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THE ORDER-PRESERVING PROPERTIES OF ESTIMATES IN POLYTOMOUS ITEM RESPONSE THEORY MODELS WITH APPROXIMATED LIKELIHOOD FUNCTIONS

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ABSTRACT. In this study, we consider the ordering properties of the estimates of the rating scale model(RSM) and related polytomous item response theory (IRT) models. First, we propose a kind of approximation to the likelihood functions for these IRT models. The approximated likelihood functions are derived from the inequality of arithmetic and geometric means. We then evaluate upper limits of the functions based on the mathematical result of Specht(1960). Next, we derive the order-preserving statistics for these polytomous IRT models. All sets of statistics are derived by using the characteristics of arrangement increasing functions (Hollander *et al.*, 1977, Marshall *et al.*, 2011). We also carry out simulation study and confirm that our order-preserving statistics work well in typical educational testing. Finally, it is shown that the order-preserving statistics of the RSM in three major three estimation methods coincide.

1 Introduction In this study, we consider the order-preserving properties of the estimates of the rating scale model (RSM;Rasch, 1960; Andrich,1978a, 1978b; Andersen,1996) and related polytomous item response theory (IRT) models. First, we introduce the RSM. Consider that a test comprises k items administered to n subjects and suppose that each item can take m categories. The response variable for the i-th subject and the j-th item becomes $X_{ijh} = \{0, 1\}$. When the i-th subject responds with an h to the j-th item, the corresponding probability of the RSM is

$$P_{ijh}(\theta_i, \boldsymbol{\alpha}_j) = P(X_{ijh} = 1; \theta_i, \boldsymbol{\alpha}_j) = \frac{\exp(w_h\theta_i + a_{jh})}{\sum_{h=1}^{m} \exp(w_h\theta_i + \alpha_{jh})}. \tag{1}$$

Here, θ_i is the ability parameter for the i-th subject, α_{jh} is the item parameter for the h-th category of the j-th item ($\boldsymbol{\alpha}_j = (\alpha_{j1}, \alpha_{j2}, \cdots, \alpha_{jm})$ and w_h is the weight coefficient for the h-th category. Note that w_h is assumed as given. In addition, $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_n)$ is an n-dimensional vector of the ability parameters and $\boldsymbol{\alpha} = (\alpha_{11}, \cdots, \alpha_{1m}, \cdots, \alpha_{k1}, \cdots, \alpha_{km})$ is a $k \times m$-dimensional vector of the item parameters. To estimate parameters in (1), we often use the maximum likelihood principle. In the RSM, the form of the likelihood function is

$$\begin{aligned} L(\boldsymbol{\theta}, \boldsymbol{\alpha}|X) &= \prod_{i=1}^{n}\prod_{j=1}^{k}\prod_{h=1}^{m} P(X_{ijh} = x_{ijh}) \\ &= \frac{\exp\left(\sum_{i=1}^{n}\theta_i\sum_{j=1}^{k}\sum_{h=1}^{m} w_h x_{ijh} + \sum_{j=1}^{k}\sum_{h=1}^{m}\alpha_{jh}\sum_{i=1}^{n} x_{ijh}\right)}{\prod_{i=1}^{n}\prod_{j=1}^{k}\sum_{h=1}^{m}\exp(w_h\theta_i + a_{jh})} \\ &= \frac{\exp\left(\sum_{i=1}^{n}\theta_i t_i + \sum_{j=1}^{k}\sum_{h=1}^{m}\alpha_{jh} r_{jh}\right)}{C(\boldsymbol{\theta}, \boldsymbol{\alpha})}, \end{aligned} \tag{2}$$

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where X is a response matrix that consists of all the response variables, $t_i = \sum_{j=1}^{k} \sum_{h=1}^{m} w_h x_{ijh}$ is the score for the i-th subject, $r_{jh} = \sum_{i=1}^{n} x_{ijh}$ is the number of subjects who response 1 to the h-th category of the j-th item and $C(\boldsymbol{\theta}, \boldsymbol{\alpha}) = \prod_{i=1}^{n} \prod_{j=1}^{k} \sum_{h=1}^{m} \exp(w_h \theta_i + \alpha_{jh})$.

In IRT, three major estimation methods have been proposed that use a the likelihood function as in (2) : joint maximum likelihood estimation (JMLE),marginal maximum likelihood estimation (MMLE; Bock and Lieberman, 1970, Thissen, 1982), and conditional maximum likelihood estimation (CMLE; Andersen, 1972). JMLE estimates $\boldsymbol{\theta}$ and $\boldsymbol{\alpha}$ simultaneously by maximizing (2) in the RSM. By contrast, CMLE and MMLE remove $\boldsymbol{\theta}$ from (2) and estimate $\boldsymbol{\alpha}$ separately. In particular, CMLE uses a conditional likelihood function in which we assume that the score t_i for the i-th subject is already given and thus remove $\boldsymbol{\theta}$ from the function. We focus on JMLE in this paper.

The RSM has many relations with other polytomous IRT models. For example, when we reparameterize $\alpha_{jh} = \sum_{p=1}^{q} v_{jhp} \eta_p$ in (2), we obtain the linear rating scale model(LRSM; Fischer and Parzer, 1991). Here, η_p is the "basic parameter," $q < m$ is a dimension of the basic parameter vector $\boldsymbol{\eta}$ and v_{jhp} is the weight coefficient which is assumed as already given. The likelihood function of the LRSM corresponding to (2) is

$$L(\boldsymbol{\theta}, \boldsymbol{\eta}|X) = \frac{\exp(\sum_{i=1}^{n} \theta_i t_i + \sum_{p=1}^{q} \eta_p r'_p)}{C(\boldsymbol{\theta}, \boldsymbol{\eta})}, \tag{3}$$

where $r'_p = \sum_{i=1}^{n} \sum_{j=1}^{k} v_{jhp} x_{ijh}$, $C(\boldsymbol{\theta}, \boldsymbol{\eta}) = \prod_{i=1}^{n} \prod_{j=1}^{k} \sum_{h=1}^{m} \exp(w_h \theta_i + \sum_{p=1}^{q} v_{jhp} \eta_p)$ and $\boldsymbol{\eta}$ is the q-dimensional vector of the basic parameters.

Another important model related to the RSM is the partial credit model (PCM; Masters, 1982). The PCM is special case of the RSM. In other words, when we substitute $w_h = h$ for the probability function of the RSM (1), we get the that of the PCM. The likelihood function of the PCM is

$$L(\boldsymbol{\theta}, \boldsymbol{\beta}|X) = \frac{\exp(\sum_{i=1}^{n} \theta_i t_i^* + \sum_{j=1}^{k} \sum_{h=1}^{m} \beta_{jh} r_{jh})}{C(\boldsymbol{\theta}, \boldsymbol{\beta})}, \tag{4}$$

where β_{jh} is the item parameter for the h-th category of the j-th item, $\boldsymbol{\beta}$ is the $k \times m$-dimensional vector of the item parameters, $t_i^* = \sum_{j=1}^{k} \sum_{h=1}^{m} h x_{ijh}$, $C(\boldsymbol{\theta}, \boldsymbol{\beta}) = \prod_{i=1}^{n} \prod_{j=1}^{k} \sum_{h=1}^{m} \exp(h\theta_i + \beta_{jh})$, and $\boldsymbol{\beta} = (\beta_{11}, \cdots, \beta_{1m}, \cdots, \beta_{11}, \cdots, \beta_{k1}, \cdots, \beta_{11}, \cdots, \beta_{km})$. In (4), by reparameterizing $\beta_{jh} = \sum_{p=1}^{q} u_{jhp} \gamma_p$, we drive the linear partial credit model (LPCM; Glas and Verhelst,1989; Fischer and Ponocny, 1994). Here, γ_p is the basic parameter and u_{jhp} is the weight coefficient which is assumed as already given. The likelihood function of the LPCM is

$$L(\boldsymbol{\theta}, \boldsymbol{\beta}|X) = \frac{\exp(\sum_{i=1}^{n} \eta_i t_i^* + \sum_{p=1}^{q} \gamma_q r_q^*)}{C(\boldsymbol{\theta}, \boldsymbol{\gamma})}, \tag{5}$$

where $r_p^* = \sum_{i=1}^{n} \sum_{j=1}^{k} \sum_{h=1}^{m} u_{jhp} x_{ijh}$, $C(\boldsymbol{\theta}, \boldsymbol{\gamma}) \prod_{i=1}^{n} \prod_{j=1}^{k} \sum_{h=1}^{m} \exp(h\theta_i + \sum_{p=1}^{q} u_{jhp} \gamma_p)$ and $\boldsymbol{\gamma}$ is the p-dimensional vector of the basic parameters.

Specht(1960) considered the upper limit of an inequality between the arithmetic mean and geometric mean. Following Seo (2000), let $y_1, \cdots, y_m \in [d, D]$ with $D \geq d > 0$. Then, this inequality is such that

$$S(z) \sqrt[m]{y_1 y_2 \cdots y_m} \geq \frac{y_1 + y_2 + \cdots + y_m}{m}, \tag{6}$$

where $z = D/d$ and $S(z)$ are defined as

$$S(z) = \frac{(z-1) z^{\frac{1}{z-1}}}{e \log z} (z > 1) \text{ and } S(1) = 1 \ (z = 0). \tag{7}$$

Here, we call $S(z)$ Specht's ratio. Then, we consider the approximation below:

$$mAM(\boldsymbol{y}) = y_1 + y_2 + \cdots + y_m \simeq m \sqrt[m]{y_1 y_2 \cdots y_m} = mGM(\boldsymbol{y}), \tag{8}$$

where $\boldsymbol{y} = (y_1, y_2, \cdots, y_m)$. We can evaluate the difference in the above approximation by using a ratio, that is

$$DR(\boldsymbol{y}) = \frac{y_1 + y_2 + \cdots + y_m}{m \sqrt[m]{y_1 y_2 \cdots y_m}}. \tag{9}$$

Here, we call $DR(\boldsymbol{y})$ the "difference ratio" between $AM(\boldsymbol{y})$ and $GM(\boldsymbol{y})$. From (6), it holds that $S(z) \geq DR(\boldsymbol{y}) \geq 1$. This means that we can evaluate the upper limit of the least difference for (9) from $S(z)$ in (6).

By substituting (8) with $y_h = \exp(w_h \theta_i + \alpha_{jh})$ for each i and j into (2), we get

$$\sum_{h=1}^{m} \exp(w_h \theta_i + \alpha_{jh}) \simeq m \sqrt[m]{\prod_{h=1}^{m} \exp(w_h \theta_i + \alpha_{jh})} \tag{10}$$

$$\begin{aligned} L(\boldsymbol{\theta}, \boldsymbol{\alpha}|X) &\simeq \frac{\exp(\sum_{i=1}^{n} \theta_i t_i + \sum_{j=1}^{k} \sum_{h=1}^{m} \alpha_{jh} r_{jh})}{m^{nk} \prod_{i=1}^{n} \prod_{j=1}^{k} \prod_{h=1}^{m} \exp(w_h \theta_i + \alpha_{jh})} \\ &= \frac{\exp(\sum_{i=1}^{n} \theta_i t_i + \sum_{j=1}^{k} \sum_{h=1}^{m} \alpha_{jh} r_{jh})}{\tilde{C}(\boldsymbol{\theta}, \boldsymbol{\alpha})} = \tilde{L}(\boldsymbol{\theta}, \boldsymbol{\alpha}|X), \end{aligned} \tag{11}$$

where $\tilde{C}(\boldsymbol{\theta}, \boldsymbol{\alpha}) = m^{nk} \prod_{i=1}^{n} \prod_{j=1}^{k} \sqrt[m]{\prod_{h=1}^{m} \exp(w_h \theta_i + \alpha_{jh})}$. These approximations of the likelihood functions can also be applied to (3), (4), and (5), which means that we can consider approximated likelihood functions to the the models related to the RSM.

In this study, we consider the ordering properties of the estimates in the RSM and the related polytomous IRT models with the approximated likelihood functions as in (11). We assume that the response matrix X is already given, all estimates derived from X exist, and each estimate is unique. Note that most conventional studies (e.g., Hemker *et al.*,1997;Van der Ark,2005, 2010) have considered the properties of other ordering: stochastic ordering (SO). In other words, they regard X as a matrix that consists of random variables and consider the ordering properties of estimators with SO.

The remainder of the paper is organized as follows. The preliminaries and main results are presented in section 2. Some performances that the approximations denoted above holds are evaluated by simulation studies in section 3. Finally, section 4 discusses our results and concludes.

2 Preliminaries and main results In this study, we use some characteristics of arrangement increasing (AI) functions (Hollander *et al.*, 1977) to consider the order preserving properties of the estimates. First, we introduce some definitions, as per Marshall *et al.* (2011), Boland and Proschan (1988) and Mori(2015).

Definition 1. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be n-dimensional vectors. We define equality $\stackrel{a}{=}$ as

$$(\boldsymbol{a}\Pi, \boldsymbol{b}\Pi) \stackrel{a}{=} (\boldsymbol{a}, \boldsymbol{b}),$$

where Π is an arbitrary $n \times n$ permutation matrix. In this definition, we find $(\boldsymbol{a}, \boldsymbol{b}) \stackrel{a}{=} (\boldsymbol{a}\Pi_1, \boldsymbol{b}\Pi_1) \stackrel{a}{=} (\boldsymbol{a}_\uparrow, \boldsymbol{b}\Pi_1) \stackrel{a}{=} (\boldsymbol{a}\Pi_2, \boldsymbol{b}\Pi_2) \stackrel{a}{=} (\boldsymbol{a}_\downarrow, \boldsymbol{b}\Pi_2)$, where Π_1 is a matrix such that $\boldsymbol{a}\Pi_1 = \boldsymbol{a}_\uparrow$ and Π_2 is a matrix such that $\boldsymbol{a}\Pi_2 = \boldsymbol{a}_\downarrow$. Here, we use the ordered vectors $\boldsymbol{a}_\uparrow$ and $\boldsymbol{a}_\downarrow$, which are vectors with the components of $\boldsymbol{a}$ arranged in ascending order and descending order, respectively. Note it is not always hold that $\boldsymbol{b}\Pi_1 = \boldsymbol{b}_\uparrow$ or $\boldsymbol{b}\Pi_2 = \boldsymbol{b}_\downarrow$. For detail, see below Example 3.

Then, we define a partial order $\overset{a}{\leq}$ for the vector arguments.

Definition 2. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be n-dimensional vectors. First, we permute $\boldsymbol{a}$ and $\boldsymbol{b}$ so that

$$(\boldsymbol{a}, \boldsymbol{b}) \overset{a}{=} (\boldsymbol{a}_\uparrow, \boldsymbol{b}'). \tag{12}$$

Here, $\boldsymbol{b}' = \boldsymbol{b}\Pi_1$ and Π_1 form the permutation matrix such that $\boldsymbol{a}\Pi_1 = \boldsymbol{a}_\uparrow$. Then, we generate a vector $\boldsymbol{b}^*_{l,m}$ from $\boldsymbol{b}'$ in (12) by interchanging the l-th and m-th component ($l < m$) of $\boldsymbol{b}$ such that $b_l > b_m$. Finally, we define the partial order $\overset{a}{\leq}$ as

$$(\boldsymbol{a}_\uparrow, \boldsymbol{b}') \overset{a}{\leq} (\boldsymbol{a}_\uparrow, \boldsymbol{b}^*_{l,m}).$$

Therefore, it holds that $(\boldsymbol{a}_\uparrow, \boldsymbol{b}_\downarrow) \overset{a}{=} (\boldsymbol{a}_\downarrow, \boldsymbol{b}_\uparrow) \overset{a}{\leq} (\boldsymbol{a}, \boldsymbol{b}) \overset{a}{\leq} (\boldsymbol{a}_\uparrow, \boldsymbol{b}_\uparrow) \overset{a}{=} (\boldsymbol{a}_\downarrow, \boldsymbol{b}_\downarrow)$.

Here we show an example for the equality $\overset{a}{=}$ and the inequality $\overset{a}{\leq}$.

Example 3. Let $\boldsymbol{a} = (7, 5, 3, 1)$ and $\boldsymbol{b} = (6, 4, 8, 2)$. Then,

$$\begin{aligned}&(\boldsymbol{a}, \boldsymbol{b}) \overset{a}{=} ((1,3,5,7),(2,8,4,6)) \overset{a}{\leq} ((1,3,5,7),(2,4,8,6)) \\ &\overset{a}{\leq} ((1,3,5,7),(2,4,6,8)) \overset{a}{=} ((7,5,3,1),(8,6,4,2)).\end{aligned}$$

Definition 4. An AI function is a function, g, with two n-dimensional vector arguments that preserve the ordering $\overset{a}{\leq}$. Thus, if g is AI, it holds that $g(\boldsymbol{a}, \boldsymbol{b}) \leq g(\boldsymbol{a}_\uparrow, \boldsymbol{b}^*_{l,m})$ for n-dimensional vectors $\boldsymbol{a}$,$\boldsymbol{b}$,$\boldsymbol{a}_\uparrow$,$\boldsymbol{b}^*_{l,m}$, such that $(\boldsymbol{a}, \boldsymbol{b}) \overset{a}{\leq} (\boldsymbol{a}_\uparrow, \boldsymbol{b}^*_{l,m})$.

Here, we find

$$g(\boldsymbol{a}_\uparrow, \boldsymbol{b}_\downarrow) = g(\boldsymbol{a}_\downarrow, \boldsymbol{b}_\uparrow) \leq g(\boldsymbol{a}, \boldsymbol{b}) \leq g(\boldsymbol{a}_\uparrow, \boldsymbol{b}_\uparrow) = g(\boldsymbol{a}_\downarrow, \boldsymbol{b}_\downarrow) \tag{13}$$

for AI function g.

Next, we prepare a general result as lemma (without proof) that describes the necessary and sufficient condition for AI functions containing summation forms.

Lemma 5. (Marshall *et al.*, 2011, p.233) If g has the form $g(\boldsymbol{a}, \boldsymbol{b}) = \sum_{i=1}^n \phi(a_i, b_i)$, then g is AI if and only if ϕ is L-superadditive.

Here, L-superadditive is the function that satisfies

$$\frac{\partial}{\partial a \partial b}\phi(a, b) \geq 0.$$

Then, we consider the log likelihoods derived from (11) for preparation:

$$\log \tilde{L}(\boldsymbol{\theta}, \boldsymbol{\alpha} | \boldsymbol{t}, \boldsymbol{r}) = \sum_{i=1}^n \theta_i t_i + \sum_{j=1}^k \sum_{h=1}^m \alpha_{jh} r_{jh} - \log \tilde{C}(\boldsymbol{\theta}, \boldsymbol{\alpha}). \tag{14}$$

As $\tilde{C}(\boldsymbol{\theta}, \boldsymbol{\alpha})$ is invariant for the rearrangement within $\boldsymbol{\theta}$ and $\boldsymbol{\alpha}$, $\tilde{C}(\boldsymbol{\theta}, \boldsymbol{\alpha}) = \tilde{C}(\boldsymbol{\theta}\Pi_1, \boldsymbol{\alpha}\Pi_2)$ for any permutation matrices Π_1 and Π_2. Thus, we can only focus on parts of log likelihood function (14) for evaluating order-preserving properties, which are

$$\sum_{i=1}^n \theta_i t_i + \sum_{j=1}^k \sum_{h=1}^m \alpha_{jh} r_{jh} = \tilde{l}_1(\boldsymbol{\theta}, \boldsymbol{t}) + \tilde{l}_2(\boldsymbol{\alpha}, \boldsymbol{r}) \tag{15}$$

Here, $\boldsymbol{t}$ and $\boldsymbol{r}$ are vectors that consist of $\{t_i\}$ and $\{r_{jh}\}$, respectively.

Now, we propose our propositions.

Proposition 6. Let $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\alpha}}$ be maximum likelihood estimates for the log likelihood function $\log \tilde{L}(\boldsymbol{\theta}, \boldsymbol{\alpha}|\boldsymbol{t}, \boldsymbol{r})$ in (14). Then, $\hat{\boldsymbol{\theta}}^*$ and $\hat{\boldsymbol{\alpha}}^*$ maximizes $\log \tilde{L}(\boldsymbol{\theta}, \boldsymbol{\alpha}|\boldsymbol{t}_\uparrow, \boldsymbol{r}_\uparrow)$ if and only if $\hat{\boldsymbol{\theta}}^* = \hat{\boldsymbol{\theta}}_\uparrow$ and $\hat{\boldsymbol{\alpha}}^* = \hat{\boldsymbol{\alpha}}_\uparrow$.

Proof. First, we assume the maximum likelihood estimates $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\alpha}}$ are already given. Then, we find that $\tilde{l}_1(\hat{\boldsymbol{\theta}}, \boldsymbol{t})$ and $\tilde{l}_2(\hat{\boldsymbol{\alpha}}, \boldsymbol{r})$ in (15) are permutation-invariant within each set of vectors $(\hat{\boldsymbol{\theta}}, \boldsymbol{t})$ and $(\hat{\boldsymbol{\alpha}}, \boldsymbol{r})$, which means that $\tilde{l}_1(\hat{\boldsymbol{\theta}}, \boldsymbol{t}) = \tilde{l}_1(\hat{\boldsymbol{\theta}}\Pi_1, \boldsymbol{t}\Pi_1)$ and $\tilde{l}_2(\hat{\boldsymbol{\alpha}}, \boldsymbol{r}) = \tilde{l}_2(\hat{\boldsymbol{\alpha}}\Pi_2, \boldsymbol{r}\Pi_2)$ for any permutation matrices, Π_1 and Π_2. From this permutation invariance and the uniqueness of the maximum likelihood estimates, we obtain

$$\tilde{l}_1(\hat{\boldsymbol{\theta}}, \boldsymbol{t}) + \tilde{l}_2(\hat{\boldsymbol{\alpha}}, \boldsymbol{r}) = \tilde{l}_1(\hat{\boldsymbol{\theta}}\Pi_1^*, \boldsymbol{t}_\uparrow) + \tilde{l}_2(\hat{\boldsymbol{\alpha}}\Pi_2^*, \boldsymbol{r}_\uparrow) = \tilde{l}_1(\hat{\boldsymbol{\theta}}^*, \boldsymbol{t}_\uparrow) + \tilde{l}_2(\hat{\boldsymbol{\alpha}}^*, \boldsymbol{r}_\uparrow),$$

where Π_1^* and Π_2^* are permutation matrices such that $\boldsymbol{t}\Pi_1^* = \boldsymbol{t}_\uparrow$ and $\boldsymbol{r}\Pi_2^* = \boldsymbol{r}_\uparrow$. Thus, we find that both $\hat{\boldsymbol{\theta}}^*$ and $\hat{\boldsymbol{\alpha}}^*$ are rearranged forms of $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\alpha}}$, respectively.

On the contrary, as $\tilde{l}_1(\hat{\boldsymbol{\theta}}, \boldsymbol{t})$ is L-superadditive for variables $\hat{\theta}_i$ and t_i, it follows that $\tilde{l}_1(\hat{\boldsymbol{\theta}}, \boldsymbol{t})$ is AI according to the Lemma 5. We find that $\tilde{l}_2(\hat{\boldsymbol{\alpha}}, \boldsymbol{r})$ is AI in the same the manner. Then, from the property of the AI functions described in (13), it holds that

$$\begin{aligned} \tilde{l}_1(\hat{\boldsymbol{\theta}}_\downarrow, \boldsymbol{t}_\uparrow) &\leq \tilde{l}_1(\hat{\boldsymbol{\theta}}^*, \boldsymbol{t}_\uparrow) \leq \tilde{l}_1(\hat{\boldsymbol{\theta}}_\uparrow, \boldsymbol{t}_\uparrow), \\ \tilde{l}_2(\hat{\boldsymbol{\alpha}}_\downarrow, \boldsymbol{r}_\uparrow) &\leq \tilde{l}_2(\hat{\boldsymbol{\alpha}}^*, \boldsymbol{r}_\uparrow) \leq \tilde{l}_2(\hat{\boldsymbol{\alpha}}_\uparrow, \boldsymbol{r}_\uparrow). \end{aligned}$$

As $\hat{\boldsymbol{\theta}}^*$ and $\hat{\boldsymbol{\alpha}}^*$ are the estimates that maximize $\tilde{l}_1(\hat{\boldsymbol{\theta}}, \boldsymbol{t}_\uparrow)$ and $\tilde{l}_2(\hat{\boldsymbol{\alpha}}, \boldsymbol{r}_\uparrow)$ respectively, it follows that $\tilde{\boldsymbol{\theta}} = \tilde{\boldsymbol{\theta}}_\uparrow$ and $\hat{\boldsymbol{\alpha}}^* = \hat{\boldsymbol{\alpha}}_\uparrow$.

Conversely, if we set $\tilde{\boldsymbol{\theta}} = \tilde{\boldsymbol{\theta}}_\uparrow$ and $\hat{\boldsymbol{\alpha}}^* = \hat{\boldsymbol{\alpha}}_\uparrow$, we find that $\tilde{l}_1(\hat{\boldsymbol{\theta}}, \boldsymbol{t}_\uparrow)$ and $\tilde{l}_2(\hat{\boldsymbol{\alpha}}, \boldsymbol{r}_\uparrow)$ reach their maximum because $\tilde{l}_1$ and $\tilde{l}_2$ are AI. Then, it is shown that $\hat{\boldsymbol{\theta}}^*$ and $\hat{\boldsymbol{\alpha}}^*$ maximize $\log \tilde{L}(\boldsymbol{\theta}, \boldsymbol{\alpha}|\boldsymbol{t}_\uparrow, \boldsymbol{r}_\uparrow)$. □

Our results in Proposition 6 hold in related models such as the LRSM, PCM and LPCM. The approximated likelihood functions corresponding to (11) in the the LRSM are

$$\tilde{L}(\boldsymbol{\theta}, \boldsymbol{\eta}|X) = \frac{\exp(\sum_{i=1}^{n} \theta_i t_i + \sum_{p=1}^{q} \eta_p r'_p)}{\tilde{C}(\boldsymbol{\theta}, \boldsymbol{\eta})}, \tag{16}$$

Here, $\tilde{C}(\boldsymbol{\theta}, \boldsymbol{\eta}) = m^{nk} \prod_{i=1}^{n} \prod_{j=1}^{k} \sqrt[m]{\prod_{h=1}^{m} \exp(w_h \theta_i + \sum_{p=1}^{q} v_{jhp}\eta_p)}$. In (16), we focus on

$$\sum_{i=1}^{n} \theta_i t_i + \sum_{p=1}^{q} \eta_q r'_q = \tilde{l}_1(\boldsymbol{\theta}, \boldsymbol{t}) + \tilde{l}_2(\boldsymbol{\eta}, \boldsymbol{r}'),$$

Then, below Proposition 6 holds.

Proposition 7. Let $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\eta}}$ be the maximum likelihood estimates for the log likelihood function $\log \tilde{L}(\boldsymbol{\theta}, \boldsymbol{\eta}|\boldsymbol{t}, \boldsymbol{r}')$ from (2). Then, $\hat{\boldsymbol{\theta}}^*$ and $\hat{\boldsymbol{\eta}}^*$ maximize $\log \tilde{L}(\boldsymbol{\theta}, \boldsymbol{\eta}|\boldsymbol{t}_\uparrow, \boldsymbol{r}'_\uparrow)$ if and only if $\hat{\boldsymbol{\theta}}^* = \hat{\boldsymbol{\theta}}_\uparrow$ and $\hat{\boldsymbol{\eta}}^* = \hat{\boldsymbol{\eta}}_\uparrow$.

Proof. The proof is done in the same way as in Proposition 1. We assume that$\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\eta}}$ are already given. $\tilde{l}_1(\boldsymbol{\theta}, \boldsymbol{t})$ and $\tilde{l}_2(\boldsymbol{\eta}, \boldsymbol{r}')$ are permutation-invariant. Then, we find that both $\hat{\boldsymbol{\theta}}^*$ and $\hat{\boldsymbol{\eta}}^*$ are rearranged forms of $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\eta}}$, respectively. As $\tilde{l}_1(\boldsymbol{\theta}, \boldsymbol{t})$ and $\tilde{l}_2(\boldsymbol{\eta}, \boldsymbol{r}')$ are AI, l_1 and l_2 reach the maximum when $\tilde{l}_1(\boldsymbol{\theta}_\uparrow, \boldsymbol{t}_\uparrow)$ and $\tilde{l}_2(\boldsymbol{\eta}_\uparrow, \boldsymbol{r}'_\uparrow)$. Consequently, $\hat{\boldsymbol{\theta}}^* = \hat{\boldsymbol{\theta}}_\uparrow$ and $\hat{\boldsymbol{\eta}}^* = \hat{\boldsymbol{\eta}}_\uparrow$. Conversely, if we set $\hat{\boldsymbol{\theta}}^* = \hat{\boldsymbol{\theta}}_\uparrow$ and $\hat{\boldsymbol{\eta}}^* = \hat{\boldsymbol{\eta}}_\uparrow$, it holds that $\hat{\boldsymbol{\theta}}^*$ and $\hat{\boldsymbol{\eta}}^*$ maximize $\log \tilde{L}(\boldsymbol{\theta}, \boldsymbol{\eta}|\boldsymbol{t}_\uparrow, \boldsymbol{r}'_\uparrow)$ because l_1 and l_2 are AI. □

The approximated likelihood functions in the PCM and LPCM are

$$\tilde{L}(\boldsymbol{\theta}, \boldsymbol{\beta}|\boldsymbol{t}^*, \boldsymbol{r}) = \frac{\exp(\sum_{i=1}^{n} \theta_i t_i^* + \sum_{j=1}^{k} \sum_{h=1}^{m} \beta_{jh} r_{jh})}{\tilde{C}(\boldsymbol{\theta}, \boldsymbol{\beta})}, \tag{17}$$

$$\tilde{L}(\boldsymbol{\theta}, \boldsymbol{\gamma}|\boldsymbol{t}^*, \boldsymbol{r}^*) = \frac{\exp(\sum_{i=1}^{n} \eta_i t_i^* + \sum_{p=1}^{q} \gamma_p r_p^*)}{\tilde{C}(\boldsymbol{\theta}, \boldsymbol{\gamma})}. \tag{18}$$

We also find that below Corollary 8 and 9 from these likelihood functions hold.

Corollary 8. Let $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\beta}}$ be the maximum likelihood estimates for the log likelihood function $\log \tilde{L}(\boldsymbol{\theta}, \boldsymbol{\beta}|\boldsymbol{t}^*, \boldsymbol{r})$ from (17). Then, $\hat{\boldsymbol{\theta}}^*$ and $\hat{\boldsymbol{\beta}}^*$ maximize $\log \tilde{L}(\boldsymbol{\theta}, \boldsymbol{\beta}|\boldsymbol{t}_\uparrow^*, \boldsymbol{r}_\uparrow)$ if and only if $\hat{\boldsymbol{\theta}}^* = \hat{\boldsymbol{\theta}}_\uparrow$ and $\hat{\boldsymbol{\beta}}^* = \hat{\boldsymbol{\beta}}_\uparrow$.

Corollary 9. Let $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\gamma}}$ be the maximum likelihood estimates for the log likelihood function $\log \tilde{L}(\boldsymbol{\theta}, \boldsymbol{\gamma}|\boldsymbol{t}^*, \boldsymbol{r}^*)$ from (18). Then, $\hat{\boldsymbol{\theta}}^*$ and $\hat{\boldsymbol{\gamma}}^*$ maximize $\log \tilde{L}(\boldsymbol{\theta}, \boldsymbol{\gamma}|\boldsymbol{t}_\uparrow^*, \boldsymbol{r}_\uparrow^*)$ if and only if $\hat{\boldsymbol{\theta}}^* = \hat{\boldsymbol{\theta}}_\uparrow$ and $\hat{\boldsymbol{\gamma}}^* = \hat{\boldsymbol{\gamma}}_\uparrow$.

3 Simulation studies In the next step, we evaluate the ranges within which approximation (10) holds in the RSM by using simulation studies. We set $n = 50, 100$, $k = 10, 20, 30$, and $m = 3, 5, 7, 9, 11$ and generate parameters from the settings below:

$$\begin{aligned} &\theta_i \sim N(0, 1^2), \ \ \alpha_{jh} \sim N(0, 1^2), w_h = \omega_h + 1, \omega_h \sim [N(0, 2^2)]^+ \\ &i = 1, 2, \cdots, n, j = 1, 2, \cdots, k, h = 1, 2, \cdots, m, \end{aligned} \tag{19}$$

where N denotes a normal distribution. Here, $[a]^+$ is a positive part of real value a, which means $[a]^+ = a$ with $a > 0$ and $[a]^+ = 0$ with $a \leq 0$. These settings are practical for educational testing. Then, we generate response matrix X, statistics t_i, and y_{jh} from (1). We also calculate a kind of "capacity factor" that is $CF = DR(\boldsymbol{y})/S(z)$ for the approximation (10). Here, $DR(\boldsymbol{y})$ and $S(z)$ are defined in (9) and (7), respectively. Finally, we evaluate Kendall's rank correlation coefficients for $(\boldsymbol{t}, \boldsymbol{\theta})$ and $(\boldsymbol{y}, \boldsymbol{\alpha})$ as efficiency indexes for the difference in approximation (10). We repeat the procedure above 1000 times.

Table 1, Table 2, and Table 3 show the medians of Kendall's correlations for $(\boldsymbol{t}, \boldsymbol{\theta})$ and $(\boldsymbol{y}, \boldsymbol{\alpha})$. First, all the correlation coefficients for $(\boldsymbol{t}, \boldsymbol{\theta})$ are quite high and stable because each t_i is a sufficient statistic for θ_i, as Andersen(1996) pointed out. We also find that the correlation coefficients for $(\boldsymbol{y}, \boldsymbol{\alpha})$ are high and depend on the size of category m. In other words, the correlations for $(\boldsymbol{y}, \boldsymbol{\alpha})$ worsen as m increases. In section 1, we found that the least difference of the upper limit for the approximation (10) was evaluated by $S(z)$ in (7) and that $S(z)$ only depends on the maximum and minimum values of the elements $(y_1, y_2, \cdots, y_m)$. Indeed, the correlation coefficients actually decrease with the size of category m under usual conditions, although the least difference corresponding to $S(z)$ in (7) does not depend on a such criterion. This is because the differences in (10) are relatively better than the least differences, as evaluated below.

Table 4, Table 5, and Table 6 present the medians of CFs. All of the CFs are very small, which means that the DRs are very small compared with the least differences. Thus, our approximation works well in these settings. Then, we evaluate more detail of the the CFs. For each k and m, the CF increases with the size of category m. This finding means that the difference by (10) worsens as m becomes large. This result is consistent with the decreasing of the correlation coefficients denoted above.

Finally, we conclude that approximation (10) shows relatively strong performance and that this approximation and the order-preserving statistics $\boldsymbol{t}$ and $\boldsymbol{y}$ are acceptable in typical educational testing.

k	10									
n	50					100				
m	3	5	7	9	11	3	5	7	9	11
Cor $(\boldsymbol{t}, \boldsymbol{\theta})$	0.879	0.908	0.914	0.916	0.913	0.888	0.908	0.916	0.920	0.920
Cor$(\boldsymbol{y}, \boldsymbol{\alpha})$	1	0.847	0.831	0.826	0.824	1	0.872	0.851	0.846	0.851

Table 1: Medians of Kendall's correlations for $(\boldsymbol{t}, \boldsymbol{\theta})$ and $(\boldsymbol{y}, \boldsymbol{\alpha})$ $(L = 10)$

k	20									
n	50					100				
m	3	5	7	9	11	3	5	7	9	11
Cor $(\boldsymbol{t}, \boldsymbol{\theta})$	0.935	0.948	0.950	0.952	0.955	0.941	0.953	0.955	0.954	0.953
Cor$(\boldsymbol{y}, \boldsymbol{\alpha})$	1	0.872	0.831	0.828	0.822	1	0.872	0.852	0.849	0.852

Table 2: Medians of Kendall's correlations for $(\boldsymbol{t}, \boldsymbol{\theta})$ and $(\boldsymbol{y}, \boldsymbol{\alpha})$ $(L = 20)$

k	30									
n	50					100				
m	3	5	7	9	11	3	5	7	9	11
Cor $(\boldsymbol{t}, \boldsymbol{\theta})$	0.954	0.963	0.964	0.966	0.966	0.956	0.966	0.967	0.968	0.969
Cor$(\boldsymbol{y}, \boldsymbol{\alpha}$	1	0.872	0.838	0.827	0.820	1	0.872	0.850	0.846	0.852

Table 3: Medians of Kendall's correlations for $(\boldsymbol{t}, \boldsymbol{\theta})$ and $(\boldsymbol{y}, \boldsymbol{\alpha})$ $(L = 30)$

k	10									
n	50					100				
m	3	5	7	9	11	3	5	7	9	11
CF	2.87×10^{-6}	3.70×10^{-6}	4.99×10^{-6}	5.95×10^{-6}	9.29×10^{-6}	4.15×10^{-7}	5.59×10^{-7}	7.71×10^{-7}	9.14×10^{-7}	1.14×10^{-7}

Table 4: Medians of the CFs for approximation (10) $(L = 10)$

k	20									
n	50					100				
m	3	5	7	9	11	3	5	7	9	11
CF	1.95×10^{-6}	4.18×10^{-6}	3.69×10^{-6}	7.75×10^{-6}	7.92×10^{-6}	2.98×10^{-7}	6.92×10^{-7}	7.32×10^{-7}	1.08×10^{-6}	1.45×10^{-7}

Table 5: Medians of the CFs for approximation (10) $(L = 20)$

k	30									
n	50					100				
m	3	5	7	9	11	3	5	7	9	11
CF	2.99×10^{-6}	3.46×10^{-6}	5.64×10^{-6}	6.16×10^{-6}	8.92×10^{-6}	1.94×10^{-7}	4.95×10^{-7}	6.48×10^{-7}	7.72×10^{-7}	1.03×10^{-6}

Table 6: Medians of the CFs for approximation (10) $(L = 30)$

4 Conclusion and discussion In this study, we considered the ordering properties of the RSM and related polytoumous IRT models in JMLE with approximation (10). We also evaluated the difference in such an approximation by using simulation study and concluded that this approximation and the order-preserving statistics proposed in this study are acceptable in typical educational testing.

For the other estimation methods in RSM, namely CMLE and MMLE, the order-preserving statistics concur with those in JMLE when (10) holds. First we consider the relations between the estimates in JMLE and CMLE. The estimates in JMLE are biased comparing with those in CMLE (e.g. Andersen,1980, Theorem 6.1) and the bias is positive. Thus, the ordering of estimates in JMLE and CMLE concur, although the estimates in JMLE are biased. Consequently, the order-preserving statistics in JMLE agree with those in CMLE.

Then, we consider the relations between the estimates in CMLE and MMLE. Andersen(1996) found that CMLE and MMLE agree when n is large, that means that estimates in the CMLE and the MMLE concur. Thus, it is clear that the ordering of estimates and the order-preserving statistics in these estimations concur. Finally, we find that the order-preserving statistics in all three estimations agree.

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OPTIMAL GROUPING OF STUDENTS IN COALITIONAL GAMES WITH THE SHAPLEY VALUE

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Abstract

In this paper, the classroom consists of three different kinds of students, and we discuss the problem how to divide these students into three person groups. The benefit of one group is the sum of three students' benefit by cooperation game. The benefit of each person is given by the Shapley value from the characteristic function we defined. Our goal is how to divide 18 students into subgroups with three persons to make the total benefit of the classroom maximal.

It is impossible to get the maximal score by using different 18 students having different potentials and six different coefficients for the combinations of three different levels of potentials. Especially, the number of combinations for dividing 18 students by 3 persons evenly is tremendous. Therefore, we can investigate some numerical examples under some limited conditions. Finally, we can obtain the theorem to make the total benefit of the classroom maximal under the limited condition.

The authors believe that this research can apply to group learning and the field of Education in the real life.

1. Introduction

There is a proverb, "Two heads are better than one". In school life, groups form spontaneously, and usually smart people study with other smart people. People who cannot be in that smart team gather and construct other groups. Seen from a big picture, in Japan, every university, high school, and even private junior high school is ranked. Each student is sent to a specific school based on their score on a paper examination which is given by each school.

I am not sure if it is good to divide students ordered by smartness for the classroom and for society, or not. The way to divide proper groups is affected by what is considered as priority. If your purpose is to make the smartest student smarter, the way we are adopting the structure of the deviation value now is obviously correct. However, to make the benefit of the entire classroom or entire society biggest, we are not sure if it is correct that the deviation value structure we have now in Japan is the best way. Therefore, we are going to talk about the structure, which makes the benefit of the entire group the biggest.

In this paper, the classroom has three different kinds of students, and we divide these students into three person groups. We assume that they cooperate and study together in groups. We anticipate that three smart students compete or work together with each other and their score should go up. Conversely, we assume, if three students who don't like to study gather, they will not gain anything. On this paper, we set one classroom with 6 smart students, 6 neutral students, and 6 not good students. We divide them into 3 persons groups, so there are 6 groups in the classroom. The benefit

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of one group is the sum of three students' benefit by cooperation game. The benefit of each person is given by the Shapley value from the characteristic function we defined. Also, in this model, we think and simulate how and where to put these groups in the classroom. If a group talks to another group, possibly they will gain something by conveying and receiving information.

2.1 Model of the classroom with 18 students

We figured out some dispositions from this problem when we construct the problem as a general form. After giving the concrete numbers to the functions and others, we find the proper structure of the classroom after finding proper groups by computational simulations.

Let X={ $X_1, X_2, X_3, X_4, X_5, X_6$ } be the set of 6 smart students.

