arbitrary, $\langle x, y - z\rangle \geq 0$, for all $y \in \Phi(x)$, that is, $x \in S_1(z)$ and hence $S_1(z)$ is closed. Since K is compact, $S_1(z)$ is compact.

Let us now show that $S_1(z)$ is convex. Let $p, q \in S_1(z)$ and $\lambda \in [0,1]$. That is, $\langle p, y - z\rangle \geq 0$, for all $y \in \Phi(p)$ and $\langle q, y' - z\rangle \geq 0$, for all $y' \in \Phi(q)$. It is enough to show that $\langle \lambda p + (1-\lambda)q, y - z\rangle \geq 0$, for all $y \in \Phi(\lambda p + (1-\lambda)q)$. Let $y \in \Phi(\lambda p + (1-\lambda)q)$. Since Φ is concave, we have $y = \lambda y_1 + (1-\lambda)y_2$, for some $y_1 \in \Phi(p), y_2 \in \Phi(q)$.

Now,

$$\begin{aligned}
&\langle \lambda p-(1-\lambda)q, y-z\rangle \\
&= \langle \lambda p-(1-\lambda)q, \lambda y_1+(1-\lambda)y_2-z\rangle \\
&= \lambda^2\langle p, y_1-z\rangle+(1-\lambda)^2\langle q, y_2-z\rangle+\lambda(1-\lambda)[\langle p, y_2-z\rangle+\langle q, y_1-z\rangle] \\
&\geq 0 \qquad [\, p, q\in S_1(z) \textit{ and assumption } (ii)],
\end{aligned}$$

which implies that $\langle \lambda p+(1-\lambda)q, y-z\rangle \geq 0$, for all $y\in\Phi(\lambda p+(1-\lambda)q)$. Thus $\lambda p+(1-\lambda)q\in S_1(z)$, for all $\lambda\in[0,1]$ and hence $S_1(z)$ is convex.

We now show that S_1 is upper semi-continuous. Let $B\subseteq K$ be closed and $\{z_\alpha\}$ be a net with $z_\alpha\in S_1^{-1}(B)$ such that $z_\alpha\longrightarrow z$ as $\alpha\longrightarrow\infty$. This implies that $S_1(z_\alpha)\cap B\neq\emptyset$, for all α. Let $y_\alpha\in S_1(z_\alpha)\cap B$ and $y_\alpha\longrightarrow y_0$ as $\alpha\longrightarrow\infty$. Since B is closed, $y_0\in B$. We have to show that $y_0\in S_1(z)$. Since $y_\alpha\in S_1(z_\alpha), \langle y_\alpha, y-z_\alpha\rangle\geq 0$, for all $y\in\Phi(y_\alpha)$ and for all α. Let $y\in\Phi(y_0)$. Since $y_\alpha\longrightarrow y_0$, by lower semi-continuity of Φ, there exist a net $y'_\alpha\in\Phi(y_\alpha)$ such that $y'_\alpha\longrightarrow y$. This implies $\langle y_\alpha, y'_\alpha-z_\alpha\rangle\geq 0, \forall\alpha$. As $\alpha\longrightarrow\infty$, we get $\langle y_0, y-z\rangle\geq 0$. Since y is arbitrary,

$$\langle y_0, y-z\rangle\geq 0, \text{ for all } y\in\Phi(y_0),$$

which implies that $y_0\in S_1(z)$. Hence $y_0\in S_1(z)\cap B$, that is, S_1 is upper semi-continuous.

Thus the set-valued mapping S is Kakutani factorizable. Now, by Theorem 2.4, $S: K\to 2^K$ has a fixed point. That is, there exists an $x_0\in K$ such that

$$x_0\in S_1(z_0), \text{ for some } z_0\in S_0(x_0)$$

which implies that there exists $x_0\in K$ and $z_0\in\Phi(x_0)$ such that

$$p(z_0-f(x_0))=d_p(f(x_0),\Phi(x_0)) \text{ and } \langle x_0, y-z_0\rangle\geq 0, \text{ for all } y\in\Phi(x_0).$$

It is worth noting that if $f(x_0)=z_0$, then x_0 becomes a solution to inverse quasi-variational inequality problem (1).

The following example illustrates our Theorem 3.1.

Example 3.2 *Let* $K=[-1,0]\subseteq\mathbb{R}$. *Let* $f(x)=e^x$ *and* $\Phi: K\to 2^K$ *be defined by* $\Phi(x)=[x,0]$, *for all* $x\in K$. *Here* $\mathbb{A}_\mathtt{p}=\{0\}$ *and it is easy to verify that all the conditions of Theorem 3.1 are satisfied and that* $x_0=0$ *is an approximate solution to inverse quasi-variational inequality problem. It is important to note that there is no solution to the inverse quasi-variational inequality problem involving these* K, f *and* Φ.

Acknowledgement: The authors thank the referee for his valuable suggestions to improve the earlier version of this paper.

References

[1] Y. Han, N. Huang, J. Lu and Y. Xioa, Existence and stability of solutions to inverse variational inequality problem, *Appl. Math. Mech. -Ed.,* **38** (2017), 749-764.

[2] D. Aussel, R. Gupta and A. Mehra, Gap functions and error bounds for inverse quasi-variational inequality problem, *J. Math. Anal. Appl.,* **407** (2013), 270-280.

[3] B. S. He and H. X. Liu, Inverse variational inequalities in economic field: applications and algorithms. https://www.paper.edu.cn/releasepaper/content/200609-260 (2006).

[4] J. Yang, Dynamic power price problem: an inverse variational inequality approach. *Journal of Industrial and management optimization,* **4** (2008), 673-684.

[5] B. S. He, A Goldstein's type projection method for a class of variant variational inequalities. *Journal of Computational Mathematics,* **17** (1999), 425-434.

[6] B. S. He, X. Z. He and H. X. Liu, Solving a class of constrained 'black-box' inverse variational inequalities. *European Journal of Operational Research,* **204** (2010), 391-401.

[7] B. S. He, Inexact implicit methods for monotone general variational inequalities. *Mathematical Programming,* **86** (1999), 199-217.

[8] Q. M. Han and B. S. He, A predict-correct method for a variant monotone variational inequality problems. *Chinese Science Bulletin,* **43** (1998), 1264-1267.

[9] R. Hu and Y. P. Fang, Well-posedness of inverse variational inequalities. *Journal of Convex analysis,* **15** (2008), 427-437.

[10] R. Hu and Y. P. Fang, Levitin-Polyak well-posedness by perturbations of inverse variational inequalities. *Optimization Letters,* **7** (2013), 343-359.

[11] X. Z. He and H. X. Liu, Inverse variational inequalities with projection-based solution methods. *European Journal of Operational Research,* **208** (2011), 12-18.

[12] P. Bhattacharyya and V. Vetrivel, An existence theorem on generalized quasi-variational inequality problem. *J. Math. Anal. Appl.*, **188** (1994), 610-615.

[13] X. Zou, D. Gong, L. Wang and Z. Chen, A novel method to solve inverse variational inequality problems based on neural networks. *Neurocomputing*, **173** (2016), 1163-1168.

[14] M. Lassonde, Fixed Point for Kakutani Factorizable Multifunctions. *J. Math. Anal. Appl.,* **152** (1990), 46-60.

[15] J.P. Aubin and I. Ekeland, Applied Non-linear Analysis, John Wiley and Sons, New york, 1984.

[16] H. H. Bauschke and P. L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, CMS Books in Mathematics, Springer New York, New York, NY, 2011.

[17] K. Nikodem, On concave and midpoint concave set-valued functions, *Glas. Mat. Ser.,* **22** (1987), 69-76.

Department of Mathematics,
Indian Institute of Technology Madras,
Chennai-600 036.
Email: deysoumitra2012@gmail.com, vetri@iitm.ac.in

A NOTE ON CERTAIN FUZZY METRIC SPACES

P.V. Subrahmanyam
10, Manimegalai 2nd Street, Pallikaranai
Chennai - 600 100, India
email: iitmpvs@gmail.com

Abstract. In this note we provide complete metrics for a large class of fuzzy sets on $\mathbb{R}$ which may not have bounded support and which may not even be measurable.

1 Introduction This note, a sequel to [9], continues to explore the problem of enlarging the scope of fuzzy numbers. Kaleva [6] consolidated the approach of Goetschel and Voxmax [5] in metrizing a larger class of fuzzy numbers, introduced by Dubois and Prade [3]. The monograph by Diamond and Kloeden [2] elaborates the contributions of Kaleva as well as their applications to differential equations besides other metrice on fuzzy sets. In the earlier publication [9], the author modified Kaleva's approach to metrize a class of fuzzy numbers that may not have bounded supports. The present note provides a method of metrizing all functions mapping $\mathbb{R}$, the real number system into [0,1], generalizing the work of Congxin Wu and Li [1].

2 A General Representation Theorem for Fuzzy Sets A general representation theorem for fuzzy subsets of an arbitrary nonvoid set is described below. Unlike in other representation theorems no assumptions involving either topology or convexity is made in the following.

Proposition 2.1. *Let X be a nonvoid set and $u : X \to [0,1]$, a function such that $u(x) = 1$ for some $x \in X$. Let $C_\alpha = [u]^\alpha = \{x \in X : u(x) \geq \alpha\}$ for each $\alpha \in [0,1]$. Then*

(i) for each $\alpha \in I$, C_α is a nonempty subset of X;

(ii) $C_\beta \subseteq C_\alpha$ for $0 \leq \alpha \leq \beta \leq 1$;

(iii) $C_\alpha = \bigcap_{i=1}^{\infty} C_{\alpha_i}$ for each sequence $\{\alpha_i\} \uparrow \alpha$ in I.

Conversely if for a nonempty set X, there is a family of nonempty sets C_α, $\alpha \in [0,1]$ satisfying the properties (i), (ii) and (iii) above, then there is a unique function $u : X \to [0,1]$, viz. a fuzzy subset u of X such that $[u]^\alpha = C_\alpha$ for each $\alpha \in [0,1]$ with $u(x) = 1$ for some $x \in X$.

Proof. Since $C_1 \neq \phi$, (i) and (ii) are clear. Let $\alpha \in [0,1]$ and $\alpha_i \uparrow \alpha$. Then $C_{\alpha_i} \supseteq C_\alpha$ for each i. So $\bigcap_{i=1}^{\infty} C_{\alpha_i} \supseteq C_\alpha$. If $x \in C_{\alpha_i}$, then $u(x) \geq \alpha_i$ for each i. So $u(x) \geq \lim \alpha_i = \alpha$. Thus $\bigcap_{i=1}^{\infty} C_{\alpha_i} \subseteq [u]^\alpha = C_\alpha$. Thus $\bigcap_{i=1}^{\infty} C_{\alpha_i} = C_\alpha$. If $\alpha = 0$, and $\alpha_i (\in [0,1]) \uparrow \alpha$, then $\alpha_i = 0$. In this case also (iii) is true.

2010 *Mathematics Subject Classification.*
Key words and phrases. Fuzzy metric outer measure measure, gauge sequential completeness.

To prove the converse, define $u : X \to I$ by $u(x) = \sup\{\alpha \in I : x \in C_\alpha\}$. Since $C_0 = X$, $u(x)$ is well-defined for each $x \in X$. $u(x) = 1$ for some x as C_1 is nonempty. If $x \in [u]^\alpha$, then $u(x) \geq \alpha$. Define $I_x = \{\beta \in I : x \in C_\beta\}$. Let $\alpha' = \sup I_x$. So $\alpha' = u(x) \geq \alpha$. By assumption (ii) $C_{\alpha'} \subseteq C_\alpha$. Thus $[u]^\alpha \subseteq C_\alpha$. On the other hand for $x \in C_\alpha$, $u(x) = \sup I_\alpha = \alpha' \geq \alpha$ and so $x \in [u]^\alpha$. Thus $C_\alpha \subseteq [u]^\alpha$ and so $[u]^\alpha = C_\alpha$ for each $\alpha \in I$. If for some $v : X \to I$, $[v]^\alpha = C_\alpha$ for each $\alpha \in I$, then $v(x) = u(x)$. Without loss of generality let $v(x) = r > u(x)$. Then $[v]^r = C_r \neq [u]^r$, a contradiction. So $u : X \to I$ is uniquely defined. □

We have a more general representation theorem.

