Online
ISSN 1346-0447

ISSN 1346-0862

Scientiae Mathematicae Japonicae

(Scientiae Mathematicae / Mathematica Japonica New Series)

Vol. 79, No. 1 Whole Number 282

June 2016

International Society for Mathematical Sciences

INTERNATIONAL SOCIETY FOR MATHEMATICAL SCIENCES
Scientiae Mathematicae Japonicae, Notices from the ISMS

The International Society for Mathematical Sciences (ISMS) is an international society consisting of mathematical scientists throughout the world.

The main activities of the ISMS are to publish (1) the (**print and online**) journal **Scientiae Mathematicae Japonicae (SCMJ)** and (2) Notices from the ISMS and to hold **assembly meeetings** in Japan and **international internet meetings** (distance symposium) of mathematical sciences (IVMS) accessible from all over the world.

SCMJ is the 21^{st} Century New Unified Series of **Mathematica Japonica (MJ)** and **Scientiae Mathematicae (SCM). MJ** was first published in 1948 and was one of the oldest mathematical journals in Japan. **SCM** was an online and print journal started in 1998 in celebration of the semi-centurial anniversary and received 26000 visits per month from 50 countries in the world. **SCMJ** contains original papers in mathematical sciences submitted from all over the world and receives **38000** visits per month now. Not only papers in pure and applied mathematics but those devoted to mathematical statistics, operations research, informatics, computer science, biomathematics, mathematical economics and other mathematical sciences are also welcome. The journal is published in January, March, May, July, September, and November in each calendar year.

The ISMS has enhanced the journal, begining from July 1995, by including excellent Research-Expository papers in the section "**International Plaza** for Mathematical Sciences " as well as original research papers. The section provides papers dealing with broad overviews of contemporary mathmatical sciences, written by experts mainly at our invitation. Papers shedding lights on open problems or new directions or new breakthroughs for future research are especially welcome.

As is shown in the **Editorial Board of SCMJ**, we have invited **many distinguished professors** of 20 countries as editors, who will receive and referee the papers of their special fields with their **high standard**.

Beginning from 2007, we make the online version of SCMJ more readable and convenient to the readers by adding the specialized contents. By this, the readers can access to the online version, in which the papers appear in the order of acceptance, from (i) the contents of the printed version, and (ii) the specialized contents of a volume. From 2007, the subscription fee of the printed version plus the online version of SCMJ becomes lower and the same of the printed version only. Therefore, the subscribers of the printed version can read the online version without no additional cost.

For benefit of the ISMS members, we publish "Notices from the ISMS" 6 times a year. We are enhancing it by adding interesting articles, including book reviewing, written by eminent professors.

The ISMS has set up a **videoconferencing system** (**IVMS**) which can connect up to twenty sites of a reserch group in the same or **different countries in the world**. Using this system, speakers of the session can write on a white board or an OHP sheet or use PowerPoint. On the other hand participants can ask questions or make comments from any connected site in the world. All these are performed similarly to the traditional meetings.

To connect with our system, you can use your own videoconferencing system only if it satisfies the International Telecommunication Union-Technical Committee Standards (ITU-T Standard).

Copyright Transfer Agreement

A copyright transfer agreement is required before a paper is published in this journal. By submitting a paper to this journal, authors are regarded to certify that the manuscript has not been submitted to nor is it under consideration for publication by another journal, conference proceedings or similar publication.

For more information, please visit http://www.jams.or.jp.

HOPF HYPERSURFACES ADMITTING ϕ-INVARIANT RICCI TENSORS IN A NONFLAT COMPLEX SPACE FORM

Kazuhiro Okumura

Received March 13, 2014

Abstract. We investigate real hypersurfaces with ϕ-invariant Ricci tensors in a nonflat complex space form $\widetilde{M}_n(c)$. In particular, we classify Hopf hypersurfaces having weakly ϕ-invariant Ricci tensor in $\widetilde{M}_n(c)$. In addition, we verify the non-existence of Hopf hypersurfaces with strongly ϕ-invariant Ricci tensor in $\widetilde{M}_n(c)$ and the non-existence of ruled real hypersurfaces with weakly ϕ-invariant Ricci tensor in $\widetilde{M}_n(c)$.

1 Introduction We denote by $\widetilde{M}_n(c)\,(n \geqq 2)$ an n-dimensional non-flat complex space form. Namely, $\widetilde{M}_n(c)$ is congruent to either a complex projective space of constant holomorphic sectional curvature $c(>0)$ or a complex hyperbolic space of constant holomorphic sectional curvature $c(<0)$. Let M^{2n-1} be a real hypersurface in $\widetilde{M}_n(c)$. It is well-known that real hypersurfaces in $\widetilde{M}_n(c)$ admitting almost contact metric structure (ϕ, ξ, η, g) induced from Kähler structure J of $\widetilde{M}_n(c)$ (see Section 2). From the viewpoint of contact geometry, real hypersurfaces are interesting in $\widetilde{M}_n(c)$. It is also well-known that there exist no *Einstein* real hypersurfaces in $\widetilde{M}_n(c)$. Thus, many geometers studied its weaker conditions and conditions related to the Ricci tensor of M^{2n-1} (See [3], [5], [7], [10], [11], [14], [15]).

In this paper, we focus on the structure tensor ϕ of M^{2n-1} and the Ricci tensor of M^{2n-1}. We define the notion of *ϕ-invariant Ricci tensor* of M^{2n-1} (for detail, see Section 5). This notion is divided into *strongly* ϕ-invariance of the Ricci tensor of M^{2n-1} or *weakly* ϕ-invariance of the Ricci tensor of M^{2n-1}. In particular, the latter is a weaker condition of Einstein real hypersurfaces.

In the theory of real hypersurfaces in $\widetilde{M}_n(c)$, Hopf hypersurfaces (namely, real hypersurfaces such that the characteristic vector ξ is a principal curvature vector at its each point) play an important role. We investigate Hopf hypersurfaces M^{2n-1} with ϕ-invariant Ricci tensors of M^{2n-1} in $\widetilde{M}_n(c)$. Note that there exist real hypersurfaces M^{2n-1} with weakly ϕ-invariant Ricci tensor of M^{2n-1} in $\widetilde{M}_n(c)$. In fact, the family of such real hypersurfaces includes real hypersurfaces of type (A) in $\widetilde{M}_n(c)$ (Theorem 1). It is known that real hypersurfaces of type (A) in $\widetilde{M}_n(c)$ have many nice geometric properties.

The purpose of this paper is to determine Hopf hypersurfaces M^{2n-1} having weakly ϕ-invariant Ricci tensor of M^{2n-1} in $\widetilde{M}_n(c)$. To do this, we shall prove that weakly ϕ-invariance of the Ricci tensor of M^{2n-1} is equivalent to the commutativity of the structure tensor ϕ of M^{2n-1} and the Ricci tensor Q of type (1, 1) of M^{2n-1} (that is, $\phi Q = Q\phi$) on a Hopf hypersurface M^{2n-1} in $\widetilde{M}_n(c)$. In addition, we shall show the non-existence of Hopf hypersurfaces M^{2n-1} with strongly ϕ-invariant Ricci tensor of M^{2n-1} in $\widetilde{M}_n(c)$.

In general, weakly ϕ-invariance of the Ricci tensor is *not* equivalent to the commutativity of the structure tensor ϕ and the Ricci tensor Q of type (1, 1) on a non-Hopf hypersurface in

2000 *Mathematics Subject Classification.* Primary 53B25; Secondary 53C40.

Key words and phrases. real hypersurfaces, non-flat complex space forms, almost contact metric structure, Ricci tensor, homogeneous real hypersurfaces, Hopf hypersurfaces, ruled real hypersurfaces.

$\widetilde{M}_n(c)$. It is natural to consider non-Hopf hypersurfaces M^{2n-1} having weakly ϕ-invariant Ricci tensor of M^{2n-1} in $\widetilde{M}_n(c)$. Ruled real hypersurfaces are typical non-Hopf hypersurfaces in $\widetilde{M}_n(c)$. So, we shall also show the non-existence of ruled real hypersurfaces M^{2n-1} with weakly ϕ-invariant Ricci tensor of M^{2n-1} in $\widetilde{M}_n(c)$.

2 Preliminaries Let M^{2n-1} be a real hypersurface with a unit local vector field $\mathcal{N}$ of a complex n-dimensional non-flat complex space form $\widetilde{M}_n(c)$ of constant holomorphic sectional curvature c. The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$ and ∇ of M^{2n-1} are related by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX,\, Y)\mathcal{N}, \tag{2.1}$$

$$\widetilde{\nabla}_X \mathcal{N} = -AX \tag{2.2}$$

for vector fields X and Y tangent to M^{2n-1}, where g denotes the induced metric from the standard Riemannian metric of $\widetilde{M}_n(c)$ and A is the shape operator of M^{2n-1} in $\widetilde{M}_n(c)$. (2.1) is called *Gauss's formula*, and (2.2) is called *Weingarten's formula*. Eigenvalues and eigenvectors of the shape operator A are called *principal curvatures* and *principal vectors* of M^{2n-1} in $\widetilde{M}_n(c)$, respectively.

It is known that M^{2n-1} admits an *almost contact metric structure* (ϕ, ξ, η, g) induced from the Kähler structure J of $\widetilde{M}_n(c)$. *The characteristic vector field* ξ of M^{2n-1} is defined as $\xi = -J\mathcal{N}$ and this structure satisfies

$$\begin{gathered} \phi^2 = -I + \eta \otimes \xi,\ \eta(X) = g(X, \xi),\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta(\phi X) = 0, \\ g(\phi X, Y) = -g(X, \phi Y) \quad \text{and} \quad g(\phi X,\, \phi Y) = g(X\,, Y) - \eta(X)\eta(Y), \end{gathered} \tag{2.3}$$

where I denotes the identity map of the tangent bundle TM of M^{2n-1}. We call ϕ and η *the structure tensor* and *the contact form* of M^{2n-1}, respectively.