Let Y={ $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6$ } be the set of 6 neutral students.

Let Z={ $Z_1, Z_2, Z_3, Z_4, Z_5, Z_6$ } be the set of 6 not good students.

Let S = $\{s_1, s_2, s_3, s_4, s_5, s_6\}$ be the set of relationship between two students.

It is assumed that $s_i \gg s_j$ for any $i < j$, so $s_1 \gg s_2 \gg s_3 \gg s_4 \gg s_5 \gg s_6$.

$\gg$ means that the relationship of s_i is better than that of s_j .

$W_j \in W=\{X, Y, Z\} \ni (j=1,2,\cdots,18)$ W is the set of all students.

W_i, X_i, Y_i, and Z_i represent people.

w_i , x_i , y_j , and z_i represent values $v(W_i)$, $v(X_i)$, $v(Y_i)$, and $v(Z_i)$ respectively.

Let s_1 be the state of the relationships between X_i and X_j.

Let s_2 be the state of the relationships between X_i and Y_j.

Let s_3 be the state of the relationships between Y_i and Y_j.

Let s_4 be the state of the relationships between X_i and Z_j.

Let s_5 be the state of the relationships between Y_i and Z_j.

Let s_6 be the state of the relationships between Z_i and Z_j.

So we have assumed the quality of the relationship is highest for good students with good students and lowest for poor students with poor students. This is probably the strongest assumption in the paper and only reasonable in some situations. In some situations, it is possible that two average (neutral) students will be able to combine and really both grow, but good students will not have much room for growth. In other situations (modeled here), two average students gain but less than two good students. So, we have the following value ordering. Another part of this assumption is that two average or neutral students gain more than a good student combined with a poor student. This would not always be true.

[Definition I]

We define the characteristic function of the reward that both W_i and W_j corporate together.

$v(W_i \cup W_j , s)= s(w_i + w_j)$, where W_i represents an arbitrary person with value $w_i=v(W_i)$. s is an arbitrary element of $S=\{s_1 \dots s_6\}$. ■

s represents the state of the relationship between two persons and is real value.

Let $G_1=(W_1, W_2, W_3)$, $G_2=(W_4, W_5, W_6), \ldots, G_6=(W_{16}, W_{17}, W_{18})$,

where $W_j \in W=\{X, Y, Z\}$ $(j=1,2,\cdots,18)$.

In this coalitional game of three players, the Shapley value of player W_i in $G_i = (W_i, W_{i+1}, W_{i+2})$ is

$$f(W_i) = \frac{2!}{3!}\{v(W_i) - v(\Phi)\} + \frac{1}{3!}\{v(W_i \cup W_{i+1}, s') - v(W_{i+1})\}$$
$$+ \frac{1}{3!}\{v(W_i \cup W_{i+2}, s'') - v(W_{i+2})\}$$
$$+ \frac{2!}{3!}\{v(W_i \cup W_{i+1} \cup W_{i+2}) - v(W_{i+1} \cup W_{i+2}, s''')\} \cdots\cdots (2\text{-}1\text{-}1)$$

$s', s'', s''' \in S$ (Φ is an empty set.)

s' depends on combination of W_i and W_j, so there are 6 possible values.

$v(W_i \cup W_{i+1} \cup W_{i+2})$ is defined as

$\frac{1}{2}\{v(W_i \cup W_{i+1}, s') + v(W_i \cup W_{i+2}, s'') + v(W_{i+1} \cup W_{i+2}, s''')\}$.

From (2-1-1),

$$f(W_i) = \frac{1}{6}\{2w_i - (w_{i+1} + w_{i+2})\} + \frac{1}{6}\{2\{v(W_i \cup W_{i+1}, s') + v(W_i \cup W_{i+2}, s'')\}$$
$$- v(W_{i+1} \cup W_{i+2}, s''')\} \cdots\cdots (2\text{-}1\text{-}2)$$

G_i's group value is defined as $F(G_i) = f(W_i) + f(W_{i+1}) + f(W_{i+2})$

The Sum of Group Values: $SGV=\sum_{i=1}^{6} F(Gi)$ $\cdots\cdots$ (2-1-3)

SGV represents the total score of whole classroom. Our objective is to find the grouping the make the SGV maximal.

2.2 The comparison of two kinds of the classroom

There are so many ways to make six groups with three people each having different values. We will observe the total value of classroom with two examples. We let $\sum_{i=1}^{6} x_i > \sum_{i=1}^{6} y_i > \sum_{i=1}^{6} z_i$ and $s_1 \geq s_2 \geq s_3 \geq s_4 \geq s_5 \geq s_6$.

[Example I]

Let us consider the situation where the groups are divided by matching abilities.

X_i and X_j have different numbers.

We let $G_1 = (X_1, X_2, X_3)$, $G_2=(X_4, X_5, X_6)$, $G_3 = (Y_1, Y_2, Y_3)$, $G_4=(Y_4, Y_5, Y_6)$,

$G_5 = (Z_1, Z_2, Z_3)$, and $G_6= (Z_4, Z_5, Z_6)$.

$$f(X_1) = \frac{1}{2}\{2x_1 - (x_2 + x_3)\} + \frac{1}{6}\{2\{(x_1 + x_2)s_1 + (x_1 + x_3)s_1\} - (x_2 + x_3)s_1\}$$
$$f(X_2) = \frac{1}{2}\{2x_2 - (x_1 + x_3)\} + \frac{1}{6}\{2\{(x_2 + x_1)s_1 + (x_2 + x_3)s_1\} - (x_1 + x_3)s_1\}$$
$$f(X_3) = \frac{1}{2}\{2x_3 - (x_1 + x_2)\} + \frac{1}{6}\{2\{(x_3 + x_1)s_1 + (x_3 + x_2)s_1\} - (x_1 + x_2)s_1\}$$
$$F(G_1)= f(X_1)+ f(X_2)+ f(X_3)=\frac{1}{2}\{(x_1 + x_2)s_1 + (x_1 + x_3)s_1 +(x_2 + x_3)s_1\} = (x_1 + x_2 +x_3)s_1$$

$F(G_2)= (x_4 + x_5 +x_6)s_1$, $F(G_3)= (y_1 + y_2 +y_3)s_3$, $F(G_4)= (y_4 + y_5 +y_6)s_3$

$F(G_5)= (z_1 + z_2 +z_3)s_6$, $F(G_6)= (z_4 + z_5 +z_6)s_6$.

Call the total value $SGV_1 = s_1 \sum_{i=1}^{6} x_i + s_3 \sum_{i=1}^{6} y_i + s_6 \sum_{i=1}^{6} z_i$ ··············(2-2-1)

[Example II]

Next, let $G_1 = (X_1, Y_1, Z_1)$, $G_2 = (X_2, Y_2, Z_2)$,…, and $G_6 = (X_6, Y_6, Z_6)$.

$f(X_1) = \frac{1}{2}\{2x_1 - (y_1 + z_1)\} + \frac{1}{6}\{2\{(x_1 + y_1)s_2 + (x_1 + z_1)s_4\} - (y_1 + z_1)s_5\}$

$f(Y_1) = \frac{1}{2}\{2y_1 - (x_1 + z_1)\} + \frac{1}{6}\{2\{(x_1 + y_1)s_2 + (y_1 + z_1)s_5\} - (x_1 + z_1)s_4\}$

$f(Z_1) = \frac{1}{2}\{2z_1 - (x_1 + y_1)\} + \frac{1}{6}\{2\{(y_1 + z_1)s_5 + (x_1 + z_1)s_4\} - (x_1 + y_1)s_2\}$

$F(G_1) = \frac{1}{2}\{(x_1 + y_1)s_2 + (x_1 + z_1)s_4 + (y_1 + z_1)s_5\}$

Call the total value for these groups

$SGV_2 = \frac{s2+s4}{2}\sum_{i=1}^{6} x_i + \frac{s2+s5}{2}\sum_{i=1}^{6} y_i + \frac{s4+s5}{2}\sum_{i=1}^{6} z_i$ ····················(2-2-2)

We show a situation where $SGV_1 > SGV_2$ and one where $SGV_1 < SGV_2$.

$SGV_1 - SGV_2 = (s_1 - \frac{s2+s4}{2})\sum_{i=1}^{6} x_i + (s_3 - \frac{s2+s5}{2})\sum_{i=1}^{6} y_i + (s_6 - \frac{s4+s5}{2})\sum_{i=1}^{6} z_i$

The first coefficient, $s_1 - \frac{s2+s4}{2}$, is bigger than 0. The third coefficient, $s_6 - \frac{s4+s5}{2}$, is smaller than 0. But the second coefficient, $s_3 - \frac{s2+s5}{2}$, is the thing we cannot know in this setting.

1) If we let the s_i-values differ by a constant increment ,

then $s_6 < s_5 = s_6+\Delta < s_4 = s_6+2\Delta < s_3 = s_6+3\Delta < s_2 = s_6+4\Delta < s_1 = s_6+5\Delta$.

And we get,

$SGV_1 - SGV_2 = (5\Delta - \frac{4\Delta+2\Delta}{2})\sum_{i=1}^{6} x_i + (3\Delta - \frac{4\Delta+\Delta}{2})\sum_{i=1}^{6} y_i + (0 - \frac{2\Delta+\Delta}{2})\sum_{i=1}^{6} z_i$

$= 2\Delta\sum_{i=1}^{6} x_i + \frac{\Delta}{2}\sum_{i=1}^{6} y_i - \frac{3\Delta}{2}\sum_{i=1}^{6} z_i > 0$

Therefore $SGV_1 > SGV_2$.

2) However, if we let $s_1 = s_2 = s_3 = s_4 = s_5 > s_6$,

$SGV_1 - SGV_2 = (s_6 - s_5)\sum_{i=1}^{6} z_i < 0$.

Therefore $SGV_1 < SGV_2$.

The two examples above show that it is hard to find the maximal SGV. That is because the total value of the classroom changes with s-values.

2.3 Finding all possible groupings

We want to make these models simple. So, now we examine all possible groupings where all the X_i's have the same value. The Y_i's and Z_i's have a single y-value and z-value respectively.

$X = (X, X, X, X, X, X)$, $Y = (Y, Y, Y, Y, Y, Y)$, $Z = (Z, Z, Z, Z, Z, Z)$

To represent a group's makeup, we use the following notation:

(number of x members in the group, number of y members, number of z members).

Group H_1 : (X,X,X)=(3,0,0), Group H_2 : (X,X,Y)=(2,1,0),

Group H_3 : (X,X,Z)=(2,0,1), Group H_4 : (X,Y,Y)=(1,2,0), Group H_5 : (Y,Y,Y)=(0,3,0), Group H_6 : (Y,Y,Z)=(0,2,1),

Group H_7 : (Y,Z,Z)=(0,1,2), Group H_8 : (X,Y,Z)=(1,1,1),

Group H_9 : (X,Z,Z)=(1,0,2), Group H_{10} : (Z,Z,Z)=(0,0,3).

The alphas in the following equation represent the number of groups of each makeup.

$$\alpha 1\begin{pmatrix}3\\0\\0\end{pmatrix}+\alpha 2\begin{pmatrix}2\\1\\0\end{pmatrix}+\alpha 3\begin{pmatrix}2\\0\\1\end{pmatrix}+\alpha 4\begin{pmatrix}1\\2\\0\end{pmatrix}+\alpha 5\begin{pmatrix}0\\3\\0\end{pmatrix}+\alpha 6\begin{pmatrix}0\\2\\1\end{pmatrix}+\alpha 7\begin{pmatrix}0\\1\\2\end{pmatrix}+\alpha 8\begin{pmatrix}1\\1\\1\end{pmatrix}+$$

$$\alpha 9\begin{pmatrix}1\\0\\2\end{pmatrix}+\alpha 10\begin{pmatrix}0\\0\\3\end{pmatrix}=\begin{pmatrix}6\\6\\6\end{pmatrix} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdot (2\text{-}3\text{-}1)$$

Equation (2-3-2) represents that the sum of alphas needs to be 6 because we have six groups. Equations (2-3-3), (2-3-4), and (2-3-5) come from the equation (2-3-1).

We will solve these with matrices.

$\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+\alpha_{10}=6$ $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdot$ (2-3-2)

$0\leqq\alpha_i\leqq 6,\ \alpha_i\in N$

$3\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_8+\alpha_9=6$ $\cdots\cdots\cdots$ (2-3-3)

$\alpha_2+2\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_8=6$ $\cdots\cdots\cdots$ (2-3-4)

$\alpha_3+\alpha_6+2\alpha_7+\alpha_8+2\alpha_9+3\alpha_{10}=6$ $\cdots\cdots\cdots$ (2-3-5)

By using a computer programming language VBA, we found there are 103 solutions meeting (2-3-2) through (2-3-5). (Appendix) These solutions correspond to possible groupings. As we did before, we find the group values for different group makeups.

$\begin{pmatrix}3\\0\\0\end{pmatrix}$ . . . $F(H_1)=3xs_1$ $\quad$ $\begin{pmatrix}2\\1\\0\end{pmatrix}$ . . . $F(H_2)=xs_1+xs_2+ys_2$

$\begin{pmatrix}2\\0\\1\end{pmatrix}$ . . . $F(H_3)=xs_1+xs_4+zs_4$ $\quad$ $\begin{pmatrix}1\\2\\0\end{pmatrix}$ . . . $F(H_4)=xs_2+ys_2+ys_3$

$\begin{pmatrix}0\\3\\0\end{pmatrix}$ . . . $F(H_5)=3ys_3$ $\quad$ $\begin{pmatrix}0\\2\\1\end{pmatrix}$ . . . $F(H_6)=ys_3+ys_5+zs_5$

$\begin{pmatrix}0\\1\\2\end{pmatrix}$ . . . $F(H_7)=ys_5+zs_5+zs_6$ $\quad$ $\begin{pmatrix}1\\1\\1\end{pmatrix}$ . . . $F(H_8)=1/2\{x(s_2+s_4)+y(s_2+s_5)+z(s_4+s_5)\}$

$\begin{pmatrix}1\\0\\2\end{pmatrix}$ . . . $F(H_9)=xs_4+zs_4+zs_6$ $\quad$ $\begin{pmatrix}0\\0\\3\end{pmatrix}$ . . . $F(H_{10})=3zs_6$

The possible groupings given by the alpha values are in the following tables. They were found by Program 3 which is given an appendix.

Also, we have the chart of all possible groupings given by alphas at an appendix. We sort the numbers of groupings by ascending order.

This chart is all Possible Groupings Given by Alphas. We set through s_1 to s_6 the characteristic numbers because this case, we assume that good student and good student can help each other the most. In other words, we assume that poor student and poor student don't cooperate each other much because they don't know the material they need to do. Finally, we sort this by highest score to lowest score.

	s_1	s_2	s_3	s_4	s_5	s_6		X	y	z	
	1.25	1.2	1.15	1.1	1.05	1		80	60	40	
NO	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	SGV
99	2	0	0	0	2	0	0	0	0	2	1254
94	1	1	0	1	1	0	0	0	0	2	1252
60	0	3	0	0	1	0	0	0	0	2	1251
76	1	0	0	3	0	0	0	0	0	2	1251
54	0	2	0	2	0	0	0	0	0	2	1250
98	2	0	0	0	1	1	1	0	0	1	1246
93	1	1	0	1	0	1	1	0	0	1	1244
59	0	3	0	0	0	1	1	0	0	1	1243
89	1	1	0	0	1	0	1	1	0	1	1242.5
96	2	0	0	0	0	3	0	0	0	1	1242
97	2	0	0	0	1	0	3	0	0	0	1242
73	1	0	0	2	0	0	1	1	0	1	1241.5
85	1	0	1	1	1	0	1	0	0	1	1241
91	1	1	0	0	1	1	0	0	1	1	1241
51	0	2	0	1	0	0	1	1	0	1	1240.5
57	0	2	1	0	1	0	1	0	0	1	1240
75	1	0	0	2	0	1	0	0	1	1	1240
92	1	1	0	1	0	0	3	0	0	0	1240
41	0	1	1	2	0	0	1	0	0	1	1239
53	0	2	0	1	0	1	0	0	1	1	1239
58	0	3	0	0	0	0	3	0	0	0	1239
71	1	0	0	1	1	0	0	1	1	1	1238.5
87	1	1	0	0	0	2	0	1	0	1	1238.5
82	1	0	1	0	2	0	0	0	1	1	1238
95	2	0	0	0	0	2	2	0	0	0	1238

49	0	2	0	0	1	0	0	1	1	1	1237.5
84	1	0	1	1	0	2	0	0	0	1	1237
90	1	1	0	0	1	0	2	0	1	0	1237
30	0	1	0	2	0	0	0	1	1	1	1236.5
40	0	1	1	1	1	0	0	0	1	1	1236
56	0	2	1	0	0	2	0	0	0	1	1236
68	1	0	0	1	0	1	0	2	0	1	1236
74	1	0	0	2	0	0	2	0	1	0	1236
80	1	0	1	0	1	1	0	1	0	1	1235.5
10	0	0	1	3	0	0	0	0	1	1	1235
46	0	2	0	0	0	1	0	2	0	1	1235
52	0	2	0	1	0	0	2	0	1	0	1235
63	1	0	0	0	1	0	0	3	0	1	1234.5
86	1	1	0	0	0	1	2	1	0	0	1234.5
38	0	1	1	1	0	1	0	1	0	1	1233.5
44	0	1	2	0	1	1	0	0	0	1	1233
72	1	0	0	1	1	0	1	0	2	0	1233
83	1	0	1	1	0	1	2	0	0	0	1233
88	1	1	0	0	0	2	1	0	1	0	1233
26	0	1	0	1	0	0	0	3	0	1	1232.5
20	0	0	2	2	0	1	0	0	0	1	1232
34	0	1	1	0	1	0	0	2	0	1	1232
50	0	2	0	0	1	0	1	0	2	0	1232
55	0	2	1	0	0	1	2	0	0	0	1232
67	1	0	0	1	0	0	2	2	0	0	1232
79	1	0	1	0	1	0	2	1	0	0	1231.5
7	0	0	1	2	0	0	0	2	0	1	1231
31	0	1	0	2	0	0	1	0	2	0	1231
45	0	2	0	0	0	0	2	2	0	0	1231
17	0	0	2	1	1	0	0	1	0	1	1230.5
69	1	0	0	1	0	1	1	1	1	0	1230.5
23	0	0	3	0	2	0	0	0	0	1	1230
66	1	0	0	0	2	0	0	0	3	0	1230
81	1	0	1	0	1	1	1	0	1	0	1230
37	0	1	1	1	0	0	2	1	0	0	1229.5

47	0	2	0	0	0	1	1	1	1	0	1229.5
43	0	1	2	0	1	0	2	0	0	0	1229
64	1	0	0	0	1	0	1	2	1	0	1229
70	1	0	0	1	0	2	0	0	2	0	1229
19	0	0	2	2	0	0	2	0	0	0	1228
29	0	1	0	1	1	0	0	0	3	0	1228
39	0	1	1	1	0	1	1	0	1	0	1228
48	0	2	0	0	0	2	0	0	2	0	1228
65	1	0	0	0	1	1	0	1	2	0	1227.5
77	1	0	1	0	0	2	1	1	0	0	1227.5
2	0	0	0	3	0	0	0	0	3	0	1227
27	0	1	0	1	0	0	1	2	1	0	1227
35	0	1	1	0	1	0	1	1	1	0	1226.5
61	1	0	0	0	0	1	1	3	0	0	1226.5
78	1	0	1	0	0	3	0	0	1	0	1226
8	0	0	1	2	0	0	1	1	1	0	1225.5
28	0	1	0	1	0	1	0	1	2	0	1225.5
18	0	0	2	1	1	0	1	0	1	0	1225
36	0	1	1	0	1	1	0	0	2	0	1225
42	0	1	2	0	0	2	1	0	0	0	1225
62	1	0	0	0	0	2	0	2	1	0	1225
9	0	0	1	2	0	1	0	0	2	0	1224
25	0	1	0	0	1	0	0	2	2	0	1224
32	0	1	1	0	0	1	1	2	0	0	1224
1	0	0	0	2	0	0	0	2	2	0	1223
102	0	1	0	0	0	0	1	4	0	0	1223
6	0	0	1	1	1	0	0	1	2	0	1222.5
15	0	0	2	1	0	1	1	1	0	0	1222.5
33	0	1	1	0	0	2	0	1	1	0	1222.5
14	0	0	2	0	2	0	0	0	2	0	1222
22	0	0	3	0	1	1	1	0	0	0	1222
4	0	0	1	1	0	0	1	3	0	0	1221.5
24	0	1	0	0	0	1	0	3	1	0	1221.5
12	0	0	2	0	1	0	1	2	0	0	1221
16	0	0	2	1	0	2	0	0	1	0	1221

5	0	0	1	1	0	1	0	2	1	0	1220
13	0	0	2	0	1	1	0	1	1	0	1219.5
101	0	0	0	1	0	0	0	4	1	0	1219
3	0	0	1	0	1	0	0	3	1	0	1218.5
21	0	0	3	0	0	3	0	0	0	0	1218
11	0	0	2	0	0	2	0	2	0	0	1217
100	0	0	1	0	0	1	0	4	0	0	1216
103	0	0	0	0	0	0	0	6	0	0	1215

When you see the difference between top 5 groups,

Highest No.99 (X,X,X),(X,X,X), (Y,Y,Y) ,(Y,Y,Y), (Z,Z,Z) ,(Z,Z,Z)

Second highest No.94 (X,X,X),(X,X,**Y**), (**X**,Y,Y) ,(Y,Y,Y), (Z,Z,Z) ,(Z,Z,Z)

Third highest No.60 (X,X,**Y**),(X,X,Y), (X,**X**,Y) ,(Y,Y,Y), (Z,Z,Z) ,(Z,Z,Z)

Fourth highest No.76 (X,X,**X**),(X,**Y**,Y), (X,**Y**,Y) ,(**X**,Y,Y), (Z,Z,Z) ,(Z,Z,Z)

Fifth highest No.54 (X,X,**Y**),(X,**X**,Y), (X,Y,Y) ,(X,Y,Y), (Z,Z,Z) ,(Z,Z,Z) .

As you can see, for creating the group with second highest score in this situation, you need to exchange one of X for one of Y on the highest grouping. And then, No.94 is created from No.99 with one exchange. Next, No.60 is created by No.94 with an exchange X for Y. By fifth highest group in this situation, the ranking changes only by exchanging X for Y.

From 6^{th} highest to lower, the ranking changes by exchanging something for Z. The group with lowest score, 103th, is (X,Y,Z),(X,Y,Z), (X,Y,Z) ,(X,Y,Z), (X,Y,Z) ,(X,Y,Z).

2-4 Some numerical calculations with different conditions

[Numerical example Ⅰ**]**

We selected a simple constant decrease in the s-values favoring the good students working together. We also selected values for the X and the Z. For getting the different result, we change the value of Y from 50 to 70 by 10.

s1	s2	s3	s4	s5	s6		x	y	z
1.25	1.2	1.15	1.1	1.05	1		80	60	40

Ranking	y=50	y=60	y=70
1	99	99	99
2	94	94	94
3	60	60	60
4	76	76	76
5	54	54	54
6	98	98	98
7	93	93	93
8	96**	59*	59*
9	97	89	89
10	59*	96**	73
11	89	97	51
12	92	73	85
13	95	85	91
14	73	91	57
15	85	51	75
16	91	57	41
17	58	75	53
18	51	92	96**
19	87	41	97
20	57	53	71
21	75	58	49
22	41	71	82
23	53	87	92
24	84	82	30
25	90	95	58

This chart is describing the ranking by ys', respectively. For example, 99 in this chart means grouping No.99, which is (X,X,X),(X,X,X), (Y,Y,Y) ,(Y,Y,Y), (Z,Z,Z) ,(Z,Z,Z).

When we change ys' from 50 to 70 by 10, we describe the top 25 groupings by descending order. We cannot see the difference above 7th but we can see the difference under the 8th. The No.59 goes up from 10th, 8th to 8th, respectively. On the other hand, No.96 goes down from 8th, 10th to 18th, respectively.

No.59 is (X,X,Y),(X,X,Y),(X,X,Y),(Y,Y,Z),(Y,Z,Z),(Z,Z,Z).

No.96 is (X,X,X),(X,X,X), (Y,Y,Z) ,(Y,Y,Z), (Y,Y,Z) ,(Z,Z,Z).

By giving y the difference, y has more advantage when Y is with Xs. Therefore, the rankings change. We can see that the degree of tops and lows don't change at all. But we can also see that some groupings around middle of the rankings change much.

[Numerical example II]

We didn't change anything but the value of s_3. Numerical example II changes the value of s_3 from 1.12 to 1.18 by 0.03.

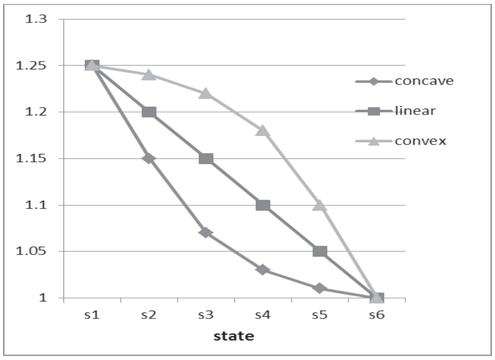
NO	s_3=1.12	s_3=1.15	s_3=1.18
1	54**	99*	99*
2	60	94	94
3	76	60	60
4	94	76	76
5	99*	54**	54**
6	59	98	98
7	93	93	82
8	58	59	91
9	98	89	85
10	51	96	89
11	92	97	93
12	73	73	96
13	89	85	97
14	96	91	71
15	97	51	57
16	41	57	75
17	53	75	73
18	87	92	59
19	57	41	40
20	75	53	49
21	95	58	80
22	85	71	41
23	91	87	53
24	46	82	84
25	52	95	90

Since s_3 is the coefficient for the relationship between Y and Y, when the value of s_3 decreases, the value of (Y,Y,Y) decreases as well. Therefore, the ranking of No.99 goes down. When s_3 is 1.12, the grouping having the highest score is No.54, which is (X,X,Y),(X,X,Y), (X,Y,Y) ,(X,Y,Y), (Z,Z,Z) ,(Z,Z,Z). The grouping of just Ys was disappeared, and groupings with X and Y have advantage more than Ys.

[Numerical example III]

Numerical example III changes the values of coefficients s's. There are three ways to change coefficients, which are concave, linear, and convex. This chart below is how we set them.

State	concave	Linear	Convex
s_1	1.25	1.25	1.25
s_2	1.15	1.2	1.24
s_3	1.07	1.15	1.22
s_4	1.03	1.1	1.18
s_5	1.01	1.05	1.1
s_6	1	1	1



This chart describes the ranking by ascending order from top to 25th.

No.99 is the top when we use concave and linear ways. No.14 is the top with convex way.

NO	concave	linear	convex
1	99	99	14
2	94	94	2
3	98	60	29
4	60	76	23
5	76	54	66
6	96	98	6
7	97	93	10
8	54	59	40
9	93	89	82
10	95	96	17
11	59	97	1
12	92	73	9
13	89	85	25
14	58	91	30
15	73	51	36

16	87	57	18
17	85	75	49
18	91	92	54
19	51	41	3
20	86	53	71
21	57	58	7
22	75	71	31
23	84	87	60
24	90	82	76
25	41	95	13

We cannot see the difference a lot between concave and linear. But when we apply the convex way, the rankings change a lot.

What we need to check out is No.14. No.14 is changed from 100th, 90th, to 1st by different settings, respectively.

No.14 is (X,X,Z), (X,X,Z),(Y,Y,Y), (Y,Y,Y),(X,Z,Z),(X,Z,Z).

s_5 (relationship between Y and Z) and s_4 (relationship between X and Z) causes this result because the values of s_4 and s_5 go up drastically. No.99 which is the top at other's setting becomes 34th with convex situation.

We have observed just groupings whose rankings are increasing or decreasing, but we found the groupings doing weird movement in rankings. For example, No.60 places 4th, 3rd, and 23th, respectively.

No.60 is (X,X,Y), (X,X,Y),(X,X,Y), (Y,Y,Y),(Z,Z,Z),(Z,Z,Z).

2-5 The Model with limited sequence $\{s_i\}$

In this part, X_i and X_j have different numbers which is x_i and x_j respectively.

Let $\{s_i\}$ be sequence of numbers with common difference d (constant),

$s_i=s+(6-i)d$, and $x_1\geq x_2 \geq\cdots\geq x_6> y_1\geq y_2\geq\cdots\geq y_6> z_1\geq z_2 \geq\cdots\geq z_6$.

We let $G_1= (X_1, X_2, X_3)$, $G_2= (X_4, X_5, X_6)$, $G_3= (Y_1, Y_2, Y_3)$, $G_4= (Y_4, Y_5, Y_6)$, $G_5= (Z_1, Z_2, Z_3)$, and $G_6= (Z_4, Z_5, Z_6)$ and call this grouping "the group of likes".

The SGV_{max} denotes the sum of group values of "the group of likes".

[Theorem]

The SGV_{max} is the maximum in the all groupings. ■

Proof:

No matter how you exchange arbitrary X_i and X_j between the two X-only groups $\{X_1, X_2, X_3\}$ and $\{X_4, X_5, X_6\}$, the value of SGV_{max} doesn't change. It is the same for $\{Y_1, Y_2,\cdots,Y_6\}$ and $\{Z_1, Z_2,\cdots,Z_6\}$. When you exchange an arbitrary X_i in G_1 or G_2 for an arbitrary Y_j in G_3 or G_4, we let

SGV'. We have already obtained SGV_{max} from Example I.

We let $\alpha = \sum_{i=1}^{6} x_i$, $\beta = \sum_{i=1}^{6} y_i$,and $\gamma = \sum_{i=1}^{6} z_i$.

$$SGV_{max} = s_1 \sum_{i=1}^{6} x_i + s_3 \sum_{i=1}^{6} y_i + s_6 \sum_{i=1}^{6} z_i$$

$$= s_1\alpha + s_3\beta + s_6\gamma = s(\alpha+\beta+\gamma) + 5d\alpha + 3d\beta \quad (\text{by} \quad s_i = s+(6-i)d \quad)$$

If X_l and Y_m belong to the same group with X_i and Y_p and Y_q belong to the same group with Y_j, we exchange X_i for Y_j to be able to get SGV' easily, where l, m, p, and q ϵ {1, 2, 3, 4, 5, 6}.

$$SGV' = s(\alpha+\beta+\gamma)+5d\alpha-(d/2)(x_l+x_m)-dx_i+3d\beta+dy_j+(d/2)(y_p+y_q)$$

$$SGV_{max} - SGV' = dx_i+(d/2)(x_l+x_m)-dy_j-(d/2)(y_p+y_q)$$

$$= d(x_i - y_j)+(d/2)\{(x_l+x_m)-(y_p+y_q)\} \quad >0 \quad (\quad \text{since } x_i > y_j \quad)$$

In "the group of likes", exchanging one of X for one of Y makes the value of SGV_{max} small.
From the same process, we can tell easily that exchanging one of X for one of Z makes the value of SGV_{max} small. Therefore, we can show that any exchange to "the group of likes" reduce SGV_{max}.

■

From numerical example Ⅰ, Ⅱ ,and III, we have predicted that the sum of group values of "likes grouping" became the maximum. But under $\sum_{i=1}^{6} x_i > \sum_{i=1}^{6} y_i > \sum_{i=1}^{6} z_i$ and $s_1 \geq s_2 \geq s_3 \geq s_4 \geq s_5 \geq s_6$., we can't prove the theorem. This proof is done with giving $\{s_i\}$ the condition of sequence of numbers with common difference.

3. Conclusion

We divide 18 students into six groups. We let each group do a coalitional game. Our purpose is that we find the benefit of whole classroom maximal. As we saw the results of numerical example I and Theorem of chapter 2-5, No.99 makes the highest benefit. That means we make groups from better students in order. However, we noticed that we have the different proper groupings from numerical example Ⅱ and Ⅲ with the different coefficient s_i. The rank of No.99 has chances to become not the highest under the condition, $s_1 > s_2 > \ldots > s_6$. We want to focus on the result with the convex way. No.14 became the highest with the convex way. Under the condition that the only relationship between Z and Z creates less benefit than others, to group Z and X creates the whole benefit bigger.

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Appendix

```
Sub MIPS()
k = -1
For a = 0 To 6
   For b = 0 To 6
        For c = 0 To 6
           For d = 0 To 6
                For e = 0 To 6
                   For f = 0 To 6
                        For g = 0 To 6
                             For h = 0 To 6
                                  For i = 0 To 6
                                     For j = 0 To 6
If WorksheetFunction.And(3 * a + 2 * b + 2 * c + d + h + i = 6, b + 2 * d + 3 * e + 2 * f + g + h = 6,
c + f + 2 * g + h + 2 * i + 3 * j = 6, a + b + c + d + e + f + g + h + i + j = 6) = True Then
   Cells(1, 1).Activate
   k = k + 1
      With Application.WorksheetFunction
           ActiveCell.Offset(0, k) = a
           ActiveCell.Offset(1, k) = b
```

```
            ActiveCell.Offset(2, k) = c
            ActiveCell.Offset(3, k) = d
            ActiveCell.Offset(4, k) = e
            ActiveCell.Offset(5, k) = f
            ActiveCell.Offset(6, k) = g
            ActiveCell.Offset(7, k) = h
            ActiveCell.Offset(8, k) = i
            ActiveCell.Offset(9, k) = j
        End With
End If
    Next j
      Next i
        Next h
          Next g
            Next f
              Next e
                Next d
                  Next c
                    Next b
                      Next a
End Sub
```

DIFFUSION APPROXIMATIONS FOR MULTICLASS FEEDFORWARD QUEUEING NETWORKS WITH ABANDONMENTS UNDER FCFS SERVICE DISCIPLINES

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Abstract. We consider multiclass feedforward queueing networks with abandonments under FCFS (first-come, first-served) service disciplines and prove a diffusion approximation theorem for the queue lengths and workloads in those networks under heavy traffic. The diffusion limit is the unique solution to a multidimensional reflected stochastic differential equation with a nonlinear drift term as the limit of abandonment-count process. The desired convergence is shown by taking the following steps: first, obtaining the stochastic boundedness of (scaled) workload in use of the feedforward property of class routing; second, proving the C-tightness of abandonment-count process; third, establishing the condition of state-space collapse; fourth, showing the C-tightness of workload. In the final step we prove the uniqueness (in law) of the solution to the limit equation for workload by reducing it to the uniqueness of a semimartingale reflecting Brownian motion via the Girsanov transformation technique.

1 Introduction.

In this paper we are concerned with multiclass feedforward queueing networks with customer abandonments in heavy traffic. Generally queueing network models have been used to analyze systems arising in a wide range of computer systems, communication networks and complex manufacturing systems. Many of those systems have stations which process more than one class of customers (or jobs) and also have complex structures of class routings after the processing of customers. So the model of multiclass queueing networks has been developed for the analysis of such systems. In particular, the heavy load of those networks is a compelling problem to solve, and thus the diffusion (or heavy-traffic) approximation of such networks has been wanted and pursued. At the same time, because it is natural to suppose that no customer has infinite patience in waiting for service in a queue, the phenomenon of customer abandonment is ubiquitous in various queue models for real applications such as telephone call centers, transmission channels and manufacturing industries, in which impatient customers faced with some waiting time leave the system without receiving service. For example, in the context of wireless communication networks, data packets are lost unless they are transmitted by some deadline.

In multiclass queueing networks (MQNs) under study, customers are categorized into $K(\geq 1)$ classes and the network is composed of $J(\geq 1)$ service stations with unlimited capacity where $J \leq K$. Customers of each class $k \in \mathbb{K}(\equiv \{1,\dots,K\})$ arrive from outside the network and they will receive service exclusively at station $j = s(k)$ where $s(\cdot)$ maps $\mathbb{K}$ onto $\mathbb{J}(\equiv \{1,2,\dots,J\})$ in a many-to-one fashion. In such networks customers change their classes on their service completions. In particular, we restrict our attention to multiclass *feedforward* queueing networks in which at the class change of a customer he either flows

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from a lower numbered station to a higher numbered one, or remains in the original station (as a new class customer). After at most a finite number of such class changes, customers will eventually leave the network. In this paper, the FCFS (firsr-come, first-served) service discipline is investigated in our multiclass feedforward queueing networks with abandonments and we establish the diffusion approximation for those networks in heavy traffic.

Related research. Diffusion approximations for (single-class) generalized Jackson queueing networks (GJNs) in heavy-traffic were established in Reiman [22] under typical moment conditions on primitive variables of the network. However, some counterexamples were found to the validity of heavy-traffic limit for multiclass queueing networks (MQNs) (cf. Dai and Wang [9]), which is in contrast with the case of GJNs. So the identification of the category of the MQNs subject to the heavy-traffic analysis has been one of the main topics in queueing theory. Due to the feature that a single server processes more than one class of customers in MQNs and also to the class-transition nature of a customer, the increased complexity is brought so that the heavy-traffic limit of scaled K-dimensional queue length vector in an MQN is understood to be difficult to obtain without additional restrictive conditions not appearing in such limits of GJNs.

In late 1990s, such problem was solved by Bramson [3] and Williams [26] for some types of MQN with important service disciplines such as FCFS, processor-sharing and buffer-priority ones. More specifically, Williams [26] established heavy-traffic limit theorems for MQNs with the limit referred to as a semimartingale reflecting Brownian motion, assuming the condition of *state-space collapse*. Loosely speaking, state-space collapse corresponds to an asymptotic-law version of Little's formula for MQNs in heavy traffic. Further, [26] indicated that state-space collapse is also a necessary condition for the heavy-traffic limit theorem in MQNs with FCFS disciplines. (Cf. Appendix B in [26]). At the same time, Bramson [3] constructed the framework on state-space collapse for MQNs in which the initial condition on *strong* state-space collapse is proved to imply *multiplicative* strong state-space collapse (cf. Theorem 1 in [3]), which forms the basis for the use of state-space collapse in [26]. In addition, [3] showed that state-space collapse is exhibited after a brief period of time under the relative compactness (tightness) of initial scaled workload (cf. Theorem 3 in [3]), which is used to prove that state-space collapse holds for a multiclass single-server queue in stationarity (cf. Katsuda [15]).

On the other hand, for the last decade, the study of a many-server queue with abandonment in the so-called Halfin-Whitt heavy-traffic regime has attracted considerable attention, because it is relevant to practical large-scale service systems such as call centers. (Cf. Dai and He [8] and references therein). Furthermore, the heavy-traffic analysis of a (single-class) single-server queue, and more generally, that of a GJN are associated with customer abandonment. (Cf. Ward and Glynn [23], [24], Reed and Ward [21] for the former study, and Huang and Zhang [13] for the latter). In particular, the works [24] and [21] identified a reflected Ornstein-Uhlenbeck process and a more general reflected diffusion process, respectively, as the heavy-traffic limit of a GI/GI/1(+GI) queue with abandonment. In all of those works, for the scaling of abandonment (or, patience time) distribution, the continuous or locally-bounded hazard-rate scaling and more generally, the locally-Lipschitz hazard-type scaling were employed because of their technical tractability. ¿From a unified point of view, those scalings are extended to the most general hazard-type one by Katsuda [17] for a G/Ph/n+GI queue in the Halfin-Whitt regime. According to such general scaling, practical and yet previously intractable examples of abandonment distribution become subject to the analysis of diffusion approximation. For instance, the Gamma distribution with scale parameter less than unity is such case. (See the introduction of Katsuda [17]).

Main result. In this paper we will state and prove a diffusion approximation for a multiclass feedforward queueing network with abandonment under the FCFS service discipline. Our main result is a generalization of two previous works [24] and [21] cited above. Specifically, we extend their diffusion approximation results via a one-dimensional Ornstein-Uhlenbeck type diffusion for a GI/GI/1+GI queue to a multiclass feedforward queueing network with GI-type abandonment. Furthermore we employ the general hazard-type scaling of abandonment distribution which includes the locally Lipschitz hazard-type scaling used in [24] and [21]. Our limit process for (scaled) workload is the unique solution to a multidimensional reflected stochastic differential equation with a nonlinear drift and the limit for queue length in each class is a constant times the limit of workload at the station serving the class, which is a consequence of state-space collapse for our queueing network with abandonment.

Methodology. In addition to the i.i.d. (independent and identically distributed) condition of primitive model variables with general probability distributions and also their parameters convergence, we impose the following four main assumptions:
(A.1) Initial condition on the weak convergence of (scaled) workload.
(A.2) Initial condition on strong state-space collapse.
(A.3) Tightness of initial queue length.
(A.4) Completely-S condition of reflection matrix in the limit equation for the workload.

To derive the diffusion approximation result from those assumptions, the following steps will be taken in our argument:
Step 1. Using assumptions (A.1) and (A.3), we show the stochastic boundedness of scaled queue length and workload in our queueing network with abandonment. In particular, the feedforward property of class routing is crucial to this step.
Step 2. For each $k \in \mathbb{K}$, the C-tightness of scaled abandonment-count process of class k is proved, using the stochastic boundedness of scaled workload in Step 1.
Step 3. According to (A.2) and Step 1, the condition corresponding to strong state-space collapse in a multiclass FCFS queueing network (without abandonment) is shown. Combining it with the condition characterizing the FCFS discipline with abandonment, we have state-space collapse for our queueing network with abandonment.
Step 4. Using the results of Step 2 and Step 3, we have the C-tightness of the sequence of scaled workloads satisfying the heavy-traffic condition, and then derive a J-dimensional reflected stochastic differential equation (SDE) satisfied by *any* limit process of the sequence.
Step 5. Observe that our limit SDE has a *nonlinear* drift term as the limit of scaled abandonment-count process due to the general hazard-type scaling of abandonment distribution. (The solution to the equation may be regarded as a semimartingale reflecting Brownian motion (SRBM) with a nonlinear drift term). Thus, applying the Girsanov transformation to the localized SDE and using (A.4), the uniqueness in law of the solution to the original SDE is achieved. Consequently we have the desired weak convergence of scaled workload to the unique solution to the SDE. The limit for queue length in each class is an immediate consequence of state-space collapse and the limit for workload at the station serving the class.