Theorem 2.2. *Let* $X = \bigcup_{n=1}^{\infty} X_n$ *be a set where* $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq X_{n+1} \subseteq \cdots$ *and each* X_i *is nonempty. Let* $u : X \to [0,1]$ *be a function such that* $u^{[1]} = \{x \in X : u(x) \geq 1\} \cap X_1 \neq \phi$. *Then*

(i) for each n, $C_{\alpha,n} = [u]^\alpha \cap X_n \neq \phi$ *for all* $\alpha \in [0,1]$*: and* $[u]^\alpha = \bigcup_{n=1}^{\infty} C_{\alpha,n}$*;*

(ii) $C_{\beta,n} \subseteq C_{\alpha,n}$ *for all* $0 \leq \alpha \leq \beta \leq 1$ *for all* n*;*

(iii) for $\alpha_i \in [0,1]$ *and* $\{\alpha_i\} \uparrow \alpha \in [0,1]$, $C_{\alpha,n} = \bigcap_{i=1}^{\infty} C_{\alpha_i,n}$ *for each* $n \in \mathbb{N}$.

Conversely if X *is the countable union of an increasing sequence of nonempty sets* (X_n) *and* $\{C_{\alpha,n} : \alpha \in [0,1], n \in \mathbb{N}\}$ *is a family of nonempty subsets satisfying (i), (ii) and (iii) above then there exists a unique* $u : X \to [0,1]$ *such that for each* $\alpha \in I$ *and* $n \in \mathbb{N}$. $[u]^\alpha \cap X_n = C_{\alpha,n}$.

Proof. The proof of (i), (ii) and (iii) is easy and omitted.

For the proof of the converse define $u : X \to [0,1]$ by $u(x) = \sup\{\alpha \in [0,1] : x \in C_{\alpha,n}$ for the smallest $n \in \mathbb{N}\}$. Since $x \in X = \bigcup_{n=1}^{\infty} X_n$, $x \in X_{n_0}$ for the least $n_0 \in \mathbb{N}$. Let $u(x) = \alpha_0$. Then $x \in C_{\alpha_0,n_0}$. Further $[u]^\alpha = \bigcup_{n\in\mathbb{N}} C_{\alpha,n}$. Since $C_{1,1}$ is nonempty $[u]^1 \neq \phi$ and for $0 \leq \alpha \leq \beta \leq 1$. $[u]^\beta \subseteq [u]^\alpha$ as $C_{\beta,n} \subseteq C_{\alpha,n}$ for all $n \in \mathbb{N}$. If $\alpha_i \uparrow \alpha$ $\alpha_i, \alpha \in [0,1]$, then by (iii) $C_{\alpha,n} = \bigcap_{i=1}^{\infty} C_{\alpha_i,n}$ for each $n \in \mathbb{N}$. So $[u]^\alpha = \bigcup_{n\in\mathbb{N}} C_{\alpha,n} = \bigcap_{i=1}^{\infty} \bigcup_{n\in\mathbb{N}} C_{\alpha_i,n} = \bigcap_{i=1}^{\infty} [u]^{\alpha_i}$.

If $u \neq v$, then for some x_0 $u(x_0) > v(x_0)$ without loss of generality. So for some r, $u(x_0) > r > v(x_0)$. Or $[u]^r = \bigcup_{n\in N} C_{r,n} \neq [v]^r$ as $x_0 \in [u]^r$, though $x_0 \notin [v]^r$. Thus u is uniquely defined. □

3 Outer Measure Spaces Let X be a nonvoid set with a hereditary σ-algebra $\mathscr{S}$. Let $\mu^* : \mathscr{S} \to [0,\infty]$ be a nonnegative extended valued countably subadditive (set) function such that $\mu(\phi) = 0$. Such a (set) function is called an outer measure and the triple $(X, \mathscr{S}, \mu^*)$ is known as an outer measure space. An outer measure μ^* is called σ-finite if $X = \bigcup_{n=1}^{\infty} X_n$ where $X_n \in \mathscr{S}$, $\mu^*(X_n)$ is finite for each $n \in \mathbb{N}$.

It is known that an outer measure for which $\mu^*(X) < +\infty$ induces metrics naturally. The following theorem is esentially due to Frechet [5] and rediscovered by Meyer and Sprinkle

[8] (see MR 0104211 21 # 2968 for [8] by F.B. Jones). Since it is not widely known, both the statement and the proof are presented here for the sake of completeness.

Theorem 3.1. *Let $(X, \mathscr{S}, \mu^*)$ be an outer measure space for which $\mu^*(X) < +\infty$, $\mathscr{S}$ being a hereditary σ-algebra on a nonvoid set X. The functions ρ and δ defined on $\mathscr{S}$ by*

$$\rho(A, B) = \mu^*(A - B) + \mu^*(B - A)$$
$$\delta(A, B) = \mu^*[(A - B) \cup (B - A)]$$

for $A, B \in \mathscr{S}$ define pseudometrics on $\mathscr{S}$. Further ρ and δ are complete pseudometrics. By defining equivalence relations $A \sim B$ in $\mathscr{S}$ if $\rho(A, B) = 0$ or $\delta(A, B) = 0$ the set of all equivalence classes in $\mathscr{S}$ becomes a complete metric space under ρ or δ. Also $\rho(A, B) = 0$ or $\delta(A, B) = 0$ if and only if $A \cup Z_1 = B \cup Z_2$ where $\mu^*(Z_i) = 0$ for $i = 1, 2$.

Proof. Clearly $\rho(A, A)$ and $\delta(A, A) = 0$ for all $A \in \mathscr{S}$. Further $\rho(A, B) = \rho(B, A)$ and $\delta(A, B) = \delta(B, A)$ for all $A, B \in \mathscr{S}$, $\rho(A, B) = 0$ implies $\mu^*(A - B) = \mu^*(B - A) = 0$. So $A \cup B = A \cup B - A = A \cup Z_1$ with $\mu^*(Z_1) = \mu^*(B - A) = 0$ and $A \cup B = B \cup A - B = B \cup Z_2$ with $Z_2 = A - B$ and $\mu^*(Z_2) = 0$. This is true for δ as well, for $\delta(A, B) = 0$ implies $\mu^*(A - B \cup B - A) = 0$ leading to $A \cup B = A \cup Z_1 = A \cup B - A = B \cup Z_2 = B \cup A - B$ with $\mu^*(Z_i) = 0$, $i = 1, 2$ as before. If $A \cup Z_1 = B \cup Z_2$ where $m^*(Z_i) = 0$ for $i = 1, 2$, $m^*(A - B) \leq m^*(B \cup Z_2 - B) = m^*(B \cap B^c \cup B^c \cap Z_2) \leq m^*(Z_2) = 0$. Similarly $m^*(B - A) \leq m^*(A \cup Z_1 - A) \leq m^*(A \cap A^c \cup Z_1 \cap A^c) \leq m^*(Z_1) = 0$. So $\rho(A, B) = m^*(A - B) + m^*(B - A) = 0$. Similarly $\delta(A, B) = m^*(A - B \cup B - A) \leq m^*(A - B) + m^*(B - A) = 0$.

For $A, B, C \in \mathscr{S}$

$$\begin{aligned} \rho(A, B) &= m^*(A - B) + m^*(B - A) \\ &\leq m^*(A - C \cup C - B) + m^*(B - C \cup C - A) \\ &\leq m^*(A - C) + m^*(C - B) + m^*(B - C) + m^*(C - A) \\ &= \rho(A, C) + \rho(C, B) \end{aligned}$$

Similarly

$$\begin{aligned} \delta(A, B) &= m^*(A - B \cup B - A) \\ &\leq m^*(A - C \cup C - B \cup B - C \cup C - A) \\ &\leq m^*(A - C \cup C - A) + m^*(C - B \cup B - C) \\ &= \delta(A, C) + \delta(C, B) \end{aligned}$$

Thus both ρ and δ are pseudometrics on $\mathscr{S}$.

We now prove that $(\mathscr{S}, \rho)$ as well as $(\mathscr{S}, \delta)$ are both complete. Let (C_n) be a sequence of sets in $\mathscr{S}$. Suppose it is Cauchy in $(\mathscr{S}, \rho)$. It suffices to show that a subsequence of (C_n) converges to an element C in $\mathscr{S}$. Choose a subsequence C_{n_i} of C_n such that $\rho(C_{n_i}, C_{n_j}) < \frac{1}{2^i}$ for $i \in \mathbb{N}$ and $j > i$.

Define $D_k = \bigcap_{i=k}^{\infty} C_{n_i}$, $E_k = \bigcup_{i=k}^{\infty} C_{n_i}$ for $k \in \mathbb{N}$, $D = \bigcup_{k=1}^{\infty} D_k = \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} C_{n_i} = \underline{\lim} C_{n_i}$ and $E = \bigcup_{k=1}^{\infty} E_k = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} C_{n_i} = \overline{\lim} C_{n_i}$. As $\mathscr{S}$ is a σ-algebra, D, E, D_k and $E_k \in \mathscr{S}$. For each $k \in \mathbb{N}$, $D_k \subseteq D \subseteq E \subseteq E_k$.

Now

$$m^*(E_k - C_{n_k}) = m^*\left(\bigcup_{i=1}^{\infty} C_{n_i} - C_{n_k}\right)$$
$$\leq \sum_{i=k}^{\infty} m^*(C_{n_i} - C_{n_k})$$
$$\leq \frac{1}{2^{k-1}}.$$

So $\rho(E_k, C_{n_k}) = m^*(E_k - C_{n_k}) \to 0$ as $k \to \infty$.

$$m^*(C_{n_k} - D_k) = m^*\left(C_{n_k} \cap \left(\bigcup_{i=k}^{\infty} C_{n_i}^c\right)\right)$$
$$\leq \sum_{i=k}^{\infty} m^*(C_{n_k} - C_{n_i}) \leq \sum_{i=k}^{\infty} \rho(C_{n_k}, C_{n_i})$$
$$\leq \frac{1}{2^{k-1}}$$

So $\rho(C_{n_k}, D_k) = m^*(C_{n_k} - D_k) \to 0$, as $k \to \infty$.

Since $\rho(E_k, D_k) \leq \rho(E_k, C_{n_k}) + \rho(C_{n_k}, D_k)$

$\lim\limits_{k\to\infty} \rho(E_k, D_k) = 0$

Since $\rho(E, D) \leq \rho(E_k, D_k)$ as

$D_k \subseteq D \subseteq E \subseteq E_k$ for all k, $\rho(E, D) = 0$

Thus $\overline{\lim} C_{n_i} = \underline{\lim} C_{n_i} = E$ or D and $\rho(E, C_{n_k}) \to 0$ as $k \to \infty$.

So (C_{n_k}) converges to $E(= D)$ in $(\mathscr{S}, \rho)$ and hence (C_n) converges to $E(= D)$ in $(\mathscr{S}, \rho)$ and hence (C_n) converges to $E(= D)$ in $(\mathscr{S}, \rho)$. Thus $(\mathscr{S}, \rho)$ is complete.