Let R be the curvature tensor of M^{2n-1} in $\widetilde{M}_n(c)$. We have the equation of Gauss given by:

$$\begin{aligned} R(X,Y)Z =& (c/4)\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &- 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY \end{aligned} \tag{2.4}$$

for all vectors X, Y and Z on M^{2n-1}.

The Ricci tensor S of type (0, 2) and the Ricci tensor Q of type (1, 1) of an arbitrary real hypersurface M^{2n-1} in $\widetilde{M}_n(c)$ ($n \geqq 2$) is expressed as:

$$\begin{aligned} S(X,Y) = g(QX,Y) = (c/4)((2n+1)g(X,Y) - 3\eta(X)\eta(Y)) \\ + (\mathrm{Trace}\, A)g(AX,Y) - g(A^2X,Y). \end{aligned} \tag{2.5}$$

3 Homogeneous Hopf hypersurfaces in $\widetilde{M}_n(c)$ We usually call M^{2n-1} a *Hopf hypersurface* if the characteristic vector ξ is a principal curvature vector at each point of M^{2n-1}. It is known that every tube of sufficiently small constant radius around each Kähler submanifold of $\widetilde{M}_n(c)$ is a Hopf hypersurface. This fact tells us that the notion of Hopf hypersurface is natural in the theory of real hypersurfaces in $\widetilde{M}_n(c)$ (see [15]).

The following lemma clarifies a fundamental property which is a useful tool in the theory of Hopf hypersurfaces in $\widetilde{M}_n(c)$ (cf. [15]).

Lemma 1. *For a Hopf hypersurface M^{2n-1} with the principal curvature δ corresponding to the characteristic vector field ξ in $\widetilde{M}_n(c)$, we have the following*:

(1) δ *is locally constant on* M^{2n-1};

(2) If X *is a tangent vector of* M^{2n-1} *perpendicular to* ξ *with* $AX = \lambda X$*, then* $(2\lambda - \delta)A\phi X = (\delta\lambda + (c/2))\phi X$.

In $\mathbb{C}P^n(c)\,(n \geqq 2)$, a Hopf hypersurface all of whose principal curvatures are constant is locally congruent to a homogeneous real hypersurface (that is, real hypersurfaces which are expressed as orbits of some subgroup of the isometry group $\mathrm{I}(\widetilde{M}_n(c))$ of $\widetilde{M}_n(c)$). Moreover, these real hypersurfaces are one of the following:

(A_1) A geodesic sphere $G(r)$ of radius r, where $0 < r < \pi/\sqrt{c}$;

(A_2) A tube of radius r around a totally geodesic $\mathbb{C}P^\ell(c)$ $(1 \leqq \ell \leqq n-2)$, where $0 < r < \pi/\sqrt{c}$;

(B) A tube of radius r around a complex hyper quadric $\mathbb{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c}\,)$;

(C) A tube of radius r around a $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n(\geqq 5)$ is odd;

(D) A tube of radius r around a complex Grassmann $\mathbb{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 9$;

(E) A tube of radius r around a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 15$.

These real hypersurfaces are said to be of types (A_1), (A_2), (B), (C), (D) and (E). Summing up real hypersurfaces of type (A_1) and (A_2), we call them real hypersurfaces of type (A). The numbers of distinct principal curvatures of these real hypersurfaces are $2, 3, 3, 5, 5, 5$, respectively. The principal curvatures of these real hypersurfaces in $\mathbb{C}P^n(c)$ are given as follows (cf. [15]):

	(A_1)	(A_2)	(B)	(C), (D), (E)
λ_1	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\right)$
λ_2	—	$-\frac{\sqrt{c}}{2}\tan\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\right)$
λ_3	—	—	—	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right)$
λ_4	—	—	—	$-\frac{\sqrt{c}}{2}\tan\left(\frac{\sqrt{c}}{2}r\right)$
δ	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$

The multiplicities of these principal curvatures are given as follows (cf. [15]):

	(A_1)	(A_2)	(B)	(C)	(D)	(E)
$m(\lambda_1)$	$2n-2$	$2n-2\ell-2$	$n-1$	2	4	6
$m(\lambda_2)$	—	2ℓ	$n-1$	2	4	6
$m(\lambda_3)$	—	—	—	$n-3$	4	8
$m(\lambda_4)$	—	—	—	$n-3$	4	8
$m(\delta)$	1	1	1	1	1	1

Remark 1. *A geodesic sphere* $G(r)$ *of radius* r $(0 < r < \pi/\sqrt{c}\,)$ *in* $\mathbb{C}P^n(c)$ *is congruent to a tube of radius* $(\pi/\sqrt{c}\,) - r$ *around totally geodesic* $\mathbb{C}P^{n-1}(c)$ *of* $\mathbb{C}P^n(c)$*. Indeed,* $\lim_{r\to\pi/\sqrt{c}} G(r) = \mathbb{C}P^{n-1}(c)$.

In $\mathbb{C}H^n(c)$ $(n \geqq 2)$, a Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following:

(A_0) A horosphere in $\mathbb{C}H^n(c)$;

($\mathrm{A}_{1,0}$) A geodesic sphere $G(r)$ of radius r, where $0 < r < \infty$;

($\mathrm{A}_{1,1}$) A tube of radius r around a totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$;

(A_2) A tube of radius r around a totally geodesic $\mathbb{C}H^{\ell}(c)\,(1 \leqq \ell \leqq n-2)$, where $0 < r < \infty$;

(B) A tube of radius r around a totally real totally geodesic $\mathbb{R}H^n(c/4)$, where $0 < r < \infty$.

These real hypersurfaces are said to be of types (A_0), ($\mathrm{A}_{1,0}$), ($\mathrm{A}_{1,1}$), (A_2) and (B). Here, type (A_1) means either type ($\mathrm{A}_{1,0}$) or type ($\mathrm{A}_{1,1}$). Summing up real hypersurfaces of types (A_0), (A_1) and (A_2), we call them hypersurfaces of type (A). A real hypersurface of type (B) with radius $r = (1/\sqrt{|c|}\,)\log_e(2+\sqrt{3}\,)$ has two distinct constant principal curvatures $\lambda_1 = \delta = \sqrt{3|c|}/2$ and $\lambda_2 = \sqrt{|c|}/(2\sqrt{3}\,)$. Except for this real hypersurface, the numbers of distinct principal curvatures of Hopf hypersurfaces with constant principal curvatures are $2,2,2,3,3$, respectively. The principal curvatures of these real hypersurfaces in $\mathbb{C}H^n(c)$ are given as follows (cf. [15]):

	(A_0)	($\mathrm{A}_{1,0}$)	($\mathrm{A}_{1,1}$)	(A_2)	(B)
λ_1	$\frac{\sqrt{\lvert c\rvert}}{2}$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$
λ_2	—	—	—	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$
δ	$\sqrt{\lvert c\rvert}$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\tanh(\sqrt{\lvert c\rvert}\,r)$

The multiplicities of these principal curvatures are given as follows (cf. [15]):

	(A_0)	($\mathrm{A}_{1,0}$)	($\mathrm{A}_{1,1}$)	(A_2)	(B)
$m(\lambda_1)$	$2n-2$	$2n-2$	$2n-2$	$2n-2\ell-2$	$n-1$
$m(\lambda_2)$	—	—	—	2ℓ	$n-1$
$m(\delta)$	1	1	1	1	1

Remark 2. *The above Hopf hypersurfaces of type* (A) *and* (B) *in* $\mathbb{C}H^n(c)$ *are homogeneous real hypersurfaces. However, there exist non-Hopf homogeneous real hypersurfaces in* $\mathbb{C}H^n(c)$ (*for detail, see* [*1*]).

4 Ruled real hypersurfaces in $\widetilde{M}_n(c)$ Next we give ruled real hypersurfaces in a non-flat complex space form $\widetilde{M}_n(c)$, which are typical examples of non-Hopf hypersurfaces. A real hypersurface M^{2n-1} is called a *ruled real hypersurface* of a non-flat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$ if the holomorphic distribution T^0M defined by $T^0M(x) = \{X \in T_xM \mid X \perp \xi\}$ for $x \in M^{2n-1}$ is integrable and each of its maximal integral manifolds is a totally geodesic complex hypersurface $M_{n-1}(c)$ of $\widetilde{M}_n(c)$. A ruled real hypersurface is constructed in the following way. Given an arbitrary regular real smooth curve γ in $\widetilde{M}_n(c)$ which is defined on an interval I we have at each point $\gamma(t)$ $(t \in \mathrm{I})$ a totally geodesic complex hypersurface $M^{(t)}_{n-1}(c)$ that is orthogonal to the plane spanned by $\{\dot\gamma(t), J\dot\gamma(t)\}$. Then we see that $M^{2n-1} = \bigcup_{t\in I} M^{(t)}_{n-1}(c)$ is a ruled real hypersurface in $\widetilde{M}_n(c)$. The following is a well-known characterization of ruled real hypersurfaces in terms of the shape operator A.

Lemma 2. *For a real hypersurface M^{2n-1} in a non-flat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$, the following conditions are mutually equivalent*:

1. M^{2n-1} *is a ruled real hypersurface*;
2. *The shape operator A of M^{2n-1} satisfies the following equalities on the open dense subset $M_1 = \{x \in M^{2n-1} | \nu(x) \neq 0\}$ with a unit vector field U orthogonal to ξ : $A\xi = \mu\xi + \nu U$, $AU = \nu\xi$, $AX = 0$ for an arbitrary tangent vector X orthogonal to ξ and U, where μ, ν are differentiable functions on M_1 by $\mu = g(A\xi, \xi)$ and $\nu = \|A\xi - \mu\xi\|$*;
3. *The shape operator A of M^{2n-1} satisfies $g(Av, w) = 0$ for arbitrary tangent vectors $v, w \in T_xM$ orthogonal to ξ_x at each point $x \in M^{2n-1}$.*

We treat a ruled real hypersurface locally, because generally this hypersurface has singularities. When we study ruled real hypersurfaces, we usually omit points where ξ is principal and suppose that ν does not vanish everywhere, namely a ruled hypersurface M^{2n-1} is usually supposed $M_1 = M^{2n-1}$.