Overview of the contents. The rest of the paper is organized as follows. In Sect. 2, we introduce some primitive variables and processes for a multiclass queueing network with abandonment under study. In terms of those primitives, we construct a piecewise deterministic Markov process for the dynamical description of our queueing network in Sect. 3. In other words, the performance measures for our network are adapted to the history of the process. In Sect. 4, we state our main result, i.e., a diffusion approximation theorem for a multiclass feedforward queueing network with abandonment, and Sect. 5 is devoted to its proof, in which the methodology mentioned above are employed. In the appendix, we put some lemmas used in the demonstration of state-space collapse in Sect. 5.

Notation. For a random variable X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the expectation of X on an event $A \in \mathcal{F}$ is denoted by $\mathbb{E}_{\mathbb{P}}[X; A]$. For a local martingale M, the optional quadratic variation process of M is denoted by $[M]$. (Cf. (1.8.3) in Liptser and Shiryayev [20]).

The symbols $\mathbb{Z}$, $\mathbb{N}$, $\mathcal{R}^1$ and $\mathcal{R}^1_+$ denote the set of integers, positive integers, real numbers and nonnegative real numbers, respectively. For $a, b \in \mathcal{R}^1$, $a \wedge b \equiv \min\{a, b\}$, $a \vee b \equiv \max\{a, b\}$, $a^+ \equiv a \vee 0$, $a^- \equiv (-a) \vee 0$, $\lfloor a \rfloor \equiv \max\{i \in \mathbb{Z} : i \leq a\}$ and $\lceil a \rceil \equiv \max\{i \in \mathbb{Z} : i < a\}$.

For $d \in \mathbb{N}$, $\mathcal{R}^d$ denotes the d-dimensional Euclidean space. Every vector in $\mathcal{R}^d$ is envisioned as a column vector. For example, $a = (a_k, k \in \mathcal{L})$ denotes the L-dimensional column vector with L the number of elements in the index set $\mathcal{L}$. The transpose of a vector or a matrix is denoted by putting a tilde on its top. The vector $e \in \mathcal{R}^d$ denotes $(1, 1, \ldots, 1)$. The norm $|u|$ of a vector $u = (u_1, \ldots, u_d) \in \mathcal{R}^d$ is defined by $|u| = |u_1| + \cdots + |u_d|$. The matrix $diag(u)$ with a vector $u = (u_1, \ldots, u_d) \in \mathcal{R}^d$ denotes the $d \times d$ diagonal matrix with (i, i)-diagonal element equal to u_i, $i = 1, \ldots, d$.

The space of functions $f : [0, \infty) \to \mathcal{R}^d$ that are right-continuous on $[0, \infty)$ and have left-hand limits in $(0, \infty)$ is denoted by $\mathbb{D}([0, \infty), \mathcal{R}^d)$ or simply by $\mathbb{D}^d$. The space $\mathbb{D}^d$ is endowed with the Skorohod J_1-topology. Similarly, the space of $\mathcal{R}^d$-valued continuous functions on $[0, \infty)$ is denoted by $\mathbb{C}([0, \infty), \mathcal{R}^d)$. For $f \in \mathbb{D}^d$ and $t > 0$, $f(t-)$ denotes its left-hand limit at t and $\Delta f(t) \equiv f(t) - f(t-)$. For a sequence of random elements $\{X^r\}_{r \geq 1}$ taking values in a metric space $\mathfrak{S}$, the symbol $X^r \Longrightarrow X$ *in* $\mathfrak{S}$ as $r \to \infty$ means the weak convergence of X^r to X in $\mathfrak{S}$ as the index r tends to infinity.

2 Multiclass feedforward queueing networks with abandonments and their Markovian description of dynamics

2.1 Model primitives In this section we first introduce some primitive random variables (r.v.'s) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to construct the model of a multiclass queueing network with abandonment studied in this paper. The network is composed of J service stations indexed by $j = 1, \ldots, J$, and the set of service stations is denoted by $\mathbb{J} = \{1, 2, \ldots, J\}$. Each of the service stations has a single server and a waiting buffer of unlimited capacity. Each customer (or job) belongs to one of K classes with $K \geq J$, indexed by $k = 1, \ldots, K$, and the set of the classes is denoted by $\mathbb{K} = \{1, 2, \ldots, K\}$. For each $k \in \mathbb{K}$, customers of class k are served at service station $s(k) \in \mathbb{J}$ exclusively. The mapping $s(\cdot)$ maps $\mathbb{K}$ onto $\mathbb{J}$ in a many-to-one fashion. In addition, we let $C(j) = \{k \in \mathbb{K} : s(k) = j\}$, $j \in \mathbb{J}$.

Customers of classes in $\mathbb{A}$, which is a non-empty subset of $\mathbb{K}$, enter the network from outside and no external arrival is allowed for any class in $\mathbb{K} - \mathbb{A}$. Upon arrival, a customer is assigned the abandonment time (or, patience time) whose probability law depends on his class, and if the time until the customer is supposed to enter service, called the offered waiting time, exceeds his abandonment time, then he will abandon the system as soon as his remaining abandonment time is exhausted. Otherwise, i.e., if the customer is supposed to receive service eventually, he is assigned the service time on his arrival, which also depends on his class. The service of customers by the server is performed according to the first-come-first-service (FCFS) discipline, i.e., in the order of their arrivals independently of their classes. (We also take the convention that customers within each class are numbered on the first-in basis). On service completion, a customer either changes his class and waits for service as the new class customer in the end of the queue, or leaves the system.

External arrivals

The *external arrival process* $E(t) = \{E_k(t), k \in \mathbb{K}\}, t \geq 0$, counts the number of arrivals at each class from outside the network. For each $k \in \mathbb{A}$, we define $E_k(\cdot)$ by

$$E_k(t) \equiv \max\{n \in \mathcal{N} : \mathcal{U}_k(n) \leq t\}$$

with $\max \phi \equiv 0$, where

$$\mathcal{U}_k(n) \equiv \sum_{i=1}^{n} u_k(i) \tag{1}$$

with $\mathcal{U}_k(0) \equiv 0$. For each $k \in \mathbb{A}$, the *external interarrival times* $\{u_k(i), i = 2, 3, \ldots\}$ are i.i.d. (independent and identically distributed) positive r.v.'s with the distribution function (d.f.)

$$F_k^u(x) \equiv \mathbb{P}(u_k(2) \leq x), \qquad x \geq 0,$$

the mean $1/\alpha_k \equiv \int_0^\infty x dF_k^u(x) > 0$, and the finite variance $a_k \equiv \int_0^\infty (x - \frac{1}{\alpha_k})^2 dF_k^u(x) \geq 0$. The r.v. $u_k(1) > 0$, corresponding to the remaining interarrival time of the customer entering first after time $t = 0$, is independent of $\{u_k(i), i = 2, 3, \ldots\}$. For each $i = 2, 3, \ldots$, the r.v. $u_k(i)$ corresponds to the interarrival time between the $(i-1)$-th customer and i-th customer in class k. For conveniece, we set

$$E_k(\cdot) \equiv 0 \quad \text{and} \quad \alpha_k = 0$$

for $k \in \mathbb{K} - \mathbb{A}$. The vector $\alpha = (\alpha_k, k \in \mathbb{K})$ is referred to as the *arrival rate.*

Service times

For each $k \in \mathbb{K}$, there are two sequences of service times, i.e., a sequence of *original* service times and a sequence of *subsequent* service times. The sequence of original service times $\{v_k^o(i), i = 1, 2, \ldots\}$ gives the (remaining) service times for class k customers who are in the system at time 0 and will eventually receive service. (There are more elements in the infinite sequence than needed). Those initial customers are assumed to have the prescribed order of arrivals at or before time 0, and if there is such i-th customer in the system, the original service time $v_k^o(i)$ is assigned to him for $i = 1, 2, \ldots$.

For each $k \in \mathbb{K}$, the *original service times* $\{v_k^o(i), i = 2, 3, \ldots\}$ are i.i.d. positive r.v.'s with

$$F_k^v(x) \equiv \mathbb{P}\big(v_k^o(2) \leq x\big), \quad x \geq 0, \tag{2}$$

the mean $m_k = \int_0^\infty x dF_k^v(x) > 0$ and the finite variance $b_k \equiv \int_0^\infty (x - m_k)^2 dF_k^v(x) \geq 0$. The constant $\mu_k \equiv 1/m_k$ is referred to as the *service rate* of class k. The r.v. $v_k^o(1)$, corresponding to the (remaining) service time of initial class k customer who arrived the longest time ago among those eventually receiving service, is independent of $\{v_k^o(i), i = 2, 3, \ldots\}$. The *cumulative original service time process* $\mathcal{V}_k^o(n), n \in \mathbb{N}, k \in \mathbb{K}$, is given by

$$\mathcal{V}_k^o(n) \equiv \sum_{i=1}^{n} v_k^o(i) \tag{3}$$

with $\mathcal{V}_k^o(0) \equiv 0$.

The *subsequent service times* $\{v_k^s(i), i = 1, 2, \ldots\}$, $k \in \mathbb{K}$, are i.i.d. positive r.v.'s with $\mathbb{P}\big(v_k^s(1) \leq x\big) = F_k^v(x), x \geq 0$. For each $k \in \mathbb{K}$, $v_k^s(i)$ corresponds to the service time assigned to the i-th class k customer among those arriving after $t = 0$ from outside or due to class change and eventually receiving service. The *cumulative subsequent service time process* $\mathcal{V}_k^s(n), n \in \mathbb{N}, k \in \mathbb{K}$, is given by

$$\mathcal{V}_k^s(n) \equiv \sum_{i=1}^{n} v_k^s(i) \tag{4}$$

with $\mathcal{V}_k^s(0) \equiv 0$.

Abandonment times

Similar to the service times above, we introduce the abandonment times in two distinct sequences, i.e., the *original* abandonment times and *subsequent* abandonment times. For each $k \in \mathbb{K}$, the original abandonment times $\{\gamma_k^o(i), i = 1, 2, \ldots\}$ is a sequence of independent positive r.v.'s which corresponds to the remaining abandonment times of the customers of class k initially at the network. (The assignment of those abandonment times to each customer is done in the same way as in service times, but distinct to that case, the abandonment time is assigned to *every* customer at the system, whether he will abandon it or not). For each $k \in \mathbb{K}$, the subsequent abandonment times $\{\gamma_k^s(i), i = 1, 2, \ldots\}$ are i.i.d. positive random variables with

$$F_k^{\gamma}(x) \equiv \mathbb{P}(\gamma_k^s(1) \le x), \qquad x \ge 0, \tag{5}$$

and correspond to the abandonment times assigned to the customers of class k arriving after $t = 0$.

Class routings

The *class-routing process* $\Phi(n) = \{\Phi^k(n), k \in \mathbb{K}\}$, $n \in \mathbb{N}$, is defined by

$$\Phi^k(n) \equiv \sum_{i=1}^{n} \phi^k(i)$$

where $\{\phi^k(i) = (\phi_l^k(i), l \in \mathbb{K}), i = 1, 2, \ldots, \}$ are i.i.d. random vectors taking values in the set $\{0, e_1, \ldots, e_K\}$ with e_k denoting the unit basis vector parallel to the k-th coordinate axis in $\mathcal{R}^K$, $k \in \mathbb{K}$. The identity $\phi^k(i) = e_l$ indicates that the i-th customer served at class k changes his class to class l after the service, and the identity $\phi^k(i) = 0$ indicates his departure from the system.
Let $P_{kl} = \mathbb{P}(\phi^k(1) = e_l)$ and $P_{k0} = \mathbb{P}(\phi^k(1) = 0)$, $k, l \in \mathbb{K}$. Then the $K \times K$ substochastic matrix $P = [P_{kl}; k, l \in \mathbb{K}]$, called the *class-routing* matrix, is assumed to have spectral radius strictly less than unity. Thus

$$Q \equiv (I - \widetilde{P})^{-1} = I + \widetilde{P} + (\widetilde{P})^2 + \cdots$$

is finite where $\widetilde{P}$ denotes the transpose of P. It is readily seen that for each $k \in \mathbb{K}$,

$$\begin{aligned} \mathbb{E}[\phi^k(1)] &= P_{k\cdot} \qquad \text{and} \\ \mathbb{C}ov[\phi^k(1)] &\equiv [\mathbb{C}ov(\phi_l^k(1), \phi_m^k(1)),\, l, m \in \mathbb{K}] \\ &= \Upsilon^k \end{aligned} \tag{6}$$

where $P_{k\cdot}$ denotes the k-th row vector of P and Υ^k denotes the $K \times K$ matrix such that

$$\Upsilon_{lm}^k = \begin{cases} P_{kl}(1 - P_{kl}) & \text{if} \quad l = m, \\ -P_{kl}P_{km} & \text{if} \quad l \ne m. \end{cases} \tag{7}$$

In this paper we will impose on the class-routing probability $\{P_{kl}, k, l \in \mathbb{K}\}$ the following condition:

Feedforward class-routing condition

For each $k, l \in \mathbb{K}$,

$$\text{if} \quad P_{kl} > 0, \quad \text{then} \quad s(k) \le s(l). \tag{8}$$

When $J = 1$(i.e., a multiclass single-server queue), condition (8) is obviously satisfied.

Remaining time processes

Associated with the interarrival, service and abandonment times introduced above, we define their remaining time processes as follows. For each $k \in \mathbb{K}$ and $t \ge 0$, let $\mathcal{R}_k^u(t)$ and $\mathcal{R}_k^v(t)$ denote the remaining interarrival time and remaining service time of class k customer at time t, respectively. (For $k \in \mathbb{K} - \mathbb{A}$, we set $\mathcal{R}_k^u(\cdot) \equiv -1$). In particular, $\mathcal{R}_k^u(0) = u_k(1), k \in \mathbb{A}$, $\mathcal{R}_l^v(0) = v_l^o(1), l \in \mathbb{K}$.

Now, for each $k \in \mathbb{K}$, let

$$Z_k(t), \quad t \ge 0, \tag{9}$$

denote the number of class k customers who are either being served or waiting in queue at time t, which is referred to as the *queue length* of class k at time t. Then the remaining abandonment time process of class $k, k \in \mathbb{K}$, is represented by

$$\mathcal{R}_k^\gamma(t) = (\mathcal{R}_{k,i}^\gamma(t), i = 1, 2, \ldots), \qquad t \ge 0,$$

in which, for each $1 \le i \le Z_k(t)$, $\mathcal{R}_{k,i}^\gamma(t)$ denotes the remaining abandonment time of i-th customer of class k at time t, and for $i \ge Z_k(t) + 1$, we set $\mathcal{R}_{k,i}^\gamma(t) \equiv -1$. In particular, $\mathcal{R}_{k,i}^\gamma(0) = \gamma_k^o(i)$ for each $1 \le i \le Z_k(0)$ and $k \in \mathbb{K}$. If the remaining abandonment time $\mathcal{R}_{k,1}^\gamma(\cdot)$ expires at $t = t_0$ and the service of the corresponding customer began before time t_0 and continues at $t = t_0$, then we set $\mathcal{R}_{k,1}^\gamma(t) \equiv 0$ for each $t \in [t_0, t_1)$ where t_1 denotes the time at which the service finishes.

Class designation processes

Relevant to the FCFS discipline investigated in this paper, we have to track the designation of the class of each customer in each service station in order to describe the dynamics of the network. For the purpose, we introduce the $\{0, 1, \ldots, 2K\}^\infty$-valued process

$$O(t) = (O_j(t),\, j \in \mathbb{J}), \qquad t \ge 0, \tag{10}$$

where

$$O_j(t) = (O_{j,i}(t),\, i \ge 1), \quad j \in \mathbb{J},$$

and for $j \in \mathbb{J}$ and $1 \le i \le \sum_{m \in C(j)} Z_m(t)$,

$$O_{j,i}(t) \equiv \begin{cases} k & \text{if } i\text{-th customer in the queue of station } j \text{ at time } t \text{ is} \\ & \text{of class } k \text{ and will eventually receive service;} \\ K + l & \text{if } i\text{-th customer in the queue of station } j \text{ at time } t \text{ is} \\ & \text{of class } l \text{ and will eventually abandon the system,} \end{cases} \tag{11}$$

and for $i \ge \sum_{m \in C(j)} Z_m(t) + 1$, we set $O_{j,i}(t) \equiv 0$. (The variable $O_{j,1}(t)$ corresponds to the class of the customer being served at time t, whenever $\sum_{m \in C(j)} Z_m(t) \ge 1$).

Note that under our assumptions on the primitives, simultaneous (exogenous or internal) arrivals of customers from different classes are allowed. So, to determine the components of the process $O(\cdot)$ without ambiguity, a rule is needed for the specification of the ordering of such customers. Following page 41 of Williams [26], we henceforth take a deterministic tie breaking rule to treat that case. For example, we adopt the convention that for customers with simultaneous arrivals, a customer of higher numbered class is ordered ahead of a customer of lower numbered class in the queue of each station.

Offered waiting times

To determine whether each customer will abandon the network or not either on his arrival to a class or at initial instant, we assign to him the *offered waiting time* as follows. For each $k \in \mathbb{K}$ and $i = 1, 2, \ldots$, the *original offered waiting time* $w_k^o(i)$ is the amount of time the i-th customer of class k initially in the system would have to wait in queue (i.e., waiting line) until getting into service if his abandonment time were infinite, with the convention that $w_k^o(i) \equiv 0$ for $i \geq Z_k(0)+1$. Thus, if $\gamma_k^o(i) \leq w_k^o(i)$, then such i-th class k customer will eventually abandon the network, and otherwise, he will receive service of class k. Similarly, for each $k \in \mathbb{K}$ and $i = 1, 2, \ldots$, the *subsequent offered waiting time* $w_k^s(i)$ is such amount of time for the i-th customer arriving at class k from outside or from other classes due to class change after $t = 0$.

Specifically, $w_k^s(i)$ is $\mathcal{G}_k^s(i)$-measurable for each $i = 1, 2, \ldots$ and $k \in \mathbb{K}$, where

$$\begin{aligned}
&\mathcal{G}_k^s(i) \\
&\equiv \sigma\{u_k(m+1), v_k^s(m), \gamma_k^s(m), m \leq i-1\} \vee \bigvee_{l \in \mathbb{K}, l \neq k} \sigma\{u_l(m), v_l^s(m), \gamma_l^s(m), m \geq 1\} \\
&\vee \bigvee_{p \in \mathbb{K}} \sigma\{v_p^o(m), \gamma_p^o(m), \phi^p(m), m \geq 1\} \vee \sigma\{O(0)\}.
\end{aligned} \tag{12}$$

Mutual independence assumption on the primitives

Finally in this subsection, we impose the following mutual independence assumption on the primitive variables introduce so far, which is fundamental to our argument in the rest of the paper:

The families of variables

$$\begin{aligned}
&\{\mathcal{R}^v(0), \mathcal{R}^\gamma(0), O(0)\}, \{\mathcal{R}_k^u(0) = u_k(1)\}, \ \ k \in \mathbb{A}, \\
&u_{k'}^*, \ \ k' \in \mathbb{A}, \quad v_1^{o,*}, \cdots, v_K^{o,*}, \\
&v_1^s, \cdots, v_K^s, \ \ \gamma_1^s, \cdots, \gamma_K^s, \ \ \phi^1, \cdots, \phi^K
\end{aligned} \tag{13}$$

are mutually independent, where

$$\begin{aligned}
&v_k^{o,*} \equiv (v_k^o(i), i \geq 2), \ \ k \in \mathbb{K}, \\
&u_{k'}^* \equiv (u_{k'}(i), i \geq 2), \ \ k' \in \mathbb{A}, \quad v_l^s \equiv (v_l^s(i), i \geq 1), \ \ l \in \mathbb{K}, \\
&\gamma_p^s \equiv (\gamma_p^s(i), i \geq 1), \ \ p \in \mathbb{K}, \quad \phi^q \equiv (\phi^q(i), i \geq 1), \ \ q \in \mathbb{K}.
\end{aligned}$$

2.2 Performance measure processes and their equation As the performance measures for our multiclass queueing network with abandonment, we define the following processes:

The K-dimensional (column) vector-valued process

$$Z(t) = (Z_k(t), k \in \mathbb{K}), \qquad t \geq 0,$$

with $Z_k(t)$ in (9) is referred to as the *queue length* process. For each $j \in \mathbb{J}$, let

$$W_j(t), \quad t \geq 0,$$

denote the total amount of immediate work (measured in units of service time) embodied by the customers in the station j at time t. Set

$$W(t) = (W_j(t), j \in \mathbb{J}), \quad t \geq 0,$$

which is referred to as the *workload* process. Also, for each $j \in \mathbb{J}$,

$$Y_j(t), \quad t \geq 0,$$

denotes the cumulative amount of time that the server at station j is idle during the time interval $(0, t]$, and set

$$Y(t) = (Y_j(t), j \in \mathbb{J})$$

that is referred to as the cumulative idle time process. To describe the dynamics of $Z(\cdot)$, $W(\cdot)$ and $Y(\cdot)$, we also introduce the following processes.

For each $k \in \mathbb{K}$ and $t \geq 0$, $A_k(t)$ denotes the total number of the (exogenous and internal) arrivals of class k customers during $(0, t]$, $D_k(t)$ denotes the total number of the service completions of class k customers during $(0, t]$, $I_k(t)$ denotes the total number of the abandonments of class k customers during $(0, t]$, and $T_k(t)$ denotes the total amount of time that the server has processed customers of class k during $(0, t]$. Furthermore, let $A_k^+(t)$ denote the number of customers who arrive at class k during $(0, t]$ and will eventually receive service (and not abandon), and let $Z_k^+(t)$ denote the number of class k customers who are either being under service or waiting in queue at time t and going to receive service.

We represent those processes in (column) vector form as

$$\begin{aligned}
A(t) &= (A_k(t), k \in \mathbb{K}), \\
A^+(t) &= (A_k^+(t), k \in \mathbb{K}), \\
D(t) &= (D_k(t), k \in \mathbb{K}), \\
I(t) &= (I_k(t), k \in \mathbb{K}), \\
T(t) &= (T_k(t), k \in \mathbb{K}), \\
Z^+(t) &= (Z_k^+(t), k \in \mathbb{K}), \quad t \geq 0.
\end{aligned}$$

Let

$$\mathfrak{X}(t) \equiv (A(t), A^+(t), D(t), I(t), T(t), W(t), Y(t), Z(t), Z^+(t)), \quad t \geq 0, \tag{14}$$

and the process $\mathfrak{X}(\cdot)$ is called the *performance measure process* for our multiclass queueing network with abandonment. Then the dynamical equation for the components of $\mathfrak{X}(t), t \geq 0$, is represented as follows:

$$A(t) = E(t) + F(t) \tag{15}$$

$$\text{with} \quad F(t) = \sum_{k=1}^{K} \Phi^k(D_k(t)), \tag{16}$$

$$Z(t) = Z(0) + A(t) - D(t) - I(t), \tag{17}$$

$$Z^+(t) = Z^+(0) + A^+(t) - D(t) \tag{18}$$

$$\text{with} \quad Z_k^+(0) \equiv \sum_{i=1}^{Z_k(0)} 1_{\{w_k^o(i) < \gamma_k^o(i)\}} \tag{19}$$

$$\text{and} \quad A_k^+(t) \equiv \sum_{i=1}^{A_k(t)} 1_{\{w_k^s(i) < \gamma_k^s(i)\}}, \quad k \in \mathbb{K}, \tag{20}$$

$$W(t) = W(0) + C\mathcal{V}^s(A^+(t)) - CT(t) \tag{21}$$

$$\text{with} \quad W(0) = C\mathcal{V}^o(Z^+(0)), \tag{22}$$

$$CT(t) + Y(t) = t, \tag{23}$$

$$\int_0^\infty W_j(s) dY_j(s) = 0, \quad \forall j \in \mathbb{J}, \tag{24}$$

for all $t \geq 0$, where $C = [C_{jk}, j \in \mathbb{J}, k \in \mathbb{K}]$ is the $J \times K$ matrix with

$$C_{jk} = \begin{cases} 1, & \text{if } j = s(k): \\ 0, & \text{otherwise.} \end{cases}$$

Associated with the *abandonment-count* process $I_k(\cdot), k \in \mathbb{K}$, we now define the process $N_k(\cdot), k \in \mathbb{K}$, by

$$N_k(t) \equiv Z_k^-(0) + A_k^-(t), \qquad t \geq 0, \tag{25}$$

where

$$Z_k^-(0) \equiv \sum_{i=1}^{Z_k(0)} 1_{\{\gamma_k^o(i) \leq w_k^o(i)\}} = Z_k(0) - Z_k^+(0), \tag{26}$$

$$A_k^-(t) \equiv \sum_{i=1}^{A_k(t)} 1_{\{\gamma_k^s(i) \leq w_k^s(i)\}} = A_k(t) - A_k^+(t). \tag{27}$$

We observe that under the FCFS service discipline, for each $k \in \mathbb{K}$, $t \geq 0$ and $\varepsilon > 0$,

$$N_k(\zeta_{s(k)}(t) - \varepsilon) \leq I_k(t) \leq N_k(t) \tag{28}$$

with

$$\zeta_j(t) \equiv \inf\{s \geq 0 : s + W_j(s) > t\}, \qquad j \in \mathbb{J}, \tag{29}$$

and

$$Z_k^-(t) \leq I_k(t + W_{s(k)}(t)) - I_k(t) \tag{30}$$

with

$$Z_k^-(t) \equiv Z_k(t) - Z_k^+(t). \tag{31}$$

2.3 Markovian description of a multiclass queueing network with abandonment

In the following we introduce the Markovian description process for a multiclass queueing network *with* abandonment in a similar way to Katsuda [15]. The process will be constructed from the primitive variables and the associated processes introduced so far. Conversely those primitives can also be represented by the description process.

Let

$$V(t) \equiv (V_k(t), k \in \mathbb{K})$$

where $V_k(t) \equiv (V_{k,i}(t), i = 1, 2, \ldots)$ with

$$V_{k,1}(t) \equiv \mathcal{R}_k^v(t),$$

for $2 \leq i \leq Z_k^+(t)$,

$$V_{k,i}(t) \equiv \begin{cases} v_k^o(D_k(t) + i), & \text{if } \quad D_k(t) + i \leq Z_k^+(0), \\ v_k^s(D_k(t) + i - Z_k^+(0)), & \qquad \text{otherwise,} \end{cases}$$

and for $i \geq Z_k^+(t) + 1$,

$$V_{k,i}(t) \equiv 0.$$

We define the stochastic process $\Xi = (\Xi(t), t \geq 0)$ by

$$\Xi(t) \equiv (O(t), \mathcal{R}^u(t), V(t), \mathcal{R}^\gamma(t)) \tag{32}$$

where

$$\begin{aligned} O(t) &= (O_j(t), j \in \mathbb{J}) = ((O_{j,i}(t), i = 1, 2, \ldots), j \in \mathbb{J}), \\ \mathcal{R}^u(t) &= (\mathcal{R}_k^u(t), k \in \mathbb{A}), \\ \mathcal{R}^\gamma(t) &= (\mathcal{R}_k^\gamma(t), k \in \mathbb{K}) = ((\mathcal{R}_{k,i}^\gamma(t), i \geq 1), k \in \mathbb{K}). \end{aligned}$$

Then $\Xi = (\Xi(t), t \geq 0)$ is a piecewise deterministic Markov process (PDMP). Generally the PDMP is a strong Markov process. (Cf. Davis [10]).

Let

$$\mathcal{F}_t^\Xi \equiv \sigma(\Xi(s); 0 \leq s \leq t), \quad t \geq 0.$$

Then $(\mathcal{F}_t^\Xi)_{t\geq 0}$ is right continuous, i.e., $\cap_{n=1}^\infty \mathcal{F}_{t+\frac{1}{n}}^\Xi = \mathcal{F}_t^\Xi$ for each $t \geq 0$. As stated in the next proposition, the performance measure processes $\mathfrak{X}(\cdot)$ is $(\mathcal{F}_t^\Xi)_{t\geq 0}$-adapted. In other words, the process $\Xi(\cdot)$ describes the dynamics of our multiclass queueing network with abandonment. For this reason, the process $\Xi(\cdot)$ is called the *Markovian description process* for the network.

We denote the probability law of Markov process $\Xi(t), t \geq 0$, starting with the value $\xi \in \mathcal{S}$ by

$$\mathsf{P}_\xi(\mathsf{E}), \quad \mathsf{E} \in \mathcal{F}_\infty^\Xi (\equiv \bigvee_{t\geq 0} \mathcal{F}_t^\Xi), \quad \xi \in \mathcal{S}, \tag{33}$$

such that $\mathsf{P}_\xi(\Xi(0) = \xi) = 1$, where $\mathcal{S}$ denotes the state space of the process $\Xi(\cdot)$. For each $\mathsf{E} \in \mathcal{F}_\infty^\Xi$, $\mathsf{P}_\xi(\mathsf{E})$ is $\mathfrak{B}(\mathcal{S})$-measurable w.r.t. ξ.

Now let $\{\theta_t\}_{t\geq 0}$ denote the family of shift transformations associated with the process $\Xi(t), t \geq 0$. Namely,

$$\Xi(t) \circ \theta_s = \Xi(t + s)$$

for each $s, t \geq 0$. Corresponding to Proposition 2.1 of Katsuda [15], we have the following proposition on the shift-transformed performance measure process. (Since the proof is done in a similar way, we omit it).

Proposition 2.1.
The performance measure process

$$\mathfrak{X}(\cdot) = (A(\cdot), A^{+}(\cdot), D(\cdot), I(\cdot), T(\cdot), W(\cdot), Y(\cdot), Z(\cdot), Z^{+}(\cdot))$$

is $(\mathcal{F}_t^{\Xi})_{t\geq 0}$*-adapted. Thus* $\mathfrak{X}(\cdot)\circ\theta_t$*,* $t \geq 0$*, is well-defined and each component of the shift transformed process is given by the following:*

$$A(t)\circ\theta_s = A(s+t) - A(s), \tag{34}$$

$$A^{+}(t)\circ\theta_s = A^{+}(s+t) - A^{+}(s), \tag{35}$$

$$D(t)\circ\theta_s = D(s+t) - D(s), \tag{36}$$

$$I(t)\circ\theta_s = I(s+t) - I(s), \tag{37}$$

$$T(t)\circ\theta_s = T(s+t) - T(s), \tag{38}$$

$$W(t)\circ\theta_s = W(s+t), \tag{39}$$

$$Y(t)\circ\theta_s = Y(s+t) - Y(s), \tag{40}$$

$$Z(t)\circ\theta_s = Z(s+t), \tag{41}$$

$$Z^{+}(t)\circ\theta_s = Z^{+}(s+t), \tag{42}$$

for any $s, t \geq 0$*.*

The quantity $Z_k^{-}(t)$, defined by (31), is the number of class k customers who are in the system at time t and will eventually abandon it. According to (41) and (42),

$$Z_k^{-}(t) = Z_k^{-}(0)\circ\theta_t \tag{43}$$

for each $t \geq 0$.

The condition characterizing the FCFS discipline with abandonment is represented as

$$D_k(t + W_{s(k)}(t)) - D_k(t) + Z_k^{-}(t) = Z_k(t) \tag{44}$$

for each $t \geq 0$ and $k \in \mathbb{K}$. In virtue of Proposition 2.1, the identity (44) is a consequence of the operation of shift transformation $\theta_t, t \geq 0$, to the initial relation

$$D_k(W_{s(k)}(0)) + Z_k^{-}(0) = Z_k(0), \qquad k \in \mathbb{K}, \tag{45}$$

and can be regarded as the extension of the FCFS characterization condition without abandonment, i.e.,

$$D_k(t + W_{s(k)}(t)) - D_k(t) = Z_k(t), \qquad t \geq 0, \quad k \in \mathbb{K},$$

that is equivalent to (2.25) in Bramson [3].

3 Heavy-traffic assumptions and scaling

In the rest of the paper we consider a sequence of multiclass FCFS queueing networks with abandonments each of which satisfies the feedforward class-routing condition (8). Each network in the sequence is indexed by r, where r tends to infinity through a sequence of values in $[1, \infty)$. (Note that the index r may possibly take non-integer values). For slight abuse of notation, denote such r-th network by $\mathfrak{X}^r(\cdot)$, whose primitive variables are defined on the probability space $(\Omega^r, \mathcal{F}^r, \mathbb{P}^r)$ for each $r \geq 1$. The number of classes K, the subset $\mathbb{A}$ of $\mathbb{K}$ with exogenous arrivals, and the map $s(\cdot) : \mathbb{K} \longrightarrow \mathbb{J}$ are fixed for all $\mathfrak{X}^r(\cdot), r \geq 1$. Also the service discipline investigated is FCFS in every network of the sequence. We put a superscript r on each of the stochastic processes, primitive variables and constants associated with them introduced so far, in order to indicate the associated network in the sequence. For example, $Z^r(\cdot)$, $A^r(\cdot)$, $A^{-,r}(\cdot)$, $v_k^{s,r}(i)$, $\gamma^{o,r}(i)$, α_k^r, etc.

On the sequence of the parameters associated with the primitive variables in $\mathfrak{X}^r(\cdot), r \geq 1$, we impose the following limit conditions:

$$\alpha_k^r \longrightarrow \alpha_k (> 0) \quad \text{as } r \to \infty, \forall k \in \mathbb{A}, \tag{46}$$

$$m_k^r \longrightarrow m_k (> 0) \quad \text{as } r \to \infty, \forall k \in \mathbb{K}, \tag{47}$$

$$a_k^r \longrightarrow a_k (> 0) \quad \text{as } r \to \infty, \forall k \in \mathbb{A}, \tag{48}$$

$$b_k^r \longrightarrow b_k (> 0) \quad \text{as } r \to \infty, \forall k \in \mathbb{K}, \tag{49}$$

$$P_{kl}^r \longrightarrow P_{kl} \quad \text{as } r \to \infty, \forall k \in \mathbb{K}, l \in \mathbb{K} \cup \{0\}, \tag{50}$$

where $P = [P_{kl}]_{k,l \in \mathbb{K}}$ is a substochastic matrix such that its spectral radius is less than unity and for each $l \in \mathbb{K} - \mathbb{A}$, there exist some $k \in \mathbb{A}$ and $m \in \mathbb{N}$ such that

$$P_{kl}^m > 0 \tag{51}$$

where $P^m \equiv [P_{kl}^m]$ with P^m denoting the m-th power of P.

We define $\lambda^r = (\lambda_k^r, k \in \mathbb{K})$ to be the unique solution to the traffic equation:

$$\lambda^r = \alpha^r + \widetilde{P}^r \lambda^r, \tag{52}$$

that is,

$$\lambda^r = Q^r \alpha^r$$

with

$$Q^r \equiv (\mathrm{I} - \widetilde{P}^r)^{-1}. \tag{53}$$

For each r and $k \in \mathbb{K}$, λ_k^r is referred to as the *nominal total arrival rate* to class k in the r-th network. It is readily seen that $\lambda = \lim_{r \to \infty} \lambda^r$ satisfies

$$\lambda_k > 0 \tag{54}$$

for each $k \in \mathbb{K}$, because of (51).
We also define

$$\rho^r \equiv CM^r \lambda^r = (\rho_j^r, j \in \mathbb{J}) \tag{55}$$

with $M^r \equiv diag(m_k^r, k \in \mathbb{K})$, which is referred to as the *traffic intensity* vector.
We impose the limit condition on the sequence $\{\rho^r\}_r$:

$$r(\rho^r - e) \longrightarrow \vartheta \tag{56}$$

as $r \to \infty$, where ϑ is some constant vector in $\mathcal{R}^J$. The condition (56) is referred to as the *heavy-traffic* condition.

In addition, to obtain the proper limit for appropriately scaled abandonment-count processes (cf. (74) below) as $r \to \infty$ under the heavy-traffic condition, we assume the following scaling condition of abandonment distribution $F_k^{\gamma,r}(x) = \mathbb{P}^r(\gamma_k^{s,r}(1) \leq x), x \geq 0, k \in \mathbb{K}, r \geq 0$:

General hazard-type scaling of abandonment distribution. (Cf. Katsuda [17]).

For each $k \in \mathbb{K}$ and $x \notin Disc(H_k)$,

$$rF_k^{\gamma,r}(rx^r) \longrightarrow H_k(x) \quad \text{as} \quad r \to \infty, \tag{57}$$

whenever $x^r \to x$ as $r \to \infty$, where $H_k(x), x \geq 0$, is a non-decreasing function and $Disc(H_k)$ is the set of discontinuities for $H_k(\cdot)$.

We impose the following uniform integrability condition:

$$\{u_k^r(2)^2\}_{r\geq 1} \quad \text{is uniformly integrable,} \tag{58}$$

$$\{v_l^{s,r}(1)^2\}_{r\geq 1} \quad \text{is uniformly integrable,} \tag{59}$$

for each $k \in \mathbb{A}$ and $l \in \mathbb{K}$. We will also assume the following three conditions on the initial primitive variables, the first two of which correspond to (3.5) in [3] and (82), (83) in [26]:
For each $k \in \mathbb{A}$, $l \in \mathbb{K}$ and $T > 0$,

$$\frac{u_k^r(1)}{r} \longrightarrow 0 \qquad \text{in pr.,} \tag{60}$$

$$\frac{v_l^{o,r}(1)}{r} \longrightarrow 0 \qquad \text{in pr.,} \tag{61}$$

$$\max_{0\leq m<rT} \left| \left\{ \widehat{\mathcal{V}}_l^{o,r}(\overline{Z}_l^{+,r}(0)) \right\} \circ \theta_{rm} \right| \longrightarrow 0 \qquad \text{in pr.,} \tag{62}$$

as r goes to infinity, where

$$\widehat{\mathcal{V}}^{o,r}(t) \equiv r^{-1}(\mathcal{V}^{o,r}(\lfloor r^2 t \rfloor) - m^r \cdot \lfloor r^2 t \rfloor), \tag{63}$$

$$\overline{Z}^{+,r}(t) \equiv r^{-2} Z^{+,r}(r^2 t). \tag{64}$$

(The convergence (62) is restated as

$$\mathbb{P}^r\Big(\max_{0\leq m<rT} \Big| \frac{1}{r} \times \sum_{i=1}^{Z_l^{+,r}(rm)} \left(v_l^{o,r}(i) \circ \theta_{rm} - m_l^r\right)\Big| > \varepsilon\Big) \longrightarrow 0, \quad \forall \varepsilon > 0, \Big)$$

as $r \to \infty$).

Concerned with the asymptotic behavior of the performance measures for our multiclass queueing network with abandonment under the heavy-traffic condition, we perform the diffusive and fluid scaling on the associated stochastic processes as follows:

Diffusion scaling.

$$\widehat{Z}^r(t) = r^{-1}Z^r(r^2t), \tag{65}$$
$$\widehat{Z}^{-,r}(t) = r^{-1}Z^{-,r}(r^2t), \tag{66}$$
$$\widehat{W}^r(t) = r^{-1}W^r(r^2t), \tag{67}$$
$$\widehat{Y}^r(t) = r^{-1}Y^r(r^2t), \tag{68}$$
$$\widehat{E}^r(t) = r^{-1}(E^r(r^2t) - \alpha^r r^2t), \tag{69}$$
$$\widehat{\mathcal{V}}^{s,r}(t) = r^{-1}(\mathcal{V}^{s,r}(\lfloor r^2t\rfloor) - m^r \cdot \lfloor r^2t\rfloor), \tag{70}$$
$$\widehat{A}^r(t) = r^{-1}(A^r(r^2t) - \lambda^r r^2t), \tag{71}$$
$$\widehat{A}^{-,r}(t) = r^{-1}A^{-,r}(r^2t), \tag{72}$$
$$\widehat{D}^r(t) = r^{-1}(D^r(r^2t) - \lambda^r r^2t), \tag{73}$$
$$\widehat{I}^r(t) = r^{-1}I^r(r^2t), \tag{74}$$
$$\widehat{N}^r(t) = r^{-1}N^r(r^2t), \tag{75}$$
$$\widehat{S}^r(t) = r^{-1}(S^r(r^2t) - \mu^r r^2t), \tag{76}$$
$$\widehat{\Phi}^{k,r}(t) = r^{-1}(\Phi^{k,r}(\lfloor r^2t\rfloor) - P^r_{k\cdot}\lfloor r^2t\rfloor). \tag{77}$$

Fluid scaling.

$$\overline{Z}^r(t) = r^{-2}Z^r(r^2t), \tag{78}$$
$$\overline{E}^r(t) = r^{-2}E^r(r^2t), \tag{79}$$
$$\overline{A}^r(t) = r^{-2}A^r(r^2t), \tag{80}$$
$$\overline{A}^{+,r}(t) = r^{-2}A^{+,r}(r^2t), \tag{81}$$
$$\overline{D}^r(t) = r^{-2}D^r(r^2t), \tag{82}$$
$$\overline{I}^r(t) = r^{-2}I^r(r^2t), \tag{83}$$
$$\overline{S}^r(t) = r^{-2}S^r(r^2t), \tag{84}$$
$$\overline{T}^r(t) = r^{-2}T^r(r^2t). \tag{85}$$

Finally in this section, we note the fundamental weak-convergence result that is based on the Donsker theorem for renewal processes (cf. Billingsley [2]) and the convergence of parameters (46)-(50):

$$\widehat{E}^r(\cdot) \Longrightarrow E^*(\cdot), \tag{86}$$
$$\widehat{\mathcal{V}}^{s,r}(\cdot) \Longrightarrow \mathcal{V}^*(\cdot) \tag{87}$$
$$\widehat{\Phi}^{k,r}(\cdot) \Longrightarrow \Phi^{k,*}(\cdot), \qquad k \in \mathbb{K}, \tag{88}$$
$$\widehat{S}^r(\cdot) \Longrightarrow S^*(\cdot), \tag{89}$$
$$\overline{S}^r_l(\cdot) \Longrightarrow \mu_l \iota(\cdot), \qquad l \in \mathbb{K}, \tag{90}$$

as $r \to \infty$, where

$$\begin{aligned}
&E^*(t) = \sqrt{\Pi} \cdot B^E(t), \\
&\mathcal{V}^*(t) = \sqrt{\Sigma} \cdot B^{\mathcal{V}}(t), \\
&\Phi^{k,*}(t) = (\Phi_1^{k,*}(t), \ldots, \Phi_K^{k,*}(t)), \\
&\Phi_l^{k,*}(t) = \sum_{m=1}^{K} \left(\sqrt{\Upsilon^k}\right)_{lm} \cdot B_m^k(t), \qquad k, l \in \mathbb{K}, \\
&\iota(t) \equiv t
\end{aligned}$$

with $B^E(\cdot)$ and $B^{\mathcal{V}}(\cdot)$ K-dimensional standard Brownian motions,

$$(B^1(\cdot), \ldots, B^K(\cdot)) = (B_1^1(\cdot), \ldots, B_K^1(\cdot), \ldots, B_1^K(\cdot), \ldots, B_K^K(\cdot))$$

a K^2-dimensional standard Brownian motion,

$$\begin{aligned}
\Pi &= diag(\alpha_1^3 a_1, \ldots, \alpha_K^3 a_K), \\
\Sigma &= diag(b_1, \ldots, b_K),
\end{aligned}$$

and Υ^k in (6) and (7) for each $k \in \mathbb{K}$. (These standard Brownian motions are mutually independent).