If (C_n) is Cauchy in $(\mathscr{S}, \delta)$ as before choose a subseqeunce (C_{n_i}) of (C_n) such that $\delta(C_{n_i}, C_k) < \frac{1}{2^c}$ for all $k \geq n_i$.

$$\delta(E_k - C_{n_k}) = m^*\left(\bigcup_{i=k}^{\infty} C_{n_i} - C_{n_k}\right)$$
$$\leq \sum_{i=k}^{\infty} m^*(C_{n_i} - C_{n_k})$$
$$\leq \frac{1}{2^{k-1}}.$$

So $\lim\limits_{k\to\infty} \delta(E_k, C_{n_k}) = 0$

Also

$$\delta(C_{n_k}, D_k) = m^*\left(C_{n_k} \cap \left(\bigcup_{i=k}^{\infty} C_{n_i}^c\right)\right)$$
$$\leq \sum_{i=k}^{\infty} m^*(C_{n_k} - C_{n_i})$$
$$\leq \sum_{i=k}^{\infty} \delta(C_{n_k}, C_{n_i})$$
$$\leq \frac{1}{2^{k-1}}$$

Thus $\lim\limits_{k\to\infty} \delta(C_{n_k}, D_k) = 0$. Now as $\delta(D_k, E_k) \le \delta(D_k, C_{n_k}) + \delta(C_{n_k}, E_k)$, $\lim\limits_{k\to\infty} \delta(D_k, E_k) = 0$. Since $\delta(E, D) = m^*(E - D) \le m^*(E_k - D_k) = \delta(E_k, D_k)$ for all k, $\delta(E, D) = 0$ or $E - D$. Thus C_{n_k} converges to $E = \overline{\lim} C_{n_i} = \underline{\lim} C_{n_i} = D$. So $\delta(E(= D), C_n) \to 0$ as $n \to \infty$. Thus $(\mathscr{S}, \delta)$ is complete. □

Considering the set of all equivalence classes of sets in $\mathscr{S}$ induced by the equivalence relation $A \sim B$ if and only if $\rho(A, B)' = 0$ or $\delta(A, B) = 0$ the metric induced by ρ or δ is complete.

In this context we recall the following definition (see Dugundji [4]).

Definition 3.2. *Let $D = \{D_\lambda : \lambda \in \Lambda\}$ be a family of pseudometrics on a nonvoid set X. The topology $\tau(D)$ with the subbase $\{B(x; d_\lambda, \epsilon)(= \{y \in X : d_\lambda(x, y) < \epsilon\}\}$ where $\epsilon > 0$ and d_λ, a pseudometric on X is called a gauge space, the family D being a gauge. The gauge is called separating if for $x, y \in X$ $x \ne y$, there exists $\lambda_0 \in \Lambda$ such that $d_{\lambda_0}(x, y) > 0$. (Clearly a gauge is separating if and only if the topology is Hausdorff).*

Definition 3.3. *Let (X, D) be a gauge space. A sequence (x_n) is called Cauchy if $d_\lambda(x_m, x_n) \to 0$ as $m, n \to \infty$ for each $\lambda \in \Lambda$. The gauge space is said to be sequentially complete if every Cauchy sequence in X is convergent.*

Remark 3.4. *A topological space is a gauge space if and only if it is a Tychonoff space. A necessary and sufficient condition for a gauge space to be metrizable is that it has a countable gauge. For these and related results Dugundji [4] may be referred.*

We have the following theorems whose straight-forward proofs are left as exercises.

Theorem 3.5. *Let $(X, \mathscr{S}, \mu^*)$ be an outer measure space with a hereditary σ-algebra $\mathscr{S}$. Suppose $X = \bigcup\limits_{n=1}^{\infty} X_n$ where $X_1 \subseteq X_2 \cdots \subseteq X_n \subseteq X_{n+1} \subseteq \cdots$ with $\mu^*(X_n) < +\infty$ for all n. Then $(\mathscr{S}, D)((\mathscr{S}, D'))$ is a Hausdorff complete metrizable gauge space. Here $D = \{\rho_n : n \in \mathbb{N}\}$, $D' = \{\delta_n : n \in \mathbb{N}\}$ where $\rho_n(A, B) = \rho(A \cap X_n, B \cap X_n)$ and $\delta_n(A, B) = \delta(A \cap X_n, B \cap X_n)$ for $n \in \mathbb{N}$, ρ and δ being the metrics defined in Theorem 3.1. Also $A, B \in \mathscr{S}$ for which $\rho_n(A, B) = 0$ ($\delta_n(A, B) = 0$) for all n are identified as equal.*

Theorem 3.6. *Let $(X, \mathscr{S}, \mu)$ be a complete measure space. Suppose $X = \bigcup\limits_{n=1}^{\infty} X_n$ where $X_1 \subseteq X_2 \cdots \subseteq X_n \subseteq X_{n+1} \subseteq \cdots$ with $\mu(X_n) < +\infty$ for all $n \in \mathbb{N}$. Then $(\mathscr{S}, D)((\mathscr{S}, D'))$ is a Hausdorff complete metrizable gauge space. Here $D = \{\rho_n : n \in \mathbb{N}\}$ and $D' = \{\delta_n : n \in \mathbb{N}\}$ where $\rho_n(A, B) = \mu(X_n \cap (A - B)) + \mu(X_n \cap (B - A))$ and $\delta_n(A, B) = \mu\{X_n \cap ((A - B) \cup (B - A))\}$ for each $n \in \mathbb{N}$. In $\mathscr{S}$ sets A, B with $\rho_n(A, B) = 0$ ($\delta_n(A, B) = 0$) for all $n \in \mathbb{N}$ are treated equivalent.*

4 Fuzzy Subsets of an Outer Measure Space

In this section we provide a metrical structure for a class of fuzzy subsets of an outer measure space $(X, \mathscr{S}, \mu^*)$ defined on a hereditary σ-algebra $\mathscr{S}$. This is described in Theorem 4.1 and Theorem 4.2 and Corollary 4.3 point out how a wide class of fuzzy subsets of $\mathbb{R}^n$ or $\mathbb{R}$ can be endowed with a complete metric.

Theorem 4.1. *Let $(X, \mathscr{S}, \mu^*)$ be an outer measure space, $\mathscr{S}$ being a hereditary σ-algebra with $X = \bigcup\limits_{n=1}^{\infty} X_n$, where $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq X_{n+1} \subseteq \cdots$ and $\mu^*(X_n) < +\infty$ for all $n \in \mathbb{N}$. Let $F_A^1(x)$ be the set of all functions $u : X \to [0, 1]$ such that $\{x \in X : u(x) \ge \alpha\} \in \mathscr{S}$ for* *each $\alpha \in [0,1]$ and u and v for which $\mu^*\{x : u(x) \neq v(x)\} = 0$ are treated as equal in $F^1_A(x)$. Further suppose there exists $A \in \mathscr{S}$ with $A \subseteq \{x : u(x) \geq 1\} \subseteq X_1$ for all u in $F^1_A(X)$ and $\mu^*(A) > 0$. Define $\Delta_n, \Delta'_n : F^1_A(X) \times F^1_A(X) \to \mathbb{R}^+$ by $\Delta_n(u,v) = \sup_{0\leq\alpha\leq 1} \rho([u]^\alpha, [v]^\alpha)$ and $\Delta'_n(u,v) = \sup_{0\leq\alpha\leq 1} \delta([u]^\alpha, [v]^\alpha)$. Then $\{F^1_A(X), \Delta_n : n \in N\}$ and $\{F^1_A(X), \Delta'_n : n \in \mathbb{N}\}$ are Hausdorff, gauge spaces which are complete metrizable spaces.*

Proof. Since $\Delta_n(u,v) = \Delta_n(v,u)$ and $\Delta_n(u,v) \leq \Delta_n(u,w) + \Delta_n(w,v)$ for all n each Δ_n is a pseudometric on $F^1_A(X)$. Further for $\Delta_n(u,v) = 0$ for all n $[u]^\alpha = [v]^\alpha$ on X for all $\alpha \in [0,1]$. Since $[u]^r = [v]^r$ for all rationals in $[0,1]$ it follows that $u = v$ almost everywhere on X with respect to μ^*. Thus $\{F'(U), \Delta_n : n \in \mathbb{N}\}$ is a Hausdorff gauge space. Let u_n be a Cauchy sequence in $\{F^1_A(U), \Delta_n\}$. Since $\sup_{\alpha\in I} \rho(X_n \cap [u_p]^\alpha, X_n \cap [u_q]^\alpha)$ is Cauchy in (X,ρ), there exists C^α_n such that $\sup_{\alpha\in I} \rho(X_n \cap [u_m]^\alpha, C^\alpha_n) \to 0$ as $m \to \infty$. Define $C^\alpha = \bigcup_{n=1}^{\infty} C^\alpha_n$. Clearly $C^\alpha \in \mathscr{S}$ for each $\alpha \in [0,1]$. Since $[u_m]^\beta$ for $\beta \geq \alpha$, $\alpha, \beta \in [0,1]$, for each $n \in \mathbb{N}$ $X_n \cap [u_m]^\beta \subseteq X_n \cap [u_m]^\alpha$. As $m \to \infty$, since $[u_m]^\beta \to C^\beta = \lim [u_m]^\beta$. $X_n \cap C^\beta \subseteq X_n \cap \lim [u_m]^\alpha = X_n \cap C^\alpha$. If $\alpha_i \uparrow \alpha$ in $[0,1]$, $\lim_{i\to\infty} [u_m]^{\alpha_i} \cap X_n = [u_m]^\alpha \cap X_n$. $\rho([u_m]^{\alpha_i} \cap X_n, C^{\alpha_i}_n) < \epsilon$ for all $m \geq m_0$ for all α. Now as $\alpha_i \uparrow \alpha$, $P([u_m]^{\alpha_i} \cap X_n, [u_m]^\alpha \cap X_n) < \epsilon$ for $i \geq i_0$. So $\rho([u_m]^{\alpha_i} \cap X_n, C^\alpha_n) \leq \rho([u_m]^{\alpha_i} \cap X_n, [u_m]^\alpha \cap X_n) + \rho([u_m]^\alpha \cap X_n, C^\alpha_n) < 2\epsilon$. So as $m \to \infty$ $\rho(C^{\alpha_i}_n, C^\alpha_n) \leq 2\epsilon$ for $i \geq i_0$

Hence $C^{\alpha_i}_n \to C^\alpha_n$ as $\lim C^{\alpha_i}_n = \bigcap_{i=1}^{\infty} C^{\alpha_i}_n$. Also $[u_n]^1 \supseteq A$ for all n. So $\lim_{n\to\infty} [u_n]^1 = C^1 \supseteq A$. Thus $\{F^1(X), \Delta_n : n \in \mathbb{N}\}$ is a Hausdorff countable gauge space which is sequentially complete. So it is metrizable and complete. One can generate the gauge topology using the metric $\Delta(u,v) = \sum_{n=1}^{\infty} \frac{\min(1, \Delta_n(u,v))}{2^n}$ or $\sum_{n=1}^{\infty} \frac{\Delta_n(u,v)}{2^n \mu^*(X_n)}$

A similar argument shows that $F^1_A(X, \mathscr{S}, \mu^*)$ with the countable gauge $\{\Delta'_n : n \geq \mathbb{N}\}$ is Hausdorff and completely metrizable and

$$\Delta'(u,v) = \sum_{n=1}^{\infty} \frac{\min(1, \Delta'_n(u,v))}{2^n} \text{ or}$$
$$= \sum_{n=1}^{\infty} \frac{\Delta'_n(u,v)}{2^n \mu^*(X_n)}$$

gives a complete metric. □

The following theorem can be proved along similar lines.