5 ϕ-invariances of the Ricci tensor and main theorem First, we define the notion of ϕ-invariance of the Ricci tensor S of M^{2n-1} in $\widetilde{M}_n(c)$. The Ricci tensor S of M^{2n-1} is called *strongly ϕ-invariant* if S satisfies

$$S(\phi X, \phi Y) = S(X, Y)$$

for all vectors X and Y on M^{2n-1}. Also it is called *weakly ϕ-invariant* if S satisfies

$$S(\phi X, \phi Y) = S(X, Y)$$

for all vectors X and Y on M^{2n-1} orthogonal to the characteristic vector ξ on M^{2n-1}.

Theorem 1. *Let M^{2n-1} be a real hypersurface in a non-flat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$. Then the following holds*:

1. *Suppose that M^{2n-1} is a Hopf hypersurface in $\widetilde{M}_n(c)$. Then M^{2n-1} has weakly ϕ-invariant Ricci tensor S of M^{2n-1} if and only if M^{2n-1} satisfies $\phi Q = Q\phi$. Moreover, M^{2n-1} is locally congruent to one of the following*:
   (a) *A real hypersurface of type* (A) *in $\widetilde{M}_n(c)$*;
   (b) *A tube of radius r around a complex hyperquadric $\mathbb{C}Q^{n-1}$ in $\mathbb{C}P^n(c)$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $\cot(\sqrt{c}\,r/2) = \sqrt{n-2} + \sqrt{n-1}$*;
   (c) *A tube of radius r around a $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$ in $\mathbb{C}P^n(c)$, where $0 < r < \pi/(2\sqrt{c}\,)$, $n\,(\geqq 5)$ is odd and $\cot(\sqrt{c}\,r/2) = (\sqrt{n-1} + 1)/\sqrt{n-2}$*;
   (d) *A tube of radius r around a complex Grassmann $\mathbb{C}G_{2,5}$ in $\mathbb{C}P^n(c)$, where $0 < r < \pi/(2\sqrt{c}\,)$, $n = 9$ and $\cot(\sqrt{c}\,r/2) = (\sqrt{8} + \sqrt{3}\,)/\sqrt{5}$*;
   (e) *A tube of radius r around a Hermitian symmetric space $SO(10)/U(5)$ in $\mathbb{C}P^n(c)$, where $0 < r < \pi/(2\sqrt{c}\,)$, $n = 15$ and $\cot(\sqrt{c}\,r/2) = (\sqrt{14} + \sqrt{5}\,)/3$*;
   (f) *A non-homogeneous real hypersurface which is a tube of radius r around an ℓ-dimensional non-totally geodesic Kähler submanifold $\widetilde{N}$ without principal curvatures $\pm(\sqrt{c}/2)\sqrt{(2\ell-1)/(2n-2\ell-1)}$, where the rank of every shape operator of $\widetilde{N}$ in the ambient space $\mathbb{C}P^n(c)$ is not greater than 2 and $\cot^2(\sqrt{c}\,r/2) = (2\ell-1)/(2n-2\ell-1)$ with $\ell = 1, \ldots, n-1$.*

2. *There does not exist a Hopf hypersurface M^{2n-1} with strongly ϕ-invariant Ricci tensor S of M^{2n-1}.*

3. *There does not exist a ruled real hypersurface M^{2n-1} with weakly ϕ-invariant Ricci tensor S of M^{2n-1}.*

Proof. From (2.5), we know that strongly ϕ-invariance of the Ricci tensor S of M^{2n-1} is equivalent to saying that

$$-\frac{c}{2}(n-1)\eta(X)\eta(Y) + (\mathrm{Trace}\,A)(g(A\phi X, \phi Y) - g(AX, Y)) - g(A^2\phi X, \phi Y) + g(A^2X, Y) = 0 \tag{5.1}$$

for all vectors X, Y on M^{2n-1}. By this equation, we obtain that weakly ϕ-invariance of the Ricci tensor S of M^{2n-1} is equivalent to saying that

$$(\mathrm{Trace}\,A)(g(A\phi X, \phi Y) - g(AX, Y)) - g(A^2\phi X, \phi Y) + g(A^2X, Y) = 0 \tag{5.2}$$

for all vectors X and Y orthogonal to ξ.

(1) First of all, we suppose that M^{2n-1} satisfies $\phi Q = Q\phi$. Then, we get

$$S(\phi X, \phi Y) = g(Q\phi X, \phi Y) = g(\phi QX, \phi Y) = -g(QX, \phi^2 Y) = g(QX, Y) = S(X, Y)$$

for any vectors X, Y orthogonal to ξ.

Next, we suppose that M^{2n-1} has weakly ϕ-invariant Ricci tensor S of M^{2n-1}. By (5.2), we have

$$(\mathrm{Trace}\,A)g(-\phi A\phi X - AX, Y) + g(\phi A^2\phi X + A^2X, Y) = 0 \tag{5.3}$$

for any vectors X, Y orthogonal to ξ. Interchanging a vector $X(\perp \xi)$ with a vector $\phi X(\perp \xi)$ in Equation (5.3), we obtain

$$(\mathrm{Trace}\,A)g((\phi A - A\phi)X, Y) - g((\phi A^2 - A^2\phi)X, Y) = 0$$

for any vectors X, Y orthogonal to ξ. This implies that

$$g((\phi Q - Q\phi)X, Y) = 0 \tag{5.4}$$

for any vectors X, Y orthogonal to ξ. On the other hand, using assumption that M^{2n-1} is a Hopf hypersurface in $\widetilde{M}_n(c)$, we obtain $\phi Q\xi = 0 = Q\phi\xi$. This, combine with (5.4), implies $\phi Q = Q\phi$.

By the works of M. Kimura [8], [9] (the case of $n \geqq 3$ in $\mathbb{C}P^n(c)$), U-H. Ki and Y. J. Suh [6] (the case of $n \geqq 3$ in $\mathbb{C}H^n(c)$) and J. T. Cho [4] (the case of $\widetilde{M}_2(c)$), we know the classification of Hopf hypersurfaces with $\phi Q = Q\phi$ in $\widetilde{M}_n(c)$. Hence, we get the classification of Hopf hypersurfaces having weakly ϕ-invariant Ricci tensor in $\widetilde{M}_n(c)$.

(2) We suppose that M^{2n-1} is a Hopf hypersurface with $A\xi = \delta\xi$ in $\widetilde{M}_n(c)$. From (5.1), we find that M^{2n-1} has strongly ϕ-invariant Ricci tensor S of M^{2n-1} if and only if M^{2n-1} satisfies the following two conditions:

(i) The Hopf hypersurface M^{2n-1} has weakly ϕ-invariant Ricci tensor S of M^{2n-1};

(ii) The Hopf hypersurface M^{2n-1} satisfies the following equation:

$$\delta^2 - (\mathrm{Trace}\,A)\delta - \frac{c}{2}(n-1) = 0. \tag{5.5}$$

Now we shall check Equation (5.5) one by one for real hypersurfaces of (1) in our Theorem.

Let M^{2n-1} be a real hypersurface of type (A_1) in $\mathbb{C}P^n(c)$. Let $x = \cot(\sqrt{c}\,r/2)$, $0 < r < \pi/\sqrt{c}$. Then we have $\delta = (\sqrt{c}/2)(x - (1/x))$, $\delta^2 = (c/4)(x^2 - 2 + (1/x^2))$ and $\mathrm{Trace}\,A = (\sqrt{c}/2)((2n-1)x - (1/x))$. These, together with Equation (5.5) we get $n = 1$, which contradicts $n \geqq 2$. Hence M^{2n-1} does not have strongly ϕ-invariant Ricci tensor S of M^{2n-1}.

Let M^{2n-1} be a real hypersurface of type (A_2) in $\mathbb{C}P^n(c)$. Let $x = \cot(\sqrt{c}\,r/2)$, $0 < r < \pi/\sqrt{c}$. Then we have $\delta = (\sqrt{c}/2)(x - (1/x))$, $\delta^2 = (c/4)(x^2 - 2 + (1/x^2))$ and $\mathrm{Trace}\,A = (\sqrt{c}/2)((2n-2\ell-1)x - (2\ell+1)(1/x))$. These, together with Equation (5.5) we get $(n-\ell-1)x^4 + \ell = 0$. However, this equation can not occur. Hence M^{2n-1} does not have strongly ϕ-invariant Ricci tensor S of M^{2n-1}.

Let M^{2n-1} be a real hypersurface of type (A_0) in $\mathbb{C}H^n(c)$. Then we have $\delta = \sqrt{|c|}$, $\delta^2 = -c$ and $\mathrm{Trace}\,A = \sqrt{|c|} + (2n-2)(\sqrt{|c|}/2)$. These, together with Equation (5.5) we get $n = 1$, which contradicts $n \geqq 2$. Hence M^{2n-1} does not have strongly ϕ-invariant Ricci tensor S of M^{2n-1}.

Let M^{2n-1} be a real hypersurface of type ($\mathrm{A}_{1,0}$) in $\mathbb{C}H^n(c)$. Let $x = \coth(\sqrt{|c|}r/2)$, $0 < r < \infty$. Then we have $\delta = (\sqrt{|c|}/2)(x + (1/x))$, $\delta^2 = -(c/4)(x^2 + 2 + (1/x^2))$ and $\mathrm{Trace}\,A = (\sqrt{|c|}/2)((2n-1)x + (1/x))$. These, together with Equation (5.5) we get $n = 1$, which contradicts $n \geqq 2$. Hence M^{2n-1} does not have strongly ϕ-invariant Ricci tensor S of M^{2n-1}. Similarly, we can show that real hypersurfaces of type ($\mathrm{A}_{1,1}$) in $\mathbb{C}H^n(c)$ do not have strongly ϕ-invariant Ricci tensor.

Let M^{2n-1} be a real hypersurface of type (A_2) in $\mathbb{C}H^n(c)$. Let $x = \coth(\sqrt{|c|}r/2)$, $0 < r < \infty$. Then we have $\delta = (\sqrt{|c|}/2)(x + (1/x))$, $\delta^2 = -(c/4)(x^2 + 2 + (1/x^2))$ and $\mathrm{Trace}\,A = (\sqrt{|c|}/2)((2n-2\ell-1)x + (2\ell+1)(1/x))$. These, together with Equation (5.5) we get $(n-\ell-1)x^4 + \ell = 0$. However, this equation can not occur. Hence M^{2n-1} does not have strongly ϕ-invariant Ricci tensor S of M^{2n-1}.