4 Main result; diffusion approximation theorem

To derive the diffusion approximation theorem for our multiclass feedforward queueing network with abandonment under the FCFS discipline, the following four main assumptions, i.e., (A.1)-(A.4), are imposed in addition to the conditions on primitive variables assumed so far:

(A.1) For some proper r.v. $W^*(0)$,

$$\widehat{W}^r(0) \Longrightarrow W^*(0) \quad \text{in} \quad \mathcal{R}^J$$

as $r \to \infty$.

(A.2) For each $k \in \mathbb{K}$,

$$\sup_{0 \le t \le W^r_{s(k)}(0)} r^{-1} |D_k^r(t) - \lambda_k^r t| \longrightarrow 0 \quad \text{in pr.}$$

as $r \to \infty$.

(A.3) The sequence $\{\widehat{Z}^r(0)\}_{r \ge 1}$ is tight in $\mathcal{R}^K$, i.e.,

$$\lim_{M \to \infty} \overline{\lim_{r \to \infty}} \mathbb{P}^r\left(|\widehat{Z}^r(0)| > M\right) = 0.$$

(A.4) (Assumption 7.1 in Williams [26]).

The matrix $R = (I + G)^{-1}$ is completely-S, where

$$G \equiv CMQ\widetilde{P}\Lambda = \lim_{r \to \infty} CM^r Q^r \widetilde{P}^r \Lambda^r$$

and $M^r \equiv diag(m_k^r, k \in \mathbb{K}), \Lambda^r \equiv diag(\lambda_k^r, k \in \mathbb{K}), r \ge 1$, and $M = \lim_{r \to \infty} M^r$, etc. (Of course, it is implicitly assumed that $I + G$ is invertible. For the definition of completely-S condition, see Definition 6.2 in Williams [26], for example).

Condition (A.2) corresponds to the initial condition on strong state-space collapse for a more general multiclass FCFS queueing network *without* abandonment. (Cf. Bramson [3], Williams [26]). While condition (A.3) is implied by (A.1) and (A.2) for such network *without* abandonment, we have to assume it in our network *with* abandonment. As established in [26], assumption (A.4) is satisfied under the asymptotically Kelly-type condition, i.e., $m_k = m_l$ if $s(k) = s(l)$. The completely-S condition on R in (A.4) is a necessary and sufficient condition for the existence and uniqueness (in law) of a semimartingale reflecting Brownian motion (SRBM) with the reflection matrix R and the data on the covariance, drift and initial measure of the Brownian motion in the SRBM. (Cf. Definition 6.1 in [26] and the references in its comment).

The following theorem is the main result in this paper. It is on the weak convergence for the sequence of scaled performance measure processes

$$\{(\widehat{W}^r(\cdot), \widehat{Y}^r(\cdot), \widehat{Z}^r(\cdot))\}_{r\geq 1}.$$

In the statement of the theorem, we use the following symbol:

$$\Gamma \equiv RC\Big\{\Lambda\Gamma_V + MQ\Big(\Gamma_E + \sum_{k=1}^{K}\lambda_k\Gamma_\Phi^k\Big)\tilde{Q}M\Big\}\tilde{C}\tilde{R}, \tag{91}$$

According to (54), we see that Γ is strictly positive definite. We also let

$$H^*(w) \equiv CMQ\Lambda \cdot H(w), \qquad w \in \mathcal{R}^J, \tag{92}$$

with $H(w) \equiv (H_k(w_{s(k)}), k \in \mathbb{K})$, $H_k(\cdot), k \in \mathbb{K}$, in (57).

Theorem 4.1. (*Diffusion approximation for a multiclass feedforward queueing network with abandonment under the FCFS discipline*).

Under the main assumptions (*A*.1), (*A*.2) *and* (*A*.3), *and also the conditions imposed on the primitive variables and processes so far, we have the weak convergence*

$$(\widehat{W}^r(\cdot), \widehat{Y}^r(\cdot), \widehat{Z}^r(\cdot)) \Longrightarrow (W^*(\cdot), Y^*(\cdot), Z^*(\cdot)) \quad in\ \mathbb{D}([0,\infty), \mathcal{R}^{2J+K}) \tag{93}$$

as $r \to \infty$, where $W^*(\cdot)$ is the unique solution to the following J-dimensional reflected stochastic differential equation:

$$W^*(t) = X^*(t) + RY^*(t), \tag{94}$$

$$X^*(t) = W^*(0) + \sqrt{\Gamma}B^*(t) + \vartheta^* t - \int_0^t H^*(W^*(u))du, \tag{95}$$

where $B^*(\cdot)$ is a J-dimensional standard Brownian motion, $\vartheta^* \equiv R\vartheta$ and $\nu(\cdot) = \mathbb{P}(W^*(0) \in \cdot)$. Furthermore,

$$Z^*(t) = \Lambda\tilde{C}W^*(t), \quad t \geq 0.$$

5 Proof of Theorem 4.1; propositions and lemmas

This section is devoted to the proof of the diffusion approximation theorem stated in the last section. We begin with the following stochastic boundedness of scaled queue length and workload in a multiclass feedforward queueing network with abandonment under any work-conserving service discipline.

5.1 Stochastic boundedness of diffusion-scaled queue length and workload In this subsection we present two propositions on the stochastic boundedness of diffusion-scaled queue length and workload in our multiclass feedforward queueing network with abandonment. Each of them plays a key role in the proof of our main theorem, specifically in proving the C-tightness of diffusion-scaled abandonment-count process and deriving state-space collapse in the network.

Proposition 5.1.
For a sequence of multiclass feedforward queueing networks with abandonments, $\{\mathfrak{X}^r\}_{r\geq 1}$, satisfying the assumptions stated so far, the sequence $\{\widehat{Z}^r(\cdot)\}_{r\geq 1}$ is stochastically bounded, i.e.,

$$\lim_{M\to\infty}\overline{\lim_{r\to\infty}}\,\mathbb{P}^r\Big(\sup_{0\leq t\leq T}|\widehat{Z}^r(t)|>M\Big)=0$$

for each $T>0$.

Proof.
Let

$$\widehat{f}^r(t)\equiv CM^rQ^r\widehat{Z}^r(t)=(\widehat{f}^r_j(t),j\in\mathbb{J})$$

where

$$\widehat{f}^r_j(t)=\widehat{f}^r_{j1}(t)+\widehat{f}^r_{j2}(t)$$

with

$$\widehat{f}^r_{j1}(t)=\sum_{k\in C(j)}m^r_k\sum_{l\in C(j)}Q^r_{kl}\widehat{Z}^r_l(t),$$
$$\widehat{f}^r_{j2}(t)=\sum_{k\in C(j)}m^r_k\sum_{l\in C(1)\cup\cdots\cup C(j-1)}Q^r_{kl}\widehat{Z}^r_l(t)$$

for each $j\in\mathbb{J}$, where we have used the feedforward class-routing condition (8). (We set $\widehat{f}^r_{12}(\cdot)\equiv 0$).

From

$$Z^r(t)=Z^r(0)+E^r(t)+\sum_{l=1}^{K}\Phi^{l,r}(S^r_l(T^r_l(t)))-S^r(T^r(t))-I^r(t)$$

with $S^r(T^r(t))\equiv(S^r_k(T^r_k(t)),k\in\mathbb{K})$, we have the following scaled identity in vector form:

$$\begin{aligned}\widehat{Z}^r(t)=&\widehat{Z}^r(0)+\widehat{E}^r(t)+\alpha^r rt+\sum_{l\in\mathbb{K}}\widehat{\Phi}^{l,r}(\overline{S}^r_l(\overline{T}^r_l(t)))-(\mathrm{I}-\widetilde{P^r})\widehat{S}^r(\overline{T}^r(t))\\&-(\mathrm{I}-\widetilde{P^r})\frac{(\mu^rT^r)(r^2t)}{r}-\widehat{I}^r(t)\end{aligned}\tag{96}$$

with the diffusion and fluid scalings given above. Multiplying (96) by CM^rQ^r from the left, we have

$$\begin{aligned}\widehat{f}^r(t)=\widehat{f}^r(0)+CM^rQ^r\Big\{\widehat{E}^r(t)+\sum_{l\in\mathbb{K}}\widehat{\Phi}^{l,r}(\overline{S}^r_l(\overline{T}^r_l(t)))\Big\}\\-CM^r\widehat{S}^r(\overline{T}^r(t))-CM^rQ^r\widehat{I}^r(t)+r(\rho^r-e)t+\widehat{Y}^r(t).\end{aligned}\tag{97}$$

Since

$$\int_0^\infty\widehat{f}^r_1(s)d\widehat{Y}^r_1(s)=\int_0^\infty\widehat{f}^r_{11}(s)d\widehat{Y}^r_1(s)=0,\tag{98}$$

from (97) we have

$$\widehat{f}_1^r(t)=\varphi\Big(\mathcal{X}_1^r(\cdot)-\sum_{k\in C(1)}m_k^r\sum_{l\in C(1)}Q_{kl}^r\widehat{I}_l^r(\cdot)\Big)(t) \tag{99}$$

where φ is the one-dimensional reflection map, i.e.,

$$\varphi(x(\cdot))(t)=x(t)+\sup_{0\le s\le t}\big(-x(s)\big)^+,\quad x\in\mathbb{D}([0,\infty),\mathcal{R}^1),\ t\ge 0, \tag{100}$$

and

$$\begin{aligned}\mathcal{X}_1^r(t)\equiv\widehat{f}_1^r(0)&+\sum_{k\in C(1)}m_k^r\sum_{l\in C(1)}Q_{kl}^r\big\{\widehat{E}_l^r(t)+\sum_{p\in\mathbb{K}}\widehat{\Phi}_l^{p,r}(\overline{S}_p^r(\overline{T}_p^r(t)))\big\}\\&-\sum_{k\in C(1)}m_k^r\widehat{S}_k^r(\overline{T}_k^r(t))+r(\rho_1^r-1)t,\qquad t\ge 0.\end{aligned} \tag{101}$$

Since each component in $\widehat{I}^r(\cdot)$ is nondecreasing, we have

$$\begin{aligned}&\widehat{f}_1^r(t)\\&=\mathcal{X}_1^r(t)-\sum_{k\in C(1)}m_k^r\sum_{l\in C(1)}Q_{kl}^r\widehat{I}_l^r(t)+\sup_{0\le s\le t}\Big(-\mathcal{X}_1^r(s)+\sum_{k\in C(1)}m_k^r\sum_{l\in C(1)}Q_{kl}^r\widehat{I}_l^r(s)\Big)^+\\&\le\mathcal{X}_1^r(t)+\sup_{0\le s\le t}(-\mathcal{X}_1^r(s))^+\\&=\varphi(\mathcal{X}_1^r(\cdot))(t).\end{aligned} \tag{102}$$

Thus, according to the Lipschitz continuity of the map φ, (A.3), the heavy-traffic condition (56), and the convergences (86)-(90), (47) and (50), we obtain

$$\lim_{M\to\infty}\overline{\lim_{r\to\infty}}\,\mathbb{P}^r\Big(\sup_{0\le t\le T}\widehat{Z}_k^r(t)>M\Big)=0 \tag{103}$$

for each $k\in C(1)$ and $T>0$.

Suppose that (103) holds for each $k\in C(1)\cup\cdots\cup C(j-1)$ with some $2\le j\le J$. Then, since

$$\int_0^\infty\widehat{f}_{j1}^r(s)\,d\widehat{Y}_j^r(s)=0,$$

we have

$$\widehat{f}_{j1}^r(t)=\varphi\Big(-\widehat{f}_{j2}^r(\cdot)+\mathcal{X}_j^r(\cdot)-\sum_{k\in C(j)}m_k^r\sum_{l\in C(1)\cup\cdots\cup C(j)}Q_{kl}^r\widehat{I}_l^r(\cdot)\Big)(t) \tag{104}$$

where

$$\begin{aligned}\mathcal{X}_j^r(t)\equiv\widehat{f}_j^r(0)&+\sum_{k\in C(j)}m_k^r\sum_{l\in C(1)\cup\cdots\cup C(j)}Q_{kl}^r\big\{\widehat{E}_l^r(t)+\sum_{p\in\mathbb{K}}\widehat{\Phi}_l^{p,r}(\overline{S}_p^r(\overline{T}_p^r(t)))\big\}\\&-\sum_{k\in C(j)}m_k^r\widehat{S}_k^r(\overline{T}_k^r(t))+r(\rho_j^r-1)t,\qquad t\ge 0.\end{aligned} \tag{105}$$

Thus, similar to the above reasoning, the inequality

$$\widehat{f}_{j1}^r(t)\le\varphi(-\widehat{f}_{j2}^r(\cdot)+\mathcal{X}_j^r(\cdot))(t) \tag{106}$$

holds so that

$$\lim_{M\to\infty}\overline{\lim_{r\to\infty}}\,\mathbb{P}^r\Big(\sup_{0\le t\le T}\widehat{Z}_k^r(t)>M\Big)=0, \tag{107}$$

is derived for each $k\in C(j)$ and $T>0$, using (103) for each $k\in C(1)\cup\cdots C(j-1)$. Consequently we have the desired result inductively.

□

Using Proposition 5.1, we also have the corresponding result for diffusion-scaled workload in the next proposition.

Proposition 5.2.

For $\{\mathfrak{X}^r\}_{r\ge1}$ in Proposition 5.1, the sequence $\{\widehat{W}^r(\cdot)\}_{r\ge1}$ is stochastically bounded, i.e.,

$$\lim_{M\to\infty}\overline{\lim_{r\to\infty}}\,\mathbb{P}^r\Big(\sup_{0\le t\le T}|\widehat{W}^r(t)|>M\Big)=0$$

for each $T>0$.

Proof.

¿From (21), (23), (67) and (68), we have

$$\widehat{W}^r(t)=\widehat{W}^r(0)+C\widehat{\mathcal{V}}^{s,r}(\overline{A}^{+,r}(t))+CM^r(\widehat{A}^r(t)-\widehat{A}^{-,r}(t))+r(\rho^r-e)t+\widehat{Y}^r(t) \tag{108}$$

with $\widehat{\mathcal{V}}^{s,r}(\cdot)$ in (70) and $\overline{A}^{+,r}(t)$ in (81).

¿From (65), (69), (71), (73), (74), (77) and (82), we see that

$$\begin{aligned}\widehat{A}^r(t)&=\widehat{E}^r(t)+\sum_{l\in\mathbb{K}}\widehat{\Phi}^{l,r}(\overline{D}_l^r(t))+\widetilde{P^r}\widehat{D}^r(t)\\&=\widehat{E}^r(t)+\sum_{l\in\mathbb{K}}\widehat{\Phi}^{l,r}(\overline{D}_l^r(t))+\widetilde{P^r}\big(\widehat{Z}^r(0)-\widehat{Z}^r(t)-\widehat{I}^r(t)+\widehat{A}^r(t)\big).\end{aligned}$$

Solving it for $\widehat{A}^r(t)$, we have

$$\widehat{A}^r(t)=Q^r\{\widehat{E}^r(t)+\sum_{l\in\mathbb{K}}\widehat{\Phi}^{l,r}(\overline{D}_l^r(t))+\widetilde{P^r}\big(\widehat{Z}^r(0)-\widehat{Z}^r(t)-\widehat{I}^r(t)\big)\}. \tag{109}$$

Substituting (109) into (108), we have

$$\begin{aligned}\widehat{W}^r(t)=\widehat{W}^r(0)&+C\widehat{\mathcal{V}}^{s,r}(\overline{A}^{+,r}(t))\\&+CM^rQ^r\big\{\widehat{E}^r(t)+\sum_{l\in\mathbb{K}}\widehat{\Phi}^{l,r}(\overline{D}_l^r(t))+\widetilde{P^r}(\widehat{Z}^r(0)-\widehat{Z}^r(t))\big\}\\&+r(\rho^r-e)t-CM^r\widehat{A}^{-,r}(t)-CM^rQ^r\widetilde{P^r}\widehat{I}^r(t)+\widehat{Y}^r(t).\end{aligned}$$

Let

$$\begin{aligned}\mathcal{Y}^r(t)\equiv\widehat{W}^r(0)&+C\widehat{\mathcal{V}}^{s,r}(\overline{A}^{+,r}(t))\\&+CM^rQ^r\big\{\widehat{E}^r(t)+\sum_{l\in\mathbb{K}}\widehat{\Phi}^{l,r}(\overline{D}_l^r(t))+\widetilde{P^r}(\widehat{Z}^r(0)-\widehat{Z}^r(t))\big\}\\&+r(\rho^r-e)t.\end{aligned}$$

Then, since

$$\int_0^\infty \widehat{W}_j^r(s)d\widehat{Y}_j^r(s) = 0, \qquad \forall j \in \mathbb{J}, \tag{110}$$

we have that for each $j \in \mathbb{J}$,

$$\begin{aligned}
\widehat{W}_j^r(t) &= \varphi\Big(\mathcal{Y}_j^r(\cdot) - \sum_{k\in C(j)} m_k^r \widehat{A}_k^{-,r}(\cdot) - \sum_{k\in C(j)} m_k^r \sum_{l\in\mathbb{K}} (Q^r\widetilde{P^r})_{kl}\widehat{I}_l^r(\cdot)\Big)(t)\\
&= \mathcal{Y}_j^r(t) - \sum_{k\in C(j)} m_k^r \widehat{A}_k^{-,r}(t) - \sum_{k\in C(j)} m_k^r \sum_{l\in\mathbb{K}} (Q^r\widetilde{P^r})_{kl}\widehat{G}_l^r(t)\\
&\quad + \sup_{0\le s\le t}\Big(-\mathcal{Y}_j^r(s) + \sum_{k\in C(j)} m_k^r \widehat{A}_k^{-,r}(s) + \sum_{k\in C(j)} m_k^r \sum_{l\in\mathbb{K}} (Q^r\widetilde{P^r})_{kl}\widehat{I}_l^r(s)\Big)^+\\
&\le \mathcal{Y}_j^r(t) + \sup_{0\le s\le t}\Big(-\mathcal{Y}_j^r(s)\Big)^+\\
&= \varphi\Big(\mathcal{Y}_j^r(\cdot)\Big)(t)
\end{aligned}$$

with $\varphi(\cdot)$ in (100), where the inequality follows from the non-decreasing property of each component in $\widehat{A}^{-,r}(\cdot)$ and $\widehat{I}^r(\cdot)$. Thus, using the Lipschitz continuity of φ, Proposition 5.1, and (A.1), we have the desired result.

□

Remark 5.1.

The conclusions of Propositions 5.1 and 5.2 are valid under any work-conserving (or non-idling) service discipline, which is embodied as (98) and (110). We note that if the stochastic boundedness condition on scaled queue length is verified for a more general multi-class queueing network, then that condition on scaled workload does hold for such network, which will be seen by mimicking the proof of Proposition 5.2.

5.2 C-tightness of diffusion-scaled abandonment-count process In this subsection, we show the C-tightness of the sequence of scaled abandonment-count processes $\{\widehat{I}_k^r(\cdot)\}_{r\ge1}, k \in \mathbb{K}$, which will be seen to follow from that of the sequence $\{\widehat{N}_k^r(\cdot)\}_{r\ge1}, k \in \mathbb{K}$, as follows.

Proposition 5.3.

For each $k \in \mathbb{K}$, the sequence $\{\widehat{I}_k^r(\cdot)\}_r$ is C-tight in $\mathbb{D}([0,\infty),\mathcal{R}^1)$.

Proof.

Similar to (29), let

$$\zeta_j^r(t) \equiv \inf\{s \ge 0 : s + W_j^r(s) > t\}, \quad t \ge 0, j \in \mathbb{J},$$

and $\overline{\zeta}_j^r(t) \equiv r^{-2}\zeta_j^r(r^2t)$. Then we have that for each $T > 0$ and $j \in \mathbb{J}$,

$$\sup_{0\le t\le T} |\overline{\zeta}_j^r(t) - t| \longrightarrow 0 \quad \text{in pr.} \tag{111}$$

as $r \to \infty$, which follows from the inequalities

$$\zeta_j^r(t) + W_j^r(\zeta_j^r(t)) \ge t \quad \text{and} \quad \zeta_j^r(t) \le t,$$

and Proposition 5.2.

¿From (28), the inequality

$$\widehat{N}_k^r(\overline{\zeta}_{s(k)}^r(t) - \frac{1}{r^3}) \leq \widehat{I}_k^r(t) \leq \widehat{N}_k^r(t) \tag{112}$$

follows, so that according to (111), the proof of C-tightness for $\{\widehat{I}_k^r(\cdot)\}_r, k \in \mathbb{K}$, is reduced to that for $\{\widehat{N}_k^r(\cdot)\}_r, k \in \mathbb{K}$, which is done in the next lemma.

□

Lemma 5.1.

For each $k \in \mathbb{K}$, the sequence $\{\widehat{N}_k^r(\cdot)\}_r$ is C-tight in $\mathbb{D}([0,\infty),\mathcal{R}^1)$.

Proof.

Assumptions (A.1) and (A.2) yield that for each $k \in \mathbb{K}$,

$$\begin{aligned}\widehat{Z}_k^{+,r}(0) &= \frac{1}{r} D_k^r(W_{s(k)}^r(0)) \\ &\Longrightarrow \lambda_k W_{s(k)}^*(0)\end{aligned}$$

as r goes to infinity, so that the tightness of $\{\widehat{Z}_k^{-,r}(0)\}_r$ follows from (A.3). Thus we are left to show the C-tightness of $\{\widehat{A}_k^{-,r}(\cdot)\}_r$, because of the identity

$$\widehat{N}_k^r(t) = \widehat{Z}_k^{-,r}(0) + \widehat{A}_k^{-,r}(t), t \geq 0.$$

¿From (27) and (75), it follows that

$$\begin{aligned}\widehat{A}_k^{-,r}(t) &= \frac{1}{r} \sum_{i=1}^{A_k^r(r^2t)} \big(1_{\{\gamma_k^{s,r}(i) \leq w_k^{s,r}(i)\}} - F_k^{\gamma,r}(w_k^{s,r}(i))\big) + \frac{1}{r} \sum_{i=1}^{A_k^r(r^2t)} F_k^{\gamma,r}(w_k^{s,r}(i)) \\ &= \widehat{\mathcal{M}}_k^{\gamma,r}(\overline{A}_k^r(t)) + \widehat{\mathcal{C}}_k^r(\overline{A}_k^r(t))\end{aligned} \tag{113}$$

where

$$\widehat{\mathcal{M}}_k^{\gamma,r}(t) \equiv \frac{1}{r} \sum_{i=1}^{\lfloor r^2 t \rfloor} \big(1_{\{\gamma_k^{s,r}(i) \leq w_k^{s,r}(i)\}} - F_k^{\gamma,r}(w_k^{s,r}(i))\big), \tag{114}$$

$$\widehat{\mathcal{C}}_k^r(t) \equiv \frac{1}{r} \sum_{i=1}^{\lfloor r^2 t \rfloor} F_k^{\gamma,r}(w_k^{s,r}(i)). \tag{115}$$

Observe that $\widehat{\mathcal{M}}_k^{\gamma,r}(\cdot)$ is a purely-discontinuous martingale since $w_k^{s,r}(i)$ is $\mathcal{G}_k^{s,r}(i)$-measurable and $\gamma_k^{s,r}(i)$ is independent of $\mathcal{G}_k^{s,r}(i)$ for each $i = 1, 2, \cdots$, where $\mathcal{G}_k^{s,r}(i)$ is given in the form (12). Then its optional quadratic variation process $\big[\widehat{\mathcal{M}}_k^{\gamma,r}\big](\cdot)$ is given by

$$\begin{aligned}[\widehat{\mathcal{M}}_k^{\gamma,r}](t) &= \sum_{0<s\leq t} |\Delta \widehat{\mathcal{M}}_k^{\gamma,r}(s)|^2 \\ &= \frac{1}{r^2} \sum_{i=1}^{\lfloor r^2 t \rfloor} \big(1_{\{\gamma_k^{s,r}(i) \leq w_k^{s,r}(i)\}} - F_k^{\gamma,r}(w_k^{s,r}(i))\big)^2.\end{aligned} \tag{116}$$

(Cf. (1.8.3) of Liptser and Shiryayev [20]).

We now show that

$$\widehat{\mathcal{M}}_k^{\gamma,r}(\cdot) \Longrightarrow 0 \quad \text{in} \quad \mathbb{D}([0,\infty),\mathcal{R}^1), \tag{117}$$

as $r \to \infty$ in a similar way to the proof of Lemma 4.3 in Dai and He [7] as follows. Observe that for each $t \geq 0$,

$$\begin{aligned}\mathbb{E}^r([\widehat{\mathcal{M}}_k^{\gamma,r}](t)) &= \frac{1}{r^2}\sum_{i=1}^{\lfloor r^2 t\rfloor} \mathbb{E}^r(F_k^{\gamma,r}(w_k^{s,r}(i)) - F_k^{\gamma,r}(w_k^{s,r}(i))^2) \\ &\leq t\,\mathbb{E}^r(\sup_{1\leq i\leq \lfloor r^2 t\rfloor} F_k^{\gamma,r}(w_k^{s,r}(i)))\end{aligned}$$

where the equality follows from the $\mathcal{G}_k^{s,r}(i)$-measurability of $w_k^{s,r}(i)$ and the independence of $\gamma_k^{s,r}(i)$ and $\mathcal{G}_k^{s,r}(i)$.
Since $A_k^r(s) \geq E_k^r(s)$ for each $s \geq 0$ and $\overline{E}_k^r(\cdot) \Longrightarrow \alpha_k\iota(\cdot)$ as $r \to \infty$, we can take an appropriate constant $t^* > 0$ such that

$$\lim_{r\to\infty} \mathbb{P}^r(A_k^r(r^2 t^*) \leq \lfloor r^2 t\rfloor) = 0. \tag{118}$$

Thus we have

$$\begin{aligned}&\overline{\lim_{r\to\infty}}\, \mathbb{E}^r\big[\sup_{1\leq i\leq\lfloor r^2t\rfloor} F_k^{\gamma,r}(w_k^{s,r}(i))\big] \\ &\leq \overline{\lim_{r\to\infty}}\, \mathbb{E}^r\big[\sup_{1\leq i\leq\lfloor r^2t\rfloor} F_k^{\gamma,r}(w_k^{s,r}(i)); A_k^r(r^2t^*) > \lfloor r^2 t\rfloor\big] \\ &\leq \overline{\lim_{r\to\infty}}\, \mathbb{E}^r\big[\sup_{1\leq i\leq A_k^r(r^2t^*)} F_k^{\gamma,r}(w_k^{s,r}(i))\big] \\ &\leq \overline{\lim_{r\to\infty}}\, \mathbb{E}^r\big[F_k^{\gamma,r}(\sup_{0\leq u\leq t^*} W_{s(k)}^r(r^2u))\big] \\ &\leq \overline{\lim_{r\to\infty}}\, \mathbb{E}^r\big[F_k^{\gamma,r}(\sup_{0\leq u\leq t^*} W_{s(k)}^r(r^2u)); \sup_{0\leq u\leq t^*} \widehat{W}_{s(k)}^r(u) \leq M\big] \\ &\quad + \overline{\lim_{r\to\infty}}\, \mathbb{P}^r\big(\sup_{0\leq u\leq t^*} \widehat{W}_{s(k)}^r(u) > M\big).\end{aligned} \tag{119}$$

According to Proposition 5.2, $\lim_{M\to\infty}$ (the second term in (119)) $= 0$, while the first term in (119) is majorized by

$$\begin{aligned}\lim_{r\to\infty} F_k^{\gamma,r}(rM) &= \lim_{r\to\infty}\frac{1}{r}\cdot rF_k^{\gamma,r}(rM) \\ &= 0\end{aligned}$$

for each fixed $M > 0$, because of (57). Therefore we have that for each $t \geq 0$,

$$\lim_{r\to\infty} \mathbb{E}^r([\widehat{\mathcal{M}}_k^{\gamma,r}](t)) = 0$$

so that the convergence (117) is established, according to Theorem 7.1.4 in Ethier and Kurtz [11].

Let $B_k^r(t) \equiv E_k^r(t) + \sum_{l=1}^K S_l^r(t), t \geq 0$. Then, since $A_k^r(t) \leq B_k^r(t)$ for each $t \geq 0$ and

$$\overline{B}_k^r(\cdot) \equiv r^{-2}B_k^r(r^2\cdot) \Longrightarrow \alpha_k\iota(\cdot) + \sum_{l=1}^K \mu_l\iota(\cdot) \tag{120}$$

as $r \to \infty$, we have

$$\widehat{\mathcal{M}}_k^{\gamma,r}(\overline{A}_k^r(\cdot)) \Longrightarrow 0 \quad \text{in} \quad \mathbb{D}([0,\infty),\mathcal{R}^1), \tag{121}$$

as $r \to \infty$.

Thus the proof of the C-tightness of $\{\widehat{A}_k^{-,r}(\cdot)\}_r$ is reduced to that of the C-tightness of $\{\widehat{\mathcal{C}}_k^r(\overline{A}_k^r(\cdot))\}_r$, and so it is enough to show the following two conditions:

$$\lim_{M\to\infty} \overline{\lim_{r\to\infty}}\, \mathbb{P}^r(\widehat{\mathcal{C}}_k^r(\overline{A}_k^r(T)) > M) = 0 \tag{122}$$

for each $T > 0$, and

$$\lim_{\delta\to 0} \overline{\lim_{r\to\infty}}\, \mathbb{P}^r(w_T(\widehat{\mathcal{C}}_k^r(\overline{A}_k^r(\cdot)),\delta) > \varepsilon) = 0 \tag{123}$$

for each $\varepsilon > 0$ and $T > 0$, where

$$w_T(x(\cdot),\delta) \equiv \sup_{\substack{0\le s,t\le T\\ |s-t|\le\delta}} |x(s)-x(t)|, \quad x(\cdot) \in \mathbb{D}([0,\infty),\mathcal{R}^d), \quad \delta>0, T>0, d\in\mathbb{N}. \tag{124}$$

(Cf. Proposition 6.3.26 in Jacod and Shiryaev [14]).

Observe that

$$\begin{aligned}\mathbb{P}^r(\widehat{\mathcal{C}}_k^r(\overline{A}_k^r(T)) > M) \le& \mathbb{P}^r(\widehat{\mathcal{C}}_k^r(\overline{A}_k^r(T)) > M, \sup_{0\le t\le T} \widehat{W}^r(t) \le L)\\ &+ \mathbb{P}^r(\sup_{0\le t\le T} \widehat{W}^r(t) > L).\end{aligned} \tag{125}$$

Then, $\lim_{L\to\infty} \overline{\lim}_{r\to\infty}$(the second term in (125))$= 0$ according to Proposition 5.2, and the first term in (125) is majorized by

$$\mathbb{P}^r(\overline{A}_k^r(T)\cdot rF_k^{\gamma,r}(rL) > M) \le \mathbb{P}^r(\overline{B}_k^r(T)\cdot rF_k^{\gamma,r}(rL) > M)$$

so that $\lim_{M\to\infty} \overline{\lim}_{r\to\infty}$(the first term in (125))$= 0$ for each fixed $L > 0$, according to (57) and (120). Thus we have (122).

Furthermore, observe that

$$\begin{aligned}&\mathbb{P}^r(w_T(\widehat{\mathcal{C}}_k^r(\overline{A}_k^r(\cdot)),\delta) > \varepsilon)\\ &\le \mathbb{P}^r(\sup_{\substack{0\le s,t\le T\\ |s-t|\le\delta}} |\widehat{\mathcal{C}}_k^r(\overline{A}_k^r(s)) - \widehat{\mathcal{C}}_k^r(\overline{A}_k^r(t))| > \varepsilon, \sup_{0\le t\le T} \widehat{W}^r(t) \le L)\\ &\quad + \mathbb{P}^r(\sup_{0\le t\le T} \widehat{W}^r(t) > L).\end{aligned} \tag{126}$$

Then, the same as above, $\lim_{L\to\infty} \overline{\lim}_{r\to\infty}$(the second term in (126))$= 0$, and the first term in (126) is less than or equal to

$$\mathbb{P}^r\Big(rF_k^{\gamma,r}(rL) \times w_T(\overline{A}_k^r(\cdot),\delta) > \varepsilon\Big).$$

Therefore, noting

$$w_T(\overline{A}_k^r(\cdot),\delta) \le w_T(\overline{E}_k^r(\cdot),\delta) + \sum_{l\in\mathbb{K}} w_T(\overline{S}_l^r(\cdot),\delta),$$

(123) is seen to be satisfied, according to (57).

Consequently we have the C-tightness of $\{\widehat{A}_k^{-,r}(\cdot)\}_r$ and so the conclusion of the lemma has been proved.

□

5.3 State-space collapse in multiclass feedforward queueing networks with abandonments under FCFS service disciplines In this subsection, under the assumption (A.2), we prove the following proposition on multiplicative strong state-space collapse and state-space collapse in a multiclass feedforward queueing network with abandonment under the FCFS service discipline.

Proposition 5.4. (*Multiplicative strong state-space collapse and state-space collapse*).

Suppose that in addition to the assumptions in Sect. 3*, conditions* (*A*.1), (*A*.2) *and* (*A*.3) *hold. Then we have the following convergences*:
For each $k \in \mathbb{K}$ *and* $T > 0$*,*

$$\frac{\sup_{0\le t\le T}\sup_{0\le s\le \widehat{W}^r_{s(k)}(t)} \mid r^{-1}D^r_k(r^2t+rs) - r^{-1}D^r_k(r^2t) - \lambda^r_k s \mid}{\sup_{0\le t\le T}\widehat{W}^r_{s(k)}(t)\vee 1} \longrightarrow 0 \quad \textit{in pr.} \tag{127}$$

as $r\to\infty$*, and also,*

$$\sup_{0\le t\le T}\left|\widehat{Z}^r_k(t) - \lambda^r_k\widehat{W}^r_{s(k)}(t)\right| \longrightarrow 0 \quad \textit{in pr.} \tag{128}$$

as $r\to\infty$.

To demonstrate the proposition, we need to modify slightly the proof of Theorem 1 in Bramson [3] by incorporating the customer abandonment to it. Specifically, to the statement of Proposition 5.1 in [3], we have to add the identity on the weak law of large numbers for $I^{r,m}(\cdot)$ that is defined in the same way as in [3] as follows.

For the performance measure process $\mathfrak{X}^r(\cdot), r\ge 1$, in (14), let

$$\mathfrak{X}^{r,m}(t) \equiv \Big\{\frac{1}{x_{r,0}}\mathfrak{X}^r(x_{r,0}t)\Big\}\circ\theta_{rm} \tag{129}$$

for $m=0,1,2,\ldots$, where $x_{r,0}\equiv |W^r(0)|\vee|Z^r(0)|\vee r$ and $\{\theta_t, t\ge 0\}$ is the shift transformation associated with Markov description process $\Xi^r(\cdot)$. For example, using Proposition 2.1, we have

$$Z^{r,m}(t) = \frac{1}{x_{r,m}}Z^r(x_{r,m}t+rm),$$
$$I^{r,m}(t) = \frac{1}{x_{r,m}}(I^r(x_{r,m}t+rm) - I^r(rm)),$$

where $x_{r,m}\equiv x_{r,0}\circ\theta_{rm} = |W^r(rm)|\vee|Z^r(rm)|\vee r$ for $m=0,1,2,\ldots$.

Proposition 5.5. (*Weak law of large numbers for* $I^{r,m}(\cdot)$).

For each $\varepsilon>0, T>0, L>0$ *and* $k\in\mathbb{K}$*,*

$$\lim_{r\to\infty}\mathbb{P}^r\Big(\max_{0\le m<rT} I^{r,m}_k(L)>\varepsilon\Big)=0.$$

Since $I^r_k(t)\le N^r_k(t)$ for each $t\ge 0$, the above proposition is a consequence of the following proposition.

Proposition 5.6.

For each $\varepsilon > 0, T > 0, L > 0$ and $k \in \mathbb{K}$,

$$\lim_{r\to\infty} \mathbb{P}^r\big(\max_{0\le m<rT} N_k^{r,m}(L) > \varepsilon\big) = 0,$$

where $N_k^{r,m}(\cdot)$ is defined as in (129).

Before giving the proof of Proposition 5.6, we define the following variables which correspond to (5.25) in Bramson [3]:

$$u_k^{max,T,r} \equiv \max\{u_k^r(i) : \mathcal{U}_k^r(i-1) \le r^2T, i = 1,2,\ldots\}, \tag{130}$$

$$\begin{aligned} v_l^{max,T,r} \equiv& \max\{v_l^{o,r}(i) : \mathcal{V}_l^{o,r}(i-1) \le r^2T, i = 1,2,\ldots, Z_l^{+,r}(0)\} \\ &\vee \max\{v_l^{s,r}(i) : \mathcal{V}_l^{o,r}(Z_l^{+,r}(0)) + \mathcal{V}_l^{s,r}(i-1) \le r^2T, i = 1,2,\ldots\} \end{aligned} \tag{131}$$

with $\max\phi \equiv 0$, for each $k \in \mathbb{A}$, $l \in \mathbb{K}$ and $T > 0$. Then we have the inequalities

$$u_k^r(1) \circ \theta_{rm} \le u_k^{max,T,r}, \tag{132}$$

$$v_l^{o,r}(1) \circ \theta_{rm} \le v_l^{max,T,r} \tag{133}$$

for each $m = 0, 1, \cdots, \lceil rT\rceil - 1$, $T > 0$, $k \in \mathbb{A}$ and $l \in \mathbb{K}$. Indeed, for each m,

$$\begin{aligned} \mathcal{U}_k^r(E_k^r(rm)) \le rm &< \mathcal{U}_k^r(E_k^r(rm)+1), \\ u_k^r(1)\circ\theta_{rm} &= \mathcal{R}_k^{u,r}(0)\circ\theta_{rm} \\ &= \mathcal{R}_k^{u,r}(rm) \\ &= \mathcal{U}_k^r(E_k^r(rm)+1) - rm, \end{aligned}$$

from which the inequality (132) follows.

The next lemma corresponds to Lemma 5.1 in [3].

Lemma 5.2.

For each $k \in \mathbb{A}$, $l \in \mathbb{K}$ and $T > 0$,

$$\frac{1}{r}u_k^{max,T,r} \longrightarrow 0 \qquad \text{in pr.,} \tag{134}$$

$$\frac{1}{r}v_l^{max,T,r} \longrightarrow 0 \qquad \text{in pr.,} \tag{135}$$

as r goes to infinity.

Proof.

We have only to prove the latter convergence (135), because the derivation of the former (134) is the same as in Lemma 5.1 in [3]. First we observe that for each δ and B_1 with $0 < \delta < B_1$,

$$\begin{aligned} &\mathbb{P}^r\big(\frac{1}{r}v_l^{max,T,r} > \varepsilon\big) \\ &\le \mathbb{P}^r\Big(\frac{1}{r}v_l^{max,T,r} > \varepsilon, \mathcal{V}_l^{o,r}(Z_l^{+,r}(0)) + \mathcal{V}_l^{s,r}(\lfloor r^2B_1\rfloor) > r^2T, Z_l^{+,r}(0) < r^2\delta\Big) \\ &\quad + \mathbb{P}^r\Big(\mathcal{V}_l^{o,r}(Z_l^{+,r}(0)) + \mathcal{V}_l^{s,r}(\lfloor r^2B_1\rfloor) \le r^2T, Z_l^{+,r}(0) < r^2\delta\Big) \\ &\quad + 2\mathbb{P}^r\big(Z_l^{+,r}(0) \ge r^2\delta\big). \end{aligned} \tag{136}$$

The second term in (136) tends to zero as r goes to infinity, since

$$\mathbb{P}^r\big(\mathcal{V}_l^{o,r}(\lfloor r^2\delta\rfloor) + \mathcal{V}_l^{s,r}(\lfloor r^2 B_1\rfloor) \le r^2 T\big) \longrightarrow 0 \tag{137}$$

as r tends to infinity for an appropriate constant $B_1 > 0$, according to the weak law of large numbers. We also have

$$\mathbb{P}^r\big(Z_l^{+,r}(0) \ge r^2\delta\big) \longrightarrow 0 \tag{138}$$

as r tends to infinity, according to assumption (A.3).