Theorem 4.2. *Let $(X, \mathscr{S}, \mu)$ be a complete measure space where $X = \bigcup_{n=1}^{\infty} X_n$ where $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq X_{n+1} \subseteq \cdots$ and $0 < \mu(X_n) < +\infty$ for all n. Let $F^1_A(X, \mathscr{S}, \mu)$ be the set of measurable functions mapping X into $[0,1]$ such that for some $A \subseteq X_1$ with $\mu(A) > 0$ and $u^{[1]} \supseteq A$. Then $F^1_M(X)$ is a Hausdorff complete metrizable gauge space with the gauge $\{\Delta_n : n \in N\}$ or $\{\Delta'_n : n \in N\}$ where*

$$\Delta_n(u,v) = \sup_{0\leq\alpha\leq 1} \rho_n([u]^\alpha, [v]^\alpha) \text{ and}$$
$$\Delta'_n(u,v) = \sup_{0\leq\alpha\leq 1} \delta_n([u]^\alpha, [v]^\alpha)$$

for $u, v \in F^1_m(X)$*.* ρ_n *and* δ_n *are as in Theorem 3.6.*

Corollary 4.3. *Let* μ^* *be the Lebesgue outer measure on* $X = \mathbb{R}^n$*. Then for any* $A \subseteq B(0;1)$ *the unit open ball with* $\mu^*(A) > 0$, $F^1_A(X, 2^X, \mu^*)$*, the set of all fuzzy subsets* u *of* $\mathbb{R}^n$ *with* $u^{[1]} \supseteq A$ *is a sequentially complete Hausdorff gauge space with the gauge* $\{\Delta_n : n \in \mathbb{N}\}$ *or* $\{\Delta'_n : n \in \mathbb{N}\}$ *where* Δ_n *and* Δ'_n *are as in Theorem 4.1.*

Corollary 4.4. *If* μ^* *is the Lebesgue measure on* $\mathbb{R}^n$ *and* $A \subseteq B(0,1)$ *the unit open ball in* $\mathbb{R}^n$ *with* $\mu(A) > 0$*. Then* $F^1_\mu(\mathbb{R}^n, \mathscr{S}, \mu)$ *the set of all fuzzy Lebesgue measurable subsets of* $\mathbb{R}^n$ *with* $[u]^1 \supseteq A$ *is a sequentially complete Hausdorff gauge space with the gauge* $\{\Delta_n : n \in \mathbb{N}\}$ *or* $\{\Delta'_n : n \in \mathbb{N}\}$ *where* Δ_n *and* Δ'_n *are as in Theorem 4.2.*

Remark 4.5. *As the space of Lebesgue outer measurable subsets of* $(0,1)$ *or the unit ball in* $\mathbb{R}^n$ *with the metrics* ρ *or* δ *induced by the Lebesgue outer measure is not separable,* $F^1_A(\mathbb{R}, 2^{\mathbb{R}})$ *or* $F^1_A(\mathbb{R}^n, 2^{\mathbb{R}^n})$ *with the gauge* $\{\Delta_n : n \in \mathbb{N}\}$ *or* $\{\Delta'_n : n \in \mathbb{N}\}$ *described in Theorem 4.1 is not separable.*

Remark 4.6. *Characteristic functions of singletons in* $\mathbb{R}$ *are identified with fuzzy real numbers in the Kaleva approach to fuzzy real numbers. However as singletons have zero Lebesgue measure, the characteristic functions of singletons are all equivalent to the zero function and so cannot be used to represent fuzzy real numbers in* $F^1(R)$*. However this can be remedied by considering the product metric space* $F^1(\mathbb{R}) \times \mathbb{R}$ *with the corresponding metric of the product space so that the real number system can be isometrically embedded in this product space. This is similar to embedding the real numbers isometrically in the complex plane or* $\mathbb{R}^2$*.*

Remark 4.7. *When* $(X, \mathscr{S}, \mu^*)$ *is a finite outer measure space, then* $F^1_U(X, \mathscr{S}, \mu^*)$*, the set of all fuzzy subsets* $u : X \to [0,1]$ *with* $[u]^1 \supseteq A$ *and* $\mu^*(A) > 0$ *is a complete metric space under the metrics*

$$\rho(u,v) = \sup_{0 \le \alpha \le 1} \{m^*([u]^\alpha - [v]^\alpha) + m^*([v]^\alpha - [u]^\alpha)]$$
$$\delta(u,v) = \sup_{0 \le \alpha \le 1} \{m^*[([u]^\alpha - [v]^\alpha) \cup ([v]^\alpha - [u]^\alpha)]$$

Finally we provide just one example to show that certain fuzzy functional equations can be solved in this setting, affording greater flexibility and scope for solving nonlinear equations involving fuzzy numbers which are neither upper semicontinuous nor convex.

Example 4.8. *Let* X *be [0,1] and* μ^* *the Lebesgue outer measure on the power set* 2^X *of* X*. Let* A *be a non-measurable subset of* X *with positive outer measure and* $F^1_A(X, \mathscr{S}, \mu^*)$ *the set of all fuzzy subsets* $u : X \to [0,1]$ *such that* $[u]^1 \supseteq A$*. Define* $T : F^1_A(X) \to F^1_A(X)$ *by* $[Tu]^\alpha = [v]^\alpha = f\{x : u(x) \ge \alpha\} \cup A$ *where* $f(x) = \frac{e^{-x}+x}{2}$*. Since* $\mu^*([Tu_1]^\alpha - [Tu_2]^\alpha) + \mu^*([Tu_2]^\alpha - [Tu_1]^\alpha) \le \frac{1}{2}\mu^*([u_1]^\alpha - [u_2]^\alpha) + \mu^*([u_2]^\alpha - [u_1]^\alpha)$ *and* $F^1_A(X)$ *is complete and* $\Delta(Tu_1, Tu_2) \le \frac{1}{2}\Delta(u_1, u_2)$*,* T *has a unique fixed point which is a solution of the functional equation* $Tu = u$ *in* $F^1_A(X)$*.*

References

[1] Congxin Wu and Fachao Li, Fuzzy metric and convergences based on the symmetric difference, Fuzzy sets and Systems, 108 (1999), 332–335.

[2] P. Diamond and P. Kloeden, Metric spaces of fuzzy sets: theory and applications, World Scientific, 1994.

[3] Didier Dubois and Henri Prade, Operations on fuzzy numbers, International Journal of Systems Science, 9(6) (1978), 613–626.

[4] James Dugundji, Topology, Allyn and Bacon Inc., 1966.

[5] M. Frechet, Sur La Distance De Deux Ensembles, Calcutta Math. Soc. Bulletin, 15 (1924), 1–8.

[6] Roy Goetschel Jr. and William Voxman, Topological properties of fuzzy numbers, Fuzzy Sets and Systems, 10(1) (1983), 87–99.

[7] Osmo Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems, 24(3) (1987), 301–307.

[8] Burnett Meyer and H.D. Sprinkle, Two nonseparable complete metric spaces defined on [0,1], Pacific J. Math., 8 (1958), 825–828.

[9] P.V. Subrahmanyam, On the space of fuzzy numbers, Scientia Math. Japonicae, Special version, 281 (2015), 1–8.

EXAMPLE OF CUBE SLICES THAT ARE NOT ZONOIDS

R.Anantharaman

Received August 6, 2018

To the memories of Som Naimpally and Joe Diestel

Abstract

Let Q be the unit cube in R^n centered at the Origin O and H a hyperplane through O.The intersection is called a central Cube slice and its study was initiated by Hadwiger, Henesley and Vaaler , continued by Ball and others. A zonoid is the range of a non atomic vector measure into R^n . In this paper, when $n = 4$ we give examples ofnon -zonoid cube slices. Let H: x + y +z +t =0 ; the slice has triangle faces and is not a zonoid. This contrasts with a result inR^3,where it follows from a classical Theorem due to Herz and Lindenstrauss that every central cube slice is a zonoid(zonotope). We also give nontrivial examples in which the slice is a zonoid. For ex. let H : ax + y + z+t=0 with a>1. If a≥ 3 , the slice is a zonotope. Otherwise it has faces that are trapeziums or pentagons and is not a zonoid.We also give other examples of the like nature.

1 Introduction

1.1 Slices Zonoids ,Zonotopes

Let us recall the result from [3]:– Let Q^n =Q =unit cube in R^n centered at Origin O; ie. Q= $\{\mathbf{x} = (\mathrm{x}_k) : |x_k| \leq \frac{1}{2}\}$.

Let H be a vector subspace of dimension n-1, ie. a plane thru the Origin with equation : H= (x= (x_k) with x .a =0) for a(non zero) vector a in R^n. The intersection of H andQ will be called central slice or, slice. Following [3] we denote by$|A|$ the appropriate volume /area of the measurable set A ,and assume n ≥2. As other examples let us note the papers [7], [8],[13]initial to this subject , and the surveys [5], [10] [14]]that treats many related topicsWenote the p th powerof L^p norm of the sinc function in [3] : for (p≥ 2):

$$I_p = \frac{1}{\pi} \int\limits_R \frac{|sint|^p}{|t||^p} dt \tag{1}$$

An upperbound for this is:- $\frac{\sqrt{2}}{\sqrt{p}}$, with equality iff $p = 2$. The lower bound is assumed by H: x_k= 0 and upper only if n=2 and H: $x + y$=0 or with $x - y$=0

Let us mention that Valler[13] considers concepts of analytic interest; his results not only prove lower bound but also apply to Minkowski's Theorem on Linear foirms.

We note that there are alsoresults on sections by centralplanes of dimension k(see [14] TH 1.2, 1.3 p 154), alsodue toBall. We treat only the case k= n-1.

This estimate is in [3]; see also [10, Ch1]. The proof of this estimatein [3, p468] is with " direct " and uses only elementary methods . The one in [10] uses Fourier methods.This integral I_p has found use in wavelets[11]

For our needs we use the more precise values also from ([3] Lemma3) below ,see eq(9), (10). In [3] this is derived ,first using Characteristic functions(= Fourier Transform) then the standard Inverse Fourier Formula .

As pointed out by an anonymous referee (of another paper)– see Acknowledgements –this I_4 is in the classic, [12] (also in [9]); see [10] for many related deeper results However we use the formula from [3] for vol of slice of cube . Our interest is more in the sliceitself .With n=4 in Sec 3we give example ofa a (central)slice that has a triangle face, and isnot nota zonoid ("$face$") defined below).On the other hand,in Sec 4 we give examples of slices that are zonoids, and othersthat have a pentagon or trapezium face and so are not.

Notation and preliminaries : We write an element of R^4 as (x, y, z, t) and use a, b, c, d as coefficients. Below we avoid the case when H is paralll to acoordinate hyperplane;inthis case the slice is a Cube of lower dimension and so a zonoid

Let a hyperplane be H : ax + by + cz + dt =0 .As in [8], we may assume that no coefficient is zero, and next they are all positive. Further we may assumethat $a \geq b \geq c \geq d$ and then by dividing by d, that H: ax + by + cz + t =0 with $a \geq b \geq c \geq 1$. In all examples of non zonoids we consider the equation [t= -1/2] to get a Face that is atriangle,trapezium or pentagon(disqualifying slice fombeig a zonoid : see beginning of Sec 3).