Let M^{2n-1} be a real hypersurface of the case of (b) in our Theorem. Then we have $\delta = \sqrt{c(n-2)}$, $\delta^2 = c(n-2)$ and $\mathrm{Trace}\,A = -\sqrt{c}/\sqrt{n-2}$. These, together with Equation (5.5) we get $n = 1$, which contradicts $n \geqq 3$.

Let M^{2n-1} be a real hypersurface of the case of (c) in our Theorem. Then we have $\delta = \sqrt{c}/\sqrt{n-2}$, $\delta^2 = c/(n-2)$ and $\mathrm{Trace}\,A = -\sqrt{c(n-2)}$. These, together with Equation (5.5) we get $n^2 - 5n + 4 = 0$, so that $n = 1, 4$, which contradicts $n \geqq 5$.

Let M^{2n-1} be a real hypersurface of the case of (d) in our Theorem. Then we have $\delta = \sqrt{3c}/\sqrt{5}$, $\delta^2 = 3c/5$ and $\mathrm{Trace}\,A = -\sqrt{5c}/\sqrt{3}$. These, together with Equation (5.5) we get $n = 21/5$, which contradicts $n = 9$.

Let M^{2n-1} be a real hypersurface of the case of (e) in our Theorem. Then we have $\delta = \sqrt{5c}/3$, $\delta^2 = 5c/9$ and $\mathrm{Trace}\,A = -3\sqrt{5c}/5$. These, together with Equation (5.5) we get $n = 37/9$, which contradicts $n = 15$.

Let M^{2n-1} be a real hypersurface of the case of (f) in our Theorem. Then M^{2n-1} has at most five distinct principal curvatures as follow: $\sqrt{c}\,\cot(\sqrt{c}\,r)$ with multiplicity 1, $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$ with multiplicity $2n-2\ell-2$, $-(\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$ with multiplicity $2\ell-2$, $(\sqrt{c}/2)\cot((\sqrt{c}\,r/2)-\theta)$ with multiplicity 1 and $(\sqrt{c}/2)\cot((\sqrt{c}\,r/2)+\theta)$ with multiplicity 1, where $(\sqrt{c}/2)\cot\theta$ is a principal curvature of the Kähler submanifold $\widetilde{N}$ (see [3], [9], [10], [12]). In this case, M^{2n-1} has either the case of $\delta = 0$ or the case of $\delta \neq 0$. When $\delta = 0$ (that is, the case of $n = 2\ell$), we have $(c/2)(n-1) = 0$, which is a contradiction. When $\delta \neq 0$, we have

$$\mathrm{Trace}\,A = \delta + \frac{c}{2\delta}(n-1). \tag{5.6}$$

It follows from (1) of Lemma 1 that the right side of Equation (5.6) is constant on M^{2n-1}.

On the other hand, the left side of Equation (5.6) is non-constant. Indeed, Trace A of M^{2n-1} is expressed as:

$$\begin{aligned}\text{Trace } A = \delta + (2n-2\ell-2)\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right) - (2\ell-2)\frac{\sqrt{c}}{2}\tan\left(\frac{\sqrt{c}}{2}r\right) \\ + \frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r-\theta\right) + \frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r+\theta\right).\end{aligned}$$

Note that $(\sqrt{c}/2)\cot((\sqrt{c}\,r/2)-\theta)+(\sqrt{c}/2)\cot((\sqrt{c}\,r/2)+\theta)$ is non-constant on M^{2n-1}. Thus, we have a contradiction. Hence, M^{2n-1} does not have strongly ϕ-invariant Ricci tensor S of M^{2n-1}.

Therefore, there exist no Hopf hypersurface M^{2n-1} with strongly ϕ-invariant Ricci tensor S of M^{2n-1} in $\widetilde{M}_n(c)$.

(3) We suppose that M^{2n-1} is a ruled real hypersurface with weakly ϕ-invariant Ricci tensor S of M^{2n-1} in $\widetilde{M}_n(c)$. It follows from (5.2) and (3) of Lemma 2 that we obtain

$$-g(A^2\phi X, \phi Y) + g(A^2X, Y) = 0$$

for all vectors X, Y orthogonal to ξ. Setting $X = Y = U$, by using Lemma 2 we have

$$0 = -g(A^2\phi U, \phi U) + g(A^2U, U) = \nu^2 \neq 0,$$

which is a contradiction. Hence, M^{2n-1} does not have weakly ϕ-invariant Ricci tensor S of M^{2n-1}. □

Remark 3. *Note that the commutativity of the structure tensor ϕ and the Ricci tensor Q of type $(1,1)$ always implies weakly ϕ-invariance of the Ricci tensor. However, in general, we do not know whether the converse holds or not.*

6 Concluding remarks

6.1 In general, there exist contact metric manifolds with strongly ϕ-invariant Ricci tensor.

For example, $\mathbb{R}^3$ with coordinates (x^1, x^2, x^3) and the contact form $\eta = (1/2)(\cos x^3\, dx^1 + \sin x^3\, dx^2)$. The characteristic vector filed ξ is defined by $\xi = 2(\cos x^3(\partial/\partial x^1)+\sin x^3(\partial/\partial x^2))$ and the metric g is given by $g_{ij} = (1/4)\delta_{ij}$, where g_{ij} are components of g. Then $\mathbb{R}^3$ has a *flat* contact metric structure (cf. [2]). Hence clearly this example admits strongly ϕ-invariant Ricci tensor.

6.2 In [13], S. Maeda and H. Naitoh investigated real hypersurfaces with *ϕ-invariant shape operators* in $\mathbb{C}P^n(c)$. The shape operator A of a real hypersurface M^{2n-1} is called *strongly ϕ-invariant* if A satisfies

$$g(A\phi X, \phi Y) = g(AX, Y)$$

for all vectors X and Y on M^{2n-1}. Also, it is called *weakly ϕ-invariant* if A satisfies

$$g(A\phi X, \phi Y) = g(AX, Y)$$

for all vectors X and Y orthogonal to the characteristic vector ξ on M^{2n-1}.

S. Maeda and H. Naitoh [13] obtained the following results:

Proposition 1. *Let M^{2n-1} be a real hypersurface M^{2n-1} with strongly ϕ-invariant shape operator A of M^{2n-1} in $\mathbb{C}P^n(c)$. Then M^{2n-1} is locally congruent to a real hypersurface of type* (A) *of radius $\pi/(2\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$.*

Proposition 2. *Let M^{2n-1} be a real hypersurface M^{2n-1} with weakly ϕ-invariant shape operator A of M^{2n-1} in $\mathbb{C}P^n(c)$. Then the following holds*:

1. *If M^{2n-1} is a Hopf hypersurface in $\mathbb{C}P^n(c)$, then M^{2n-1} is locally congruent to a real hypersurface of type* (A) *in $\mathbb{C}P^n(c)$.*
2. *If the holomorphic distribution $T^0M = \{X \in TM : X \perp \xi\}$ is integrable, then M^{2n-1} is locally congruent to a ruled real hypersurface in $\mathbb{C}P^n(c)$.*

By using the discussion of [13], we know that there exists no real hypersurface M^{2n-1} in $\mathbb{C}H^n(c)$ such that the shape operator A of M^{2n-1} is strongly ϕ-invariant. In addition, for real hypersurfaces in $\mathbb{C}H^n(c)$, Proposition 2 also holds.

From our theorem, ruled real hypersurfaces do not have weakly ϕ-invariant Ricci tensor in $\widetilde{M}_n(c)$. However, ruled real hypersurfaces have weakly ϕ-invariant shape operator in $\widetilde{M}_n(c)$.

6.3 We shall consider the notion of *ϕ-invariant curvature tensor R* of M^{2n-1} in $\widetilde{M}_n(c)$. The curvature tensor R of a real hypersurface M^{2n-1} is called *strongly ϕ-invariant* if R satisfies

$$R(\phi X, \phi Y) = R(X, Y)$$

for all vectors X and Y on M^{2n-1}. Also, it is called *weakly ϕ-invariant* if R satisfies

$$R(\phi X, \phi Y) = R(X, Y)$$

for all vectors X and Y orthogonal to the characteristic vector ξ on M^{2n-1} .

From our theorem and S. Maeda and H. Naitoh's work [13], real hypersurfaces of type (A) in $\widetilde{M}_n(c)$ have both weakly ϕ-invariant Ricci tensor and weakly ϕ-invariant shape operator. Now we investigate whether there exists a real hypersurface of type (A) in $\widetilde{M}_n(c)$ having weakly ϕ-invariant curvature tensor R or not.

Proposition 3. *There does not exist a real hypersurface M^{2n-1} of type* (A) *admitting weakly ϕ-invariant curvature tensor R of M^{2n-1} in $\widetilde{M}_n(c)$ $(n \geqq 3)$.*

Proof. We suppose that a real hypersurface M^{2n-1} admitting weakly ϕ-invariant curvature tensor R of M^{2n-1}. By (2.4), we know that weakly ϕ-invariance of the curvature tensor R of M^{2n-1} is equivalent to saying that

$$g(A\phi Y, Z)A\phi X - g(A\phi X, Z)A\phi Y - g(AY, Z)AX + g(AX, Z)AY = 0 \tag{6.1}$$

for $\forall X, Y \perp \xi$ and $\forall Z \in TM$.