Furthermore, the first term in (136) is majorized by

$$\begin{aligned}
&\mathbb{P}^r\Big(\frac{1}{r} \times \max_{1\le i\le \lfloor r^2\delta\rfloor} v_l^{o,r}(i) \vee \max_{1\le i\le \lfloor r^2 B_1\rfloor} v_l^{s,r}(i) > \varepsilon\Big)\\
&\le \mathbb{P}^r\Big(\frac{1}{r} v_l^{o,r}(1) > \varepsilon\Big) + (\lfloor r^2\delta\rfloor + \lfloor r^2 B_1\rfloor)\cdot \frac{1}{(r\varepsilon)^2}\eta(r\varepsilon)\\
&\longrightarrow 0
\end{aligned} \tag{139}$$

as r goes to infinity, where

$$\eta(R) \equiv \sup_r \mathbb{E}^r\big[v_l^{o,r}(2)^2; v_l^{o,r}(2) > R\big], \quad R > 0,$$

and the convergence to zero follows from assumptions (61) and (59). So the proof is completed.

□

Proof of Proposition 5.6.

First we observe that according to (25) and Proposition 2.1,

$$\begin{aligned}
N_k^{r,m}(t) &= \Big\{\frac{1}{x_{r,0}} Z_k^{-,r}(0)\Big\} \circ \theta_{rm} + \Big\{\frac{1}{x_{r0}} A_k^{-,r}(x_{r,0}t)\Big\} \circ \theta_{rm}.\\
&\le \frac{1}{r} Z_k^{-,r}(rm) + \Big\{\frac{1}{x_{r,0}} A_k^{-,r}(x_{r,0}t)\Big\} \circ \theta_{rm},
\end{aligned} \tag{140}$$

and also that

$$\max_{0\le m< rT} \frac{1}{r} Z_k^{-,r}(rm) \le \sup_{0\le t\le T} \widehat{Z}_k^{-,r}(t). \tag{141}$$

Using the inequality (30), we see that for each $t \ge 0$,

$$\widehat{Z}_k^{-,r}(t) \le \widehat{I}_k^r(t + r^{-1}\widehat{W}_{s(k)}^r(t)) - \widehat{I}_k^r(t).$$

Thus, using Propositions 5.2 and 5.3, we have

$$\widehat{Z}_k^{-,r}(\cdot) \Longrightarrow 0 \tag{142}$$

as $r \to \infty$, which yields

$$\lim_{r\to\infty} \mathbb{P}^r\Big(\max_{0\le m< rT} \frac{1}{r} Z_k^{-,r}(rm) > \frac{\varepsilon}{2}\Big) = 0,$$

according to (141). So, in virtue of (140), it suffices to show that for each $k \in \mathbb{K}$,

$$\lim_{r\to\infty} \mathbb{P}^r\Big(\max_{0\le m<rT}\big\{\frac{1}{x_{r,0}}A_k^{-,r}(x_{r,0}L)\big\}\circ\theta_{rm} > \frac{\varepsilon}{2}\Big) = 0 \tag{143}$$

in order to obtain the conclusion of the lemma.

Now we have that for each $\delta > 0$ and $M > 0$,

$$\begin{aligned}
&\mathbb{P}^r\Big(\max_{0\le m<rT}\big\{\frac{1}{x_{r,0}}A_k^{-,r}(x_{r,0}L)\big\}\circ\theta_{rm} > \frac{\varepsilon}{2}\Big)\\
&\le \mathbb{P}^r\Big(\max_{0\le m<rT}\big\{\frac{1}{x_{r,0}}A_k^{-,r}(x_{r,0}L)\big\}\circ\theta_{rm} > \frac{\varepsilon}{2}, \frac{|u^{max,T,r}|}{r}\le\delta,\\
&\quad \max_{p\in\mathbb{K}}\max_{0\le m<rT}\Big|\widehat{\mathcal{V}}_p^{o,r}(\overline{Z}_p^{+,r}(0))\Big|\circ\theta_{rm}\le\delta, \sup_{0\le t\le T+L}|\widehat{W}^r(t)|\le M, \sup_{0\le t\le T}|\widehat{Z}^r(t)|\le M\Big)\\
&\quad + \mathbb{P}^r\Big(\frac{|u^{max,T,r}|}{r} > \delta\Big) + \mathbb{P}^r\Big(\max_{p\in\mathbb{K}}\max_{0\le m<rT}\Big|\widehat{\mathcal{V}}_p^{o,r}(\overline{Z}_p^{+,r}(0))\Big|\circ\theta_{rm} > \delta\Big)\\
&\quad + \mathbb{P}^r\Big(\sup_{0\le t\le T+L}|\widehat{W}^r(t)| > M\Big) + \mathbb{P}^r\Big(\sup_{0\le t\le T}|\widehat{Z}^r(t)| > M\Big)
\end{aligned} \tag{144}$$

where $\widehat{\mathcal{V}}_p^{o,r}(\cdot), p \in \mathbb{K}$, and $\overline{Z}_p^{+,r}(\cdot), p \in \mathbb{K}$, are given in (63) and (64), respectively. According to Lemma 5.2,

$$\lim_{r\to\infty}(\text{the second term in (144)}) = 0,$$

and according to assumption (62),

$$\lim_{r\to\infty}(\text{the third term in (144)}) = 0.$$

Further, according to Propositions 5.1 and 5.2,

$$\lim_{M\to\infty}\overline{\lim_{r\to\infty}}(\text{the fourth term in (144)}) = 0$$

and

$$\lim_{M\to\infty}\overline{\lim_{r\to\infty}}(\text{the fifth term in (144)}) = 0.$$

Observe that in addition to (132) and (133),

$$|\widehat{Z}^r(0)|\circ\theta_{rm} \le \sup_{0\le t\le T}|\widehat{Z}^r(t)|, \tag{145}$$

$$\sup_{0\le t\le L}|\widehat{W}^r(t)|\circ\theta_{rm} \le \sup_{0\le t\le T+L}|\widehat{W}^r(t)|, \tag{146}$$

for each $0 \le m < rT$. Then, in use of the Markov property of $\Xi^r(\cdot)$, we see that

$$\begin{aligned}
&\text{(the first term in (144))} \\
&\le \mathbb{P}^r\Big(\bigcup_{0\le m<rT} \Big\{ \{ \frac{1}{x_{r,0}} A_k^{-,r}(x_{r,0}L) \} \circ \theta_{rm} > \frac{\varepsilon}{2}, \frac{|u^{max,T,r}|}{r} \le \delta, \\
&\qquad \max_{p\in\mathbb{K}} \Big| \widehat{\mathcal{V}}_p^{o,r}(\overline{Z}_p^{+,r}(0)) \Big| \circ \theta_{rm} \le \delta, \sup_{0\le t\le L} |\widehat{W}^r(t)| \circ \theta_{rm} \le M, |\widehat{Z}^r(0)| \circ \theta_{rm} \le M \Big\} \Big) \\
&\le \sum_{0\le m<rT} \mathbb{E}^r \big[\mathbb{P}^r(\cdots\cdots\cdots\cdots \mid \mathcal{F}^r_{rm}) \big] \\
&= \sum_{0\le m<rT} \mathbb{E}^r \Big[\mathsf{P}^r_{\Xi^r(rm)} \Big(\frac{1}{x_{r,0}} A_k^{-,r}(x_{r,0}L) > \frac{\varepsilon}{2}, \frac{|u^r(1)|}{r} \le \delta, \\
&\qquad\qquad \max_{p\in\mathbb{K}} \Big| \widehat{\mathcal{V}}_p^{o,r}(\overline{Z}_p^{+,r}(0)) \Big| \le \delta, \sup_{0\le t\le L} |\widehat{W}^r(t)| \le M, |\widehat{Z}^r(0)| \le M \Big) \Big]
\end{aligned}$$

Thus, in order to show (143), it is enough to prove that for each $\varepsilon > 0$,

$$\begin{aligned}
&\mathsf{P}^r_*\Big(\frac{1}{x_{r,0}} A_k^{-,r}(x_{r,0}L) > \frac{\varepsilon}{2}, \frac{|u^r(1)|}{r} \le \delta, \max_{p\in\mathbb{K}} \Big| \widehat{\mathcal{V}}_p^{o,r}(\overline{Z}_p^{+,r}(0)) \Big| \le \delta, \\
&\quad \sup_{0\le t\le L} |\widehat{W}^r(t)| \le M, |\widehat{Z}^r(0)| \le M \Big) < \frac{\varepsilon}{r}
\end{aligned} \tag{147}$$

if r is sufficiently large independently of the initial value $*$.

Observe that

$$\frac{1}{x_{r,0}} A_k^{-,r}(x_{r,0}L) = \frac{1}{x_{r,0}} \sum_{i=1}^{A_k^r(x_{r,0}L)} 1_{\{\gamma_k^{s,r}(i) \le w_k^{s,r}(i)\}}$$

and if $\sup_{0\le t\le L} |\widehat{W}^r(t)| \le M$, $|\widehat{Z}^r(0)| \le M$ and $r > M > 1$, then

$$\begin{aligned}
w_k^{s,r}(i) &\le \sup_{0\le t\le x_{r,0}L} W^r_{s(k)}(t) \le \sup_{0\le t\le rML} W^r_{s(k)}(t) \le r \sup_{0\le t\le L} |\widehat{W}^r(t)| \\
&\le rM
\end{aligned}$$

for each $i = 1, 2, \cdots, A_k^r(x_{r,0}L)$. Thus, if $r > M > 1$, then the left-hand side of (147) is dominated by

$$\begin{aligned}
&\mathsf{P}^r_*\Big(\frac{1}{x_{r,0}} \sum_{i=1}^{A_k^r(x_{r,0}L)} 1_{\{\gamma_k^{s,r}(i) \le rM\}} > \frac{\varepsilon}{2}, \max_{p\in\mathbb{K}} \Big| \widehat{\mathcal{V}}_p^{o,r}(\overline{Z}_p^{+,r}(0)) \Big| \le \delta, \frac{|u^r(1)|}{r} \le \delta \Big) \\
&\le \mathsf{P}^r_*\Big(\frac{1}{x_{r,0}} \sum_{i=1}^{\lfloor c x_{r,0} \rfloor} 1_{\{\gamma_k^{s,r}(i) \le rM\}} > \frac{\varepsilon}{2} \Big) \\
&\quad + \mathsf{P}^r_*\Big(A_k^r(x_{r,0}L) > c x_{r,0}, \max_{p\in\mathbb{K}} \Big| \widehat{\mathcal{V}}_p^{o,r}(\overline{Z}_p^{+,r}(0)) \Big| \le \delta, \frac{|u^r(1)|}{r} \le \delta \Big) \\
&\equiv (i) + (ii),
\end{aligned} \tag{148}$$

where c is any positive constant. (The value of c will be appropriately determined below).

We first evaluate the term (i) in (148). Note that

$$\begin{aligned}
&\frac{1}{x_{r,0}}\sum_{i=1}^{\lfloor cx_{r,0}\rfloor} 1_{\{\gamma_k^{s,r}(i)\le rM\}}\\
&=\frac{1}{x_{r,0}}\sum_{i=1}^{\lfloor cx_{r,0}\rfloor}\big(1_{\{\gamma_k^{s,r}(i)\le rM\}}-F_k^{\gamma,r}(rM)\big)+\frac{1}{x_{r,0}}F_k^{\gamma,r}(rM)\lfloor cx_{r,0}\rfloor.
\end{aligned}$$

Then, since $F_k^{\gamma,r}(rM)\to 0$ as $r\to\infty$ because of (57), we have that

$$\mathsf{P}_*^r\Big(\frac{1}{x_{r,0}}F_k^{\gamma,r}(rM)\lfloor cx_{r,0}\rfloor>\frac{\varepsilon}{4}\Big)=1_{\left\{x_{r,0}^{-1}F_k^{\gamma,r}(rM)\lfloor cx_{r,0}\rfloor>\varepsilon/4\right\}}=0$$

for sufficiently large r independently of the value $*$.

Further, we have that for each $\varepsilon>0$,

$$\begin{aligned}
&\mathsf{P}_*^r\Big(\sup_{0\le t\le c}\big|\sum_{i=1}^{\lfloor x_{r,0}t\rfloor}\big(1_{\{\gamma_k^{s,r}(i)\le rM\}}-F_k^{\gamma,r}(rM)\big)\big|>x_{r,0}\frac{\varepsilon}{4}\Big)\\
&\le\frac{4^4}{(x_{r,0}\varepsilon)^4}\mathsf{E}_*^r\Big[\Big\{\sum_{i=1}^{\lfloor x_{r,0}c\rfloor}\big(1_{\{\gamma_k^{s,r}(i)\le rM\}}-F_k^{\gamma,r}(rM)\big)\Big\}^4\Big]\\
&\le\frac{4^4}{(x_{r,0}\varepsilon)^4}\cdot 3(x_{r,0}c)^2\le\frac{768c^2}{(x_{r,0})^2\varepsilon^4}\le\frac{768c^2}{r^2\varepsilon^4},
\end{aligned}$$

where the first inequality is due to Doob's submartingale inequality. Therefore, if r is sufficiently large such that

$$\frac{1}{r}<\frac{\varepsilon^5}{768c^2},$$

then

$$\text{the term }(i)\le\frac{\varepsilon}{r}. \tag{149}$$

We next evaluate the term (ii) in (148). Because of (149), it is enough to show that for each $k\in\mathbb{K}$, there exists some constant $c>0$ such that

$$\mathsf{P}_*^r\Big(A_k^r(x_{r,0}L)\ge cx_{r,0},\max_{p\in\mathbb{K}}\Big|\widehat{\mathcal{V}}_p^{o,r}(\overline{Z}_p^{+,r}(0))\Big|\le\delta,\frac{|u^r(1)|}{r}\le\delta\Big)\le\frac{\varepsilon}{r}, \tag{150}$$

if r is sufficiently large independently of $*$. Because of (15) and (16), we have only to show that for each $k\in\mathbb{A}$ and $l\in\mathbb{K}$, there exists some constant $c,c_1,c_2,c_3>0$ such that

$$\mathsf{P}_*^r\Big(E_k^r(x_{r,0}L)\ge\frac{1}{2}cx_{r,0},\frac{|u^r(1)|}{r}\le\delta\Big)\le c_1\frac{\varepsilon}{r}, \tag{151}$$

$$\mathsf{P}_*^r\Big(F_l^r(x_{r,0}L)\ge\frac{1}{2}cx_{r,0},\max_{p\in\mathbb{K}}\Big|\widehat{\mathcal{V}}_p^{o,r}(\overline{Z}_p^{+,r}(0))\Big|\le\delta\Big)\le(c_2+c_3)\frac{\varepsilon}{r}, \tag{152}$$

if r is sufficiently large independently of $*$. Using Lemma 7.2 in the Appendix, we immediately have (151) with $c\ge 2\sup_r\alpha_k^r\dot{L}+1$ and any $\varepsilon\in(0,1)$. Further, using Lemmas 7.3 and 7.4, we have (152) with $c\ge 4\sum_{p=1}^K\sup_r P_{pl}^r\mu_p+1$. Therefore (150) has been established so that the conclusion of the lemma follows.

□

Proof of Proposition 5.4.

According to Proposition 5.5, the methodology employed in Bramson [3], specifically the contents of Sect. 5 and Sect. 6 in [3], also applies to the demonstration of multiplicative strong state-space collapse, i.e., (127) in our multiclass feedforward queueing network with abandonment under the FCFS service discipline. Thus, using Proposition 5.2, we see that strong state-space collapse holds, i,e.,

$$\sup_{0\le t\le T}\ \sup_{0\le s\le \widehat{W}^r_{s(k)}(t)} \mid r^{-1}D^r_k(r^2t+rs)-r^{-1}D^r_k(r^2t)-\lambda^r_k s\mid \longrightarrow 0 \quad \text{in pr.}$$

as $r\to\infty$, for each $k\in\mathbb{K}$. In particular, we have

$$\sup_{0\le t\le T} \mid r^{-1}D^r_k(r^2t+r\widehat{W}^r_{s(k)}(t))-r^{-1}D^r_k(r^2t)-\lambda^r_k\widehat{W}^r_{s(k)}(t)\mid \longrightarrow 0 \quad \text{in pr.} \tag{153}$$

as $r\to\infty$.

On the other hand, because of the FCFS service discipline with abandonment, we have

$$r^{-1}D^r_k(r^2t+r\widehat{W}^r_{s(k)}(t))-r^{-1}D^r_k(r^2t)+\widehat{Z}^{-,r}_k(t)=\widehat{Z}^r_k(t) \tag{154}$$

for each $k\in\mathbb{K}$. Also recall that for each $T>0$,

$$\sup_{0\le t\le T}\widehat{Z}^{-,r}_k(t)\longrightarrow 0 \quad \text{in pr.} \tag{155}$$

as $r\to\infty$, as established in (142). Then, combining (155) with (153) and (154), we have the condition of state-space collapse (128).

□

5.4 Proof of the diffusion approximation theorem (i.e., Theorem 4.1) Before presenting the proof of the theorem, we show the next lemma on the fluid limits of $\{A^r(\cdot)\}_r$ and $\{D^r(\cdot)\}_r$, which corresponds to Lemma 8.2 in Williams [26].

Lemma 5.3.

For each $k\in\mathbb{K}$ and $T>0$,

$$\sup_{0\le t\le T}|\overline{A}^r_k(t)-\lambda_k t|\longrightarrow 0 \quad \textit{in pr.},$$

$$\sup_{0\le t\le T}|\overline{D}^r_k(t)-\lambda_k t|\longrightarrow 0 \quad \textit{in pr.},$$

as $r\to\infty$.

Proof.

¿From (17), (78), (80), (82) and (83), we have

$$\overline{Z}^r_k(t)=\overline{Z}^r_k(0)+\overline{A}^r_k(t)-\overline{D}^r_k(t)-\overline{I}^r_k(t)$$

for each $k\in\mathbb{K}$ and $t\ge 0$. Because of Propositions 5.1 and 5.3, we see that for each $T>0$,

$$\sup_{0\le t\le T}\overline{Z}^r_k(t)\longrightarrow 0 \quad \text{in pr.},$$

$$\sup_{0\le t\le T}\overline{I}^r_k(t)\longrightarrow 0 \quad \text{in pr.}$$

as $r \to \infty$. So we have that for each $T > 0$,

$$\sup_{0 \le t \le T} |\overline{A}_k^r(t) - \overline{D}_k^r(t)| \longrightarrow 0 \quad \text{in pr.} \tag{156}$$

as $r \to \infty$.

Fix any $t \ge 0$. Then, according to (120), $\{\overline{A}_k^r(t)\}_r$ is tight in $\mathcal{R}^1$ for each $k \in \mathbb{K}$, which yields that for any subsequence $\{r'\}$ of $\{r\}$, there exists some further subsequence $\{r''\}$ of $\{r'\}$ such that

$$\overline{A}_k^{r''}(t) \Longrightarrow a_k(t) \quad \text{in} \quad \mathcal{R}^1$$

as $r'' \to \infty$, for some r.v. $a_k(t)$. Thus we also have

$$\overline{D}_k^{r''}(t) \Longrightarrow a_k(t) \quad \text{in} \quad \mathcal{R}^1$$

as $r'' \to \infty$, because of (156). Therefore, from (15), (46) and (86), it follows that

$$a_k(t) = \alpha_k t + \sum_{l=1}^{K} P_{lk} a_l(t)$$

for each $k \in \mathbb{K}$, which implies $a_k(t) = \lambda_k t, k \in \mathbb{K}$. Consequently we have proved that

$$\overline{A}_k^r(t) \Longrightarrow \lambda_k t \quad \text{in} \quad \mathcal{R}^1,$$

$$\overline{D}_k^r(t) \Longrightarrow \lambda_k t \quad \text{in} \quad \mathcal{R}^1$$

as $r \to \infty$, for each $t \ge 0$ and $k \in \mathbb{K}$. Therefore, in virtue of Polya's theorem (cf. Problem 5.3.2 in Liptser and Shiryayev [20]), we obtain the conclusion.

□

The next lemma identifies the weak limit of scaled abandonment-count process as a functional of the limit of scaled workload process, which is similar in form to the case of heavy-traffic limit for a many-server queue with abandonment under the hazard-type scaling of abandonment distribution (cf. Lemma 2.7 in Katsuda [17]), with the difference of multiplicative constant due to our multiclass setting.

Lemma 5.4.

Suppose that

$$\widehat{W}^r(\cdot) \Longrightarrow W^*(\cdot) \quad \textit{in} \quad \mathbb{D}([0,\infty), \mathcal{R}^J),$$

as $r \to \infty$. Then we have that

$$\widehat{I}_k^r(\cdot) \Longrightarrow \lambda_k \int_0^{\cdot} H_k(W^*_{s(k)}(u))du \quad \textit{in} \quad \mathbb{D}([0,\infty), \mathcal{R}^1), \tag{157}$$

as $r \to \infty$, for each $k \in \mathbb{K}$.

Proof.

According to (25), (112) and (155), we have that for each $k \in \mathbb{K}$,

$$\sup_{0 \le t \le T} |\widehat{I}_k^r(t) - \widehat{A}_k^{-,r}(t)| \longrightarrow 0 \quad \text{in pr.}$$

as $r \to \infty$. Thus, because of (113) and (121),

$$\sup_{0 \le t \le T} |\widehat{I}_k^r(t) - \widehat{C}_k^r(\overline{A}_k^r(t))| \longrightarrow 0 \quad \text{in pr.} \tag{158}$$

as $r \to \infty$, with $\widehat{\mathcal{C}}_k^r(\cdot)$ in (115).

Observing that according to (115),

$$\int_0^t rF_k^{\gamma,r}(r\widehat{W}_{s(k)}^r(u-))d\overline{A}_k^r(u) \leq \widehat{\mathcal{C}}_k^r(\overline{A}_k^r(t)) \leq \int_0^t rF_k^{\gamma,r}(r\widehat{W}_{s(k)}^r(u))d\overline{A}_k^r(u)$$

for each $t \geq 0$, we have the convergence (157) in virtue of (57), (158) and Lemma 5.3 in the same way as in the proof of Lemma 2.7 in [17].

□

Proof of Theorem 4.1.

The first half of the proof uses an analogous argument to the proof of Theorem 7.1 in Williams [26] as follows.
From (21), (67) and (68), we have that for each $j \in \mathbb{J}$,

$$\begin{aligned}
\widehat{W}_j^r(t) &= \widehat{W}_j^r(0) + \sum_{k\in C(j)} \frac{1}{r} \sum_{i=1}^{A_k^{+,r}(r^2t)} (v_k^{s,r}(i) - m_k^r) + \sum_{k\in C(j)} \frac{1}{r} m_k^r (A_k^r(r^2t) - A_k^{-,r}(r^2t)) \\
&\quad - rt + \widehat{Y}_j^r(t) \\
&= \widehat{W}_j^r(0) + \sum_{k\in C(j)} \widehat{\mathcal{V}}_k^{s,r}(\overline{A}_k^{+,r}(t)) + \sum_{k\in C(j)} m_k^r \widehat{A}_k^r(t) - \sum_{k\in C(j)} m_k^r \widehat{\mathcal{M}}_k^{\gamma,r}(\overline{A}_k^r(t)) \\
&\quad - \sum_{k\in C(j)} m_k^r \widehat{\mathcal{C}}_k^r(\overline{A}_k^r(t)) + r(\rho_j^r - 1)t + \widehat{Y}_j^r(t)
\end{aligned} \tag{159}$$

with $\widehat{\mathcal{V}}_k^{s,r}(\cdot)$ in (70), $\widehat{\mathcal{M}}_k^{\gamma,r}(\cdot)$ in (114) and $\widehat{\mathcal{C}}_k^r(\cdot)$ in (115) for each $k \in \mathbb{K}$. In vector form, (159) is represented as

$$\begin{aligned}
\widehat{W}^r(t) &= \widehat{W}^r(0) + C\widehat{\mathcal{V}}^{s,r}(\overline{A}^{+,r}(t)) + CM^r\widehat{A}^r(t) - CM^r\widehat{\mathcal{M}}^{\gamma,r}(\overline{A}^r(t)) \\
&\quad - CM^r\widehat{\mathcal{C}}^r(\overline{A}^r(t)) + r(\rho^r - e)t + \widehat{Y}^r(t).
\end{aligned} \tag{160}$$

On the other hand, using (109), we have

$$\begin{aligned}
CM^r\widehat{A}^r(t) &= CM^rQ^r\{\widehat{E}^r(t) + \sum_{l=1}^K \widehat{\Phi}^{l,r}(\overline{D}_l^r(t))\} \\
&\quad - CM^rQ^r\widetilde{P}^r(\widehat{Z}^r(t) - \widehat{Z}^r(0)) - CM^rQ^r\widetilde{P}^r\widehat{I}^r(t) \\
&= CM^rQ^r\{\widehat{E}^r(t) + \sum_{l=1}^K \widehat{\Phi}^{l,r}(\overline{D}_l^r(t))\} - CM^rQ^r\widetilde{P}^r(\widehat{\epsilon}^r(t) - \widehat{\epsilon}^r(0)) \\
&\quad - G^r(\widehat{W}^r(t) - \widehat{W}^r(0)) - CM^rQ^r\widetilde{P}^r\widehat{I}^r(t)
\end{aligned} \tag{161}$$

where

$$\begin{aligned}
&\widehat{\epsilon}^r(t) = (\widehat{\epsilon}_k^r(t), k \in \mathbb{K}) \quad \text{with} \quad \widehat{\epsilon}_k^r(t) \equiv \widehat{Z}_k^r(t) - \lambda_k^r\widehat{W}_{s(k)}^r(t), k \in \mathbb{K}, \\
&G^r \equiv CM^rQ^r\widetilde{P}^r\Lambda^r.
\end{aligned}$$

Therefore, substituting (161) into (160) and using assumption (A.4), we have

$$\widehat{W}^r(t) = \widehat{X}^r(t) + R^r\widehat{Y}^r(t) \tag{162}$$

for sufficiently large r, where

$$\begin{aligned} R^r &\equiv (1+G^r)^{-1}, \\ \widehat{X}^r(t) &\equiv \widehat{W}^r(0) + R^r(\widehat{\xi}^r(t) + \widehat{\eta}^r(t) + \widehat{\zeta}^r(t)) \end{aligned} \tag{163}$$

with

$$\begin{aligned} \widehat{\xi}^r(t) &\equiv C\widehat{\mathcal{V}}^{s,r}(\overline{A}^{+,r}(t)) + CM^rQ^r\big\{\widehat{E}^r(t) + \sum_{l=1}^{K}\widehat{\Phi}^{l,r}(\overline{D}_l^r(t))\big\} \\ &\quad - C\widehat{\mathcal{M}}^{\gamma,r}(\overline{A}^r(t)), \\ \widehat{\eta}^r(t) &\equiv r(\rho^r - e)t - CM^rQ^r\widetilde{P}^r\widehat{I}^r(t) - CM^r\widehat{\mathcal{C}}^r(\overline{A}^r(t)), \\ \widehat{\zeta}^r(t) &\equiv CM^rQ^r\widetilde{P}^r(\widehat{\epsilon}^r(0) - \widehat{\epsilon}^r(t)). \end{aligned}$$

Using (86), (87), (88), (121) and Lemma 5.3, we see that

$$\widehat{\xi}^r(\cdot) \Longrightarrow \xi^*(\cdot) \qquad \text{in} \quad \mathbb{D}([0,\infty),\mathcal{R}^J) \tag{164}$$

as r goes to infinity, where

$$\xi^*(t) = C\mathcal{V}^*(\lambda t) + CMQ\big\{E^*(t) + \sum_{l=1}^{K}\Phi^{l,*}(\lambda_l t)\big\}. \tag{165}$$

Applying the oscillation inequality in Williams [25] to (162) as in (120) of Williams [26], we have that for $w_T(x(\cdot),\delta)$ in (124),

$$Osc(x(\cdot),I) \equiv \sup_{u,v\in I}|x(u)-x(v)|, \qquad I\subset\mathcal{R}^1,$$

and sufficiently large r,

$$\begin{aligned} w_T(\widehat{W}^r(\cdot),\delta) &= \sup_{u\in[0,T-\delta]} Osc\big(\widehat{W}^r(\cdot),[u,u+\delta]\big) \\ &\le const\cdot \sup_{u\in[0,T-\delta]} Osc\big(\widehat{X}^r(\cdot),[u,u+\delta]\big) \\ &= const\cdot w_T(\widehat{X}^r(\cdot),\delta), \end{aligned}$$

so that

$$w_T(\widehat{W}^r(\cdot),\delta) \le const\cdot\big\{w_T(\widehat{\xi}^r(\cdot),\delta) + w_T(\widehat{\eta}^r(\cdot),\delta) + w_T(\widehat{\zeta}^r(\cdot),\delta)\big\}$$

for each $T>0$ and $\delta>0$, because of (163).

The convergence (164) implies

$$\lim_{\delta\to 0}\overline{\lim_{r\to\infty}}\,\mathbb{P}^r\big(w_T(\widehat{\xi}^r(\cdot),\delta)>\varepsilon\big) = 0, \quad \forall\varepsilon>0.$$

(Cf. Proposition 6.3.26 in Jacod and Shiryaev [14]).

In addition, from the heavy-traffic condition (56) and the C-tightness of both $\{\widehat{I}_k^r(\cdot)\}_r$ and $\{\widehat{\mathcal{C}}_k^r(\overline{A}_k^r(\cdot))\}_r$ already established, we have

$$\lim_{\delta\to 0}\overline{\lim_{r\to\infty}}\,\mathbb{P}^r\Big(w_T(\widehat{\eta}^r(\cdot),\delta)>\varepsilon\Big)=0,\quad \forall\varepsilon>0.$$

Therefore, by virtue of the condition of state-space collapse (128), we have

$$\lim_{\delta\to 0}\overline{\lim_{r\to\infty}}\,\mathbb{P}^r\Big(w_T(\widehat{W}^r(\cdot),\delta)>\varepsilon\Big)=0,\quad \forall\varepsilon>0. \tag{166}$$

Combining (166) with Proposition 5.2, we obtain the C-tightness of $\{\widehat{W}^r(\cdot)\}_r$.

Let $W^*(t), t\geq 0$, be any limit process of the sequence $\{\widehat{W}^r(\cdot)\}_r$, and suppose that a subsequence $\{r'\}$ of $\{r\}$ satisfies

$$\widehat{W}^{r'}(\cdot)\Longrightarrow W^*(\cdot)\quad\text{in}\quad \mathbb{D}([0,\infty),\mathcal{R}^J),$$

as $r'\to\infty$. Then, according to Lemma 5.4, we have that for each $k\in\mathbb{K}$,

$$\widehat{I}_k^{r'}(\cdot)\Longrightarrow \lambda_k\int_0^{\cdot} H_k(W^*_{s(k)}(u))du\qquad\text{in}\quad \mathbb{D}([0,\infty),\mathcal{R}^1) \tag{167}$$

and

$$\widehat{\mathcal{C}}_k^{r'}(\overline{A}_k^{r'}(\cdot))\Longrightarrow \lambda_k\int_0^{\cdot} H_k(W^*_{s(k)}(u))du\qquad\text{in}\quad \mathbb{D}([0,\infty),\mathcal{R}^1) \tag{168}$$

as r' goes to infinity. Therefore, using (167), (168) and (56), we have

$$\widehat{\eta}^{r'}(\cdot)\Longrightarrow \eta^*(\cdot) \tag{169}$$

as r' goes to infinity, where

$$\eta^*(t)=\vartheta t-CMQ\Big(\lambda_k\int_0^t H_k(W^*_{s(k)}(u))du,k\in\mathbb{K}\Big),\qquad t\geq 0.$$

Consequently, substituting assumption (A.1), (164), (128) and (169) into (163) and using assumption (A.4), we have

$$\widehat{X}^{r'}(\cdot)\Longrightarrow X^*(\cdot) \tag{170}$$

as r' goes to infinity, where

$$X^*(t)=W^*(0)+R(\xi^*(t)+\eta^*(t)),\qquad t\geq 0. \tag{171}$$

Therefore, *any* limit process $W^*(\cdot)$ of the C-tight sequence $\{\widehat{W}^r(\cdot)\}_r$ is a semimartingale reflecting Brownian motion (SRBM) with a nonlinear drift term, i.e., (94) and (95). Applying the Girsanov transformation technique to the localized version of such SRBM (cf. the proof of Theorem 2.1 in Katsuda [17], for example), we can reduce the uniqueness in law of $W^*(\cdot)$ to that of SRBM, so that the desired convergence

$$(\widehat{W}^r(\cdot),\widehat{Y}^r(\cdot))\Longrightarrow (W^*(\cdot),Y^*(\cdot)), \tag{172}$$

as $r\to\infty$ has been shown. Combining (172) with the result on state-space collapse, i.e., Proposition 5.4, we also have the convergence

$$\widehat{Z}^r(\cdot)\Longrightarrow Z^*(\cdot)$$

as $r\to\infty$, where $Z^*(\cdot)=(\lambda_k W^*_{s(k)}(\cdot),k\in\mathbb{K})$, so the proof of the theorem has been completed.

□

6 Final remarks

As an example of our diffusion approximation with the unstable random behavior of abandonment time near the origin, consider a GI/GI/1+GI queue for which the abandonment time is distributed according to the Gamma distribution

$$G_p(x) = \int_0^x g_p(u)du, x \geq 0, \quad g_p(u) = (\Gamma(p))^{-1}u^{p-1}e^{-u}, u \geq 0,$$

with $p \in (0,1)$. Then, its hazard-rate function $h_p(x) = g_p(x)/(1 - G_p(x))$ is not locally bounded so that the diffusion approximation result in the literature such as [24] and [21] is inapplicable. However, in virtue of our general hazard-type scaling, our main result does hold in ths case.

In this paper we impose the feedforward routing condition on our multiclass queueing networks (MQNs) and the only place where it is used is the proof of the stochstic boundedness of queue length. So, if it is established without such restriction, our main result is valid for general MQNs with abandonment.

One of the most important studies around diffusion approximations of queueing systems is the application of such approximations to the validation of steady-state approximations of those systems. Gamarnik and Zeevi [12] is a seminal work of the study, in which steady-state approximations for generalized Jackson networks have been validated under the condition of the existence of moment generating functions for primitive model variables. It is also noted that such relatively restrictive assumption can be relaxed to moment condition of p-th order with $p \geq 2$ by the work of Budhiraja and Lee [5] in conjunction with the appendix of Krichagina and Taksar [19]. Furthermore, the author's works [15, 16] used the Lyapunov function method of [12] and the framework on the uniform moment bounds of the Markov state process in [5], respectively, to study such steady-state analysis of a multiclass single-server queue in heavy traffic under various service disciplines.

In this paper we have proved the diffusion approximation theorem for multiclass feedforward queueing networks with abandonments under FCFS service disciplines, and so we are interested in steady-state approximations of those networks as an application of our theorem. Restricting our attention to a multiclass single-server queue with $\vartheta < 0$ (in heavy-traffic condition (56)), we are able to validate such approximation of the queue with abandonment in a similar fashion to [15] and [16], in which conditions (A.1), (A.2) and (A.3) of this paper may be verified to hold in stationarity. However, checking the case with $\vartheta \geq 0$ remains unresolved and is worth pursuing in future research. More specifically, it is solved if the following two tasks are done:
(i) To seek a sufficient condition for the stability of multiclass feedforward queueing networks with abandonments.
To be expected from the literature (cf. Baccelli et al. [1], Dai [6]), the condition is such that the traffic intensity at each station may possibly be greater than unity in such a way that its excess over unity can be balanced out by the effect of abandonment;
(ii) To show the tightness of stationary workload and queue length in the queue with abandonment for the verification of conditions (A.1), (A.2) and (A.3) in stationarity.
As concerns the issue (i), in his recent work [18] the author has given a stability condition for those networks, which involves the total probability mass of abandonment time in addition to the model parameters of networks.

7 Appendix

This appendix corresponds to Sect. 5 of Bramson [3] in which hydrodynamically scaled performance measure processes for multiclass queueing networks are asymptotically estimated as approximately Lipschitz continuous. Different from [3], our argument employs such scaling in association with the shift transformation of the description process $\Xi(\cdot)$ in Sect. 3 and uses its Markov property to obtain such asymptotic estimation of performance measure processes in our queueing network.

The next lemma corresponds to Proposition 4.2 in Bramson [3] and plays a fundamental role in proving the rest of the lemmas as in [3].

Lemma 7.1.

Suppose that the sequence of r.v.'s $\{X^r(i), i \geq 1\}$ is i.i.d. for each $r \geq 1$, and $\{X^r(1)^2\}_{r\geq 1}$ is uniformly integrable. Let $\mathcal{S}^r(i) \equiv \sum_{j=1}^{i} X^r(j), i \geq 1$, and $\mu_X^r \equiv \mathbb{E}^r[X^r(1)]$, $r \geq 1$. Then, for each $\varepsilon > 0$,

$$\sup_r \mathbb{P}^r\big(\max_{1\leq i\leq n} |\mathcal{S}^r(i) - i\mu_X^r| > \varepsilon n\big) < \frac{\varepsilon}{n}$$

if n is sufficiently large.

Lemma 7.2.

For each $\varepsilon > 0$ and $k \in \mathbb{K}$, there exist constants $\delta_1 > 0$ and $c_1 > 0$ such that

$$\mathsf{P}_*^r\Big(\sup_{0\leq t\leq x_{r,0}L} |E_k^r(t) - \alpha_k^r t| > x_{r,0}\varepsilon, \frac{|u^r(1)|}{r} \leq \delta_1\Big) < c_1 \cdot \frac{\varepsilon}{r} \tag{173}$$

if r is sufficiently large independently of $*$, where $\mathsf{P}_*^r(\cdot)$ is the probability law of Markov process $\Xi^r(\cdot)$ starting with the value $*$ for each $r \geq 1$. (Cf. (33)).

Proof.

First observe that the inequality

$$\sup_{0\leq t\leq x_{r,0}L} |E_k^r(t) - \alpha_k^r t| \geq x_{r,0}\varepsilon$$

implies that there exists some $t \in [0, x_{r,0}L]$ such that either

$$E_k^r(t) \geq \alpha_k^r t + x_{r,0}\varepsilon \tag{174}$$

or

$$E_k^r(t) \leq \alpha_k^r t - x_{r,0}\varepsilon. \tag{175}$$

The inequality (174) and condition (46) implies that

$$\mathcal{U}_k^r(\lfloor \alpha_k^r t + x_{r,0}\varepsilon \rfloor) - \lfloor \alpha_k^r t + x_{r,0}\varepsilon \rfloor \frac{1}{\alpha_k^r} \leq -\frac{x_{r,0}\varepsilon}{2\alpha_k} \tag{176}$$

if r is sufficiently large. Similarly, the inequality (175) implies that

$$\mathcal{U}_k^r(\lfloor \alpha_k^r t - x_{r,0}\varepsilon \rfloor + 1) - (\lfloor \alpha_k^r t - x_{r,0}\varepsilon \rfloor + 1)\frac{1}{\alpha_k^r} > \frac{x_{r,0}\varepsilon}{2\alpha_k} \tag{177}$$

if r is sufficiently large.