In ex 3.1 we consider the case when a=b=c(= 1); and as mentioned above show that the slice has triangular faces and so not a zonoid(" face" defined below).The sections of this sliceby planes[t= -c] with $0 < c < 1/2$ are hexagons .These tend to the triangle face as c tends to 1/2 . We may feel that " cube slices in R^4 are never nontrivial zonoids". Hencein ex 4.1 we consider H: a x + y + z + t =0 with $a > 1$ Now the slice is a zonoid if $a \geq 3$ and is a paralletope;in the contrary cases the slice has pentagon faces and isnot a zonoid . In Ex4.2 we consider H:a(x +y)+ z +t=0 ; the slice is not a zonoid onaccount of trapezium faces.In Ex 4.3 we have H: a(x + y) +z +t=0and slice has pentagon faces In ex4.4 Webriefly indicate special cases of H: ax + by + z +t=0 witha $> b >$1. As the methods in these examples is same asthe one in Ex 3.1 we donot give details. In Ex.4.4 we consider the case of H:$ax + by + cz + t = 0$.Slice is a paralleotope in case $a \geq b + c + 1$ and $b \geq c + 1$. If (i) and (ii) both fail then the slice has pentagon faces and is not a zonoid

We give these as samples ; and do not consider every possible case .Roughly , the non zonoids prevail in our list of examples.

DiagramThey will help.

Our **methods** are elementary and can be found for ex in [6]. We do use the formula for vol of slices from [3] (see also [10] ch1) referred to above.

We notethatthat in al of our examples we use the face [t= -1/2] of the cube , this is also a faceof the slice The "*domain*" C of face is found first , then an affine map T to determine theFace T(C). The points in C are found by checking the x and y intercepts of lines involved satisfy theconditions for slice:–$|x|$, $|y|$ and $|z|$ are all$\leq$ 1/2. This condition must be satisfied by all coordinates of points in the Face (of slices) that we find, and leads tothe conditions imposed on the coefficients of H.

Let us first describe the result on slices

. **1.2Theorem ([3][5] ,[7], [8] [13] ,[14])**In all dimensions the measure of cube slice is between 1 and $\sqrt{2}$; these arebest.

1.3Zonoids

Returning to the title of this paper, about Zonoids:– Our concern is :– When is a slice a zonoid? We do not have a complete characterization of this .Instead let us concentrate in R^4, and give examples of non zonoid slices as well as those that are zonoids and a consequence(known) for I_4 . We recall from [6] with X=R^n :– . A zonoid is range of a non atomic vector measure and above all the classical Liapunovs Theorem:- A zonoid is compactand convex A zonotope is sum of segments(each centered at the Origin) For our purpose we need the classic result of Herz and Lindenstrauss from [6]:– The closed unit ball in every 2dimensional normed space is a zonoid

2 Zonoids and Zonotopes

2.1 Theorem[6]

i)If H is 2 dimensional then every such slice is a zonotope ii) In all dimensions every projection of Q is a zonotope

Proof:

(i) This follows from the classic result due to Herz and Lindenstrauss quoted above and the result from ([6]) :- inR^2 every centrally symmetric polygon is always a sum of segments

(ii) This is in [6] and can also be verified directly. Hence the Theorem.

Remark 2.2:

For much more about projections see, [4].

3 Example of slice that is not a zonoid

A notedbefore,in contrast(Th 2.1, part i) to the situation in R^3 we offer an example of a slice in R^4 that is not a zonoid. Reasons to disqualify it from being a zonoid are the useful facts, all from [6] :–If K is a zonoid then

(i) K has center of symmetry c say .In fact by definition of " K is a Zonoid "(see Introduction)

K=$\mu(\sum)$ for a (vector measure) μ then c= $\frac{1}{2}\mu(S)$ will do For, with A^c = complement of set A, we have $\frac{1}{2}(\mu(A)+\mu(A^c)=1/2\mu(S)$ =c for every A in domain$\sum$ of μ

(ii) faces are translates of zonoids of lower dimension and

(iii) Since it has no center of symmetry,the triangle is not a zonoid; neither is a trapezium (trapezoid) or a pentagon

(iv)Hence any compact ,convex, balanced set that has a triangular,(or a trapezium face) cannot be a zonoid. Thus, the Octohedron in R^3 is nota zonoid, for it has triangular faces. There are deeper non zonoids for ex the 1976 result due to LE Dor (for ex[10]):–If $1<p<2$ and $n\geq 3$ then the closed unit Balls of the spaces l_n^p are not zonoiids

We give, in Th3.4, a version of(ii) from [2]:– a face (defined below) is a translate of some zonoid of lower dimension . We need this version in the Th 3.4 to produce non zonoid slices in our examples.

Let us recall fom [6] theterm , Face of acompact convex set K in a real(normed space) X. Let us use H for any hyperplane (not necessarily thru O)

As above a hyperplane is

$$H=(x\epsilon X:(x,x^*)=\alpha)\ , \tag{2}$$

where x^* isa non zero functional in X^* and $\alpha$is a real number .

The set K is on one side" of this H if

$$sup[(x,x*):x\epsilon K]\leq\alpha, \tag{3}$$

A similar condition holds with inf replacing sup and by$\geq$ replacing $\leq$;and H supports K if K is to one side of H as in eq(3,) H $\cap K\neq\phi$and K is not entirely contained in H. Finally the (compact convex) set H$\cap$ K is called a Face of K .

Below we use the fact that an affinemap preserves convexity.

Let X, Y be real Banach spaces .Then a mapT: X Y is affineif T (ax + by)= aT(x) + bT(y) for every x, y in X and a , b 0 with a + b =1.ie; the definition of Linear map is now restricted to line segments in domain.

Let us recall ; K= $H\cap Q$ is the slice correponding to H[$t=-1/2$] In the next (and other) examples all we need is that the relevant , y, z coordinates of our pints are limited by $|x|\leq 1/2$ etc .

3.1Example with triangle face

Let us recall H is given in R^4 by

$$x+y+z+t=0, \tag{4}$$

The slice (i) has triangular faces and so is not a zonoid

(ii) the intersections of slice with t= -c , 0$\leq c<\frac{1}{2}$

are hexagons ;these are sections (iii) These tend to the above triangle as c tends to 1/2

proof(i)

Substituting t = -1/2 in eq(4) of H , for any **x** =(x, y, z, t) in this H we have

x = x(1,0,0 -1) + y (0,1,0,-1) +z(0,0, 1, -1) is the linear combination x u + y v + z w. (these 3 vectors u, v, w are Linearly independent)

First consider the 2 dimensional set S in slice,in span of vectors u and v. Starting with A (u/2) on the x axis and going counterclockwise,we see

that S is a hexagon with vertices A (u/2), B(v/2), C((v-u)/2, A'= -A,

B'= -B , C= -C. Further it is regular all sides have length $1/\sqrt{2}$ and that this = sum of 3 segments , OA , OC and OB'. This set S is in plane z=0 Now we consider the 3rd term in above eq for x ; we note that the vector D= w/2 cannot be added to A or B as the sum will leave the cube We consider the Hyperplane $H_1 =$ (x, y, z, t): t= -1/2) or, simply by t = -1/2 and claim that

(a) this plane supports the slice K and that

(b)the face F $=H_1 \cap K$ is convex triangle.

As noted above, (b) disqualifies the slice from being a zonoid

Let us verify the claims. Now (a) follows directly from def. of Q.In fact for every element in Q we have $t \geq -1/2$ ie., Q is to one side of H_1; so is the slice .Further , the elements A , B, are in the slice, and also lie inH_1 , hence in Face F. The Origin O is in slice K not in H_1; ie. the sliceis not entirely contained inH_1.Hence H_1 is a supporting hyperplane of the slice as claimed.

For claim (b) we may write any element in the Face as

x = (x, y, z, t)=(x, y, $\frac{1}{2} - x - y, -1/2)$,

since we use t= - 1/2 in eq (4) of H and

we get z= $1/2 - -x - y$.

As x is in Q we need $|x|$ and $|y|$ and also $|z|$ from above ≤1/2 and so

$$|1/2 - x - y| \leq 1/2 \ , \tag{5}$$

This last translates to

$$0 \leq x + y \leq 1 \ , \tag{6}$$

Geometrically, we note that the last inequality gives two boundary lines of "*domainC*" sayL_1 := x+y =1 , and L_2:= x+ y=0. We sketch these lines; as the x-intercept ofL_1 exceed the bound 1/2 let us consider its intersection with the line x =1/2 toget point (1/2, 1/2).; intersection of L_2 with the line y=1/2 gives (-1/2, 1/2). This line with y=-1/2 gives (1/2, -1/2)

These result in a (convex right angled) triangle C in x-y plane with above vertices P(1/2, -1/2) , Q(1/2, 1/2) and R (- 1/2, 1/2)

Now let us define a map Tfrom Cto F by

$$T(x, y) = (x, y, 1/2 - x - y, -1/2) \tag{7}$$

and C is its domain. Then we may verify that, T is affine and that T(C)=F. Further as observed before statement of this example, affine map preserves convexity, and so the image T (C) = convex hull of the 3 points

$(p_1,p_2$, $p_3)$ where p_1 = T(P)= (1 /2 , -1/2 ,1/2, -1/2) , p_2 =
T(Q)= (1 /2, 1/2, -1/2, -1/2) andp_3= T(R)= (- 1 /2, ,1/2, 1 /2,-1/2). These points are not collinear, form a triangle and we conclude that the face F is a triangle , completing Claim (b) and proof of (i) .

We need to prove (ii) and (iii) .

Recall K = slice ; now we let $0 < c < 1/2$ and Section K_c = K $\cap$[t = - c].

Use t= - c in eq (4) H; any x in Kc is then of

the form $\mathbf{x}$ = (x, y , -x-y+c, -c) with the conditions

$|x|$, $|y|$ and $|x + y - c| \leq 1/2$.

Similarly to above (6) this last translates to

$$-1/2 + c \leq x + y \leq 1/2 + c \; , \tag{8}$$

As in part (i) we draw the "*boundary*" lines L_1 , L_2 from eq (8) . Again , both the x and y- intercepts of L_1 fail the bounds of 1/2; however L_2 passes (noting the limits on c)Then we find the vertices of our "*domainC*" by intersecting L_1 and L_2 with the lines y=1/2, y= -1/2, x= 1/2 and x= - 1/2. . We get a hexagon(domain). Its 6 vertices are shown in a Chart in next Theorem 3.3 and as follows:–

p_6 =(c, -1/2) on lines y = -1/2 and L_2 , p_1 =
(1 / 2, -1/2) andp_2 =(1/2 , c) on line x= 1 / 2 andL_1 .Next p_3 =(c, 1 / 2) on lines L_1 and y=1/2 and p_4 =(- 1/2, 1/2) then p_5 = (-1/2, c) on lines L_2 and x= -1/2

These 6 points (p_i) form a hexagon making
the new domain C of map T defined analogous to eq(7) in part (i) above.
As there we see that the section K_c = T(C) is also a hexagon.
Finally let c tend to 1 / 2 ; then we see from above that the
following vertices coincide:- $p_2 = p_3$ =(1/2, 1/2) ,$p_4 = p_5$
=(-1/2, 1 /2) and p_6 = (1/2, -1/2) = p_1 . Correspondingly (as in
case i above) we verify that the section T(C) becomes the triangle in
part(i) completing thereby proof of (ii) and the example

Remark 3.2. Above we used the hyper plane given by the equation,
t= - 1 /2 and found that the face of slice given by it is triangular
; we may instead consider t = 1 / 2 Further, the equation defining H is
symmetric with respect to the four variables x, y, z, t. Hence we may conclude that there are 8 triangular faces. We do not know what are the remaining faces and we think there are 4 more but not triangles .

For the next result, we follow [3]Lemma 3 (see also[10] ch1) and recall from Introduction eq(1) theintegral I_p :

$\frac{1}{\pi}\int_R \frac{|sint|^p}{|t||^p} dt.$

Here p is an integer ≥2, and we have from the result in [3] above,
the formula forthe exact value of slice :-

$$\|H \cap Q| = \frac{1}{\pi}\int_R g(t)dt, \tag{9}$$

where g is the finite product

$$g(t) = \prod_{1}^{N} \frac{sina_i t}{a_i t}, \tag{10}$$

and the sequence (a_i) (of coordinatesof vector normal to H) is normalized in l_2 andalso each $|a_j| \leq 1/2$

To find volume of slice S we use Cavaleri s principle = Fubini's Theorem .