Let M^{2n-1} be a real hypersurface of type (A_1) in $\mathbb{C}P^n(c)\,(n \geqq 3)$. We take a local field of orthogonal frame $\{e_1, e_2, \dots, e_{n-1}, \phi e_1, \phi e_2, \dots \phi e_{n-1}, \xi\}$ in M^{2n-1} such that

$$Ae_i = (\sqrt{c}/2)\cot(\sqrt{c}r/2)e_i, \quad A\phi e_i = (\sqrt{c}/2)\cot(\sqrt{c}r/2)\phi e_i \quad (1 \leqq i \leqq n-1).$$

We can put $X = e_i, Y = e_j, Z = e_j$ in Equation (6.1) satisfying $e_i \neq e_j, \phi e_i \neq e_j$. Then we have $\cot^2(\sqrt{c}r/2) = 0$, which is a contradiction. Hence M^{2n-1} does not have weakly ϕ-invariant curvature tensor R of M^{2n-1}. Similarly, real hypersurfaces M^{2n-1} of types (A_0) and (A_1) in $\mathbb{C}H^n(c)\,(n \geqq 3)$ do not admit ϕ-invariant curvature tensor R of M^{2n-1}.

Let M^{2n-1} be a real hypersurface of type (A_2) in $\mathbb{C}P^n(c)\,(n \geqq 3)$. We take a local field of orthogonal frame $\{e_1, e_2, \dots, e_{2n-2}, \xi\}$ in M^{2n-1} such that

$$\begin{aligned} Ae_i &= (\sqrt{c}/2)\cot(\sqrt{c}r/2)e_i \quad (1 \leqq i \leqq 2n-2\ell-2), \\ Ae_j &= -(\sqrt{c}/2)\tan(\sqrt{c}r/2)e_j \quad (2n-2\ell-1 \leqq j \leqq 2n-2). \end{aligned}$$

We set $X = e_i, Y = e_j, Z = e_j$ $(1 \leqq i \leqq 2n-2\ell-2, 2n-2\ell-1 \leqq j \leqq 2n-2)$ in Equation (6.1). Note that $\phi V_{\lambda_1} = V_{\lambda_1} = \{X \in TM : AX = \lambda_1 X\}, \phi V_{\lambda_2} = V_{\lambda_2} = \{X \in TM : AX = \lambda_2 X\}$ and $V_{\lambda_1} \oplus V_{\lambda_2} = T^0 M = \{X \in TM : X \perp \xi\}$, where $\lambda_1 = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2), \lambda_2 = -(\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$. Then we obtain $\cot(\sqrt{c}\,r/2)\tan(\sqrt{c}\,r/2) = 0$, which is a contradiction. Hence M^{2n-1} does not have weakly ϕ-invariant curvature tensor R of M^{2n-1}. Similarly, real hypersurfaces M^{2n-1} of type (A_2) in $\mathbb{C}H^n(c)\,(n \geqq 3)$ does not have ϕ-invariant curvature tensor R of M^{2n-1}.

Therefore real hypersurfaces of type (A) in $\widetilde{M}_n(c)\,(n \geqq 3)$ do not admit ϕ-invariant curvature tensor. $\square$

Acknowledgments The author would like to thank Professor Sadahiro Maeda for his valuable suggestions and encouragement during the preparation of this paper. The author would also like to thank Professor Yasuhiko Furihata for his valuable comments. Thanks are due to the referee for helpful comments.

References

[1] J. Berndt and H. Tamaru, *Cohomogeneity one actions on noncompact symmetric spaces of rank one*, Trans. Amer. Math. Soc. **359** (2007), 3425–3439.

[2] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics **203**, 2010.

[3] T. E. Cecil and P. J. Ryan, *Focal sets real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), 481–499.

[4] J. T. Cho, *Ricci operators and structural Jacobi operators on real hypersurfaces in a complex space form*, Taiwanese J. Math. **14(4)** (2010), 1325–1336.

[5] T. A. Ivey and P. J. Ryan, *Hopf hypersurfaces of small Hopf principal curvature in* $\mathbb{C}H^2$, Geom. Dedicata. **141** (2009), 147–161.

[6] U-H. Ki and Y. J. Suh, *On real hypersurfaces of a complex space form*, Math. J. Okayama **32** (1990), 207–221.

[7] H. S. Kim and P. J. Ryan, *A classification of pseudo-Einstein hypersurfaces in* $\mathbb{C}P^2$, Differential Geom. Appl. **26** (2008), 106–112.

[8] M. Kimura, *Some real hypersurfaces of a complex projective space*, Saitama Math. J. **5** (1987), 1–5.

[9] M. Kimura, *Correction to: "Some real hypersurfaces of a complex projective space"*, Saitama Math. J. **10** (1992), 33–34.

[10] M. Kimura and S. Maeda, *Lie derivatives on real hypersurfaces in a complex projective space*, Czechoslovak Math. J. **45** (1995), 135–148.

[11] M. Kon, *Pseudo-Einstein real hypersurfaces in complex space forms*, J. Differential Geom. **14** (1979), 339–354.

[12] S. Maeda, *Ricci tensors of real hypersurfaces in a complex projective space*, Proc. Amer. Math. Soc. **122** (1994), 1229–1235.

[13] S. Maeda and H. Naitoh, *Real hypersurfaces with ϕ-invariant shape operator in a complex projective space*, Glasgow Math. J. **53** (2011), 347–358.

[14] S. Montiel, *Real hypersurfaces of a complex hyperbolic space* J. Math. Soc. Japan **37** (1985), 515–535.

[15] R. Niebergall and P. J. Ryan, *Real hypersurfaces in complex space forms*, in: Tight and Taut submanifolds, T.E. Cecil and S.S. Chern (eds.), Cambridge Univ. Press, 1998, 233–305.

Communicated by *Yasunao Hattori*

COLORINGS FOR SET-VALUED MAPS

Atsushi Ebisawa and Hisao Kato

Received March 28, 2014 ; revised November 27, 2014

Abstract. We know many results about colorability for single-valued maps. But we know a few results about colorability for set-valued maps. In this paper we generalize some results on colorability for single-valued maps to those for set-valued maps. Especially, our main result is a generalization of E. K. van Douwen's result, which insists that every fixed-point free continuous closed map $f : X \to X$ with $\sup\{|f^{-1}(x)| : x \in X\} < \infty$ on a finite-dimensional paracompact space X is colorable. In fact, we prove the following: Let X be a finite-dimensional paracompact space and $f : X \to \mathcal{F}_k(X)$ a fixed-point free upper semi-continuous map, where $\mathcal{F}_k(X)$ is the family of non-empty subsets of X with at most k elements. Suppose that $\sup\{|f^{-1}(x)| : x \in X\} < \infty$ and $\bigcup\{f(x) : x \in F\}$ is closed in X for any closed subset F of X. Then f is colorable.

1 Introduction

All spaces under discussion are regular. We will discuss some set-valued versions of results about colorability for single-valued maps.

We define some notions about colorability of single-valued maps as follows: Let X be a subset of a space Y and $f : X \to Y$ a single-valued map. For a subset A of X, A is called a *color* of f if $A \cap f(A) = \emptyset$ and a *bright color* of f if $\overline{A}^Y \cap \overline{f(A)}^Y = \emptyset$, where $\overline{A}^Y$ denotes the closure of A in Y. Also we call a finite closed cover of X consisting of colors of f a *coloring* of f and we say that f is *colorable* if there is a coloring of f. Similarly, we define a *bright coloring* of f and say that f is *brightly colorable* if there is a bright coloring of f.

The following shows the essential meaning of colorability for single-valued maps:

Proposition 1.1. *Let X be a closed subspace of a normal space Y and let $f : X \to Y$ be a fixed-point free continuous map. Then, the following are equivalent:*

(1) *f is brightly colorable.*

(2) *The Stone-Čech extension $\beta f : \beta X \to \beta Y$ of f is fixed-point free.*

Also the following results for single-valued maps are known:

Proposition 1.2. *Let X be a compact subspace of a space Y and let $f : X \to Y$ be a fixed-point free continuous map. Then f is colorable.*

Theorem 1.3. *([5]) Let X be a closed subspace of a locally compact separable metrizable space Y with $\dim Y \le n$ and let $f : X \to Y$ be a fixed-point free continuous map. Then, f is brightly colorable.*

2014 *Msthematics Subject Classification.* 54F45, 54C60, 54H25 .
Key words and phrases. set-valued map, color, fixed-point free, extension.

Theorem 1.4. *([6]) Let X be a paracompact space with $\dim X \leq n$ and let $f : X \to X$ be a fixed-point free continuous closed map such that $l = \sup\{|f^{-1}(x)| : x \in X\} < \infty$. Then, f is colorable with at most $(l+1)(n+1)+1$ colors.*

Theorem 1.5. *([2]) Let X be a separable metrizable space with $\dim X \leq n$ and let $f : X \to X$ be a fixed-point free homeomorphism. Then, f is colorable with at most $n+3$ colors.*

In this paper we generalize these results for single-valued maps to some results for set-valued maps. To start our discussion we give a topology of the space consisting of closed subsets (see [9] in detail).

For a space X we define the hyperspace 2^X of X as the family of all non-empty closed subsets of X and endow 2^X with the Vietoris topology, which has

$$\langle \mathcal{U} \rangle = \left\{ A \in 2^X : A \subset \bigcup \mathcal{U} \text{ and } A \cap U \neq \emptyset \text{ for any } U \in \mathcal{U} \right\},$$

where $\mathcal{U}$ is a finite family of open subsets of X, as the basic open subsets of 2^X. Also let $\mathcal{K}(X)$ and $\mathcal{F}_k(X)$ for $k \in \mathbb{N}$ denote the family of non-empty compact subsets of X and the family of non-empty finite subsets of X with at most k elements, respectively.

Let X and Y be spaces and $f : X \to 2^Y$ a set-valued map. For $A \subset X$ we write $f(A) = \bigcup\{f(x) : x \in A\}$. Also for $y \in Y$, $B \subset Y$ and $\mathcal{B} \subset 2^Y$ we write $f^{-1}(y) = \{x \in X : y \in f(x)\}$, $f^{-1}(B) = \{x \in X : f(x) \cap B \neq \emptyset\}$ and $f^{-1}[\mathcal{B}] = \{x \in X : f(x) \in \mathcal{B}\}$. Also $f : X \to 2^Y$ is upper semi-continuous if for $x \in X$ and an open set V of Y with $f(x) \subset V$, $f^{-1}[\langle\{V\}\rangle](= \{x' \in X : f(x') \subset V\})$ is open in X.