Thus, noting that for each $t \in [0, x_{r,0}L]$,

$$\lfloor \alpha_k^r t + x_{r,0}\varepsilon \rfloor < x_{r,0}(\alpha_k L + 1) \tag{178}$$

if r is sufficiently large, where we suppose $\varepsilon \in (0, \frac{1}{2})$, and using (176), (177) and (178), we have

$$\begin{aligned} &\mathsf{P}^r_*\Big(\sup_{0\le t\le x_{r,0}L} |E^r_k(t) - \alpha^r_k t| \ge x_{r,0}\varepsilon, \frac{|u^r(1)|}{r} \le \delta\Big) \\ &\le \mathsf{P}^r_*\Big(\max_{1\le i\le \lfloor x_{r,0}(\alpha_k L+1)\rfloor} \Big|\mathcal{U}^r_k(i) - \frac{1}{\alpha^r_k} i\Big| > \frac{x_{r,0}\varepsilon}{2\alpha_k}, \frac{|u^r(1)|}{r} \le \delta\Big). \end{aligned} \tag{179}$$

Suppose that the constant $\delta_1 > 0$ satisfies the inequality

$$\delta_1 \le \frac{\varepsilon}{4\alpha_k} - \frac{2}{\alpha_k r}$$

for sufficiently large r satisfying $\frac{\varepsilon}{4\alpha_k} - \frac{2}{\alpha_k r} > 0$. Then, when $\frac{|u^r(1)|}{r} \le \delta_1$, we have that for each $i \ge 1$,

$$\begin{aligned} \Big|\mathcal{U}^r_k(i) - \frac{1}{\alpha^r_k} i\Big| &\le u^r_k(1) + \frac{1}{\alpha^r_k} + \Big|\sum_{j=2}^{i}\Big(u^r_k(j) - \frac{1}{\alpha^r_k}\Big)\Big| \\ &\le \delta_1 r + \frac{2}{\alpha_k} + \Big|\sum_{j=2}^{i}\Big(u^r_k(j) - \frac{1}{\alpha^r_k}\Big)\Big| \\ &\le \frac{x_{r,0}\varepsilon}{4\alpha_k} + \Big|\sum_{j=2}^{i}\Big(u^r_k(j) - \frac{1}{\alpha^r_k}\Big)\Big|, \end{aligned}$$

where we set $\sum_{j=2}^{i} \cdots \equiv 0$ when $i = 1$. Therefore, applying Lemma 7.1 and observing that $x_{r,0}$ is a function of $*$ on the event inside P_*, we have that the display (179) with $\delta = \delta_1$ is dominated by

$$\begin{aligned} &\mathsf{P}^r_*\Big(\max_{2\le i\le \lfloor x_{r,0}(\alpha_k L+1)\rfloor} \Big|\sum_{j=2}^{i}\Big(u^r_k(j) - \frac{1}{\alpha^r_k}\Big)\Big| > \frac{x_{r,0}\varepsilon}{4\alpha_k}\Big) \\ &\le \frac{x_{r,0}\varepsilon}{4\alpha_k} \times \frac{1}{(\lfloor x_{r,0}(\alpha_k L+1)\rfloor - 1)^2} \\ &\le \frac{x_{r,0}\varepsilon}{\alpha_k} \times \frac{1}{\{x_{r,0}(\alpha_k L+1)\}^2} \\ &\le \frac{1}{\alpha_k(\alpha_k L+1)^2} \times \frac{\varepsilon}{r}, \end{aligned}$$

if r is sufficiently large. Letting $c_1 \equiv \frac{1}{\alpha_k(\alpha_k L+1)^2}$, we have the conclusion of the lemma.

□

The next lemma corresponds to Lemma 5.2 in Bramson [3], and it will be used in the proof of Lemma 7.4 below.

Lemma 7.3.

For each $\varepsilon > 0$ and $k \in \mathbb{K}$, there exist constants $\delta > 0$ and $c_2 > 0$ such that

$$\begin{aligned} &\mathsf{P}^r_*\Big(\sup_{t_1,t_2\in[0,x_{r,0}L]} \big(|D^r_k(t_2) - D^r_k(t_1)| - \mu_k|t_2 - t_1|\big) \ge x_{r,0}\varepsilon, |\widehat{\mathcal{V}}^{o,r}_k(\overline{Z}^{+,r}_k(0))| < \delta\Big) \\ &< c_2 \cdot \frac{\varepsilon}{r}, \end{aligned} \tag{180}$$

if r is sufficiently large independently of $*$. In particular,

$$\mathsf{P}^r_*\Big(D^r_k(x_{r,0}L) \geq 2\mu_k x_{r,0}L,\, |\widehat{\mathcal{V}}^{o,r}_k(\overline{Z}^{+,r}_k(0))| < \delta\Big) < c_2 \cdot \frac{\varepsilon}{r}, \tag{181}$$

if r is sufficiently large independently of $*$.

Proof.
Let

$$\xi^r_k(s) \equiv \max\{n \in \mathbb{N} : \mathcal{V}^r_k(n) \leq s\}, \qquad s \geq 0, \quad k \in \mathbb{K},$$

where

$$\mathcal{V}^r_k(n) \equiv \mathcal{V}^{o,r}_k(Z^{+,r}_k(0) \wedge n) + \mathcal{V}^{s,r}_k((n - Z^{+,r}_k(0))^+). \tag{182}$$

Then

$$D^r_k(t) = \xi^r_k(T^r_k(t)), \qquad t \geq 0, \quad k \in \mathbb{K}.$$

Since

$$D^r_k(t_2) - D^r_k(t_1) - \mu^r_k(t_2 - t_1) \leq \xi^r_k(T^r_k(t_2)) - \xi^r_k(T^r_k(t_1)) - \mu^r_k(T^r_k(t_2) - T^r_k(t_1))$$

for each $t_1, t_2 \in [0, x_{r,0}L]$ such that $t_1 \leq t_2$, we have

$$\begin{aligned}
&\sup_{t_1,t_2 \in [0,x_{r,0}L]} \{|D^r_k(t_2) - D^r_k(t_1)| - \mu^r_k|t_2 - t_1|\} \\
&\leq \sup_{s_1,s_2 \in [0,x_{r,0}L]} \{|\xi^r_k(s_2) - \xi^r_k(s_1)| - \mu^r_k|s_2 - s_1|\}. \\
&\leq 2 \sup_{s \in [0,x_{r,0}L]} |\xi^r_k(s) - \mu^r_k s|.
\end{aligned}$$

Thus, it suffices to show

$$\mathsf{P}^r_*\Big(\sup_{s \in [0,x_{r,0}L]} \big|\xi^r_k(s) - \mu^r_k s\big| \geq \frac{x_{r,0}\varepsilon}{2}, |\widehat{\mathcal{V}}^{o,r}_k(\overline{Z}^{+,r}_k(0))| < \delta\Big) < c_2 \cdot \frac{\varepsilon}{r}, \tag{183}$$

if r is sufficiently large independently of $*$. In the same way as in the derivation of (179), the left-hand side of (183) is majorized by

$$\mathsf{P}^r_*\Big(\max_{1 \leq i \leq \lfloor x_{r,0}(\mu_k L + 1)\rfloor} \Big|\mathcal{V}^r_k(i) - m^r_k i\Big| > \frac{x_{r,0}\varepsilon}{4\mu_k}, |\widehat{\mathcal{V}}^{o,r}_k(\overline{Z}^{+,r}_k(0))| < \delta\Big). \tag{184}$$

Suppose that the constant $\delta > 0$ satisfies the inequality $\delta < \frac{\varepsilon}{8\mu_k}$. Then, when $i > Z^{+,r}_k(0)$ and $|\widehat{\mathcal{V}}^{o,r}_k(\overline{Z}^{+,r}_k(0))| < \delta$, we have

$$\begin{aligned}
\big|\mathcal{V}^r_k(i) - m^r_k i\big| &\leq r\delta + \big|\mathcal{V}^{s,r}_k(i - Z^{+,r}_k(0)) - m^r_k(i - Z^{+,r}_k(0))\big| \\
&\leq \frac{x_{r,0}\varepsilon}{8\mu_k} + \big|\mathcal{V}^{s,r}_k(i - Z^{+,r}_k(0)) - m^r_k(i - Z^{+,r}_k(0))\big|.
\end{aligned} \tag{185}$$

Therefore we have

$$\begin{aligned}
(184) &\leq \mathsf{P}^r_*\Big(\max_{1 \leq j \leq \lfloor x_{r,0}(\mu_k L+1)\rfloor - Z^{+,r}_k(0)} \Big|\mathcal{V}^{s,r}_k(j) - m^r_k j\Big| > \frac{x_{r,0}\varepsilon}{8\mu_k}\Big) \\
&\leq \frac{x_{r,0}\varepsilon}{8\mu_k} \times \frac{1}{(\lfloor x_{r,0}(\mu_k L + 1)\rfloor - Z^{+,r}_k(0))^2} \\
&\leq \frac{x_{r,0}\varepsilon}{2\mu_k} \times \frac{1}{(x_{r,0}\mu_k L)^2} \\
&\leq \frac{1}{2\mu_k^3 L^2} \times \frac{\varepsilon}{r}
\end{aligned}$$

if r is sufficiently large, where the second inequality is a consequence of the application of Lemma 7.1 and the third inequality follows from $Z_k^{+,r}(0) \le x_{r,0}$. Consequently the conclusion (180) follows with $c_2 = \frac{1}{2\mu_k^3 L^2}$.

Substituting $t_1 = 0$ and $t_2 = x_{r,0}L$ into (180) and letting $\varepsilon \in (0, \mu_k L)$, we immediately have (181).

□

Lemma 7.4.
For each $\varepsilon > 0$ and $k \in \mathbb{K}$, there exist constants $\delta > 0$ and $c_3 > 0$ such that

$$\mathsf{P}_*^r\Big(\sup_{0\le t\le x_{r,0}L}\Big|F_k^r(t) - \sum_{l=1}^K P_{lk}^r D_l^r(t)\Big| > x_{r,0}\varepsilon, \max_{p\in\mathbb{K}}|\widehat{\mathcal{V}}_p^{o,r}(\overline{Z}_p^{+,r}(0))| < \delta\Big) < c_3\cdot\frac{\varepsilon}{r} \tag{186}$$

if r is sufficiently large independently of $*$, where $F^r(\cdot)$ is given in (16).

Proof.

$$\begin{aligned}
&(\text{The left-hand side of (186)})\\
&= \mathsf{P}_*^r\Big(\sup_{0\le t\le x_{r,0}L}\Big|\sum_{l=1}^K\sum_{i=1}^{D_l^r(t)}(\phi_k^{l,r}(i) - P_{lk}^r)\Big| > x_{r,0}\varepsilon, \max_{p\in\mathbb{K}}|\widehat{\mathcal{V}}_p^{o,r}(\overline{Z}_p^{+,r}(0))| < \delta\Big)\\
&\le \mathsf{P}_*^r\Big(\max_{l\in\mathbb{K}}\sup_{0\le t\le x_{r,0}L}\Big|\sum_{i=1}^{D_l^r(t)}(\phi_k^{l,r}(i) - P_{lk}^r)\Big| > \frac{x_{r,0}\varepsilon}{K}, \max_{p\in\mathbb{K}}|\widehat{\mathcal{V}}_p^{o,r}(\overline{Z}_p^{+,r}(0))| < \delta\Big)\\
&\le \sum_{l=1}^K \mathsf{P}_*^r\Big(\sup_{0\le t\le x_{r,0}L}\Big|\sum_{i=1}^{D_l^r(t)}(\phi_k^{l,r}(i) - P_{lk}^r)\Big| > \frac{x_{r,0}\varepsilon}{K}, |\widehat{\mathcal{V}}_l^{o,r}(\overline{Z}_l^{+,r}(0))| < \delta\Big).
\end{aligned}\tag{187}$$

Each term in the summation w.r.t. l in (187) is dominated by

$$\begin{aligned}
&\mathsf{P}_*^r\Big(D_l^r(x_{r,0}L) \ge 2\mu_l x_{r,0}L, |\widehat{\mathcal{V}}_l^{o,r}(\overline{Z}_l^{+,r}(0))| < \delta\Big)\\
&\qquad + \mathsf{P}_*^r\Big(\sup_{0\le t\le x_{r,0}L}\Big|\sum_{i=1}^{D_l^r(t)}(\phi_k^{l,r}(i) - P_{lk}^r)\Big| > \frac{x_{r,0}\varepsilon}{K}, D_l^r(x_{r,0}L) < 2\mu_l x_{r,0}L\Big)
\end{aligned}\tag{188}$$

According to Lemma 7.3,

$$(\text{the first term in (188)}) < c_2\frac{\varepsilon}{r}$$

if r is sufficiently large independently of $*$. Furthermore we have that

$$\begin{aligned}
&(\text{the second term in (188)})\\
&\le \mathsf{P}_*^r\Big(\max_{1\le j\le\lfloor 2\mu_l x_{r,0}L\rfloor}\Big|\sum_{i=1}^{j}(\phi_k^{l,r}(i) - P_{lk}^r)\Big| > \frac{x_{r,0}\varepsilon}{K}\Big)\\
&\le \mathsf{P}_*^r\Big(\max_{1\le j\le\lfloor 2\mu_l x_{r,0}L\rfloor}\Big|\sum_{i=1}^{j}(\phi_k^{l,r}(i) - P_{lk}^r)\Big| > \lfloor 2\mu_l x_{r,0}L\rfloor\cdot\frac{\varepsilon}{2\mu_l LK}\Big)\\
&\le \frac{\varepsilon}{2\mu_l LK}\cdot\frac{1}{\lfloor 2\mu_l x_{r,0}L\rfloor} \le \frac{\varepsilon}{(\mu_l L)^2 K x_{r,0}} < \frac{\varepsilon}{(\mu_l L)^2 K r}
\end{aligned}$$

if r is sufficiently large independently of $*$, where the third inequality follows from the application of Lemma 7.1.

Consequently we have the conclusion of the lemma with $c_3 \equiv Kc_2 + (\min_{l\in\mathbb{K}} \mu_l \cdot L)^{-2}$. $\square$

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CLASSIFICATION OF CONTRACTIBLE SPACES BY C^*-ALGEBRAS AND THEIR K-THEORY

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Abstract. We consider contractible spaces and the corresponding C^*-algebras to show that contractible spaces are classifiable or not (up to homeomorphisms) by the C^*-algebras and their K-theory.

1 Introduction We consider contractible spaces and the corresponding C^*-algebras to show that contractible spaces in some cases are classifiable or not (up to homeomorphism classes or manifold classes with some operations like jointed sums) by the C^*-algebras or their K-theory. Note that contractible spaces are homotopically identified with a point.

For the classification program in our sense, we introduce several notions for C^*-algebras and spaces and also do for several examples. As a summary, we obtain several tables as classification results as collections, and the overview obtained from these tables as maps would be useful for further study in this topic.

Refer to several textbooks [1], [2], [4], or [8] about C^*-algebras and their K-theory and in particular, contractible C^*-algebras. Beyond or extending several facts on them, we further go into studying targeted ones in details in a way this time.

See also [7] for another classification result for some topological manifolds by C^*-algebras and their K-theory, with the same sprit as in this paper.

Let us begin with **some notations** as follows.

For a compact Hausdorff sapce X, we denote by $C(X)$ the C^*-algebra of all continuous, complex-valued functions on X with the uniform (or supremum) norm and pointwise operations.

For a non-compact, locally compact Hausdorff sapce X, we denote by $C_0(X)$ the C^*-algebra of all continuous, complex-valued functions on X, vanishing at infinity. We denote by $X^+ = X \cup \{\infty\}$ the one-point compactification of X. We may say that a non-compact, locally compact Hausdorff space X^- is the one-point **un-compactification** of a compact Hausdorff space X if $X^- \cup \{\infty\} = X$.

We write $\mathfrak{A} \cong \mathfrak{B}$ if two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are $*$-isomorphic. We write $X \approx Y$ if two spaces X and Y are homeomorphic. Use the same symbol $\cong$ for (K-theory) group isomorphisms as well.

2 Contractible, spaces and C^*-algebras A topological space X is said to be **contractible** (in X) if there is a point p in X such that the identity map $\mathrm{id}_X : X \to X$ is homotopic to the constant map id_p on X, which sends elements of X to the point p. Namely, there is a continuous path of continuous maps (f_t) of X (to X) for $t \in [0,1] = I$ the interval such that $f_0 = \mathrm{id}_X$ and $f_1 = \mathrm{id}_p$ and the map $F(t,x) = f_t(x)$ is continuous on the product space $I \times X$. The map F is called a homotopy for X.

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Note that a contractible space may or may not be compact. For instance, the Euclidean space $\mathbb{R}^n$ as well as any convex subspace are all contractible by convexity. Note also that a contractible space is path-connected by definition.

We may say that a topological space X is **identically contractible** if X is contractible by a continuous path of homeomorphisms $(f_t)_{0\leq t<1}$ of X and $f_1 = \mathrm{id}_p$.

A C^*-algebra $\mathfrak{A}$ is said to be **contractible** (to zero) if the identity map $\mathrm{id}_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}$ is homotopic to the zero map $\mathrm{id}_0 = 0 : \mathfrak{A} \to \mathfrak{A}$ by a (norm or uniform) continuous path of $*$-homomorphisms (φ_t) of $\mathfrak{A}$ (to $\mathfrak{A}$) for $t \in [0,1] = I$ such that $\varphi_0 = \mathrm{id}_{\mathfrak{A}}$ and $\varphi_1 = \mathrm{id}_0$ and the map $\Phi(t,a) = \varphi_t(a)$ is continuous on the product space $I \times \mathfrak{A}$. We may call the map Φ a (norm or uniform) C^*-homotopy for $\mathfrak{A}$.

We also say that a C^*-algebra $\mathfrak{A}$ is **identically contractible** (to zero) if $\mathfrak{A}$ is contractible (to zero) by a continuous path of $*$-isomorphisms $(\varphi_t)_{0\leq t<1}$ of $\mathfrak{A}$ and $\varphi_1 = \mathrm{id}_0$.

We say that a C^*-algebra $\mathfrak{A}$ is **contractible** to $\mathbb{C}$ if the identity map $\mathrm{id}_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}$ is homotopic to a 1-dimensional representation (or character) $\chi : \mathfrak{A} \to \mathbb{C}1$ in $\mathfrak{A}$ by a continuous path of $*$-homomorphisms of $\mathfrak{A}$.

We also say that a C^*-algebra $\mathfrak{A}$ is **identically contractible** to $\mathbb{C}$ if $\mathfrak{A}$ is contractible to $\mathbb{C}$ by a continuous path of $*$-isomorphisms $(\varphi_t)_{0\leq t<1}$ of $\mathfrak{A}$ and $\varphi_1 = \chi$.

Note that a non-trivial C^*-algebra contractible to $\mathbb{C}$ is not simple.

Furthermore, we say that a C^*-algebra $\mathfrak{A}$ (especially when $\mathfrak{A} = C(X)$ or $C_0(X)$) is **weakly contractible** (to zero), weakly identically contractible (to zero), weakly contractible to $\mathbb{C}$, and weakly identically contractible to $\mathbb{C}$, respectively, if $\mathfrak{A}$ is contractible (to zero), identically contractible (to zero), contractible to $\mathbb{C}$, and identically contractible to $\mathbb{C}$, by a **pointwise** continuous C^*-homotopy Φ for $\mathfrak{A}$ (with respect to X), respectively. In these cases, we may call such a homotopy Φ either a **weak** homotopy, a weakly continuous path, or a pointwise continuous path for $\mathfrak{A}$.

A homotopy (f_t) for a space X induces directly a homotopy (φ_t) for $C(X)$ (or $C_0(X)$) as the composition as $\varphi_t(g) = g \circ f_t \in C(X)$, which we call the **induced** homotopy.

Indeed, as a summary, with (1) below certainly known ([2]),

Proposition 2.1. (1) *A unital C^*-algebra is not contractible. Equivalently, if a C^*-algebra is contractible, then it is non-unital.*

(2) *If a compact Hausdorff space X is contractible* (*in X*) *by a homotopy, then $C(X)$ is contractible to $\mathbb{C}$ by the induced homotopy.*

(3) *Similarly, if a compact Hausdorff space X is identically contractible by a homotopy, then $C(X)$ is identically contractible to $\mathbb{C}$ by the induced homotopy. The converse in this case also holds.*

(4) *Moreover, if a non-compact, locally compact Hausdorff space X is contractible* (*in X*) *by a homotopy, then $C_0(X)$ is weakly contractible to $\mathbb{C}$ by the induced homotopy.*

(5) *Similarly, if a non-compact, locally compact Hausdorff space X is identically contractible by a homotopy, then $C_0(X)$ is weakly identically contractible to $\mathbb{C}$ by the induced homotopy. The converse in this case also holds.*

Proof. For (1). Note that $*$-homomorphisms φ_t of a unital C^*-algebra $\mathfrak{A}$ (to $\mathfrak{A}$) are always unital, which can not be homotopic to the zero map on $\mathfrak{A}$. Because the constant map $1 = \varphi_t(1) \in \mathfrak{A}$ on $[0,1)$ converges continuously to $1 \in \mathfrak{A}$ at $1 \in [0,1]$.

For (2). Let (f_t) be a continuous path between id_X and id_p for some $p \in X$. Define a continuous path of $*$-homomorphisms of $C(X)$ by $\varphi_t(g) = g \circ f_t$ for $g \in C(X)$ and $t \in [0,1]$.

Indeed,

$$\begin{aligned}\varphi_t(\lambda g + h) &= (\lambda g + h) \circ f_t = (\lambda g \circ f_t) + (h \circ f_t) = \lambda \varphi_t(g) + \varphi_t(h),\\ \varphi_t(g \cdot h) &= (g \cdot h) \circ f_t = (g \circ f_t) \cdot (h \circ f_t) = \varphi_t(g) \cdot \varphi_t(h),\\ \varphi_t(g)^* &= (g \circ f_t)^* = \overline{g \circ f_t} = g^* \circ f_t = \varphi_t(g^*)\end{aligned}$$

for $g, h \in C(X)$ and $\lambda \in \mathbb{C}$, where the overline is the complex conjugate. Note that $\varphi_0(g) = g \circ \mathrm{id}_X = g$ and $\varphi_1(g)(x) = (g \circ \mathrm{id}_p)(x) = g(p)$ for any $x \in X$, so that $\varphi_1(g) = g(p)1 \equiv \chi_p(g)$ the character as the evaluation map at $p \in X$. Note also that the following norm estimate holds:

$$\begin{aligned}\|\Phi(t, g) - \Phi(s, h)\| &= \|\varphi_t(g) - \varphi_s(h)\|\\ &\leq \|\varphi_t(g) - \varphi_s(g)\| + \|\varphi_s(g) - \varphi_s(h)\|\\ &\leq \|g \circ f_t - g \circ f_s\| + \|g - h\|,\end{aligned}$$

which can be small enough when (t, g) and (s, h) are close enough on $[0, 1] \times \mathfrak{A}$. Because X is compact, so that a continuous function $g \in C(X)$ is uniformly continuous on X. In particualr, when $s = 1$, note that

$$\|g \circ f_t - g \circ f_1\| = \sup_{x \in X} |g(f_t(x)) - g(p)|,$$

which goes to zero as $t \to 1$.

For (3). The same as above shows that if X is identically contractible, then $C(X)$ is identically contractible to $\mathbb{C}$.

Conversely, let $(\varphi_t)_{0 \leq t < 1}$ be a continuous path of $*$-isomorphisms of $C(X)$ between $\varphi_0 = \mathrm{id}_{C(X)}$ and a character $\varphi_1 = \chi_p$ for some $p \in X$, by the Gelfand transform (see [4]). In fact, it is a well known fact that the space $C(X)^\wedge$ of all 1-dimensional represetations of $C(X)$ is identifed with the space X. Define a continuous path of homeomorphisms $f_t : X \to X$, induced from the following diagram to make it commutaive:

$$\begin{array}{ccc} C(X) \cong \varphi_t(C(X)) & \xrightarrow{\chi_x} & \mathbb{C} \\ {\scriptstyle \varphi_t}\uparrow & & \| \\ C(X) & \xrightarrow{\chi_{f_t(x)}} & \mathbb{C}, \end{array}$$

since $\chi_x \circ \varphi_t$ for any x is written as χ_y for some $y \in X$, and set $y = f_t(x)$. Note that $\chi_{f_t(x)} \to \chi_{f_s(y)}$ as $(t, x) \to (s, y) \in I \times X$ in weak $*$-topology, if and only if for any $g \in C(X)$,

$$\begin{aligned}|\chi_{f_s(y)}(g) - \chi_{f_t(x)}(g)| &= |(\chi_y \circ \varphi_s)g - (\chi_x \circ \varphi_t)(g)|\\ &= |\varphi_s(g)(y) - \varphi_t(g)(x)|,\end{aligned}$$

which certainly goes to zero as $(t, x) \to (s, y)$, by continuity for the homotopy (φ_t). Note also that

$$(g \circ f_t)(x) = \chi_{f_t(x)}(g) = \varphi_t(g)(x).$$

For (4) and (5). Even if X is a non-compact, Hausforff space, the proof for this case is the similar as given above. Note that the space $C_0(X)^\wedge$ of all 1-dimensional representations of $C_0(X)$ is identified with X. Note also that for any $x \in X$,

$$\begin{aligned}|[\Phi(t, g) - \Phi(s, h)](x)| &= |[\varphi_t(g) - \varphi_s(h)](x)|\\ &\leq |[\varphi_t(g) - \varphi_s(g)](x)| + |[\varphi_s(g) - \varphi_s(h)](x)|\\ &\leq |(g(f_t(x)) - g(f_s(x))| + \|g - h\|,\end{aligned}$$

which can be small enough when (t, g) and (s, h) are close enough on $[0,1]\times\mathfrak{A}$. In particualr, when $s = 1$, note that

$$|[g \circ f_t - g \circ f_1](x)| = |g(f_t(x)) - g(p)|,$$

which goes to zero as $t \to 1$. Note that the uniform continuity for Φ is not expected from the assumption because the norm for the difference above can be non-zero constant, but the difference converges to zero pointwise (see the examples below). It is always assumed from the assumption in this case that only the pointwise continuity for Φ holds, which implies that the estimate above evaluated at $x \in X$ goes to zero, pointwise on X. □

Remark. More generally, when (φ_t) is a continuous path of $*$-homomorphisms of $C(X)$ between $\mathrm{id}_{C(X)}$ and χ_p for some $p \in X$, each image $\varphi_t(C(X))$ as a quotient of $C(X)$ is a commutative C^*-subalgebra of $C(X)$, so that $\varphi_t(C(X))$ is isomorphic to $C(X_t)$ for some compact Hausdorff space X_t, which can be viewed as a closed subspace of X, from which, one can define a continuous path of continuous maps $f_t : X_t \to X_t$ in X, induced from the following diagram to make it commutative (only on X_t):

$$\begin{array}{ccc} \varphi_t(C(X)) \cong C(X_t) & \xrightarrow{\chi_x} & \mathbb{C} \\ {\scriptstyle \varphi_t}\big\uparrow & & \big\| \\ C(X) \supset \varphi_t(C(X)) & \xrightarrow{\chi_{f_t(x)}} & \mathbb{C}. \end{array}$$

If each f_t extends to X, then the extension of (f_t) to X gives a continuous path of continuous maps of X between id_X and id_p.

Furthermore, since a compact Hausdorff space X is normal, there is a continuous extension to X of a 1-dimensional closed interval valued, continuous function on a closed subset such as X_t of X, by Tietze-Urysohn extension theorem in general topology.

Example 2.2. • The C^*-algebra $C(I)$ on the closed interval $I = [0,1]$ is unital (so that not contractible) but weakly identically contractible to $\mathbb{C}$, by the C^*-homotopy induced by a homotopy in $[0,1]$.

If we define $\varphi_t(g)(x) = g((1-t)x) \in \mathbb{C}$ for $g \in C(I)$ and $t, x \in I$. Then (φ_t) is a continuous path of $*$-isomorphisms of $\mathfrak{A} = C(I)$ between $\mathrm{id}_{\mathfrak{A}}$ and χ_0, so that $\mathfrak{A}$ is weakly identically contractible to $\mathbb{C}$. Also, define $h_t(x) = (1-t)x \in [0, 1-t] \approx [0,1]$ for $t, x \in I$, so that $\varphi_t(\mathfrak{A}) \cong \mathfrak{A}$ for $t \in [0,1)$. Then (h_t) is a continuous path of homeomorphisms of $[0,1]$ such that $f_0 = \mathrm{id}_X$ and id_0, so that $[0,1]$ is identically contractible (to 0).

• The (interval) C^*-algebra $I\mathfrak{A} = C(I, \mathfrak{A})$ over a C^*-algebra $\mathfrak{A}$, of all $\mathfrak{A}$-valued, continuous functions on I, viewed as the C^*-tensor product $C(I) \otimes \mathfrak{A}$, is weakly identicallty contractible to $\mathbb{C}$. In particular, $I\mathbb{C} = C(I)$. If $\mathfrak{A}$ is unital, then $C(I) \otimes \mathfrak{A}$ is unital and not contractible.

Note that $\|f \otimes a\| = \|f\|\|a\|$ for $f \otimes a \in I\mathbb{C} \otimes \mathfrak{A}$. Hence, the (norm) homotopy (φ_t) for $I\mathbb{C}$ to $\mathbb{C}$ is extended trivially as $\varphi_t(f \otimes a) = \varphi_t(f) \otimes a$.

• The C^*-algebra $C_0([0,1))$ on the half open interval $[0,1)$ (non-compact), viweded as the cone $C\mathbb{C} \cong C_0([0,1), \mathbb{C}) \cong C_0([0,1)) \otimes \mathbb{C}$ over $\mathbb{C}$, is non-unital and weakly identically contractible to $\mathbb{C}$ by the induced C^*-homotopy by a homotopy in $[0,1)$ (and is certainly contractible, but soon later discussed in the example given below).

Indeed, if we define $\psi_t(g)(x) = g(\frac{x}{1-t})$ for $g \in C_0([0,1))$, $t \in [0,1)$, and $x \in [0, 1-t)$, and $\psi_1(g)(x) = g(0)$. Then (ψ_t) is a weakly continuous path of $*$-isomorphisms of $\mathfrak{A} = C_0([0,1))$ between $\mathrm{id}_{\mathfrak{A}}$ and χ_0, so that $\mathfrak{A}$ is weakly identically contractible to $\mathbb{C}$. Also, define $h_t(x) = \frac{x}{1-t} \in [0,1)$ for $t \in [0,1)$ and $x \in [0, 1-t) \approx [0,1)$, and $h_1(x) = 0$, so that $\psi_t(\mathfrak{A}) \cong \mathfrak{A}$ for

$t \in [0,1)$. Then (h_t) is a continuous path of homeomorphisms of $[0,1)$ such that $f_0 = \mathrm{id}_X$ and $f_1 = \mathrm{id}_0$, so that $[0,1)$ is identically contractible (to 0).

Furthermore, now let $g(x) = x$ for $x \in [0, \frac{1}{2}]$ and $g(x) = 1 - x$ for $x \in [\frac{1}{2}, 1)$ and $g \in C_0([0,1))$. Then the norm $\|\psi_t(g)\| = \|g\| = 1$, but $\chi_0(g) = g(0) = 0$.

If a (compact or non-compact) space X is contractible to a point $p \in X$, then we define $\mathfrak{I}_p$ to be the closed ideal of all continuous functions of ($C(X)$ or $C_0(X)$) on X vanishing at the point p. Note that $\mathfrak{I}_p$ is isomorphic to $C_0(X \setminus \{p\})$.

As a generalization from the case of $C_0([0,1))$ as a closed ideal of $C([0,1])$,

Proposition 2.3. *If a compact Hausdorff space X is contractible to a point $p \in X$, then the closed ideal $\mathfrak{I}_p = C_0(X \setminus \{0\})$ is contractible to zero.*

As well, in this case, $\mathfrak{I}_p \otimes \mathfrak{A}$ for any C^*-algebra $\mathfrak{A}$ is contractible to zero.

Proof. As shown above, it follows that $C(X)$ is contractible to $\mathbb{C}$ (at $p \in X$). Therefore, $\mathfrak{I}_p$ is contractible to zero (at $p \in X$).

Since $\mathfrak{I}_p$ is contractible, so is $\mathfrak{I}_p \otimes \mathfrak{A}$ by the same reason as in the example above. □

Remark. Even if a non-compact, Hausdorff space X is contractible to a point $p \in X$, the closed ideal $\mathfrak{I}_p$ is not necessarily contractible. For instance, let $X = [0,1)$. Then X is contractible to $\{0\}$, but $\mathfrak{I}_0 = C_0((0,1))$ is not contractible. However, $\mathfrak{I}_0$ in this case is weakly contractible to $\mathbb{C}$ since $(0,1)$ is contractible and non compact. Note also that $(0,1)^+ \approx \mathbb{T}$ the one-dimensional torus, which is not contractible.

We now define that a non-compact topological space X is **extended contracible** (in the one-point compactification $X^+ = X \cup \{\infty\}$ of X) if the identity map $\mathrm{id}_{X^+} : X^+ \to X^+$ is homotopic to the constant map id_∞ on X^+, which sends elements of X^+ to the point ∞. We write F^+ for the corresponding homotopy on $I \times X^+$ and call it the extended homotopy for X^+.

Possibly, the most important thing to notice at this moment is that

Proposition 2.4. (1) *Let X be a non-compact, locally compact Hausdorff space. Then X is extended contractible in X^+ in our sense if and only if X^+ is contractible.*

(2) *If X is extended contractible in X^+ in our sense, in other words, if X is a one-point un-compactification of a contractible space, then $C_0(X)$ is contractible to zero.*

(3) *The direct product of finitely many, extended contractible, non-compact locally compact Hausdorff spaces is also extended contractible.*

Proof. By definition, the first statement (1) holds.

The second statement (2) follows from that $C_0(X) \cong \mathfrak{I}_\infty$ in $C(X^+)$.

For the third (3), if $X_1, \cdots, X_n$ are extended contractible, non-compact locally compact Hausdorff spaces, then $(\Pi_{i=1}^n X_i)^+$ is contractible because the coordinante homotopy in X_i^+ extends in $(\Pi_{i=1}^n X_i)^+$ as a product of the homotopies □

Example 2.5. • Let $\mathfrak{A} = C_0([0,1))$. Then $\mathfrak{A}$ is contractible (to zero) as in the references ([2], [4], and [8]).

Indeed, define $\varphi_t(g)(x) = g(t + x(1-t)) \in \mathbb{C}$ for $x \in [0,1)$ and $t \in [0,1]$. Then $\varphi_0(g)(x) = g(x)$ and $\varphi_1(g)(x) = g(1) = 0$, and φ_t for $t \in [0,1)$ are $*$-isomorphisms of $\mathfrak{A}$. Also the space $[0,1)$ is contractible (but to $1 \notin [0,1)$, however in $[0,1]$), because the maps on $[0,1)$ defined by $f_t(x) = t + x(1-t) \in [t,1) \approx [0,1)$ give a continuous path of homeomorphisms of $[0,1)$ such that $f_0 = \mathrm{id}_X$ and $f_1 = \mathrm{id}_1$.

Therefore, $[0,1)$ is extended contractible in $[0,1)^+ = [0,1]$ and $C_0([0,1))$ is identically contractible.

Remark. Note that a contractible space in the 1-dimensional closed interval $I = [0,1]$ is always identically contractible. Moreover, any 1-dimensional contractible space in I is homeomorphic to either $I = [0,1]$, $I_1 = [0,1)$, or $I_{0,1} = (0,1)$. Furthermore, $I = [0,1]$ is a 1-dimensional compact manifold with boundary $\partial I = \{0,1\}$, and $I_1 = [0,1)$ is a 1-dimensional non-compact manifold with boundary $\partial I_1 = \{0\}$, and $I_{0,1} = (0,1)$ is a 1-dimensional non-compact (or open) manifold without boundary.

On the other hand, an extended contractible space may or may not be connected.

Example 2.6. Let $X = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ be a union of half open intervals. Then X is non-connected and is viewed as the one-point un-compactification of $[0,1]$ a contractible space. Hence $C_0(X)$ is contractible to zero. Note that $C_0(X) \cong C_0([0, \frac{1}{2}]) \oplus C_0((\frac{1}{2}, 1])$ with both components contractible to zero.

Just as the 1-dimensional case of connected sums of topological manifolds, one can define (but) a **non-connected sum** of two contractible spaces X and Y in $[0,1]$, denoted as $X\#_pY$, for a point p viewed in the interiors X° and Y° of X and Y respectively, where X is viewed in the line of a Euclidean space and the boundary ∂X is $X \setminus X^\circ$. More precisely, $X\#_pY$ is defined by removing a point in the interiors X° and Y° of X and Y respectively, each identified with a point p, to make disjoint unions $X \setminus \{p\} = X_p^1 \sqcup X_p^2$ and $Y \setminus \{p\} = Y_p^1 \sqcup Y_p^2$ and by gluing X_p^1 and Y_p^1 together with p and gluing X_p^2 and Y_p^2 together with p to make two lines $X_p^1 \cup \{p\} \cup Y_p^1$ and $X_p^2 \cup \{p\} \cup Y_p^2$, where each p in these unions are assumed to be distinct. By definition, the non-connected sum $X\#_pY$ is a disjoint union of two contractible line segments L_j $(j = 1,2)$ in $[0,1]$, so that $X\#_pY = L_1 \sqcup L_2$. Note that $X\#_pY$ is not contractible, and $C(X\#_pY) \cong C(L_1) \oplus C(L_2)$, and $C_0(X\#_pY) \cong C_0(L_1) \oplus C_0(L_2)$ where L_1 or L_2 may be compact and that $X\#_pY$ is compact if and only if X and Y are compact.

Example 2.7. We have $[0,1]\#_p[0,1] \approx [0,1] \sqcup [0,1] \equiv \sqcup^2[0,1]$, and $[0,1)\#_p[0,1) \approx [0,1] \sqcup (0,1)$, and $(0,1)\#_p(0,1) \approx \sqcup^2(0,1)$, and $[0,1]\#_p[0,1) \approx [0,1] \sqcup [0,1)$, and $[0,1]\#_p(0,1) \approx \sqcup^2[0,1)$, and $[0,1)\#_p(0,1) \approx [0,1)\#_p(0,1)$.

Note that only the case $X = [0,1]\#_p(0,1) \approx \sqcup^2[0,1)$ is extended contractible, with $X^+ \approx [0,1]$.

Moreover, we can define inductively a **successive** non-connected sum of n contractible spaces $X_1, \cdots, X_n$ in $[0,1]$ as

$$\#_{p_i}^n X_i \equiv (\cdots((X_1\#_{p_1}X_2)\#_{p_2}X_3)\cdots\#_{p_{n-1}}X_n,$$

where each point p_k is identified with both a point of the interior of $\#_{p_i}^{k-1}X_i$ and a point of the interior of X_{k+1}. The operation taking a non-connected sum is associative. Namely, for example, $(X_1\#_{p_1}X_2)\#_{p_2}X_3 \approx X_1\#_{p_1}(X_2\#_{p_2}X_3)$, where for this we may assume that $p_2 \in X_2$. Note that the points p_1 and p_2 and the points p_k in more general may or may not be the same. Even if $p_i = p_j$ in $[0,1]$ with $i \neq j$, the attached points corresponding to p_i and p_j are assumed to be distinct. Therefore, we always have

$$\#_{p_i}^n X_i \approx L_1 \sqcup L_2 \sqcup \cdots \sqcup L_n \equiv \sqcup_i^n L_i,$$

where each L_i is a contractible space in $[0,1]$.

Proposition 2.8. *Let $X_1, \cdots, X_n$ be contractible spaces in $[0,1]$. Then a disconnected sum $\#_{p_i}^n X_i$ is a non-contractible, locally compact Hausforff space, and is compact if and only if each X_i is compact. We have $\partial(\#_{p_i}^n X_i) = \cup_i \partial X_i$.*

A non-compact $\#_{p_i}^n X_i$ is extended contractible if and only if $\#_{p_i}^n X_i$ is homeomorphic to the disjoint union $\sqcup^n[0,1)$. Hence, $C_0(\sqcup^n[0,1)) \cong \oplus^n C_0([0,1))$ is contractible to zero.

Proof. The first part is clear.

For the second, note that if a non-compact $\#_{p_i}^n X_i$ contains a X_i, homeomorphic to $(0,1)$, then the one-point compactification $(\#_{p_i}^n X_i)^+$ contains a circle embedded as a subset, so that it can not be contractible. □

Recall that the connected sum $M\#N$ of two topological manifolds M and N of dimension $d \geq 2$ is obtained by removing the d-dimensional closed unit ball B viewed in M and N and attaching $M \setminus B$ and $N \setminus B$ together with the boundary ∂B of B along. Note that ∂B is not contractible. Hence $M\#N$ is always not contractible even when M and N are contractible.

On the other hand, one can also define a **pointed jointed sum** of two spaces X and Y, denoted as $X \sqcup_p Y$, for a point p viewed in X and Y. More precisely, $X\#_p Y$ is defined by joining X and Y at p in the disjoint union $X \sqcup Y$. By definition, if X and Y are contractible, then the pointed jointed sum $X \sqcup_p Y$ is contractible.

Moreover, we can define inductively a **successive** pointed jointed sum of n spaces $X_1, \cdots, X_n$ as

$$\sqcup_{p_i}^n X_i \equiv (\cdots((X_1 \sqcup_{p_1} X_2) \sqcup_{p_2} X_3) \cdots \sqcup_{p_{n-1}} X_n,$$

where each point p_k is identified with both a point of $\sqcup_{p_i}^{k-1} X_i$ and a point of X_{k+1}. By definition, if $X_1, \cdots, X_n$ are contractible, then a successive pointed jointed sum $\sqcup_{p_i}^n X_i$ is contractible. To have associativity for successive pointed jointed sums, such as

$$(X_1 \sqcup_{p_1} X_2) \sqcup_{p_2} X_3 \approx X_1 \sqcup_{p_1} (X_2 \sqcup_{p_2} X_3),$$

we may assume that each p_i is in X_i. We **assume** this associativity in what follows. Note that homeomorphism classes of pointed jointed sums do depend on both the way of arrangement (or permutation with respect to i) of X_i and the choice (distinct or not) of the points p_i in general. For instance,

$$([0,1] \sqcup_{\frac{1}{3}} (0,1)) \sqcup_{\frac{1}{2}} [0,1] \not\approx ((0,1) \sqcup_{\frac{1}{3}} [0,1]) \sqcup_{\frac{1}{2}} [0,1].$$

Proof. Indeed, consider the interval $[\frac{1}{3}, \frac{1}{2}]$ viewed in the middle intervals. The jointed points $\frac{1}{3}$ and $\frac{1}{2}$ emit three intervals closed or open at the other end points respectively (2 closed and 1 open at $\frac{1}{3}$ and $\frac{1}{2}$ and 2 open and 1 closed at $\frac{1}{3}$ and 3 closed at $\frac{1}{2}$), whose respective parts in the jointed sums are not homeomorphic respectively. □

Proposition 2.9. *Let $X_1, \cdots, X_n$ be contractible spaces. A pointed jointed sum $\sqcup_{p_i}^n X_i$ is a contractible, locally compact Hausforff space, and is compact if and only if each X_i is compact, and $\partial(\sqcup_{p_i}^n X_i) = \cup_i \partial X_i$.*

A non-compact $\sqcup_{p_i}^n X_i$ is extended contractible if and only if its boundary has only one point.