Let $|A(c)|$= area of the section of S by plane [t= -c]. Then the

vol of slice =2 $\int_0^{1/2} |A(c)|dc.$

We saw in ex 3.1 that A (c) is a hexagon . We give the details in the next result;

3.3Theorem (i)The volume of the slice in Ex3.1 is 4/3 (ii) $I_4 = 2/3$

Proof(i) We refer to part (b)in ex3.1 and list the vertices of the hexagons in domain C as well as in the range T(C).

Recall T(x, y)= (x, y, c-x-y, -c) with $0 < c < 1/2$.

Domain C .... rangeT(C)

p_1(1/2, -1/2)................... P_1(1/2, -1/2, c, -c)

p_2 (1/2, c) .... P_2 (1/2, c ,-1/2 , -c)

p_3 (c , 1/2) .... P_3 (c , 1/2, -1/2, -c)

p_4 (-1/2, 1/2) .... $P_4(-1/2, 1/2, c, -c)$

$p_5(-1/2, c)$..........................P5(-1/2, c, 1/2, -c)

p_6(c, -1/2) ....P_6 (c, -1/2, 1/2 , -c)

We claim that area of domain C=

$$|A(c)| = (3/4 - c^2), \tag{11}$$

In the following we use formula for area of trapezium by rule

(1 /2) h (a + b) where h is the height and a, b are lengths of parallel sides.

Let us use the chart for domain C first then use it to get the image.

The domain C = two trapeziumsT_1 andT_2 ; these are the top and at bottom respy. Namely, T_1 has vertices, p_5, p_2, p_3 and p_4

and T_2 has vertices, p_6, p_1,p_2 andp_5.

Then we have

$|T_1|$= $\frac{1}{2}(1 + c + 1/2)(1/2 - c) = 1/2(3/2 + c)(1/2 - c) and$

$|T_2|$= 1/2 (1 -c +1/2) (1/2 + c)= 1 /2(3/2 -c) (1/2 +c)

Adding them we get the eq (11) for A(c).

To get the area of image T(C) observe that the domain C is the projection on plane(z=0) of the wanted T(C) .

For the factor needed we note that the unit normal to H is

n=(1/2 , 1/2 , 1 /2 , 1 /2) and that e_3 = (0,0,1, 0).

Using the dot product $n.e_3$ we see that area of Projection

= 1/2 area of T (C). Thus the area of T (C) = $2(3/4 - c^2)$ from above.

We integrate from c=0 to 1 / 2 to get

$2\int_0^{1/2} (\frac{3}{4}$ - c $^2)dc = 2/3$

Taking into account alsothe part t= 1/2 to 0 we get 2(2/3) =4/3 as claimed

Part (ii) :Recalling H: x+ y + z+t =0 and the coefficients, normalized , we apply the formula from [3] quotedabove in eq (9) , (10) to get vol of slice=

$\frac{1}{\pi}\int_R \frac{(sint/2)^4}{(t/2)^4}dt.$

(as in part (i) we used each a_i coefficient is 1/2 due to normalizing them in eq of H) Now a change of variable gives

$\frac{2}{\pi}\int_R (sint/t)^4$ dt $= 2I_4$.

From part (i) we have $2I_4 = 4/\ 3$

and so part (ii) and the Theorem

Above,in example of a non zonoid slice we used the important fact about faces of a zonoid from [6] (innext Theorem) The following proof is different from the one in[6] which uses Every Zonoid is a zonoid of moments .This approach is not suitable for our purpose; hence we give a proof (in[2]) in next result. We see in the proof that it is more meaningful incase the Face isnot a singleton, ie. when the composed measure $x^*o\mu$ is not equivalent to μ

3.4Theorem[6][2]]Let K =$\mu(\sum)$ be a zonoid in X=R^n, and H a supporting Hyperplane given by x* in X*.Then the face F=$K\cap H$ is a translate of a zonoid of lower dimension . In fact there are μ almost disjoint sets S_0 and S_1 such that

(i)$x^*o\mu(S_1)$=sup$\{x^*o\mu(E) : E\epsilon\sum\}$ and every set E inS_0 is $x^*o\mu$ - null

(ii F= $\mu(S_1)$ + $\mu_{S0}(\sum)$.

Proof: With β= sup$x^*\mu(\sum)$ we have,from definition of F

$$F = \{x : x = \mu(E) s.t. x^*(x) = \beta\}, \tag{12}$$

Let S^+ be such that $x^*o\mu(S^+) = \beta$.

We will writeS^+ =S_0 ∪ S_1 as stated in the Theorem.

To do this let usnote that the signed measure $x^*o\mu << \mu$

;consider those E that are $x^*o\mu$ - null but not $\mu - null$.

(ifthere are no such sets E then S_0may be taken to be ∅).

Othewise consider a maximal pairwise disjoint family ofsuch sets; this family is countable, so that theirunion is in $\sum$. Call this set S_0 and let $S_1 = S^+ - S_0$ .Then

(i) follows from the fact that

x^*o $\mu(S_0)$ =0 by the construction of S_0 and so

$\beta =$ x *o $\mu(S^+) =$ x * o$\mu(S_0)$+x *$o\mu(S_1)$

$= x^*$o $\mu(S_1)$. we see that second part in (i) follows by construction again.

As for part(ii) we have from Eq (12) if x= $\mu(E)$ ϵ F then

$x^*o\mu$ (E) $= \beta$.

We need to write x =μ (E) asthe sum, μ (E)= μ (S_1) + μ (A) for some set A⊂S_0 To do this,first we claim that(ae—x*$o\mu$) this E⊂ S^+ . Ifnot we can argue to contradict to the fact that S = S^+ ∪S^- is a Hahndecomposition of the underlying set S in terms of $x^*o\mu$

Again we can argue that S_1 - E is μ null;from it being $x^*o\mu$ null, and then on (subsets of) S_1 these two measures are equivalent by construction.

Hence we have E= E $\cap$ S_1 $\cup E\cap S_0, andso$
$\mu(E) = \mu(E\cap)S_1) + \mu(E\cap S_0) =$
$\mu(S_1) + \mu(A) with A = (E\cap S_0)\subset S_0$ as claimed.
Hence the Theorem

4 Examples of non zonoids with pentagon faces and some zonoids a >1

.

As in Introduction welet H: ax + by+ cz + t=0 be a hyperplane in R^4,with a≥b ≥c ≥1.

We donot consider all cases but hope the following are of interest. There are non trivial cases of zonoid slices . As the methods are same as the one in the earlier ex 3.1 we only summarise the results It seems the non zonoid slices dominate:–

In the next ex. we donotknow if the converse is truein this gnerality. Hencewe give some special cases ofthe eq of H in the EXs 4.2 and on.In all cases for the Face we use as before the support hyperplane of Q [t= $\frac{-1}{2}$]

4.1 H: general caseabove

If a $\geq b + c + 1$ then the sliceis a zonotope.

Proceeding as in Ex3.1, we find the " domain" for the face .For this we have the boundary lines L_1 to be ax + by = $\frac{c+1}{2}$ and L_2 to be ax+by =$\frac{-c+1}{2}$

Firstwe note bothx andy-intercepts of L_2 are always (regardless ofthis condition) $\frac{1}{2}$ in absolute value. As forL_1 this condition gives the x-intercepttobe $\leq \frac{1}{2}$ in absolute velue. In the following " domain the vertex p_2 depends on this condition, ie. its " $|x|$: satisfies the limits $\leq\frac{1}{2}$.

Withthe condition above we have now the chart

Domain

p_1 $(\frac{b-c+1}{2a}, \frac{-1}{2})$

p_2 $(\frac{c+1+b}{2a}, \frac{-1}{2})$

p_3 $(\frac{c+1-b}{2a}, \frac{1}{2})$

p_4 $(\frac{1-b-c}{2a}, \frac{1}{2})$

Next the corresponding points on the Face:–

FaceT(x,y)=(x, y, $\frac{1/2 - ax - by}{c}$, $\frac{-1}{2}$)

P_1 $(\frac{b-c+1}{2a}, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{2})$

P_2 $(\frac{c+1+b}{2a}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2})$

$P_3\ (\dfrac{c+1-b}{2a}, \dfrac{1}{2}, \dfrac{-1}{2}, \dfrac{-1}{2})$
$P_4\ (\dfrac{1-b-c}{2a}, \dfrac{1}{2}, \dfrac{1}{2}, \dfrac{-1}{2})$

It is seen that this domain is a parallogram with the parallel sides(so is the Face):

$(p_1p_2) = (p_4p_3) = (\dfrac{c}{a}, 0)$ and
$(p_2\ p_3) =$ (p$_1$ p_4) $=(\dfrac{b}{a}$. -1)

Likewise, it can be verified using the map "T" , that so is the Face.

Further, the sections of theslicebby planes with eqs t= $-c_1$, with 0 <c$_1$ $< 1/2$ are parallograms that are congruent to the one for the Face. Hence it follows (using symmetry) thatthe slice is a zonotope.

4.2 H : a x +y + z+ t=0

In one direction this is a special caseofEx 4.1 ;however due tolimitation of eq ofH we can state " iff" and we give details :–

In this case if $a\geq 3$ then the slice isa zonoid ; it is a paralleotope if not the slice has pentagon faces and is not a zonoid.

Case $a\geq 3$: Face is a ;paralleogram ; so is every parallel section congruent to it

Letus note that analogously to eq(6) above we replace x by ax there. Thus the x-intercept of the line with equation $ax + y = 1$ is x= $1/a$.The condition $x\leq 1/2$ now holds (due tothe condition on a). This forces the "Domain" to be a paralleogram as we now state. As before we use (x,y) forpoints p_i and

x= T(x,y)= (x, y, 1/2- (ax +y), -1/2) for points P_i :

Domain C(x,y) Face T(C)

p_1(1/2a, -1/2) P_1(1/2a, - 1/2, 1/2, -1/2)
p_2(3/2a, -1/2) P_2 (3/2a, -1/2, -1/2, -1/2)
p_3 (1/2a, 1/2) P_3(1/2a, 1/2, - 1/2, -1/2)
p_4 (-1/2a, 1/2) P_4(- 1/2a, 1/2, 1/2, -1/2)

We see that the opposite sides are parallel and have equal length , so that the Face is a rhombus .Further so is any section by plane[t= -c] with $0 < c < 1/2$, the area does not depend on c and equals $\sqrt{1 + 2a^{-}2}$

case $a < 3$ in this case we can verify the " domain" tobe apentagon; so is the face and slice is not a zoniod

4.3 H: a(x +y) +z +t =0 with $a\geq 2$

The Face$[t = -1/2]$ is a trapezium again, slice not a zonoid

4.4 H:ax +by+ z +t =0 (compare ex4.1) Face is a paralleogram in case $b + 2\leq a$. The parallel sections$[t = -c]$ are congruent parallograms, and the slice isa parallotope. Otherwise Face is a pentagon, slice is not a zonoid

4.5 H: ax + by + z +t =0 The slice is a zonotope if (i) a $\geq b + c + 1$ and (ii) $b\geq c + 1$.

In case (i) and (ii) both fail Face is a pentagon and slice is nota zonoid.