When $X \subset Y$ we define some notions about colorability of set-valued maps as follows: A map $f : X \to 2^Y$ is called a *fixed-point free* map if $x \notin f(x)$ for any $x \in X$. For a subset A of X, A is called a *color* of f if $A \cap f(A) = \emptyset$ and called a *bright color* of f if $\overline{A}^Y \cap \overline{f(A)}^Y = \emptyset$. Also we call a finite closed cover of X consisting of colors of f a *coloring* of f and we say that f is *colorable* if there is a coloring of f. Similarly, we define a *bright coloring* of f and say that f is *brightly colorable* if there is a bright coloring of f.

Any space X can be embedded to 2^X by the inclusion $\iota : X \to 2^X$ defined by $x \mapsto \{x\}$. Hence all results for set-valued maps are also true for single-valued maps. The proofs are modifications of proofs for single-valued versions in [5], [6] and [2].

Also let (A, B) be a pair of disjoint closed subsets of a space X. A subset S of X is called a partition between A and B if there is a pair (U, V) of disjoint open subsets of X such that $A \subset U$, $B \subset V$ and $X \setminus S = U \cup V$.

2 Results

First, we present a generalization of Proposition 1.2.

Proposition 2.1. *Let X be a compact subspace of a space Y and let $f : X \to 2^Y$ be a fixed-point free and upper semi-continuous map. Then, f is colorable.*

Proof. By compactness of X it is sufficient to show that for each $x \in X$ there is an open neighborhood of x in X such that its closure is a color of f. Take $x \in X$. Then $x \notin f(x)$ since f is fixed-point free. By regularity of Y there are two open neighborhoods U and V of x and $f(x)$ in Y, respectively, such that $U \cap V = \emptyset$. Since f is upper semi-continuous, $f^{-1}[\langle\{V\}\rangle]$ is open in X. By regularity of X there is an open neighborhood W of x in X such that $\overline{W} \subset U \cap f^{-1}[\langle\{V\}\rangle]$. This is as required. □

Next, we consider a generalization of Theorem 1.3.

Theorem 2.2. *([4]) Let X be a closed subspace of $\mathbb{R}^n$ and let $f : X \to \mathcal{F}_k(\mathbb{R}^n)$ be a fixed-point free continuous map. Then, f is brightly colorable.*

Applying Theorem 2.2, we obtain a generalization of Theorem 1.3 as follows.

Theorem 2.3. *Let X be a closed subset of a locally compact separable metrizable space Y with $\dim Y \leq n$ and let $f : X \to \mathcal{F}_k(Y)$ be a fixed-point free continuous map. Then, f is brightly colorable.*

Proof. We may assume that Y is closed in $\mathbb{R}^{2n+1}$ since any n-dimensional locally compact separable metrizable space can be embedded in $\mathbb{R}^{2n+1}$ as a closed subset. Therefore, this proof is completed by Theorem 2.2. $\square$

Remark. For Theorem 2.2 we know that for $n, k \in \mathbb{N}$ there is a minimal integer $K(n,k)$ such that every fixed point free continuous map $f : X \to \mathcal{F}_k(\mathbb{R}^n)$ is colorable with at most $K(n,k)$ colors (see [4]). So we can see that $K(2n+1,k)$ plays the same part for Theorem 2.3. But it is not clear about the exact values.

To show our main result we define the *order* and give a lemma.

Let X be a space and $\mathcal{U}$ a family of subsets of X and $n \in \{0,1,2,...\}$. We define the order of $\mathcal{U}$, which is denoted by $\operatorname{ord}\mathcal{U}$, as follows:

$$\operatorname{ord}\mathcal{U} \leq n \text{ if } \sup\Big\{\big|\{U \in \mathcal{U} : x \in U\}\big| : x \in X\Big\} \leq n.$$

Remark. In many books $\operatorname{ord}\mathcal{U} \leq n$ is defined by $\big|\{U \in \mathcal{U} : x \in U\}\big| \leq n+1$ for any $x \in X$. But in this paper we use the above definition to see inequalities about the order easily.

Lemma 2.4. *([6]) Let X be a normal space. Let $\{G_i : i = 1,...,k\}$ be a family of closed subsets of X with $\operatorname{ord}\{G_i : i = 1,...,k\} \leq \dim X + 1$ and $\{W_i : i = 1,...,k\}$ an open cover of X such that $G_i \subset W_i$ for $i = 1,...,k$. Then, there is an open cover $\{V_i : i = 1,...,k\}$ of X such that $\operatorname{ord}\{\overline{V_i} : i = 1,...,k\} \leq \dim X + 1$ and $G_i \subset V_i$ and $\overline{V_i} \subset W_i$ for $i = 1,...,k$.*

The following theorem is a generalization of Theorem 1.4.

Theorem 2.5. *Let X be a paracompact space with $\dim X \leq n$ and let $f : X \to \mathcal{F}_k(X)$ be a fixed-point free upper semi-continuous map. Suppose that $l = \sup\big\{|f^{-1}(x)| : x \in X\big\} < \infty$ and $f(F)$ is closed in X for any closed subset F of X. Then, f is colorable with at most $(k+l)(n+1)+1$ colors.*

Proof. First, fix $x \in X$. Since f is fixed-point free, there are two open neighborhoods U_x and V_x of x and $f(x)$ in X, respectively, such that $U_x \cap V_x = \emptyset$. $f^{-1}[\langle\{V_x\}\rangle]$ is an open neighborhood of x in X since f is upper semi-continuous. Put $W_x = U_x \cap f^{-1}[\langle\{V_x\}\rangle]$. Then $W_x \cap f(W_x) = \emptyset$.

Put $\mathcal{W} = \{W_x : x \in X\}$. Then $\mathcal{W}$ covers X. So by paracompactness of X there is a locally finite closed refinement $\mathcal{A}$ of $\mathcal{W}$. List $\mathcal{A}$ as $\{A_\xi : \xi < \kappa\}$ for some ordinal number κ. Observe that $A_\xi \cup f(A_\xi) = \emptyset$ for each $\xi < \kappa$.

Next, put $p = (k+l)(n+1)+1$ and for each $\xi < \kappa$ we will construct inductively a closed cover $\{B_{\xi,i} : i = 1, ..., p\}$ of A_ξ in a way such that if

$$C_{\eta,i} = \bigcup_{\xi<\eta} B_{\xi,i} \quad \text{for } i = 1, ..., p$$

then for all $\eta < \kappa$ we have

$$(1_\eta) \qquad C_{\eta,i} \cap f(C_{\eta,i}) = \emptyset \quad \text{for } i = 1, ..., p,$$

$$(2_\eta) \qquad \mathrm{ord}\big\{C_{\eta,i} : i = 1, ..., p\big\} \leq n+1.$$

We note that $C_{\eta,i}$ is closed in X for each $\eta < \kappa$ and $i = 1, ..., p$ since $\mathcal{A}$ is locally finite.

The construction: For $\eta = 0$ (1_0) and (2_0) hold since $C_{0,i} = \emptyset$ for $i = 1, ..., p$.

When constructing $\{B_{\xi,i} : i = 1, ..., p\}$ for an $\eta < \kappa$ and each $\xi < \eta$, we may assume (1_η) and (2_η) to hold. Now we will construct $\{B_{\eta,i} : i = 1, ..., p\}$. For $i = 1, ..., p$ define

$$D_i = f^{-1}(C_{\eta,i}) \cup f(C_{\eta,i}).$$

Then D_i is closed in X since f is upper semi-continuous. To see that

(a) $\{A_\eta \backslash D_i : i = 1, ..., p\}$ covers A_η

we claim that

$$\bigcap_{i=1}^{p} D_i = \emptyset.$$

By (2_η) and $|f(x)| \leq k$, $|f^{-1}(x)| \leq l$ for all $x \in X$ we have

$$\mathrm{ord}\{f^{-1}(C_{\eta,i}) : i = 1, ..., p\} \leq k(n+1),$$
$$\mathrm{ord}\{f(C_{\eta,i}) : i = 1, ..., p\} \leq l(n+1).$$

Indeed, for the first when we put $f(x) = \{x_1, ..., x_k\}$ for each $x \in X$, $\big|\{i : x_j \in C_{\eta,i}\}\big| \leq n+1$ for $j = 1, ..., k$ by (2_η). Hence

$$\begin{aligned}\big|\{i : x \in f^{-1}(C_{\eta,i})\}\big| &= \Big|\bigcup_{j=1}^{k}\{i : x_j \in C_{\eta,i}\}\Big| \\ &\leq \sum_{j=1}^{k}\big|\{i : x_j \in C_{\eta,i}\}\big| \\ &\leq k(n+1).\end{aligned}$$

Similarly, we can verify the second.

Thus, from the definition of D_i

$$\begin{aligned}\mathrm{ord}\{D_i : i = 1, ..., p\} &\leq \mathrm{ord}\{f^{-1}(C_{\eta,i}) \cup f(C_{\eta,i}) : i = 1, ..., p\} \\ &\leq k(n+1) + l(n+1) \\ &= (k+l)(n+1).\end{aligned}$$

So $\bigcap_{i=1}^{p} D_i = \emptyset$ and (a) holds.

By (1_η)

(b) $C_{\eta,i} \cap D_i = \emptyset$ for $i = 1, ..., p$.

So because $\dim A_\eta \leq n$, $\text{ord}\{A_\eta \cap C_{\eta,i} : i = 1, ..., p\} \leq n+1$ and $A_\eta \cap C_{\eta,i} \subset A_\eta \backslash D_i$ for $i = 1, ..., p$, by Lemma 2.4 there is a relatively open cover $\{O_i : i = 1, ..., p\}$ of A_η such that

(c) $A_\eta \cap C_{\eta,i} \subset O_i$, $\overline{O_i} \subset A_\eta \backslash D_i$ for $i = 1, ..., p$,

(d) $\text{ord}\big\{\overline{O_i} : i = 1, ..., p\big\} \leq n+1$.