Moreover, if each X_i is identically contractible, then $\sqcup_{p_i}^n X_i$ is identically contractible.

Proof. Note that for a non-compact $\sqcup_{p_i}^n X_i$, if $\partial(\sqcup_{p_i}^n X_i)$ has more than one point, then the one-point compactfication $(\sqcup_{p_i}^n X_i)^+$ contains a circle embedded as a subset and thus the compactification is not contractible.

Since each X_i is identically contractible by a homotopy, so is the jointed sum $\sqcup_{p_i}^n X_i$ by taking the (simultaneous) homotopy induced by the homotopies of X_i □

Let M and N be topological manifolds of dimension $d \geq 1$ (or greater than d). We define a d-dimensional **balled jointed sum** of M and N to be obtained by identifying the d-dimensional closed unit balls B viewed in M and N, and to be denoted by $M \sqcup_B N$.

Note that the 1-dimensional closed unit ball is the closed interval $[-1,1]$. Also, a pointed jointed sum may be defined to be a **zero**-dimensional jointed sum. Moreover, one can define inductively a **successive** d-dimensional (or at most) balled jointed sum of topological manifolds $M_1, \cdots, M_n$ of dimension d (or greater than d) by

$$\sqcup_{B_i}^n M_i \equiv (\cdots((M_1 \sqcup_{B_1} M_2) \sqcup_{B_2} M_3) \cdots) \sqcup_{B_{n-1}} M_n,$$

where each B_i is a d-dimensional (or at most) closed unit ball viewed in M_i and M_{i+1}. Note that the dimension d may not be constant as $\dim B_i = d_i$ for i. By definition, if $M_1, \cdots, M_n$ are contractible, then $\sqcup_{B_i}^n M_i$ is also contractible, but only a space, not a manifold in general. To have associativity for successive balled jointed sums, such as

$$(M_1 \sqcup_{B_1} M_2) \sqcup_{B_2} M_3 \approx M_1 \sqcup_{B_1} (M_2 \sqcup_{B_2} M_3),$$

we may assume that each B_i is in M_i. We **assume** this associativity in what follows. Note that homeomorphism classes of balled jointed sums do depend on both the way of arrangement (or permutation with respect to i) of M_i and the choice (distinct or not) of the balls B_i in general.

As a collection, we obtain

Table 1: Classification for contractible spaces and examples by C^*-algebras

C^*-algebras \ Spaces	Compact	Non-compact, contractible
Contractible to zero (non-unital)	No	Extended contractible: $I_1 = [0,1)$, $I_1^d = \Pi^d I_1$, $(\sqcup_{p_i}^{n-1} I) \sqcup_{p_{n-1}} I_1$, $(\sqcup_{B_i}^{n-1} I^d) \sqcup_{B_{n-1}} I_1^d$
Non-contractible to zero (unital or non-unital)	Contractible: $I = [0,1]$, I^d $\sqcup_{p_i}^n I$, $\sqcup_{B_i}^n I^d$	Non-extended contractible: $I_{0,1} = (0,1)$, $I_{0,1}^d \approx \mathbb{R}^d$, $\sqcup_{p_i}^{n+m+l} X_i$, $\sqcup_{B_i}^{n+m+l} X_i^d$ $(m+l \geq 2)$ $(X_i = I, I_1, I_{0,1}$ n, m, l copies$)$

Remark. There are non-contractible spaces whose C^*-algebras are contractible to zero, such as disjoint unions of extended contractible, non-compact locally compact Hausdorff spaces like $\sqcup^n [0,1)$.

It follows from the Table 1 that

Corollary 2.10. *The being or not being contractible to zero for C^*-algebras* (*together with unitalness or non-unitalness for C^*-algebras*) *classifies contractible spaces to be compact or non-compact and to be extended contractible or not.*

Remark. Note that compactness and non-compactness for spaces just correspond to unitalness and non-unitalness for C^*-algebras, respectively.

Now let X be a topological space. Denote by $\partial\overline{X}$ the boundary of $\overline{X}$, which is equal to $\overline{X} \setminus (\overline{X})^\circ$, where $\overline{X}$ is the closure of X in a suitable topology (or a suitable compactification of X along $\partial\overline{X}$) and $(\overline{X})^\circ$ is the interior of $\overline{X}$, where note that we mostly deal with topological spaces X viewed as (homeomorphically bounded) subsets with relative topology in Euclidean spaces and take their closures $\overline{X}$ in there. We may say that $\partial\overline{X} \setminus X = \overline{X} \setminus X$ is the **attached boundary** of X and $\overline{X}$ is the **flat** compactification of X.

Example 2.11. Let $I = [0, 1]$. Then $\partial I = \{0, 1\}$ and $\partial I \setminus I = \emptyset$, and also $\partial(I^d) \setminus I^d = \emptyset$.

Let $I_1 = [0, 1)$. Then $\overline{I_1} = [0, 1]$, $\partial\overline{I_1} = \{0, 1\}$ and $\partial\overline{I_1} \setminus I_1 = \overline{I_1} \setminus I_1 = \{1\}$.

Let $I_{0,1} = (0, 1)$. Then $\overline{I_{0,1}} = [0, 1]$, $\partial\overline{I_{0,1}} = \{0, 1\}$ and $\partial\overline{I_{0,1}} \setminus I_{0,1} = \{0, 1\}$.

We have $\overline{I_1^2} = [0, 1]^2$, and $\partial(\overline{I_1^2}) \setminus I_1^2 = (\{1\} \times [0, 1]) \cup ([0, 1] \times \{1\}) \approx [0, 1]$, which is contractible and has covering dimension one.

We have $\overline{I_{0,1}^2} = [0, 1]^2$, and $\partial(\overline{I_{0,1}^2}) \setminus I_{0,1}^2 \approx S^1$ the 1-dimensional sphere, which is not contractible and has covering dimension one.

Table 2: Classification for examples of contractible spaces by boundaries

Attached boundaries	Contractible spaces
No	Compact: $I = [0, 1]$, I^d, $\sqcup_{p_i}^n I$, $\sqcup_{B_i}^n I^d$
One point	Non-compact: $I_1 = [0, 1)$, $(\sqcup_{p_i}^{n-1} I) \sqcup_{p_{n-1}} I_1$ $(n \geq 2)$
Contractible, dimension $d - 1$	Non-compact: I_1^d, $(\sqcup_{B_i}^{n-1} I^d) \sqcup_{B_{n-1}} I_1^d$
Two points $m + 2l$ points	Non-compact: $I_{0,1} = (0, 1)$, $\sqcup_{p_i}^{n+m+l} X_i$, $\sqcup_{B_i}^{n+m+l} X_i$ $(m + 2l \geq 2)$ $(X_i = I, I_1, I_{0,1}$ n, m, l copies, resp)
Non-contractible, dim $d - 1$	$I_{0,1}^d \approx \mathbb{R}^d$ $(d \geq 2)$
Non-contractible, dim $d - 1$, $m + l$ components	$\sqcup_{B_i}^{n+m+l} X_i^d$ $(m + l \geq 2, d \geq 2)$ $(X_i = I, I_1, I_{0,1}$ n, m, l copies, resp)

It follows from the Tables 1 and 2 that

Corollary 2.12. *The being contractible and being unital or not for C^*-algebras, together with attached boundaries for spaces as similar invariants, and with dimension and pointed or balled jointedness for spaces or manifolds classify* (*up to homeomorphisms in part*) *1-dimensional, contractible manifolds and d-dimensional, jointed sums of d-dimensional contractible, their product manifolds, as in the collection lists above.*

Remark. The homeomorphism classes of the spaces $\sqcup_{p_i}^{n+m+l} X_i$ with $X_i = I$, I_1, or $I_{0,1}$ n, m, l copies respectively do depend on how to take the points p_j. For instance, all p_j may be the unique point, like $p_j = \frac{1}{2}$. Namely, the homeomorphism classes depend on that p_j are mutually, the same or different and as well their positions, in general. The similar things hold for $\sqcup_{B_i}^{n+m+l} X_i^d$.

It follows from the Table 3 (at the top of the next page) that

Corollary 2.13. *The being either unital and identically contractible to $\mathbb{C}$ or being non-unital and weakly identically contractible to $\mathbb{C}$ for C^*-algebras classifies contractible spaces to be compact or not to be.*

3 K-theory We now consider K-theory (abelian) groups for C^*-algebras.

It is known that if a C^*-algebra $\mathfrak{A}$ is contractible to zero, then the K-theory groups $K_0(\mathfrak{A})$ and $K_1(\mathfrak{A})$ both are zero, Note that the K-theory groups are homotopy invariant. In fact, the zero C^*-algebra $\{0\}$ has K_0 zero and the unitization $\{0\}^+ = \mathbb{C}$ has K_1 zero, so that the zero C^*-algebra has K_1 zero.

In particular,

Table 3: Classification for identically contractible spaces and examples by C^*-algebras

C^*-algebras \ Spaces	Compact	Non-compact, contractible
Unital, identically contractible to $\mathbb{C}$	Contractible: I^d, $\sqcup_{p_i}^n I^d$, $\sqcup_{B_i}^n I^d$	No
Non-unital, weakly identically contractible to $\mathbb{C}$	No	Extended: I_1^d, $(\sqcup_{p_i}^{n-1} I^d) \sqcup_{p_{n-1}} I_1^d$, $(\sqcup_{B_i}^{n-1} I^d) \sqcup_{B_{n-1}} I_1^d$ Non-extended: $I_{0,1}^d$, $\sqcup_{p_i}^{n+m+l} X_i$, $\sqcup_{B_i}^{n+m+l} X_i^d$ $(m+2l \geq 2)$ $(X_i = I, I_0, I_{0,1}$ n, m, l copies)

Example 3.1. Since $C_0([0,1)) = C\mathbb{C}$ the cone over $\mathbb{C}$ is contractible, it follows that $K_0(C_0([0,1))) \cong 0$ and $K_1(C_0([0,1))) \cong 0$. The same holds by replacing $[0,1)$ with $(\sqcup_{p_i}^{n-1} I) \sqcup_{p_{n-1}} I_1$ and also by $C\mathbb{C}$ with $C\mathfrak{A} \cong C_0([0,1)) \otimes \mathfrak{A}$ for any C^*-algebra $\mathfrak{A}$.

As a contrast, with (1) below certainly known ([8]),

Proposition 3.2. (1) *Let X be a contractible, compact space. Then*

$$K_0(C(X)) \cong \mathbb{Z} \quad \textit{and} \quad K_1(C(X)) \cong 0.$$

(2) *For a non-comapct space X, we have*

$$K_0(C_0(X)) \cong K_0(C(X^+))/\mathbb{Z} \quad \textit{and} \quad K_1(C_0(X)) \cong K_1(C(X^+)).$$

(3) *If a non-compact space X is extended contractible, then we have*

$$K_0(C_0(X)) \cong 0 \quad \textit{and} \quad K_1(C_0(X)) \cong 0.$$

Proof. The first statement (1) holds because $K_j(C(X)) \cong K_j(\mathbb{C})$ for $j = 0, 1$.

For the second (2), there is the short exact sequence of C^*-algebras:

$$0 \to C_0(X) \to C(X^+) \to \mathbb{C} \to 0$$

that splits, where the section from $\mathbb{C}$ to $C(X^+)$ is given by sending $1 \in \mathbb{C}$ to $1 \in C(X^+)$. The associated six-term exact sequence of K-theory groups implies that

$$K_j(C(X^+)) \cong K_j(C_0(X)) \oplus K_j(\mathbb{C})$$

for $j = 0, 1$, with $K_0(\mathbb{C}) \cong \mathbb{C}$ and $K_1(\mathbb{C}) = 0$.

The third (3) follows from (1) and (2) above. □

Example 3.3. We have $K_0(C([0,1])) \cong \mathbb{Z}$ and $K_0(C([0,1])) \cong 0$. Since a compact, pointed or balled, jointed sums $J = \sqcup_{p_i}^n I$ or $J = \sqcup_{B_i}^n I^d$ contractible, thus $K_0(C(J)) \cong \mathbb{Z}$ and $K_1(C(J)) \cong 0$.

There is the following short exact sequence of C^*-algebras:

$$0 \to C_0((0,1)) \to C_0([0,1)) \to \mathbb{C} \to 0,$$

which is not splitting, but the six-term exact sequence of K-theory groups, associated, becomes:

$$\begin{array}{ccccc} K_0(C_0((0,1))) & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \\ \partial \uparrow & & & & \downarrow \partial \\ 0 & \longleftarrow & 0 & \longleftarrow & K_1(C_0((0,1))) \end{array}$$

with the maps ∂ as the up and down arrows in the left and right, respectively, the index map and the exponential map (as a dual of the index map), and hence $K_0(C_0((0,1))) \cong 0$ and $K_1(C_0((0,1))) \cong \mathbb{Z}$.

The converses of (1) and (3) in the proposition above do not hold for contractible spaces.

Example 3.4. Let $X = \mathbb{R}^{2n}$ be the $2n$-dimensional Euclidean space, for $n \geq 1$, which is contractible but non-compact. Then

$$K_0(C_0(\mathbb{R}^{2n})) \cong K_0(\mathbb{C}) \cong \mathbb{Z} \quad \text{and} \quad K_1(C_0(\mathbb{R}^{2n})) \cong K_1(\mathbb{C}) \cong 0$$

by Bott periodicity of K-theory groups. Also, X^+ is homeomorphic to S^{2n} the $2n$-dimensional sphere, which is not contractible, because $K_0(C(S^{2n})) \cong \mathbb{Z}^2$ and $K_1(C(S^{2n})) \cong 0$, so that X is not extended contractible.

Let $X = \mathbb{R}^{2n} \times [0,1)$ the product space. Then

$$K_j(C_0(X)) \cong K_j(C_0(\mathbb{R}^{2n}) \otimes C_0([0,1))) \cong K_j(C_0([0,1))) \cong 0$$

for $j = 0, 1$. Also, X^+ is homeomophic to $S^{2n} \sqcup_1 I_1$, which is not contractible, because $S^{2n} \sqcup_1 I_1$ is homotopic to S^{2n}, so that X is not extended contractible.

Proposition 3.5. *Let $\#_{p_i} X_i$ be the successive non-connected sum of n contractible spaces $X_1, \cdots, X_n$ in $[0,1]$, with $\#_{p_i} X_i \approx \sqcup_{i=1}^n L_i$. Then*

$$K_j(C_0(\#_{p_i} X_i)) \cong \oplus_{i=1}^n K_j(C_0(L_i))$$

for $j = 0, 1$.

Proposition 3.6. *Let $X \sqcup_p Y$ be the (pointed) jointed sum of two spaces X, Y. If $X \sqcup_p Y$ is compact, then*

$$K_0(C(X \sqcup_p Y)) \cong K_0(C_0(X \setminus \{p\})) \oplus K_0(C_0(Y \setminus \{p\})) \oplus \mathbb{Z},$$
$$K_1(C(X \sqcup_p Y)) \cong K_1(C_0(X \setminus \{p\})) \oplus K_1(C_0(Y \setminus \{p\})),$$

and if $X \sqcup_p Y$ is not compact, then

$$K_0(C_0(X \sqcup_p Y)) \cong K_0(C_0(X \setminus \{p\})) \oplus K_0(C_0(Y \setminus \{p\})),$$
$$K_1(C_0(X \sqcup_p Y)) \cong [K_1(C_0(X \setminus \{p\})) \oplus K_1(C_0(Y \setminus \{p\}))]/\mathbb{Z}.$$

Proof. There is the following short exact sequence of C^*-algebras:

$$0 \to C_0(X \setminus \{p\}) \oplus C_0(Y \setminus \{p\}) \to C_0(X \sqcup_p Y) \to \mathbb{C} \to 0,$$

which splits only when $X \sqcup_p Y$ is compact, where the quotient map is the evaluation map at p. It follows that if $X \sqcup_p Y$ is compact, then

$$K_j(C(X \sqcup_p Y)) \cong K_j(C_0(X \setminus \{p\})) \oplus K_j(C_0(Y \setminus \{p\})) \oplus K_j(\mathbb{C})$$

for $j = 0, 1$. If $X \sqcup_p Y$ is not compact, then the induced quotient map from $K_0(C_0(X \sqcup_p Y))$ to $K_0(\mathbb{C})$ is zero, so that it follows from exactness of the six-term exact sequences of K-theory groups that

$$K_0(C_0(X \sqcup_p Y)) \cong K_0(C_0(X \setminus \{p\})) \oplus K_0(C_0(Y \setminus \{p\}))$$

and

$$K_1(C_0(X \sqcup_p Y)) \cong [K_1(C_0(X \setminus \{p\})) \oplus K_1(C_0(Y \setminus \{p\}))]/K_1(\mathbb{C}).$$

□

Moreover

Proposition 3.7. *Let $\sqcup_{p_i}^n X_i$ be the successive (pointed) jointed sum of n path-connected spaces $X_1, \cdots, X_n$. If $\sqcup_{p_i}^n X_i$ is compact, then*

$$K_0(C(\sqcup_{p_i}^n X_i)) \cong \oplus_{i=1}^n K_0(C_0(X_i \setminus \{p_{i-1}\})) \oplus \mathbb{Z},$$
$$K_1(C(\sqcup_{p_i}^n X_i)) \cong \oplus_{i=1}^n K_1(C_0(X_i \setminus \{p_{i-1}\})).$$

If $\sqcup_{p_i}^n X_i$ is not compact, then

$$K_0(C_0(\sqcup_{p_i}^n X_i)) \cong \oplus_{i=1}^n K_0(C_0(X_i \setminus \{p_{i-1}\})),$$
$$K_1(C_0(\sqcup_{p_i}^n X_i)) \cong [\oplus_{i=1}^n K_1(C_0(X_i \setminus \{p_{i-1}\}))]/\mathbb{Z}.$$

Proof. There is a homotopy between $X = \sqcup_{p_i}^n X_i$ and the jointed sum $Y = \sqcup_p^n X_i$ with the common point p as in the case where $p_i = p_{i+1}$ (identified) for $1 \le i \le n-2$. Then there is the following short exact sequence of C^*-algebras:

$$0 \to \oplus_{i=1}^n C_0(X_i \setminus \{p_{i-1}\}) \to C_0(Y) \to \mathbb{C} \to 0,$$

which splits only when Y is compact, where the quotient map is the evaluation map at the common point p and $X_i \setminus \{p\} \approx X_i \setminus \{p_{i-1}\}$. It follows that if Y is compact (if and only if X is compact), then

$$K_j(C(Y)) \cong [\oplus_{i=1}^n K_j(C_0(X_i \setminus \{p_{i-1}\}))] \oplus K_j(\mathbb{C})$$

for $j = 0, 1$. If Y is not compact, then the induced quotient map from $K_0(C_0(Y))$ to $K_0(\mathbb{C})$ is zero, so that it follows from exactness of the six-term exact sequences of K-theory groups that

$$K_0(C_0(Y)) \cong \oplus_{i=1}^n K_0(C_0(X_i \setminus \{p_{i-1}\}))$$

and $K_1(C(Y)) \cong [\oplus_{i=1}^n K_1(C_0(X_i \setminus \{p_{i-1}\}))]/K_1(\mathbb{C})$. □

As examples,

Example 3.8. Let $X = \sqcup_{p_i}^n I_1$ be a (pointed) jointed sum of n copies of $I_1 = [0, 1)$ ($n \ge 2$). Then

$$K_0(C_0(\sqcup_{p_i}^n I_1)) \cong 0 \quad \text{and} \quad K_1(C_0(\sqcup_{p_i}^n I_1)) \cong \mathbb{Z}^{n-1}.$$

This also holds for $n = 1$, with $\sqcup^1 I_1 = I_1$ and $\mathbb{Z}^0 = 0$.

Proof. There is a homotopy between X and $\sqcup_0^n I_1$ the jointed sum of n copies of I_1 at the common zero point 0. Because if $I_1 = [0, p_i) \cup [p_i, 1)$ and $[0, p_i)$ does not contain other p_j, then it is homotopic to $[p_i, 1)$ in X. We continue this process inductively and finitely to obtain the required homotopy.

When $n = 2$, X is homotopic to $(0, 1) \approx \sqcup_0^2 I_1$.

When $n = 3$, there is the following short exact sequence:

$$0 \to C_0((0, 1)) \to C_0(\sqcup_0^3 I_1) \to C_0(\sqcup_0^2 I_1) \to 0,$$

where $\sqcup_0^2 I_1$ in the quotient is homeomoprhic to $(0, 1)$ and closed in $\sqcup_0^3 I_1$, and its complement is $(0, 1)$ in the ideal. The six-term exact sequence of K-theory groups, associated, becomes:

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(C_0(\sqcup_0^3 I_1)) & \longrightarrow & 0 \\ \partial \big\uparrow & & & & \big\downarrow \partial \\ \mathbb{Z} & \longleftarrow & K_1(C_0(\sqcup_0^3 I_1)) & \longleftarrow & \mathbb{Z}. \end{array}$$

It follows that $K_0(C_0(\sqcup_0^3 I_1)) \cong 0$ and $K_1(C_0(\sqcup_0^3 I_1)) \cong \mathbb{Z}^2$.

By induction, we assume that $K_0(C_0(\sqcup_0^n I_1)) \cong 0$ and $K_1(C_0(\sqcup_0^n I_1)) \cong \mathbb{Z}^{n-1}$. Then there is the following short exact sequence:

$$0 \to C_0((0,1)) \to C_0(\sqcup_0^{n+1} I_1) \to C_0(\sqcup_0^n I_1) \to 0$$

since $\sqcup_0^n I_1$ is closed in $\sqcup_0^{n+1} I_1$ and its complement is $(0,1)$. The six-term exact sequence of K-theory groups, associated, becomes:

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(C_0(\sqcup_0^{n+1} I_1)) & \longrightarrow & 0 \\ \partial\big\uparrow & & & & \big\downarrow\partial \\ \mathbb{Z}^{n-1} & \longleftarrow & K_1(C_0(\sqcup_0^{n+1} I_1)) & \longleftarrow & \mathbb{Z}. \end{array}$$

It follows that $K_0(C_0(\sqcup_0^{n+1} I_1)) \cong 0$ and $K_1(C_0(\sqcup_0^{n+1} I_1)) \cong \mathbb{Z}^n$.

There is also the following short exact sequence:

$$0 \to C_0(\sqcup^n I_{0,1}) \to C_0(\sqcup_0^n I_1) \to \mathbb{C} \to 0,$$

which is not splitting, with $C_0(\sqcup_0^n I_1) \cong \oplus^n C_0((0,1))$. The six-term exact sequence of K-theory groups, associated, becomes:

$$\begin{array}{ccccc} \oplus^n 0 & \longrightarrow & K_0(C_0(Z)) & \longrightarrow & \mathbb{Z} \\ \partial\big\uparrow & & & & \big\downarrow\partial \\ 0 & \longleftarrow & K_1(C_0(Z)) & \longleftarrow & \oplus^n \mathbb{Z} \end{array}$$

and $K_0(C_0(Z)) \cong 0$ and $K_1(C_0(Z))) \cong \mathbb{Z}^{n-1}$. □

Example 3.9. Let $X = \sqcup_{p_i}^n I_{0,1}$ be a (pointed) jointed sum of n copies of $I_{0,1} = (0,1) \approx \mathbb{R}$ $(n \geq 2)$. Then

$$K_0(C_0(\sqcup_{p_i}^n I_{0,1})) \cong 0 \quad \text{and} \quad K_1(C_0(\sqcup_{p_i}^n I_{0,1})) \cong \mathbb{Z}^{2n-1}.$$

This also holds for $n = 1$, with $\sqcup^1 I_{0,1} = I_{0,1}$.

Proof. There is a homotopy between X and $\sqcup_0^{2n} I_1$ the jointed sum at the common zero point 0. By Proposition 3.7 above, we obtain the conclusion. □

Example 3.10. Let $X = \sqcup_{p_i}^{n+m+l} X_i$ be a (pointed) jointed sum of $X_i = I$, I_1, or $I_{0,1}$, with n copies of I, m copies of I_1, and l copies of $I_{0,1}$. Then

$$K_0(C_0(\sqcup_{p_i}^{n+m+l} X_i)) \cong 0 \quad \text{and} \quad K_1(C_0(\sqcup_{p_i}^{n+m+l} X_i)) \cong \mathbb{Z}^{m+2l-1}.$$

Proof. There is a homotopy between X and $\sqcup_0^{m+2l} I_1$ the jointed sum at the common zero point 0, as considered above. By Proposition 3.7 above, we obtain the conclusion. □

As 2-dimensional analogues as examples,

Example 3.11. Let $X = \sqcup_{p_i}^n (I^2)^-$ be a (pointed) jointed sum of n copies of $(I^2)^-$ the one-potint uncompactification of the 2-direct product of $I = [0,1]$. Then

$$K_0(C_0(X)) \cong 0 \quad \text{and} \quad K_1(C_0(X)) \cong \mathbb{Z}^{n-1}.$$

Proof. To determine $K_j(C_0(X))$, it is enough to compute $K_j(C_0((I^2)^- \setminus \{p_i\}))$. Then one can show that the space $(I^2)^- \setminus \{p_i\}$ is homotopic to $(0,1)$. Because p_i is different from the removed point (say q_i) of each I^2 to make $(I^2)^-$, and that I^2 is homotopic to a 1-dimensional closed interval with end points identified with p_i and q_i, so that $(I^2)^- \setminus \{p_i\}$ is homotopic to the interior of the interval. □

Quite similarly, as higher-dimensional analogues as examples,

Example 3.12. Let m be a positive integer with $m \geq 2$. Let $X = \sqcup_{p_i}^n (I^m)^-$ be a (pointed) jointed sum of n copies of $(I^m)^-$ the one-potint uncompactification of the m-direct product of $I = [0,1]$. Then

$$K_0(C_0(X)) \cong 0 \quad \text{and} \quad K_1(C_0(X)) \cong \mathbb{Z}^{n-1}.$$

Moreover,

Example 3.13. Let $X = \sqcup_{p_i}^n \mathbb{R}^2$ be a (pointed) jointed sum of n copies of $\mathbb{R}^2$. Then

$$K_0(C_0(X)) \cong \mathbb{Z}^n \quad \text{and} \quad K_1(C_0(X)) \cong \mathbb{Z}^{n-1}.$$

Proof. Note that $\mathbb{R}^2$ is viewed as $(S^2)^-$, so that $(S^2)^- \setminus \{p_i\}$ is homeomorphic to $S^1 \times \mathbb{R}$, where the removed two points from S^2 may be assumed to be north and south poles in S^2. Then we have $K_j(C_0(S^1 \times \mathbb{R})) \cong K_{j+1}(C(S^1)) \cong \mathbb{Z}$ for $j = 0, 1 \pmod 2$. □

Similarly,

Example 3.14. Let $X = \sqcup_{p_i}^n \mathbb{R}^{2m}$ be a (pointed) jointed sum of n copies of $\mathbb{R}^{2m}$. Then

$$K_0(C_0(X)) \cong \mathbb{Z}^n \quad \text{and} \quad K_1(C_0(X)) \cong \mathbb{Z}^{n-1}.$$

Proof. Note that $\mathbb{R}^{2m}$ is viewed as $(S^{2m})^-$, so that $(S^{2m})^- \setminus \{p_i\}$ is homeomorphic to $S^{2m-1} \times \mathbb{R}$, where we may assume that the removed two points from S^{2m} are north and south poles in S^{2m}. Then we have $K_j(C_0(S^{2m-1} \times \mathbb{R})) \cong K_{j+1}(C(S^{2m-1})) \cong \mathbb{Z}$ for $j = 0, 1$ (mod 2). □

On the other hand,

Example 3.15. Let $X = \sqcup_{p_i}^n \mathbb{R}^{2m+1}$ be a (pointed) jointed sum of n copies of $\mathbb{R}^{2m+1}$. Then

$$K_0(C_0(X)) \cong 0 \quad \text{and} \quad K_1(C_0(X)) \cong \mathbb{Z}^{2n-1}.$$

Proof. Note that $\mathbb{R}^{2m+1}$ is viewed as $(S^{2m+1})^-$, so that $(S^{2m+1})^- \setminus \{p_i\}$ is homeomorphic to $S^{2m} \times \mathbb{R}$, where we may assume that the removed two points from S^{2m+1} are north and south poles in S^{2m+1}. Then we have $K_j(C_0(S^{2m} \times \mathbb{R})) \cong K_{j+1}(C(S^{2m}))$ for $j = 0, 1$ (mod 2) and $K_0(C(S^{2m})) \cong \mathbb{Z}^2$ and $K_1(C(S^{2m})) \cong 0$. □

Furthermore,

Example 3.16. Let $X = \sqcup_{p_i}^{n+m} X_i$ be a (pointed) jointed sum of X_i of n Euclidean spaces with dimensions even and m Euclidean spaces with dimensions odd. Then

$$K_0(C_0(X)) \cong \mathbb{Z}^n \quad \text{and} \quad K_1(C_0(X)) \cong \mathbb{Z}^{n+2m-1}.$$

Next, we consider the balled case.

Proposition 3.17. *Let $M \sqcup_B N$ be the d-dimensional (balled) jointed sum of two topological manifolds M, N of dimension d (or greater than d). If $M \sqcup_B N$ is compact, then*

$$K_0(C(M \sqcup_B N)) \cong K_0(C_0(M \setminus B)) \oplus K_0(C_0(N \setminus B))) \oplus \mathbb{Z},$$
$$K_1(C(M \sqcup_B N)) \cong K_1(C_0(M \setminus B))) \oplus K_1(C_0(N \setminus B))),$$

and if $M \sqcup_B N$ is not compact, then

$$K_0(C_0(M \sqcup_B N)) \cong K_0(C_0(M \setminus B))) \oplus K_0(C_0(N \setminus B))),$$
$$K_1(C_0(M \sqcup_B N)) \cong [K_1(C_0(M \setminus B)) \oplus K_1(C_0(N \setminus B))]/\mathbb{Z}.$$

Proof. The proof is exactly the same as that for Proposition 3.6. Note that $K_j(C(B)) \cong K_j(\mathbb{C})$ for $j = 0, 1$ and the d-dimensional closed ball B is contractible. □

Moreover,

Proposition 3.18. *Let $\sqcup_{B_i}^n M_i$ be the successive d-dimensional (balled) jointed sum of path-connected, topological manifolds $M_1, \cdots, M_n$ of dimension d (or greater than d). If $\sqcup_{B_i}^n M_i$ is compact, then*

$$K_0(C(\sqcup_{B_i}^n M_i)) \cong \oplus_{i=1}^n K_0(C_0(M_i \setminus B_{i-1})) \oplus \mathbb{Z},$$
$$K_1(C(\sqcup_{B_i}^n M_i)) \cong \oplus_{i=1}^n K_1(C_0(M_i \setminus B_{i-1})).$$

If $\sqcup_{B_i}^n M_i$ is not compact, then

$$K_0(C_0(\sqcup_{B_i}^n M_i)) \cong \oplus_{i=1}^n K_0(C_0(M_i \setminus B_{i-1})),$$
$$K_1(C_0(\sqcup_{B_i}^n M_i)) \cong [\oplus_{i=1}^n K_1(C_0(M_i \setminus B_{i-1}))]/\mathbb{Z}.$$

Proof. The proof is exactly the same as that for Proposition 3.7. □

Example 3.19. Let $M = \sqcup_{B_i}^n I_1^d$, with $I_1 = [0, 1)$ and $n \geq 2$. Then

$$K_0(C_0(M)) \cong 0 \quad \text{and} \quad K_1(C_0(M)) \cong \mathbb{Z}^{n-1}.$$

If $M = I_1^d$, then $K_0(C_0(M)) \cong 0 \cong K_1(C_0(M))$.

Proof. We compute $K_j(C_0(I_1^d \setminus B_i))$. Since each ball B_i is contractible, there is the following short exact sequence of C^*-algebras:

$$0 \to C_0(I_1^d \setminus B_i) \to C_0(I_1^d) \to \mathbb{C} \to 0.$$

Since $C_0(I_1^d) \cong \otimes^d C_0(I_1)$ is a contractible C^*-algebra, hence $K_j(C_0(I_1^d)) \cong 0$ for $j = 0, 1$. Note also that the space I_1^d is extended contractible since $(I_1^d)^+$ is contractible. It follows from the six-term exact sequence of K-theory groups that

$$K_0(C_0(I_1^d \setminus B_i)) \cong 0 \quad \text{and} \quad K_1(C_0(I_1^d \setminus B_i)) \cong \mathbb{Z}.$$

□

Example 3.20. Let $M = \sqcup_{B_i}^n I_{0,1}^d$, with $I_{0,1} = (0, 1)$. If d is even, then

$$K_0(C_0(M)) \cong \mathbb{Z}^n \quad \text{and} \quad K_1(C_0(M)) \cong \mathbb{Z}^{n-1},$$

and if d is odd, then $K_0(C_0(M)) \cong 0$ $K_1(C_0(M)) \cong \mathbb{Z}^{2n-1}$.

Proof. We compute $K_j(C_0(I_{0,1}^d \setminus B_i))$. Since each ball B_i is contractible, there is the following short exact sequence of C^*-algebras:

$$0 \to C_0(I_{0,1}^d \setminus B_i) \to C_0(I_{0,1}^d) \to \mathbb{C} \to 0.$$

Since $C_0(I_{0,1}^d) \cong \otimes^d C_0(\mathbb{R}) = S^d\mathbb{C}$, we have $K_0(S^d\mathbb{C}) \cong \mathbb{Z}$ and $K_0(S^d\mathbb{C}) \cong 0$ if d is even and $K_0(S^d\mathbb{C}) \cong 0$ and $K_0(S^d\mathbb{C}) \cong \mathbb{Z}$ if d is odd. It follows from the six-term exact sequence of K-theory groups that if d is even, then

$$K_0(C_0(I_{0,1}^d \setminus B_i)) \cong \mathbb{Z} \quad \text{and} \quad K_1(C_0(I_{0,1}^d \setminus B_i)) \cong \mathbb{Z},$$

and if d is odd, then $K_0(C_0(I_{0,1}^d \setminus B_i)) \cong 0$ and $K_1(C_0(I_{0,1}^d \setminus B_i)) \cong \mathbb{Z}^2$. $\square$

Furthermore, combining Examples 3.19 and 3.20 with Proposition 3.18 we obtain

Example 3.21. Let $M = \sqcup_{B_i}^{n+m+l} X_i^d$, where X_i are n, m, l copies of $I, I_1, I_{0,1}$ respectively. If $m + l \geq 1$, then M is non-compact, and if d is even, then

$$K_0(C_0(M)) \cong \mathbb{Z}^l \quad \text{and} \quad K_1(C_0(M)) \cong \mathbb{Z}^{m+l-1}$$

and if d is odd, then $K_0(C_0(M)) \cong 0$ and $K_1(C_0(M)) \cong \mathbb{Z}^{m+2l-1}$.

Table 4: Classification for contractible spaces by K-theory of C^*-algebras

K-theory of C^*-algebras	Contractible spaces
$K_0 = 0$, $K_1 = 0$	Non-compact, extended contractible: I_1, $(I^n)^- \approx I_1^n$ $(n \geq 2)$,
$K_0 = \mathbb{Z}$, $K_1 = 0$	Compact: I^n Noncompact, non-extended: $I_{0,1}^{2n} \approx \mathbb{R}^{2n}$
$K_0 = \mathbb{Z}^n$, $K_1 = \mathbb{Z}^{n-1}$	$\sqcup_{p_i}^n \mathbb{R}^{2m}$ (pointed), $\sqcup_{B_i}^n I_{0,1}^{2m}$ (balled)
$K_0 = 0$, $K_1 = \mathbb{Z}$	Noncompact, non-extended: $I_{0,1}^{2n+1} \approx \mathbb{R}^{2n+1}$, $\sqcup_p^2 I_1$, $\sqcup_p^2 (I^m)^-$ (pointed), $\sqcup_B^2 I_1^d$ (balled)
$K_0 = 0$, $K_1 = \mathbb{Z}^{n-1}$	$\sqcup_{p_i}^n I_1$, $\sqcup_{p_i}^n (I^m)^-$ (pointed), $\sqcup_{B_i}^n I_1^d$ (balled)
$K_0 = 0$, $K_1 = \mathbb{Z}^{m+2l-1}$,	$\sqcup_{p_i}^{n+m+l} X_i$, $\sqcup_{B_i}^{n+m+l} X_i^{2d+1}$, $(X_i = I, I_1, I_{0,1}$ n, m, l copies, $m, l \geq 1)$,
$K_0 = 0$, $K_1 = \mathbb{Z}^{2n-1}$	$\sqcup_{p_i}^n \mathbb{R}$, $\sqcup_{p_i}^n \mathbb{R}^{2m+1}$ (pointed), $\sqcup_{B_i}^n I_{0,1}^{2m+1}$ (balled)
$K_0 = \mathbb{Z}^l, K_1 = \mathbb{Z}^{m+2l-1}$,	$\sqcup_{B_i}^{n+m+l} X_i^{2d}$, $(X_i = I, I_1, I_{0,1}$ n, m, l copies, $m, l \geq 1)$,
$K_0 = \mathbb{Z}^n, K_1 = \mathbb{Z}^{n+2m-1}$,	$\sqcup_{p_i}^{n+m} X_i$, $\sqcup_{B_i}^{n+m} X_i$ (dim mixed), with $X_i = \mathbb{R}^{2n_i}$ $(1 \leq i \leq n)$, $X_i = \mathbb{R}^{2m_i+1}$ $(n+1 \leq i \leq n+m)$

It follows from the Table 4 that

Corollary 3.22. *The ranks of K-theory groups for C^*-algebras (together with compactness of spaces and dimension of spaces and that of balls in (generic) jointed sums and with jointedness (jointed or not) and with arrangement (or permutation) in jointed sums) classify contractible spaces as in the table (up to homeomorphisms) and to be compact, non-compact and extended, or non-compact and non-extended.*

Remark. Similarly, one can obtain almost the same table for identically contractible spaces.

In the statements above and below, to obtain classification results up to homeomorphisms we may **assume** that pointed or balled jointed sums are generic, i.e., points or balls involved are mutually distinct.

Recall ([5] or [6]) that the Euler characteristic $\chi(\mathfrak{A})$ of a C^*-algebra $\mathfrak{A}$ is defined to be the (formal) difference:

$$\chi(\mathfrak{A}) = \mathrm{rank}_{\mathbb{Z}} K_0(\mathfrak{A}) - \mathrm{rank}_{\mathbb{Z}} K_1(\mathfrak{A}) \in \mathbb{Z} \cup \{\pm\infty\} \cup \{\infty - \infty\}$$

of the $\mathbb{Z}$-ranks of the free abelian direct summands of the K-theory groups of $\mathfrak{A}$. In particular, it is shown that $\chi(C(X)) = \chi(X)$, where $\chi(X)$ is the Euler characteristic of a compact space (or a finite cell complex) X in homology (or cohomology) for spaces.

What's more, it is deduced from the table 4 above that

Table 5: Classification for contractible spaces by the Euler characteristic

Euler numbers of C^*-algebras	Contractible spaces
Zero: $\chi = 0 - 0 = 0$	Non-compact, extended contractible: I_1, $(I^n)^- \approx I_1^d$ $(n \geq 2)$
Positive: $\chi = 1 - 0 = 1 > 0$ $\chi = n - (n-1) = 1 > 0$	Compact: I^n Noncompact, non-extended: (even dim): $I_{0,1}^{2n} \approx \mathbb{R}^{2n}$, $\sqcup_{p_i}^n \mathbb{R}^{2m}$ (pointed), $\sqcup_{B_i}^n I_{0,1}^{2m}$ (balled)
Negative: $\chi = 0 - 1 = -1 < 0$	Noncompact, non-extended: (odd dim): $I_{0,1}^{2n+1} \approx \mathbb{R}^{2n+1}$, 2-fold: $\sqcup_p^2 I_1$, $\sqcup_p^2 (I^m)^-$ (pointed), $\sqcup_B^2 I_1^d$ (balled)
$\chi = 0 - (n-1) = 1 - n < 0$ $\chi = 0 - (m + 2l - 1)$ $= 1 - m - 2l < 0$ $\chi = 0 - (2n - 1)$ $1 - 2n < 0$	n-fold: $\sqcup_{p_i}^n I_1$, $\sqcup_{p_i}^n (I^m)^-$ (pointed), $\sqcup_{B_i}^n I_1^d$ (balled) $\sqcup_{p_i}^{n+m+l} X_i$, $\sqcup_{B_i}^{n+m+l} X_i^{2d+1}$, $(X_i = I, I_1, I_{0,1}$ n, m, l copies, $m, l \geq 1)$, n-fold (odd dim): $\sqcup_{p_i}^n \mathbb{R}$, $\sqcup_{p_i}^n \mathbb{R}^{2m+1}$ (pointed), $\sqcup_{B_i}^n I_{0,1}^{2m+1}$ (balled)
$\chi = l - (m + 2l - 1)$ $= 1 - m - l < 0$	$\sqcup_{B_i}^{n+m+l} X_i^{2d}$, $(X_i = I, I_1, I_{0,1}$ n, m, l copies, $m, l \geq 1)$
$\chi = n - (n + 2m - 1)$ $= 1 - 2m < 0$	$\sqcup_{p_i}^{n+m} X_i$, $\sqcup_{B_i}^{n+m} X_i$ (dim mixed), with $X_i = \mathbb{R}^{2n_i}$ $(1 \leq i \leq n)$, $X_i = \mathbb{R}^{2m_i+1}$ $(n+1 \leq i \leq n+m)$

It follows from the Table 5 that

Corollary 3.23. *The numbers or signs* (*being positive, zero, or negative*) *of the Euler characteristic for C^*-algebras* (*together with compactness, dimension, jointedness of spaces, and arrangement* (*or permutation*) *in* (*generic*) *jointed sums*) *classify contractible spaces as in the table* (*up to homeomorphisms*) *and to be compact, non-compact and extended, or non-compact and non-extended.*

Remark. Our classification tables obtained as collections in this paper would be useful for further classification of contractible spaces in more general, with more examples as representatives to be added.