If (i) fails but(ii) is true then the Face is a hexagon

Remark 4.5 In the last case we dont know if the slice is a zonoid

Acknowledgements:

The author is thankful to an anonymous referee(of another related paper) for the reference [12] and the information there about the Sinc Integral I_4

AnnouncementThe abstract of this paper wasannounced in the Proceedings of Ramanujan Birthday Conference held on Dec 22nd 2017 in IITMadras, CHennai, India

References

[1] R.Anantharaman, *Keith Ball's Cube Slicing Theorem* unpublished-manuscript

[2] , R.Anantharamanand KM Garg, *Properties of a residual set of vector measures*, Proc. Conf. Shebrooke, Canada,Lecture Notes in Math;Springer-Verlag,**1033**(1982),12-35

[3] K Ball, *Cube slicing in* R^n, Proc. AMS **97**,3(1986)465-473.

[4] ...., *Shadows of Convex Bodies*, Trans. AMS.**327** (2)(1991)891-901

[5] , ... *An Elementary Introduction to modern convex geometry*, Math Sci.Re. Inst.Publ.**31** Cambridge Univ Press , Cambridge (1997)

[6] , E.Bolker, *A class of convex bodies*, Trans AMS **145** (1969),323-345

[7] , H.Hadwiger, *Gutterperiodische Pungtmengen und Isoperimetrie*, Monash. Math. 76(1972), 410-418

[8] , D.Henseley, *Slicing the cube in* R^n *and Probability*, Proc.AMS.73(1979)95-100

[9] , E.Hewitt, KO Stromberbg *Real and Abstract Analysis* Springer-Verlag, Berlin-Heidelberg-NewYork1965 and later eds

[10] A.Koldobsky, V.Yaskin, *The Interface between Convex Geometry and Fourier Analysis*, CBMS . AMS . Providence ,(RI)**108**,(2008) .

[11] D. Labate, G Weiss and E Wilson, *Wavelets* , Notices of AMS,**60** (1) (2013), 66-76

[12] G.Polya, G.Szego, *Problems and Thorems in Analysis,English Trans* vol **I** ,Springer-Verlag(1972) .

[13] Vaaler, *A geometric inequality with application to linear forms*, Pacific J. Math. **83** (1979)543-553

[14] C.Zong, *What isknown about unit cubes* , Bull AMS. textbf42(2)(2005),181-211

date first done : Dec 20th 2016

Retired from SUNY/College @Old Westbury Old Westbury NY 11568-0210
e mail rajan.anantharaman@gmail.com

address: 3032 Briar Oak Dr Duluth GA 30096

MR Classification: Primary 52 A 20, Secondary 42 A38, 52A40

INTERNATIONAL ADVISORS

BOARD OF EDITORS

CZECH REPUBLIC

(a) **Milan Vlach**
(b) Charles University, Faculty of Mathematics and Physics
Malostranske namesti 25, 118 00 Prague 1, Czech Republic
(b') milan.vlach@mff.cuni.cz and mvlach@ksi.ms.mff.cuni.cz
(c) Game theory, Fair division, Optimization

FINLAND

(a) **Arto Salomaa**
(b) Jaanintie 34 A 26, 20540 Turku, Finland
(b') asalomaa@utu.fi
(c) Formal languages, automata, computability, cryptography

(a) **Jarkko Kari**
(b) Department of Mathematics and Statistics, FI-20014 University of Turku, Finland
(b') jkari@utu.fi
(c) automata theory, cellular automata, tilings, symbolic dynamics

GERMANY

(a)**Klaus Denecke**
(b) 14542 Werder (Havel), Germany, Zanderweg 3
(b') klausdenecke@hotmail.com
(c) General Algebra, Discrete Mathematics, Multiple-valued Logic, Ordered Sets and Lattices, Theory of Semigroups

GREECE

(a) **Maria Fragoulopoulou**
(b) Department of Mathematics, University of Athens, Panepistimiopolis, Athens 157 84, Greece
(b') fragoulop@math.uoa.gr
(c) Non-normed Topological Algebras, Topological Algebras with an Involution, Unbounded Operator Theory, Tensor products of Topological Algebras and Topological *-Algebras.

HUNGARY

(a) **Gyula Maksa**
(b) Institute of Mathematics, University of Debrecen, H-4002 Debrecen, Pf. 400, Hungary
(b') maksa@science.unideb.hu
(c) Functional equations

(a) **Kálmán Györy**
(b) University of Debrecen, Institute of Mathematics, 4010 Debrecen, Hungary
(b') gyory@science.unideb.hu
(c) Number Theory (mainly Diophantine and Algebraic Number Theory)

(a) **Pál Dömösi**
(b) Institute of Mathematics and Informatics, Nyíregyháza University, Nyíregyháza, Sóstói út 31/B, H-4400, Hungary
(b') domosi.pal@nye.hu
(c) Theoretical Computer Science, Algebra

(c) Mathematical logic, set theory, category theory

(a) **John B Conway**
(b) Professor Emeritus, George Washington University,Phillip Hall 801 22nd St. NW
Washington, DC 20052, U.S.A
(b') Conway@gwu.edu
(c) Functional Analysis and Operator Theory

(a) **Paul Cull**
(b) Computer Science, Kelley Engineering Center, Oregon State University, Corvallis, OR 97331, USA
(b') pc@cs.orst.edu
(c) Difference Equations and Dynamical Systems, Computer Science (Theory, Algorithms, Networks), Mathematical Biology (Population Models, Neural Nets)

(a) **W. Wistar Comfort**
(b) Department of Mathematics, Wesleyan University, Wesleyan Station, Middletown, CT USA 06459
(b') wcomfort@wesleyan.edu
(c) Topological theory of topological groups, General (set-theoretic) topology

JAPAN

(a) **Mariko Yasugi**
(b) non-public
(b') yasugi@cc.kyoto-su.ac.jp
(c) Logic Oriented Mathematics

(a) **Haruo Maki**
(b) non-public
(b') makih@pop12.odn.ne.jp
(c) (Topological) digital n-spaces (n>0), Generalized closed sets (after Levine),
Operation theory in topology (in the sense of Kasahara and Ogata)

(a) **Kohzo Yamada**
(b) Faculty of Education, Shizuoka Univ., 836 Ohya, Shizuoka 422-8529, Japan
(b') kohzo.yamada@shizuoka.ac.jp
(c) General Topology

(a) **Yasunao Hattori**
(b) Shimane Univ., Matsue, Shimane 690-8504, Japan
(b') hattori@riko.shimane-u.ac.jp
(c) General Topology

(a) **Yoshikazu Yasui**
(b) Department of Modern Education, Faculty of Education, Kio University, 4-2-2, Umami-naka, Koryo-cho,
Kitakaturagi-gun, Nara, 635-0832, Japan
(b') y.yasui@kio.ac.jp
(c) General Topology

(a) Eiichi Nakai
(b) Department of Mathematics, Ibaraki University, Mito, Ibaraki 310-8512, Japan
(b') eiichi.nakai.math@vc.ibaraki.ac.jp
(c) Real analysis, harmonic analysis, Fourier analysis, function spaces, singular and fractional integrals

(a) **Jun Kawabe**
(b) Division of Mathematics and Physics, Shinshu University, 4-17-1 Wakasato, Nagano 380-8553, Japan
(b') jkawabe@shinshu-u.ac.jp
(c) Measure and integration, Vector measure, Nonadditive measure

(a) **Shizu Nakanishi**
(b) non-public
(b’) shizu.nakanishi@nifty.ne.jp
(c) measures and integrations

(a) **Jun Ichi Fujii**
(b) Department of Educational Collaboration(Science, Mathematics and Information),Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan
(b') fujii@cc.osaka-kyoiku.ac.jp
(c) Operator Theory

(a) **Masaru Nagisa**
(b) Department of Mathematics and Informatics, Graduate School of Science, Chiba University, Yayoi-cho, Chiba, 263-8522, Japan
(b’) nagisa@math.s.chiba-u.ac.jp
(c) operator algebra, operator theory

(a) **Hiroyuki Osaka**
(b) Graduate School of Science and Engineering, Ritsumeikan University, 1-1-1 Noji-higashi, Kusatsu, Shiga 525-8577 Japan
(b') osaka@se.ritsumei.ac.jp
(c) Operator Theory and Operator Algebras

(a) **Masatoshi Fujii**
(b) non-public
(b’) mfujii@cc.osaka-kyoiku.ac.jp
(c) Operator Theory

(a) **Wataru Takahashi**
(b) Keio Research and Education Center for Natural Science,Keio University,Kouhoku-ko,Yokohama 223-8521, Japan
(b’) wataru@is.titech.ac.jp, wataru@a00.itscom.net
(c) Nonlinear Functional Analysis

(a) **Shigeo Akashi**
(b) Department of Information Sciences, Faculty of Science and Technology, Tokyo University of Science, 2641, Yamazaki, Noda-City, Chiba-Prefecture, 278-8510, Japan
(b’) akashi@is.noda.tus.ac.jp
(c) Information Theory, Entropy Analysis, Applied Mathematics, Functional Analysis

(a) **Yoshitsugu Kabeya**
(b) Department of Mathematical Sciences, Osaka Prefecture University, 1-1, Gakuen-cho, Naka-ku, Sakai, Osaka 599-8531, Japan
(b’) kabeya@ms.osakafu-u.ac.jp
(c) Partial Differential Equations, Ordinary Differential Equations

(a) **Atsushi Yagi**
(b) Dept. of Applied Physics, Graduate School of Engineering, Osaka Univ., 2-1 Yamadaoka, Suita, Osaka 565-0871, Japan
(b’) yagi@ap.eng.osaka-u.ac.jp
(c) Nonlinear partial differential equations, Infinite-dimensional dynamical systems

(a) **Yoshimasa Nakamura**
(b) Graduate School of Informatics, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan
(b)’ ynaka@i.kyoto-u.ac.jp

(c) integrable systems, numerical linear algebra, special functions

(a) **Yasumasa Fujisaki**
(b) Department of Information and Physical Sciences, Graduate School of Information Science and Technology, Osaka University, 1-5 Yamadaoka, Suita, Osaka 565-0871, Japan
(b') fujisaki@ist.osaka-u.ac.jp
(c) Control Systems Theory

(a) **Naruhiko Aizawa**
(b) Department of Physical Science, Graduate School of Science, Osaka Prefecture University,Sakai,Osaka 599-8531,Japan
(b')aizawa@p.s.osakafu-u.ac.jp
(c) representation theory

(a) **Hisao Nagao**
(b) non-public
(b') nagao.hisao@aqua.plala.or.jp
(c) Multivariate Analysis, Sequential Analysis, Jackknife Statistics and Bootstrap Method

(a) **Masamori Ihara**
(b) non-public
(b') ihara@osakac.ac.jp
(c) Structural equations modeling, Statistical Quality Managment, Factor analysis, Multivariate Analysis

(a) **Masanobu Taniguchi**
(b) Dept. of Applied Mathematics, School of Fundamental Science & Engineering, Waseda University, 3-4-1, Okubo, Shinjuku-ku, Tokyo,169-8555, Japan, Tel & Fax: 03-5286-8386
(b') taniguchi@waseda.jp
(c) Statistical Inference for Stochastic Processes

(a) **Masao Kondo**
(b) non-public
(b') kondo@sci.kagoshima-u.ac.jp
(c) Time Series Analysis

(a) **Masao Fukushima**
(b) Dept. of Systems and Mathematical Science, Faculty of Science and Engineering, Nanzan University, Nagoya, Aichi 466-8673, Japan
(b)' fuku@nanzan-u.ac.jp
(c) Mathematical Programming, Nonlinear Optimization

(a) **Ryusuke Hohzaki**
(b) Department of Computer Science, National Defense Academy, 1-10-20, Hashirimizu, Yokosuka, 239-8686, Japan
(b') hozaki@cc.nda.ac.jp
(c) Reviewable area: Operations Research, Search theory, Game theory