Define $B_{\eta,i} = \overline{O_i}$ for $i = 1, ..., p$. Then, $C_{\eta+1,i} = C_{\eta,i} \cup B_{\eta,i} = C_{\eta,i} \cup \overline{O_i}$ for $i = 1, ..., p$. We check $(1_{\eta+1})$ and $(2_{\eta+1})$. For $(2_{\eta+1})$ we obtain

$$\begin{aligned}\text{ord}\{C_{\eta+1,i} : i = 1, ..., p\} &= \text{ord}\{C_{\eta,i} \cup \overline{O_i} : i = 1, ..., p\} \\ &= \text{ord}\{(C_{\eta,i} \backslash A_\eta) \cup \overline{O_i} : i = 1, ..., p\} \\ &\leq n+1\end{aligned}$$

by (2_η), (d) and the first part of (c). For $(1_{\eta+1})$ it is sufficient to prove that

$$\begin{aligned}C_{\eta,i} \cap f(C_{\eta,i}) &= \emptyset, \\ C_{\eta,i} \cap f(B_{\eta,i}) &= \emptyset, \\ B_{\eta,i} \cap f(C_{\eta,i}) &= \emptyset, \\ B_{\eta,i} \cap f(B_{\eta,i}) &= \emptyset\end{aligned}$$

for $i = 1, ..., p$. The first and fourth are trivial from (1_η) and the property of A_η. Also $B_{\eta,i} \cap f^{-1}(C_{\eta,i}) = \emptyset$ if and only if $C_{\eta,i} \cap f(B_{\eta,i}) = \emptyset$. Thus, the second and third hold by the second part of (c). This completes the construction of $B_{\xi,i}$.

Finally, define

$$C_i = \bigcup_{\eta<\kappa} C_{\eta,i} \quad \text{for } i = 1, ..., p.$$

It is easy to see that $\mathcal{C} = \{C_i : i = 1, ..., p\}$ is a closed cover of X consisting of colors of f. Consequently, $\mathcal{C}$ is as required. □

When X is compact, Theorem 2.5 implies the following corollary.

Corollary 2.6. *Let X be a compact space with $\dim X \leq n$ and let $f : X \to \mathcal{F}_k(X)$ be a fixed-point free and upper semi-continuous map. Suppose that $l = \sup\big\{|f^{-1}(x)| : x \in X\big\} < \infty$. Then f is colorable with at most $(k+l)(n+1)+1$ colors.*

Proof. By compactness of X, $f(F)$ is closed in X for any closed subset F of X. So this is shown from Theorem 2.5. □

The numbers of colors in the above results are not sharp. Here we consider reducing the numbers of colors.

Lemma 2.7. *Let X be a separable metrizable space with $\dim X \leq n$ and let $f : X \to 2^X$ be an upper semi-continuous map such that $f(F)$ is closed in X and $\dim f^{-1}(F) = \dim F$ for any closed subset F of X. Let φ and $\varphi_i (i = 1, 2, ...)$ denote one of the map f and the inclusion ι. Assume that $\mathcal{S} = \{S_i : i \in \mathbb{N}\}$ is a family of closed subsets of X such that*

$$\dim(\varphi_{i_1}(S_{i_1}) \cap \cdots \cap \varphi_{i_k}(S_{i_k})) \leq n-k$$

whenever $i_1 < \cdots < i_k$ and $k = 1, ..., n+1$. Then for every pair (G, H) of disjoint closed subsets of X there is a partition S between G and H in X such that

$$\dim(\varphi_{i_1}(S_{i_1}) \cap \cdots \cap \varphi_{i_{k-1}}(S_{i_{k-1}}) \cap \varphi(S)) \leq n - k$$

whenever $i_1 < \cdots < i_{k-1}$ and $k = 1, ..., n+1$.

Proof. Put $X_k = \bigcup\{\varphi_{i_1}(S_{i_1}) \cap \cdots \cap \varphi_{i_k}(S_{i_k}) : i_1 < \cdots < i_k\}$ for $k = 1, ..., n$. Write $X_0 = X$. Then X_k is an F_σ-subset of X. By assumptions of $\mathcal{S}$ we have $\dim X_k \leq n - k$. So there is an F_σ-subset Z of X with $\dim Z = 0$ and $\dim(X_k \backslash Z) \leq n - k - 1$ for $k = 1, ..., n$. Since f is upper semi-continuous, $f^{-1}(Z)$ is an F_σ-subset of X. By assumption of f we have $\dim(Z \cup f^{-1}(Z)) = 0$. Hence there is a partition S between G and H in X such that $S \cap (Z \cup f^{-1}(Z)) = \emptyset$.

Then

$$\varphi_{i_1}(S_{i_1}) \cap \cdots \cap \varphi_{i_{k-1}}(S_{i_{k-1}}) \cap S \subset X_{k-1} \backslash Z,$$
$$\varphi_{i_1}(S_{i_1}) \cap \cdots \cap \varphi_{i_{k-1}}(S_{i_{k-1}}) \cap f(S) \subset X_{k-1} \backslash Z,$$

whenever $i_1 < \cdots < i_{k-1}$ and $k = 1, ..., n+1$. Therefore,

$$\begin{aligned}\dim((\varphi_{i_1}(S_{i_1}) \cap \cdots \cap \varphi_{i_{k-1}}(S_{i_{k-1}}) \cap \varphi(S)) &\leq n - (k-1) - 1 \\ &= n - k.\end{aligned}$$

So S is as required. □

Lemma 2.8. *Let X be a separable metrizable space with $\dim X \leq n$ and let $f : X \to 2^X$ be an upper semi-continuous map such that $f(F)$ is closed in X and $\dim f^{-1}(F) = \dim F = \dim f(F)$ for any closed subset F of X. Let φ and $\varphi_i (i = 1, 2, ...)$ denote one of the map f and the inclusion ι. Let $\mathcal{U} = \{U_i : i = 1, ..., m\}$ be an open cover of X and $\mathcal{K} = \{K_i : i = 1, ..., m\}$ be an closed shrinking of $\mathcal{U}$. Then there is a closed cover $\mathcal{L} = \{L_i : i = 1, ..., m\}$ of X such that $K_i \subset L_i \subset U_i$ for $i = 1, ..., m$ and*

$$\varphi_{i_1}(\partial L_{i_1}) \cap \cdots \cap \varphi_{i_{n+1}}(\partial L_{i_{n+1}}) = \emptyset$$

whenever $1 \leq i_1 < \cdots < i_{n+1} \leq m$.

Proof. The proof will be done by induction.

First, we define L_1. Since $\dim X \leq n$, there is a partition S_1 between K_1 and $X \backslash U_1$ in X such that $\dim S_1 \leq n - 1$. By assumption of f we have $\dim f(S_1) \leq n - 1$. Now $X \backslash S_1$ is the disjoint union of two open subsets V_1 and W_1 in X such that $K_1 \subset V_1$ and $X \backslash U_1 \subset W_1$. Define $L_1 = \overline{V_1}$. Then $\partial L_1 \subset S_1$ and so $\dim \varphi(\partial L_1) \leq n - 1$.

Next, assume that for some $r \in \{1, ..., m\}$ L_i is defined for $i = 1, ..., r-1$ such that the family $\{\partial L_i : i = 1, ..., r-1\}$ has the property

$$(*) \qquad \dim(\varphi_{i_1}(\partial L_{i_1}) \cap \cdots \cap \varphi_{i_k}(\partial L_{i_k})) \leq n - k,$$

whenever $1 \leq i_1 < \cdots < i_k \leq r - 1$ and $k = 1, ..., n+1$. From Lemma 2.7 there is a partition S_r between K_r and $X \backslash U_r$ in X such that the property $(*)$ holds for the family $\{\partial L_i : i = 1, ..., r-1\} \cup \{S_r\}$. Now $X \backslash S_r$ is the disjoint union of two open subsets V_r and W_r in X such that $K_r \subset V_r$ and $X \backslash U_r \subset W_r$. Define $L_r = \overline{V_r}$. Then $\partial L_r \subset S_r$ and so the property $(*)$ holds for $\{\partial L_i : i = 1, ..., r\}$. This completes the construction of L_i.

Take $1 \leq i_1 < \cdots < i_{n+1} \leq m$. Then

$$\dim(\varphi_{i_1}(\partial L_{i_1}) \cap \cdots \cap \varphi_{i_{n+1}}(\partial L_{i_{n+1}})) \leq n - (n+1) = -1$$

and so

$$\varphi_{i_1}(\partial L_{i_1}) \cap \cdots \cap \varphi_{i_{n+1}}(\partial L_{i_{n+1}}) = \emptyset.$$

Consequently, $\mathcal{L} = \{L_i : i = 1, ..., m\}$ is as required. $\square$

Lemma 2.9. *([2]) Let $\mathcal{K} = \{K_i : i = 1, ..., k\}$ be a finite closed cover of a space X. Define $L_i = \overline{K_i \backslash (\bigcup_{j=1}^{i-1} K_j)}$ for $i = 1, ..., m$. Then $\mathcal{L} = \{L_i : i = 1, ..., k\}$ has the following properties:*

(1) *$L_s \cap L_t = \partial L_s \cap \partial L_t$ for $s \neq t$.*

(2) *If $\partial L_{i_1} \cap \cdots \cap \partial L_{i_m} \neq \emptyset$, then $\partial K_{i_1} \cap \cdots \cap \partial K_{i_{m-1}} \neq \emptyset$ whenever $1 \leq i_1 < \cdots < i_m \leq k$.*

The following theorem is a generalization of Theorem 1.5.

Theorem 2.10. *Let X be a separable metrizable space with $\dim X \leq n$ and let $f : X \to \mathcal{F}_k(X)$ be a fixed-point free upper semi-continuous map such that $l = \sup\big\{|f^{-1}(x)| : x \in X\big\} < \infty$. Suppose that $f(F)$ is closed in X and $\dim f^{-1}(F) = \dim F = \dim f(F)$ for any closed subset F of X. Then f is colorable with at most $kn + k + l + 1$ colors.*

Proof. f is colorable by Theorem 2.5. So there is a coloring $\mathcal{A} = \{A_i : i = 1, ..., r\}$ of f for some $r \in \mathbb{N}$. Assume that $r > kn + k + l + 1$. Because A_i and $f(A_i)$ are disjoint closed subsets of X for each $i = 1, ..., r$ and X is normal, there are two open neighborhoods U_i and V_i of A_i and $f(A)$ in X, respectively, such that $U_i \cap V_i = \emptyset$ for each $i = 1, ..., r$. Since f is upper semi-continuous, $f^{-1}[\langle\{V_i\}\rangle]$ is an open neighborhood of A_i in X for each $i = 1, ..., r$. Put $B_i = U_i \cap f^{-1}[\langle\{V_i\}\rangle]$ for each $i = 1, ..., r$. Then $\mathcal{B} = \{B_i | i = 1, ..., r\}$ is an open cover of X such that $A_i \subset B_i$ and $B_i \cap f(B_i) = \emptyset$ for $i = 1, ..., r$.