Once more,

Corollary 3.24. *Our classfication tables say that contractible spaces restricted to examples viewed as representatives of equivalence classes by homeomorphisms are classifiable by their corresponding C^*-algebras and K-theory data, plus, compactness, dimension, pointed or balled jointedness for spaces, and arrangement* (*or permutation*) *in* (*generic*) *jointed sums, as complete invariants.*

Remark. The covering dimension for spaces as an invariant can be replaced by the real rank for C^*-algebras ([3]). Being compact for spaces corresponds to being unital for their corresponding C^*-algebras. Also, being jointed for spaces corresponds to being jointed for their corresponding C^*-algebras, and arrangement (or permutation) in jointed sums for spaces corresponds to that in jointed sums for their corresponding C^*-algebras.

Corollary 3.25. *Both the ranks of K-theory groups for C^*-algebras and the Euler characteristic for C^*-algebras can not classify jointedness for spaces, and as well, can not do pointed or balled jointed sums of contractible spaces, up to arrangement* (*or permutation*), *in general, except that all the components in jointed sums are the same.*

However, if restricted to this exceptional case, and further restricted with dimension fixed in spaces and balls in (*generic*) *jointed sums, the ranks and the Euler characteristic together with compactness and jointedness for spaces can be complete invariants to classify the contractible spaces as in the lists above, up to homeomorphisms.*

Consequently, we obtain

Corollary 3.26. *Let M, N be product manifolds of finitely many 1-dimensional contractible manifolds. Then the d and d'-dimensional* (*with $d, d' \geq 0$*), *jointed sums $\sqcup_{B_i}^n M$ and $\sqcup_{B_i'}^m N$ are homeomorphic,* (*which is equivalent to that*

$$C(\sqcup_{B_i}^n M) \cong C(\sqcup_{B_i'}^m N) \quad or \quad C_0(\sqcup_{B_i}^n M) \cong C_0(\sqcup_{B_i'}^m N),$$

where both M and N are compact or not), *if and only if*

$$K_j(C(\sqcup_{B_i}^n M)) \cong K_j(C(\sqcup_{B_i'}^m N)) \quad or \quad K_j(C_0(\sqcup_{B_i}^n M)) \cong K_j(C_0(\sqcup_{B_i'}^m N))$$

for $j = 0, 1$, and $n = m$ (*jointedness*), *and $\dim M = \dim N$ and $\dim B_i = d = d' = \dim B_i'$ for every i.*

Furthermore, the K-theory group isomorphisms can be replaced by

$$\chi(C(\sqcup_{B_i}^n M)) = \chi(C(\sqcup_{B_i'}^m N)) \quad or \quad \chi(C_0(\sqcup_{B_i}^n M)) = \chi(C_0(\sqcup_{B_i'}^m N)),$$

with the same other conditions.

Proof. As a note, suppose that there is a homeomorphism $\varphi : X \to Y$ of locally compact Hausdorff spaces. Then there is a $*$-isomorphism $\psi : C_0(Y) \to C_0(X)$ defined by $\psi(f) = f \circ \varphi$ for $f \in C_0(Y)$. The converse also holds by that X is the spectrum of $C_0(X)$ by the Gelfand transform. □

4 Noncommutative jointed sums We may say that a **jointed sum** of two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ with a common quotient $\mathfrak{D}$ is defined to be the pull back C^*-algebra $\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}$ as

$$\begin{tikzcd}
\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B} = \{(a,b) \in \mathfrak{A} \oplus \mathfrak{B} \,|\, \varphi(a) = \psi(b)\} \arrow[r, "\rho"] \arrow[d, "\pi"'] & \mathfrak{B} \arrow[d, "\psi"] \\
\mathfrak{A} \arrow[r, "\varphi"] & \mathfrak{D}
\end{tikzcd}$$

where $\varphi : \mathfrak{A} \to \mathfrak{D}$ and $\psi : \mathfrak{B} \to \mathfrak{D}$ are quotient maps and $\pi : \mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B} \to \mathfrak{A}$ and $\rho : \mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B} \to \mathfrak{B}$ are natural projections.

The Mayer-Vietoris sequence for K-theory of C^*-algebras (see [1]) is the following six-term diagram:

$$\begin{tikzcd}
K_0(\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \arrow[r, "(\pi_*, \rho_*)"] & K_0(\mathfrak{A}) \oplus K_0(\mathfrak{B}) \arrow[r, "\psi_* - \varphi_*"] & K_0(\mathfrak{D}) \arrow[d] \\
K_1(\mathfrak{D}) \arrow[u] & K_1(\mathfrak{A}) \oplus K_1(\mathfrak{B}) \arrow[l, "\psi_* - \varphi_*"'] & K_1(\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \arrow[l, "(\pi_*, \rho_*)"']
\end{tikzcd}$$

In particular, it follows that

Proposition 4.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be contractible C^*-algebras with a common quotient $\mathfrak{D}$ that is contractible to $\mathbb{C}$. Then*

$$K_0(\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \cong 0 \quad \text{and} \quad K_1(\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \cong \mathbb{Z}.$$

Proof. Indeed, the Mayer-Vietoris sequence becomes in this case:

$$\begin{tikzcd}
K_0(\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \arrow[r, "(\pi_*, \rho_*)"] & 0 \oplus 0 \arrow[r, "\psi_* - \varphi_*"] & \mathbb{Z} \arrow[d] \\
0 \arrow[u] & 0 \oplus 0 \arrow[l, "\psi_* - \varphi_*"'] & K_1(\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}). \arrow[l, "(\pi_*, \rho_*)"']
\end{tikzcd}$$

□

Now suppose that the jointed sum C^*-algebra $\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}$ and a C^*-algebra $\mathfrak{C}$ have a common quotient E. Then one can define a **successive** jointed sum of three C^*-algebras

$$(\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \oplus_E \mathfrak{C}$$

as the successive pull back C^*-algebra. Note that the associativity for successive jointed sums may not hold or not be defined in general. To have the associativity as

$$(\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \oplus_E \mathfrak{C} \cong \mathfrak{A} \oplus_{\mathfrak{D}} (\mathfrak{B} \oplus_E \mathfrak{C})$$

we further need to assume that E is a common quotient of $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}$.

Proposition 4.2. *Let $(\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \oplus_E \mathfrak{C}$ be a successive jointed sum C^*-algebra of contractible C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ with quotients $\mathfrak{D}$ and E that are contractible to $\mathbb{C}$. Then*

$$K_0((\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \oplus_E \mathfrak{C}) \cong 0 \quad \text{and} \quad K_1((\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \oplus_E \mathfrak{C}) \cong \mathbb{Z}^2.$$

Proof. Indeed, the Mayer-Vietoris sequence becomes in this case:

$$\begin{array}{ccccc} K_0((\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \oplus_E \mathfrak{C}) & \xrightarrow{(\pi_*,\rho_*)} & 0 \oplus 0 & \xrightarrow{\psi_* - \varphi_*} & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \xleftarrow{\psi_* - \varphi_*} & \mathbb{Z} \oplus 0 & \xleftarrow{(\pi_*,\rho_*)} & K_1((\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \oplus_E \mathfrak{C}), \end{array}$$

where $\pi : (\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \oplus_E \mathfrak{C} \to \mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}$ and $\rho : (\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \oplus_E \mathfrak{C} \to \mathfrak{C}$ by the same symbols as for $\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}$, for convenience. $\square$

Inductively, one can define a successive jointed sum of C^*-algebras $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ with quotients $\mathfrak{D}_1, \cdots, \mathfrak{D}_{n-1}$ as

$$\oplus_{\mathfrak{D}_i}^n \mathfrak{A}_i \equiv (\cdots((\mathfrak{A}_1 \oplus_{\mathfrak{D}_1} \mathfrak{A}_2) \oplus_{\mathfrak{D}_2} \mathfrak{A}_3) \cdots) \oplus_{\mathfrak{D}_{n-1}} \mathfrak{A}_n.$$

Note that the associativity for the successive jointed sums may not hold or not be defined in general. To have the associativity as in the 3-fold case, we further need to assume that the quotients are more common to have this as in the 3-fold case.

Proposition 4.3. *Let $\oplus_{\mathfrak{D}_i}^n \mathfrak{A}_i$ be a successive jointed sum C^*-algebra of contractible C^*-algebras $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ with quotients $\mathfrak{D}_1, \cdots, \mathfrak{D}_{n-1}$ that are contractible to $\mathbb{C}$. Then*

$$K_0(\oplus_{\mathfrak{D}_i}^n \mathfrak{A}_i) \cong 0 \quad \textit{and} \quad K_1(\oplus_{\mathfrak{D}_i}^n \mathfrak{A}_i) \cong \mathbb{Z}^{n-1}.$$

Proof. We use induction by the same way as in the proof above. $\square$

Corollary 4.4. *The jointed sum of two contractible C^*-algebras with a common quotient that is contractible to $\mathbb{C}$ is not contractible.*

As well, the successive jointed sum of n contractible C^*-algebras with successive common quotients that are contractible to $\mathbb{C}$ is not contractible.

Remark. Since a contractible C^*-algebra $\mathfrak{A}$ has K-theory groups zero, the Künneth formula in K-theory for C^*-algebras implies that any tensor product of $\mathfrak{A}$ with any other C^*-algebra $\mathfrak{B}$ has K-theory groups zero if $\mathfrak{A}$ or $\mathfrak{B}$ is in the bootstrap category.

What's more. As an interest, we obtain

Proposition 4.5. *Let $\mathfrak{A}$ be a contractible C^*-algebra. Then any C^*-tensor product $\mathfrak{A} \otimes \mathfrak{B}$ with any C^*-algebra $\mathfrak{B}$ is contractible.*

It follows that $K_j(\mathfrak{A} \otimes \mathfrak{B}) \cong 0$ for $j = 0, 1$.

Proof. There is a continuous homotopy (φ_t) between the identity map $\mathrm{id}_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}$ and the zero map $0 : \mathfrak{A} \to \mathfrak{A}$, with $\varphi_1 = \mathrm{id}_{\mathfrak{A}}$ and $\varphi_0 = 0$. For any simple tensor $a \otimes b \in \mathfrak{A} \otimes \mathfrak{B}$, we define maps $\psi_t : \mathfrak{A} \otimes \mathfrak{B} \to \mathfrak{A} \otimes \mathfrak{B}$ by $\psi_t(a \otimes b) = \varphi_t(a) \otimes b$, which extends to $*$-homomorphism from $\mathfrak{A} \otimes \mathfrak{B}$ to $\mathfrak{A} \otimes \mathfrak{B}$. Then (ψ_t) gives a continuous homotopy between the identity map $\mathrm{id}_{\mathfrak{A} \otimes \mathfrak{B}} : \mathfrak{A} \otimes \mathfrak{B}$ and the zero map $0 : \mathfrak{A} \otimes \mathfrak{B} \to \mathfrak{A} \otimes \mathfrak{B}$.

Indeed, any element $x \in \mathfrak{A} \otimes \mathfrak{B}$ is approximated by finite sums of simple tensors, so that $x = \lim_{n\to\infty} \sum_{k=1}^n a_k \otimes b_k \equiv \lim_{n\to\infty} s_n$. Then define

$$\psi_t(x) = \lim_{n\to\infty} \psi_t(s_n) = \lim_{n\to\infty} \psi_t(\sum_{k=1}^n a_k \otimes b_k) = \lim_{n\to\infty} \sum_{k=1}^n \varphi_t(a_k) \otimes b_k,$$

which is well defined. Then

$$\begin{aligned} &\|\psi_t(x) - \psi_s(x)\| \\ &\leq \|\psi_t(x) - \psi_t(s_n)\| + \|\psi_t(s_n) - \psi_s(s_n)\| + \|\psi_s(s_n) - \psi_s(x)\|, \end{aligned}$$

which is arbitrary small when n is large enough and $|t - s|$ is small enough. $\square$

Remark. As for examples of noncommutative jointed sums, see the commutative cases in the previous sections. One (principal case) of noncommutaive cases can be also obtained as taking tensor products of C^*-algebras $\mathfrak{A}_i$ with commutative C^*-algebras $C_0(X_i)$ and taking their jointed sums, with quotients (of $\mathfrak{A}_i$ or $C_0(X_i)$) involved to be assumed. If the K-theory groups of $\mathfrak{A}_i$ are computable, then so are the K-theory groups of the jointed sums. As the other cases, tensor products may be replaced by other operations such as crossed products of C^*-algebras with suitable actions and free products of C^*-algebras.

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MULTIPLIERS WITH CLOSED RANGE ON FRÉCHET ALGEBRAS

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ABSTRACT. In this paper, we determine several equivalent conditions pertaining to closed range multipliers defined on a semisimple Fréchet locally m-convex algebra. Moreover, we give a complete description of the point spectrum and the residual spectrum of multipliers.

1. INTRODUCTION

The investigation of closed range multipliers, in the context of commutative semisimple Banach algebras was initiated by Glicksberg [8] in 1971, whereby he raised the following question: If T is a multiplier on a commutative semisimple Banach algebra A, whether a factorization $T = PB$, where P is an idempotent and B an invertible multiplier, is necessary and sufficient to ensure the closedness of TA? This problem was partially resolved by Host and Parreau [12] for a particular situation of the group algebra $L^1(G)$, where G is a locally compact abelian group. Various equivalent conditions have been determined in [17] for a multiplier T defined on a semisimple Banach algebra to have closed range.

It is quite natural to ask whether the above characterization of closed range multipliers holds for a semisimple Fréchet locally m-convex algebra A. In this paper, we consider this problem and establish several equivalent conditions pertaining to closed range multipliers on A. Precisely, we prove that if A has a bounded approximate identity, then TA is a closed ideal with a bounded approximate identity if and only if T admits a factorization $T = PB$ with P an idempotent and B an invertible multiplier. Moreover, if A is also a Fréchet locally C*-algebra then T has closed range if and only if $T^2A = TA$. Also, in this case, T is injective if and only if it is surjective.

Finally, we discuss the spectral properties of multipliers defined on a simisimple commutative Fréchet locally m-convex algebra A. The investigation of spectral properties of a multiplier T defined on $L^1(G)$ was initiated by Zafran [22]. Successively this problem was studied by several other authors in the framework of commutative semisimple Banach algebras. We study this problem in the more abstract situation of (non-normed) topological algebras. We show that if the maximal ideal space $\Delta(A)$ is discrete, then the point spectrum is completely characterized by $\sigma_p(T) = \mu^T(\Delta(A))$. Under the assumption that socle of A is dense in A, we establish that the residual spectrum of T is empty.

2. CLOSED RANGE MULTIPLIERS

Before investigating certain features of a multiplier with closed range, we need to establish our preliminaries. A Hausdorff topological algebra A whose topology is generated by a family $\{p_\alpha : \alpha \in \Lambda\}$ of seminorms is called a *locally convex* algebra. Moreover, if each seminorm p_α is also submultiplicative, i.e.,

$$p_\alpha(xy) \leq p_\alpha(x)\, p_\alpha(y), \text{ for all } x, y \in A,$$

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then A is called a *locally m-convex* algebra. Usually, a complete metrizable locally convex (resp. locally m-convex) algebra is called a *Fréchet locally convex* (resp. *Fréchet locally m-convex*) algebra.

Given a semisimple Fréchet locally convex algebra A, then following [13], a mapping $T : A \to A$ is said to be a *multiplier* if $x\,(Ty) = (Tx)y$ holds for all $x, y \in A$. We denote the set of all multipliers on A by $M(A)$. Since A is semisimple, any $T \in M(A)$ turns out to be linear and the identity $x\,(Ty) = T(xy)$ holds for any $x, y \in A$. Using the closed graph theorem, the definition of a multiplier, and the semisimplicity of A, one can show that all multipliers are necessarily continuous and hence bounded (see for instance, [13], Corollary 2.3). Moreover, $M(A)$ is a closed subalgebra of $B(A)$ with respect to the strong operator topology, where $B(A)$ denotes the algebra of all continuous (or bounded) linear operators on A. Also, $M(A)$ is commutative (see for instance, [13], Theorem 2.4) and has an identity element. An application of the identity $x\,(Ty) = T(xy)$ for all $x,\ y \in A$, yields that both TA and $\ker T$ are two sided ideals of A, where TA and $\ker T$ denote the range and kernel of T, respectively.

In this work, we want to study closed range multipliers on A. In [12], Host and Parreau have established that if $A = L^1(G)$, where G is a locally compact abelian group, and if T is a multiplier on $L^1(G)$, then TA is closed if and only if $T = PB$, where P is an idempotent and T an invertible multiplier. Thus they partially resolved the interesting problem due to Glicksberg [8] whether the factorization $T = PB$ is necessary and sufficient to ensure the closedness of TA for any multiplier T on a semisimple commutative Banach algebra A. Various equivalent conditions have been determined in [1] , [17] and [21] under which a multiplier T has closed range. Our aim is to consider this problem for a more general situation in (non-normed) topological algebras.

We recall that an operator $T \in B(A)$ has a *generalized inverse* (abbreviated as *g-inverse*), if there is an operator $S \in B(A)$ such that $T = TST$ and $S = STS$. The operator T is also called *relatively regular* [10]. We want to make a few observations about these operators.

Remark 1. *(i) There is no loss of generality in requiring only that $T = TST$. In fact, if $T = TST$, then $S' = STS$ will satisfy $T = TS'T$, as well as $S' = S'TS'$.*

(ii) If $T = TST$ and $S = STS$, then TS and ST are idempotents and hence projections for which $TS\,(A) = T\,(A)$ and $\ker T = \ker ST$. Indeed, $(TS)^2 = TSTS = TS$ and $(ST)^2 = STST = ST$. Moreover, from $T(A) = TST(A) \subseteq TS(A) \subseteq T(A)$ and $\ker T \subseteq \ker(ST) \subseteq \ker(TST) = \ker T$, we obtain $TS(A) = T(A)$ and $\ker(ST) = (I - ST)A = \ker T$, where I denotes the identity element in $B(A)$.

(iii) Generally speaking, a generalized inverse of T is rarely uniquely determined. For instance, if $T = TST$, then S can be anything on $\ker(T)$. But there is at most one generalized inverse which commutes with the given $T \in B(A)$. In fact, if S and S' are g-inverses of T, both commuting with T, then $TS' = TSTS' = ST$, and hence $S' = S'TS' = S'TS = STS = S$.

The following result has been proved in [21].

Theorem 2.1. *Let A be a semisimple Fréchet locally m-convex algebra and $T \in M(A)$. Then the following statements are equivalent.*

(1) *T has a g-inverse $S \in B(A)$ such that $ST = TS$.*
(2) *T has a g-inverse $S \in B(A)$ such that $TS \in M(A)$.*
(3) *T has a g-inverse $S \in B(A)$ such that TS commutes with T.*
(4) *T has a g-inverse $S \in M(A)$.*
(5) *$TA \oplus \ker T = A$.*
(6) *$T^2A = TA$ and $\ker T^2 = \ker T$.*
(7) *$T = PB = BP$, where $B \in M(A)$ is invertible and $P \in M(A)$ is idempotent.*

(8) *T is decomposably regular in $M(A)$, i.e., $T = TCT$, where C is an invertible multiplier.*

We see from the preceding theorem that if T$\in M(A)$ has a commuting g-inverse then this must be a multiplier. One fact about multipliers on semisimple algebras that we shall use below is that they satisfy the relation $\ker T^2 = \ker T$. In fact, if $T^2x = 0$ then $0{=}T^2x^2 = T(xTx) = (Tx)^2$, hence $Tx = 0$. An immediate consequence of this is that $TA \cap \ker T = \{0\}$.

Corollary 2.2. *Let A be a semisimple Fréchet locally m-convex algebra and $T \in M(A)$. If $T^2A = TA$, then TA is closed.*

Proof. For the proof see [21]. □

We remark that the converse of Corollary 2.2 may not be true even in the case of general Banach algebras. For instance, consider the disc algebra $A = A(D)$ of all complex valued continuous functions on the closed unit disc D which are analytic in the interior of D. Let $g \in A(D)$ be such that $g(z) = z$ for each $z \in D$, and let T_g be the corresponding multiplication operator. Clearly, $T_g \in M(A)$ and $T_gA = \{f \in A : f(0) = 0\}$, $T_g^2A = \{f \in A : f(0) = f'(0) = 0\}$. Obviously T_gA is closed, but $T_gA \neq T_g^2A$.

Let A be a Fréchet locally m-convex algebra whose topology is generated by a family $\{p_n : n \in \mathbb{N}\}$ of submultiplicative seminorms. A net $\{e_\alpha : \alpha \in I\}$ in A is called a *bounded approximate identity* (abbreviated as *bai*) if $p_n(e_\alpha) \leq 1$ for all $n \in \mathbb{N}$ and for all $\alpha \in I$, $\lim\limits_\alpha e_\alpha x = \lim\limits_\alpha xe_\alpha = x$ for all $x \in A$. Following Inoue [15], A is called a *Fréchet locally C^*-algebra* if it has an involution $*$ satisfying $p_n(x^*x) = (p_n(x))^2$ for all $n \in \mathbb{N}$ and $x \in A$. It is well-known that every Fréchet locally C*-algebra has a bai (see [15, Theorem 2.6] and [6, Theorem 4.5]).

Theorem 2.3. *Let A be a semisimple Fréchet locally m-convex algebra with a bounded approximate identity and $T \in M(A)$. Then TA is a closed ideal with a bounded approximate identity if and only if T admits a factorization $T = PB$, where P is an idempotent multiplier and B an invertible multiplier.*

Proof. Let $\{e_\alpha\}$ be a bounded approximate identity of A. Assume that $T \in M(A)$ has a factorization $T = PB$, where $P \in M(A)$ is idempotent and $B \in M(A)$ is invertible. Since $TA = PA$, it follows immediately that TA is a closed ideal. Also, the bounded net $\{Pe_\alpha\}$ is subset of TA. Hence $xPe_\alpha = P(xe_\alpha) \to Px = x$, for all $x \in TA$.

Conversely assume that TA is a closed ideal with a bounded approximate identity. Then using the generalized version of the Cohen's factorization theorem ([5], p. 610), for each $x \in TA$, there exist y, z in TA such that $x = yz$, i.e., $TA = (TA)^2$ which implies $T^2A \subseteq TA = (TA)^2$. On the other hand, for any $x, y \in A$, we have $(Tx)(Ty) = T(xTy) = T^2(xy) \in T^2A$, and so $(TA)^2 \subseteq T^2A$. Hence $TA = T^2A$. The desired factorization $T = PB$ follows from the preceding theorem. □

Corollary 2.4. *Let A be a semisimple Fréchet locally m-convex algebra with a bounded approximate identity and $T \in M(A)$. Then the conditions (1) to (8) of Theorem 2.1 are equivalent to the following condition: (9) TA is a closed ideal with a bounded approximate identity.*

Note that every Fréchet locally C*-algebra is semisimple (cf. [6, Corollary 5.6] and [7, Lemma 8.14(ii)]). Now we remark that Theorem 3.6 [21] follows immediately as a simple corollary of the preceding theorem. Precisely, we have:

Corollary 2.5. *Let A be a Fréchet locally C^*-algebra and $T \in M(A)$. Then TA is closed if and only if $T^2A = TA$.*

Corollary 2.6. *Let A be a semisimple Fréchet locally m-convex algebra and $T \in M(A)$. If $T^2A = TA$, then T is injective if and only if it is surjective.*

Proof. Let T be surjective. Since $TA \cap \ker T=\{0\}$, it follows that $\ker T = \{0\}$, that is, T is injective. Conversely, assume that $\ker T = \{0\}$. Since, by assumption, $T^2A = TA$, it follows from Theorem 2.1 that $TA \oplus \ker T = A$. Hence $TA = A$, that is, T is surjective. □

Now we see, by virtue of Corollary 2.4, that if T is a multiplier on a semisimple Fréchet locally m-convex algebra with a bounded approximate identity such that TA is a closed ideal with a bounded approximate identity, then T is injective if and only if it is surjective. In particular, we obtain a result of [20] which states that a closed range multiplier on a Fréchet locally C*-algebra is injective if and only if it is surjective.

3. Spectral Properties of Multipliers

In this section we investigate certain spectral properties of multipliers defined on a semisimple commutative Fréchet locally m-convex algebra A. Denote the set of all non-zero continuous multiplicative linear functionals on A by $\Delta(A)$. In what follows, we assume that $\Delta(A)$ is non-empty and point-separating, without mentioning it explicitly. For any $x \in A$, define the Gelfand transform $\widehat{x}$ of x by $\widehat{x}(f) = f(x)$ for each $f \in \Delta(A)$. The space $\Delta(A)$ is equipped with the Gelfand topology, i.e., the induced topology inherited from the weak* topology of A^*. We shall use the following result of [13] frequently.

Theorem 3.1. *There is a continuous function $\mu^T : \Delta(A) \to \mathbb{C}$ corresponding to each $T \in M(A)$ defined by $\mu^T(f) = f \circ T(x)$, where x is chosen such that f(x)=1, satisfying the relation $(\widehat{Ty})(f) = \widehat{y}(f)\mu^T(f)$, for all $y \in A$ and all $f \in \Delta(A)$.*

Now we need to recall the definition of the socle of a semisimple commutative Fréchet locally m-convex algebra A, an ideal that plays an important role in our subsequent discussion. A *minimal idempotent* of A is a non-zero idempotent e such that eAe is a division algebra. Note that if e is a minimal idempotent element, then $eAe = \mathbb{C}e$ ([3], p. 292). The set of all minimal idempotents of A is denoted by E_A. It is well-known that an ideal J of A is a minimal ideal if and only if $J = eA$ for some $e \in E_A$ (see for instance, [4]). The *socle* of A, denoted by $\mathrm{soc}(A)$, is defined as the sum of all minimal ideals of A, or (0) if there are none. In what follows, we assume that the ideal $\mathrm{soc}(A)$ does exist, without mentioning it explicitly. The socle of A can be characterized in a simple way as:

$$\mathrm{soc}(A) = \left\{ \sum_{k=1}^{n} e_k A : e_k \in E_A,\, n \in \mathbb{N} \right\} = \mathrm{span}(E_A).$$

An important class of topological algebras consists of those which have a dense socle. For instance, consider the algebra $A = H(D)$ of all holomorphic functions defined on the open disc $D = \{z \in \mathbb{C} : |z| < 1\}$ with point-wise addition and scalar multiplication. With the Cauchy-Hadamard product and the compact-open topology, it is a semisimple commutative Fréchet locally m-convex algebra possessing an orthogonal basis $\{e_n : n \geq 0\}$, where $e_n(z) = z^n$ for $z \in D$. The element $e(z) = \sum_{n=0}^{\infty} z^n$ is the identity element of $H(D)$. Note that e_nA is a minimal ideal of A, for all $n \in \mathbb{N}$. Moreover, A is the direct sum of these minimal ideals, i.e., $\mathrm{soc}(A)$ is dense in A (see [14], Chapter III, p. 97).

Similarly, the algebra $A = s$ of all complex sequences with coordinate-wise operations is a semisimple commutative Fréchet locally m-convex algebra with identity and possessing an orthogonal basis $\{\mathrm{e}_n : n \geq 1\}$(see [14], Example 3.4, Chapter II). In this case, $\mathrm{soc}(A)$ is also dense in A. In fact, the socle is dense in every Hausdorff topological algebra possessing an orthogonal basis. Moreover, $\Delta(A)$ is homeomorphic with the discrete space of natural numbers $\mathbb{N}$ (see [14], Theorem 3.12, Chapter III). We now prove the following:

Theorem 3.2. *Let A be a semisimple commutative Fréchet locally m-convex algebra. If $\overline{\mathrm{soc}(A)} = A$, then $\Delta(A)$ is discrete.*

Proof. First we observe that $\widehat{A} = \{\widehat{a} : a \in A\}$ separates the points of $\Delta(A)$. In fact, if $f, g \in \Delta(A)$ such that $f \neq g$, then there exists $x_0 \in A$ with $f(x_0) \neq g(x_0)$. Therefore, it implies that $\widehat{x_0}(f) \neq \widehat{x_0}(g)$. Hence there is no $h \in \Delta(A)$ at which $\widehat{x}$ vanishes for all $x \in \mathrm{soc}(A)$. Thus if $\mathrm{f}_0 \in \Delta(A)$, then there exists an element $x \in \mathrm{soc}(A)$ for which $\widehat{x}(f_0) = 1$. Therefore, $\{h \in \Delta(A) : |\widehat{x}(h) - \widehat{x}(f_0)| < \frac{1}{2}\} = \{f_0\}$ is a weak*-neighborhood of f. This implies that $\Delta(A)$ is discrete. □

We denote by $\mathcal{C}_c(\Delta(A))$ the algebra of all $\mathbb{C}$-valued continuous functions on $\Delta(A)$ endowed with the topology of compact convergence. Now by combining Theorem 3.2 with [9, Theorem 4.2], we get:

Corollary 3.3. *Let A be a unital semisimple commutative Fréchet locally m-convex algebra. If $\overline{\mathrm{soc}(A)} = A$, then $A = \mathcal{C}_c(\Delta(A))$, with respect to a topological algebraic isomorphism.*

A locally m-convex (resp. Fréchet locally m-convex) algebra A whose topology is generated by a family $\{p_\alpha : \alpha \in \Lambda\}$ of submultiplicative seminorms is called a *uniform locally m-convex* (resp. *uniform Fréchet locally m-convex*) algebra if $p_\alpha\left(x^2\right) = \left(p_\alpha(x)\right)^2$, for all $x \in A$, $\alpha \in \Lambda$. Every uniform locally m-convex algebra is commutative and semisimple (see [18, p. 275, Lemma 5.1]). Moreover, from [9, Corollary 5.4(ii)] and Theorem 3.2, we get:

Corollary 3.4. *A unital uniform Fréchet locally m-convex algebra with dense socle is a Banach algebra.*

We showed in Section 2 that the converse of Corollary 2.2 may not be true even in the case of Banach algebras, but it is true for Fréchet locally C*-algebras (see Corollary 2.5). A similar result proved in [2] states that if A is a semisimple commutative Fréchet locally m-convex algebra and $T \in M(A)$, then T^2A is closed if and only if $TA \oplus \ker T$ is closed. Note that a Fréchet locally m-convex algebra is simply called a Fréchet algebra in [2]. Now we remark that Theorem 5 [2] follows directly from Theorem 2.1. More precisely, we have:

Corollary 3.5. *Let A be a semisimple commutative Fréchet locally m-convex algebra with $T \in M(A)$ and $\overline{\mathrm{soc}(A)} = A$. Then T is a product of an idempotent multiplier and an invertible multiplier if and only $TA \oplus \ker T = A$.*

Observe that two conditions on A, it being a commutative algebra and having the dense socle, in Theorem 5 [2] can be relaxed by virtue of Theorem 2.1.

In the sequel, we denote by $\sigma_p(T)$ and $\sigma_r(T)$ the point spectrum and the residual spectrum of T, respectively. Recall that A is said to be *regular* if for each closed subset E of $\Delta(A)$ in the Gelfand topology and $f_0 \in \Delta(A)\backslash E$, there exists an element x in A such that $\widehat{x}(f_0) = 1$ and $\widehat{x}(f) = 0$ for all $f \in E$ (see for instance, [18], p. 332). We remark that if $\Delta(A)$ is discrete, then clearly A is regular. We recall that the *ascent* $p(T)$ of an operator T is defined as the smallest non-negative integer p, whenever it exists, such that $\ker T^p = \ker T^{p+1}$.

Theorem 3.6. *Let A be a semisimple commutative Fréchet locally m-convex algebra and $T \in M(A)$. Then*

(1) $\sigma_p(T) \subseteq \mu^T(\Delta(A)) \subseteq \sigma_p(T) \cup \sigma_r(T)$.

(2) *For any $\lambda \in \sigma(T)$ we have $p(\lambda I - T) \leq 1$.*

Proof. (1) Let $\lambda \in \sigma_p(T)$. Then there exists a none-zero element x of A such that $(\lambda I - T)(x) = 0$. Therefore, $(\widehat{(\lambda I - T)(x)}) = (\lambda - \mu^T)\,\widehat{x} = \widehat{0}$. Since A is semisimple and $\widehat{x} \neq \widehat{0}$ there exists $f_0 \in \Delta(A)$ such that $\widehat{x}(f_0) \neq 0$. Thus it follows, from above that $(\lambda - \mu^T) f_0 = 0$, and so $\mu^T(f_0) = \lambda$. That is, $\lambda \in \mu^T(\Delta(A))$.

To prove the second inclusion, let T^* denote the topological dual of T. Then for each $f \in \Delta(A)$, we have $(T^* f)x = f(Tx) = (\widehat{Tx})(f) = \mu^T(f)\,\widehat{x}(f) = \mu^T(f)\,f(x)$, (using Theorem 3.1), for all x$\in A$. Therefore, $T^* f = \mu^T(f)\,f$, and hence $\mu^T(f)$ is an eigenvalue of T^*. Since the inclusion $\sigma_p(T^*) \subseteq \sigma_p(T) \cup \sigma_r(T)$ holds by virtue of Theorem 2.16.5 [11], the desired inclusion follows immediately.

(2) Let $x \in \ker(\lambda I - T)^2$, where $x \neq 0$. Since $(\lambda I - T)^2 \in M(A)$ and $\mu^{(\lambda I - T)^2} = (\lambda - \mu^T)^2$, it follows that $0 = (\widehat{(\lambda I - T)^2(x)})(f) = (\lambda - \mu^T)^2(f) \cdot \widehat{x}(f)$, for all $f \in \Delta(A)$ (using Theorem 3.1). Hence $(\lambda - \mu^T)(f) \cdot \widehat{x}(f) = 0$ for each $f \in \Delta(A)$. Therefore, $\widehat{(\lambda I - T)(x)} = \widehat{0}$. Since A is semisimple, $(\lambda I - T)(x) = 0$, and so $x \in \ker(\lambda I - T)$. Thus $\ker(\lambda I - T)^2 \subseteq \ker(\lambda I - T)$. Since the reverse inclusion is trivial, we conclude that $p(\lambda I - T) \leq 1$. □

Remark 2. *To every $T \in M(A)$ the corresponding function μ^T may not be bounded, in general. However, if M(A) is a Q-algebra, then the function μ^T is bounded since $\mu^T(\Delta(A)) \subseteq \sigma_p(T) \cup \sigma_r(T) \subseteq \sigma(T)$ and every element in a Q-algebra has compact spectrum* [19]. *Note that it would be interesting to investigating whether property Q on A could pass onto $M(A)$ and vice versa?*

Now we give a complete description of the point spectrum of $T \in M(A)$.

Theorem 3.7. *Let A be a semisimple commutative Fréchet locally m-convex algebra and $T \in M(A)$. If $\Delta(A)$ is discrete, then we have $\sigma_p(T) = \mu^T(\Delta(A))$.*

Proof. By virtue of Theorem 3.6, it remains only to show that $\mu^T(\Delta(A)) \subseteq \sigma_p(T)$. Let f_0 be fixed in $\Delta(A)$. Since, by assumption $\Delta(A)$ is discrete and hence A is regular, there exists an element x in A such that $\widehat{x}(f_\circ) = 1$ and $\widehat{x}$ vanishes identically on the set $\Delta(A)\backslash\{f_0\}$. Therefore, $(\widehat{[\mu^T(f_0)I - T]x})(f) = (\mu^T(f_0) - \mu^T(f)) \cdot \widehat{x}(f) = 0$ for each $f \in \Delta(A)$ and so $[\mu^T(f_0)\,I - T]x = 0$, because A is semisimple. Since $x \neq 0$, we obtain $\mu^T(f_0) \in \sigma_p(T)$. Hence $\sigma_p(T) = \mu^T(\Delta(A))$. □

Under the assumption that $\overline{\mathrm{soc}(A)} = A$, we now give a complete description of the residual spectrum of $T \in M(A)$.

Theorem 3.8. *Let A be a semisimple commutative Fréchet locally m-convex algebra with dense socle. Then $\sigma_r(T) = \emptyset$.*

Proof. Assume on the contrary that $\sigma_r(T) \neq \emptyset$. Let $\lambda \in \sigma_r(T)$. Then by Theorem 3.7, $\lambda \notin \sigma_p(T)$ implies that $\lambda \neq \mu^T(f)$ for each $f \in \Delta(A)$. For any $x \in E_A$ there exists $f_0 \in \Delta(A)$ such that $\widehat{x}(f_0) = 1$ and $\widehat{x}$ vanishes identically on $\Delta(A)\backslash\{f_0\}$. Set $y = (\lambda - \mu^T(f_0))^{-1} x$, then we have $[\widehat{(\lambda I - T)y}](f) = \widehat{x}(f)$ for all f in $\Delta(A)$ and so $(\lambda I - T)y = x$, that is, $E_A \subseteq (\lambda I - T)(A) \subseteq A$. Since, by hypothesis, we have $A = \overline{\mathrm{span}\{E_A\}}$ which implies $A = \overline{(\lambda I - T)(A)}$ and so $\lambda \notin \sigma_r(T)$, a contradiction. Hence $\sigma_r(T) = \emptyset$. □

Finally we give an application of our previous results: Let A denote a Hausdorff topological algebra with an orthogonal basis $\{x_i\}$. Then A is commutative ([14], Corollary 1.4, Chapter III), proper ([14], Proposition 1.6, Chapter III), semisimple ([14], Corollary 2.5, Chapter III), and has dense socle ([14], Theorem 4.3, Chapter III). Also, each coordinate functional λ_i determined by the basis $\{x_i\}$ via $x = \sum_{i=1}^{\infty} \lambda_i(x)x_i$, is continuous, i.e., $\{x_i\}$ is a Schauder basis ([14] Theorem 1.12, Chapter III). Further, each λ_i is a multiplicative linear functional ([14], p. 79). Moreover, $\Delta(A)$ is homeomorphic with the discrete space of natural numbers $\mathbb{N}$ ([14] Theorem 3.12, Chapter III). To each $T \in M(A)$, there corresponds a sequence $\{\mu_i^T\}$ of complex numbers defined by $\mu_i^T = \mu^T(\lambda_i)$ for all $i \geq 1$, and moreover it is completely described by: $Tx = \sum_{i=1}^{\infty} \lambda_i(x)\mu_i^T x_i$, for all $x \in A$ ([14], p. 225).

Corollary 3.9. *Let A be a locally m-convex algebra with an orthogonal basis $\{x_i\}$ and $T \in M(A)$. Then we have $\sigma_p(T) = \{\mu_i^T : i \geq 1\}$ and $\sigma_r(T) = \emptyset$.*

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Submission to the SCMJ

In September 2012, the way of submission to Scientiae Mathematicae Japonicae (SCMJ) was changed. Submissions should be sent electronically (in PDF file) to the editorial office of International Society for Mathematical Sciences (ISMS).

(1) Preparation of files and Submission

a. Authors who would like to submit their papers to the SCMJ should make source files of their papers in LaTeX2e using the ISMS style file (scmjlt2e.sty) Submissions should be in PDF file compiled from the source files. Send the PDF file to s1bmt@jams.jp .

b. Prepare a Submission Form and send it to the ISMS. The required items to be contained in the form are:

1. Editor's name whom the author chooses from the Editorial Board (http://www.jams.or.jp/hp/submission_f.html)and would like to take in charge of the paper for refereeing.
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3. Authors' names.
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5. Membership number in case the author is an ISMS member.

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9. **Special fields**	
10. **Membership category**	

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ISMS Publications: We published **Mathematica Japonica (M.J.)** in print, which was first published in 1948 and has gained an international reputation in about sixty years, and its offshoot **Scientiae Mathematicae (SCM)** both online and in print. In January 2001, the two publications were unified and changed to **Scientiae Mathematicae Japonicae (SCMJ)**, which is the "21st Century New Unified Series of Mathematica Japonica and Scientiae Mathematicae" and published both online and in print. Ahead of this, the online version of SCMJ was first published in September 2000. The whole number of SCMJ exceeds 270, which is the largest amount in the publications of mathematical sciences in Japan. The features of SCMJ are:

1) About 80 eminent professors and researchers of not only Japan but also 20 foreign countries join the Editorial Board. The accepted papers are published both online and in print. SCMJ is reviewed by Mathematical Review and Zentralblatt from cover to cover.
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Table 1: Membership Dues for 2015

Categories	**Domestic**	**Overseas**	**Developing countries**
1-year Regular member	¥8,000	US$80 , Euro75	US$50, Euro47
1-year Students member	¥4,000	US$50 , Euro47	US$30 , Euro28
Life member*	Calculated as below*	US$750 , Euro710	US$440, Euro416
Honorary member	Free	Free	Free

(Regarding submitted papers,we apply above presented new fee after April 15 in 2015 on registoration date.) * Regular member between 63 - 73 years old can apply the category.

(73 − age) × ¥3,000

Regular member over 73 years old can maintain the qualification and the privileges of the ISMS members, if they wish.

Categories of 3-year members were abolished.

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Notices from the ISMS