(a) **Hiroaki Ishii**
(b) Department of Mathematical Sciences, School of Science and Technology, Kwansei Gakuin University
2-1 Gakuen, Sanda, Hyogo 669-1337, Japan
(b') ishiihiroaki@kwansei.ac.jp
(c) Operations Research and Fuzzy Theory, especially Mathematical Programming (Stochastic Programming, Combinatorial Optimization, Fuzzy Programming), Scheduling Theory, Graph and Network Theory, Inventory control, Mathematical evaluation method

(a) **Junzo Watada**
(b) Universiti Teknologi PETRONAS Department of Computer & Information Sciences 32610 Seri Iskandar,Perak Darul Ridzuam,Malaysia Office Phone:
+60-5-368-7517 Mobile:+60-13-598-0208
Professor Emeritus,Waseda University,Japan
(b') junzow@osb.att.ne.jp
(c) Fuzzy systems, Management Engineering

(a) **Kensaku Kikuta**
(b) School of Business Administration, University of Hyogo,
8-2-1 Gakuen-nishi-machi, Nishi-ku, Kobe City 651-2197 JAPAN
(b') kikuta@biz.u-hyogo.ac.jp
(c) Game Theory, Operations Research,

(a) **Wuyi Yue**
(b) Dept. of Intelligence and Informatics, Faculty of Intelligence and Informatics, Konan University, 8-9-1 Okamoto, Higashinada-ku , Kobe 658-8501, JAPAN
(b') yue@konan-u.ac.jp
(c) Queueing Networks, Performance Analysis and Modeling, Communications Networks, Operations Research, Markov Processes, Probabilistic Methods, Systems Engineering

(a) **Hiroaki Sandoh**
(b) Faculty of Policy Studies Kwansei Gakuin University 2-1, Gakuen, Sanda-shi, Hyogo 669-1337 Japan
(b') sandoh@kwansei.ac.jp
(c) Operations Research and Management Science, Stochastic modeling

(a) Katsunori Ano
(b) Department of Mathematical Sciences, Shibaura Institute of Technology, 307 Fukasaku Minuma-ku Saitama-city, 337-8570, Japan
(b') k-ano@shibaura-it.ac.jp
(c) Optimal Stopping, Mathematical Finance, Applied Probability

(a) **Koyu Uematsu**
(b) Graduate School of Management and Information Sciense Faculty of Global Business ,Osaka International University 6-21-57 Tohdacho, Moriguchi-Shi, Osaka,570-8555,Japan
(b') uematsu@oiu.jp
(c) Stochastic Process and its Applications,Reliability Analysis,and Game Theory

(a) **Yoshiki Kinoshita**
(b) Dept. of Information Sciences , Faculty of Science, Kanagawa University, Tsuchiya 2946, Hiratsuka-shi, Kanagawa 259-1293, Japan
(b') yoshiki@kanagawa-u.ac.jp
(c) Software Science, Programming language semantics

(a) **Shunsuke Sato**
(b) non-public
(b')ss_22362@nifty.com
(c) Mathematical biology in general

(a)**Tadashi Takahashi**
(b)Department of Intelligence and Informatics, Konan University, 8-9-1 Okamoto, Higashinada, Kobe, Hyogo 658-8501, Japan
(b') takahasi@konan-u.ac.jp
(c)Mathematics Education

Submission to the SCMJ

In September 2012, the way of submission to Scientiae Mathematicae Japonicae (SCMJ) was changed. Submissions should be sent electronically (in PDF file) to the editorial office of International Society for Mathematical Sciences (ISMS).

(1) Preparation of files and Submission

a. Authors who would like to submit their papers to the SCMJ should make source files of their papers in LaTeX2e using the ISMS style file (scmjlt2e.sty) Submissions should be in PDF file compiled from the source files. Send the PDF file to s1bmt@jams.jp .

b. Prepare a Submission Form and send it to the ISMS. The required items to be contained in the form are:

1. Editor's name whom the author chooses from the Editorial Board (http://www.jams.or.jp/hp/submission_f.html)and would like to take in charge of the paper for refereeing.
2. Title of the paper.
3. Authors' names.
4. Corresponding author's name, e-mail address and postal address (affiliation).
5. Membership number in case the author is an ISMS member.

Japanese authors should write 3 and 4 both in English and in Japanese.

At http://www.jams.or.jp/hp/submission_f.html, the author can find the Submission Form. Fulfill the Form and sent it to the editorial office by pushing the button "transmission". Or, without using the Form, the author may send an e-mail containing the items 1-5 to s1bmt@jams.jp

(2) Registration of Papers

When the editorial office receives both a PDF file of a submitted paper and a Submission Form, we register the paper. We inform the author of the registration number and the received date. At the same time, we send the PDF file to the editor whom the author chooses in the Submission Form and request him/her to begin the process of refereeing. (Authors need not send their papers to the editor they choose.)

(3) Reviewing Process

a. The editor who receives, from the editorial office, the PDF file and the request of starting the reviewing process, he/she will find an appropriate referee for the paper.

b. The referee sends a report to the editor. When revision of the paper is necessary, the editor informs the author of the referee's opinion.

c. Based on the referee report, the editor sends his/her decision (acceptance of rejection) to the editorial office.

(4) a. Managing Editor of the SCMJ makes the final decision to the paper valuing the editor's decision, and informs it to the author.

b. When the paper is accepted, we ask the author to send us a source file and a PDF file of the final manuscript.

c. The publication charges for the ISMS members are free if the membership dues have been paid without delay. If the authors of the accepted papers are not the ISMS members, they should become ISMS members and pay ¥6,000 (US$75, Euro55) as the membership dues for a year, or should just pay the same amount without becoming the members.

Items required in Submission Form

1. Editor's name who the authors wish will take in charge of the paper
2. Title of the paper
3. Authors' names

3'. 3. in Japanese for Japanese authors

4. Corresponding author's name and postal address (affiliation)

4'. 4. in Japanese for Japanese authors

5. ISMS membership number
6. E-mail address

Call for ISMS Members

Call for Academic and Institutional Members

Discounted subscription price: When organizations become the Academic and Institutional Members of the ISMS, they can subscribe our journal Scientiae Mathematicae Japonicae at the yearly price of US$225. At this price, they can add the subscription of the online version upon their request.

Invitation of two associate members: We would like to invite two persons from the organizations to the associate members with no membership fees. The two persons will enjoy almost the same privileges as the individual members. Although the associate members cannot have their own ID Name and Password to read the online version of SCMJ, they can read the online version of SCMJ at their organization.

To apply for the Academic and Institutional Member of the ISMS, please use the following application form.

Application for Academic and Institutional Member of ISMS

Subscription of SCMJ Check one of the two.	☐Print (US$225) ☐Print ＋ Online (US$225)
University (Institution)	
Department	
Postal Address where SCMJ should be sent	
E-mail address	
Person in charge	Name: Signature:
Payment Check one of the two.	☐Bank transfer ☐Credit Card (Visa, Master)
Name of Associate Membership	1.
	2.

Call for Individual Members

We call for individual members. The privileges to them and the membership dues are shown in "**Join ISMS !**" on the inside of the back cover.

Items required in Membership Application Form

1. Name
2. Birth date
3. Academic background
4. Affiliation
5. 4's address
6. Doctorate
7. Contact address
8. E-mail address
9. Special fields
10. Membership category (See **Table 1** in "**Join ISMS !**")

Individual Membership Application Form

1. Name	
2. Birth date	
3. Academic background	
4. Affiliation	
5. 4's address	
6. Doctorate	
7. Contact address	
8. E-mail address	
9. Special fields	
10. Membership category	

Contributions (Gift to the ISMS)

We deeply appreciate your generous contributions to support the activities of our society.

The donation are used (1) to make medals for the new prizes (Kitagawa Prize, Kunugi Prize, and ISMS Prize), (2) to support the IVMS at Osaka University Nakanoshima Center, and (3) for a special fund designated by the contributors.

Your remittance to the following accounts of ours will be very much appreciated.

(1) Through a post office, remit to our giro account (in Yen only):
No. 00930-1-11872, Japanese Association of Mathematical Sciences (JAMS)
or send International Postal Money Order (in US Dollar or in Yen) to our address:
International Society for Mathematical Sciences
2-1-18 Minami Hanadaguchi, Sakai-ku, Sakai, Osaka 590-0075, Japan

(2) A/C 94103518, ISMS
CITIBANK, Japan Ltd., Shinsaibashi Branch
Midosuji Diamond Building
2-1-2 Nishi Shinsaibashi, Chuo-ku, Osaka 542-0086, Japan

****

Payment Instructions:

Payment can be made through a post office or a bank, or by credit card. Members may choose the most convenient way of remittance. Please note that we do not accept payment by bank drafts (checks). For more information, please refer to an invoice.

Methods of Overseas Payment:

Payment can be made through (1) a post office, (2) a bank, (3) by credit card, or (4) UNESCO Coupons.

Authors or members may choose the most convenient way of remittance as are shown below. Please note that **we do not accept payment by bank drafts (checks)**.

(1) Remittance through a post office to our giro account No. 00930-1-11872 or send International Postal Money Order to our postal address (2) Remittance through a bank to our account No. 94103518 at Shinsaibashi Branch of CITIBANK (3) **Payment by credit cards** (AMEX, VISA, MASTER or NICOS), or (4) Payment by UNESCO Coupons.

Methods of Domestic Payment:

Make remittance to:

(1) Post Office Transfer Account - 00930-3-73982 or

(2) Account No.7726251 at Sakai Branch, SUMITOMO MITSUI BANKING CORPORATION, Sakai, Osaka, Japan.

All of the correspondences concerning subscriptions, back numbers, individual and institutional memberships, should be addressed to the Publications Department, International Society for Mathematical Sciences.

Join ISMS !

ISMS Publications: We published **Mathematica Japonica (M.J.)** in print, which was first published in 1948 and has gained an international reputation in about sixty years, and its offshoot **Scientiae Mathematicae (SCM)** both online and in print. In January 2001, the two publications were unified and changed to **Scientiae Mathematicae Japonicae (SCMJ)**, which is the "21st Century New Unified Series of Mathematica Japonica and Scientiae Mathematicae" and published both online and in print. Ahead of this, the online version of SCMJ was first published in September 2000. The whole number of SCMJ exceeds 270, which is the largest amount in the publications of mathematical sciences in Japan. The features of SCMJ are:

1) About 80 eminent professors and researchers of not only Japan but also 20 foreign countries join the Editorial Board. The accepted papers are published both online and in print. SCMJ is reviewed by Mathematical Review and Zentralblatt from cover to cover.
2) SCMJ is distributed to many libraries of the world. The papers in SCMJ are introduced to the relevant research groups for the positive exchanges between researchers.
3) **ISMS Annual Meeting:** Many researchers of ISMS members and non-members gather and take time to make presentations and discussions in their research groups every year.

The privileges to the individual ISMS Members:

(1) No publication charges
(2) Free access (**including printing out**) to the online version of SCMJ
(3) Free copy of each printed issue

The privileges to the Institutional Members:

Two associate members can be registered, free of charge, from an institution.

Table 1: Membership Dues for 2018

Categories	**Domestic**	**Overseas**	**Developing countries**
1-year Regular member	¥8,000	US$80， Euro75	US$50， Euro47
1-year Students member	¥4,000	US$50， Euro47	US$30， Euro28
Life member*	Calculated as below*	US$750， Euro710	US$440， Euro416
Honorary member	Free	Free	Free

(Regarding submitted papers,we apply above presented new fee after April 15 in 2015 on registoration date.) * Regular member between 63 - 73 years old can apply the category.

(73－age) × ¥3,000

Regular member over 73 years old can maintain the qualification and the privileges of the ISMS members, if they wish.

Categories of 3-year members were abolished.

CONTENTS

Notices from the ISMS