Define $g : X \to 2^X$ by $g(x) = f(f(x))$ for $x \in X$. Since f is upper semi-continuous, g is upper semi-continuous. For any closed subset F of X, $g(F) = f(f(F))$ and $g^{-1}(F) = f^{-1}(f^{-1}(F))$. Hence $g(F)$ is closed in X and $\dim g^{-1}(F) = \dim F = \dim g(F)$ by assumptions of f. These enable us to apply Lemma 2.8 as φ and $\varphi_i (i = 1, 2, ...)$ denote one of the map g and the inclusion ι. So there is a closed cover $\mathcal{C} = \{C_i : i = 1, ..., r\}$ of X such that $A_i \subset C_i \subset B_i$ for $i = 1, ..., r$ and

$$(\sharp) \qquad \varphi_{i_1}(\partial C_{i_1}) \cap \cdots \cap \varphi_{i_{n+1}}(\partial C_{i_{n+1}}) = \emptyset,$$

whenever $1 \leq i_1 < \cdots < i_{n+1} \leq r$. Define $D_i = \overline{C_i \backslash (\bigcup_{j=1}^{i-1} C_j)}$ and let $\mathcal{D} = \{D_i : i = 1, ..., r\}$. Observe that $\mathcal{D}$ is a coloring of f.

Take $x \in D_r$ and put $f^{-1}(x) = \{y_1, ..., y_l\}$ and $f(x) = \{z_1, ..., z_k\}$. Define m, p_a and q_b for $a = 1, ..., l$, $b = 1, ..., k$ as follows:

$$\begin{aligned}
&m = |\{i : (f^{-1}(x) \cup f(x)) \cap D_i \neq \emptyset\}|,\\
&p_1 = |\{i : y_1 \in D_i\}|,\\
&p_a = |\{i : \{y_1, ..., y_{a-1}\} \cap D_i = \emptyset \text{ and } y_a \in D_i\}| \ (a \geq 2),\\
&q_1 = |\{i : f^{-1}(x) \cap D_i = \emptyset \text{ and } z_1 \in D_i\}|,\\
&q_b = |\{i : (f^{-1}(x) \cup \{z_1, ..., z_{b-1}\}) \cap D_i = \emptyset \text{ and } z_b \in D_i\}| \ (b \geq 2).
\end{aligned}$$

Without lost of generality we may assume that $p_a \geq 1$, $q_b \geq 1$ for $a = 1, ..., l$, $b = 1, ..., k$. Note that no indices i overlap in the definition of p_a and q_b for $a = 1, ..., l$, $b = 1, ..., k$ i.e.

$$m = \sum_{a=1}^{l} p_a + \sum_{b=1}^{k} q_b$$

By Lemma 2.9

$$\begin{aligned} y_a &\in \partial C_i \quad \text{for at least } p_a - 1 \text{ indices } i, \\ z_b &\in \partial C_i \quad \text{for at least } q_b - 1 \text{ indices } i, \end{aligned}$$

for $a = 1, ..., l$, $b = 1, ..., k$. So

$$f(x) \subset g(\partial C_i) \quad \text{for at least } \sum_{a=1}^{l}(p_a - 1) \text{ indices } i.$$

Hence for $b = 1, ..., k$

$$z_b \in \varphi(\partial C_i) \quad \text{for at least } \sum_{a=1}^{l}(p_a - 1) + (q_b - 1) \text{ indices } i.$$

By the property ($\sharp$) for $b = 1, ..., k$

$$\sum_{a=1}^{l}(p_a - 1) + (q_b - 1) \leq n.$$

Since $p_a - 1 \geq 0$ for $a = 1, ..., l$,

$$\begin{aligned} m &= \sum_{a=1}^{l} p_a + \sum_{b=1}^{k} q_b \\ &= \sum_{a=1}^{l}(p_a - 1) + l + \sum_{b=1}^{k}(q_b - 1) + k \\ &\leq \sum_{b=1}^{k}\Big(\sum_{a=1}^{l}(p_a - 1) + (q_b - 1)\Big) + k + l \\ &\leq \sum_{b=1}^{k} n + k + l \\ &= kn + k + l. \end{aligned}$$

Now since $r > kn + k + l + 1$, there is a $j(x) \in \{1, ..., r-1\}$ such that $x \notin f^{-1}(D_{j(x)}) \cup f(D_{j(x)})$. Because $f^{-1}(D_{j(x)})$ and $f(D_{j(x)})$ are closed in X, there is an open neighborhood W_x of x in X such that $W_x \subset B_r \backslash (f^{-1}(D_{j(x)}) \cup f(D_{j(x)}))$.

Put $\mathcal{W} = \{W_x : x \in D_r\}$. By paracompactness of D_r there is a locally finite closed refinement $\mathcal{K} = \{K_s : s \in S\}$ of $\mathcal{W}$, where S is an index set. Define $\psi : S \to \{1, ..., r-1\}$ as it satisfies that $K_s \subset B_r \backslash (f^{-1}(D_{\psi(s)}) \cup f(D_{\psi(s)}))$. Put $E_j = \bigcup\{K_s : j = \psi(s)\}$ and $F_j = D_j \cup E_j$ for $j = 1, ..., r-1$. Then $\mathcal{F} = \{F_j : j = 1, ..., r-1\}$ is a coloring of f consisting of $r-1$ colors. In fact, since $\mathcal{K}$ is locally finite, E_j is closed in X and so F_j is closed in X. To show that F_j is a color of f for each $j = 1, ..., r-1$ we check the followings:

$$\begin{aligned} D_j \cap f(D_j) &= \emptyset, \\ D_j \cap f(E_j) &= \emptyset, \\ E_j \cap f(D_j) &= \emptyset, \\ E_j \cap f(E_j) &= \emptyset \end{aligned}$$

for $j = 1, ..., r-1$. The second can be replaced by $E_j \cap f^{-1}(D_j) = \emptyset$. Therefore, all hold since $\mathcal{D}$ is the coloring of f and $E_j \subset B_r \setminus (f^{-1}(D_j) \cup f(D_j))$.

We have reduced the number of colors by one, under the assumption that this number is greater than $kn + k + l + 1$. Inductively, the coloring of f can be reduced to a coloring of f with $kn + k + l + 1$ colors. □

When X is compact, Theorem 2.10 implies the following corollary by the same way as Corollary 2.6.

Corollary 2.11. *Let X be a compact metrizable space with $\dim X \leq n$ and let $f : X \to \mathcal{F}_k(X)$ be a fixed-point free upper semi-continuous map such that $l = \sup\{|f^{-1}(x)| : x \in X\} < \infty$. Suppose that $\dim f^{-1}(F) = \dim F = \dim f(F)$ for any closed subset F of X. Then f is colorable with at most $kn + k + l + 1$ colors.*

We would like to finish the paper by mentioning a relation between colorability and the Stone-Čech compactification. Let X be a normal space. Then the Stone-Čech compactification βX of X is equivalent to the Wallman compactification of X with respect to the Wallman base consisting of all closed subsets of X. Hence $\overline{F}^{\beta X} \cap \overline{G}^{\beta X} = \emptyset$ for any pair (F, G) of disjoint closed subsets of X. Also if F is closed in X, $\beta F = \overline{F}^{\beta X}$. So we may assume that $\beta F \subset \beta X$.

The following is a generalization of Proposition 1.1.

Proposition 2.12. *Let X be a closed subspace of a normal space Y and let $f : X \to \mathcal{K}(Y)$ be a fixed-point free continuous map. Then, the following are equivalent:*

(1) *f is brightly colorable.*

(2) *The Stone-Čech extension $\beta f : \beta X \to 2^{\beta Y}$ of f is fixed-point free.*

Proof. We will show that (1) implies (2). Since $2^{\beta Y}$ is compact and $\mathcal{K}(Y) \subset 2^{\beta Y}$, there is a continuous extension $\beta f : \beta X \to 2^{\beta Y}$ of f. Take $z \in \beta X$ to show that βf is fixed-point free . By (1) there is a bright coloring $\mathcal{C}$ of f. Then $\overline{\mathcal{C}}^{\beta Y}$ is a finite cover of βX and hence there is a $C \in \mathcal{C}$ such that $z \in \overline{C}^{\beta Y}$. Because C is a bright color of f, $C \cap \overline{f(C)}^{Y} = \emptyset$. By the property of the Stone-Čech compactification $\overline{C}^{\beta Y} \cap \overline{f(C)}^{\beta Y} = \emptyset$. By continuity of f

$$\beta f(z) \subset \beta f(\overline{C}^{\beta Y}) \subset \overline{\beta f(C)}^{\beta Y} \subset \overline{f(C)}^{\beta Y}.$$

Thus, $z \notin \beta f(z)$.

Next, we will show that (2) implies (1). Since βf is fixed-point free continuous and βX is compact, βf is colorable from Proposition 2.1. So there is a coloring $\mathcal{C}$ of βf. Then the restriction of $\mathcal{C}$ to X is as required. □

This shows that colorability for set-valued continuous maps with compact values is similar to that for single-valued continuous maps.

REFERENCES

[1] J. W. Alexander, *Order sets, complexes and the problem of compactification*, Proc. Natl. Acad. Sci., 25 (1939), 296-298.

[2] J. M. Aarts, R. J. Fokkink and H. Vermeer, *Variations on a theorem of Lusternik and Schnirelmann*, Topology, 35 (1996), 1051-1056.

[3] J. M. Aarts and T. Nishiura, *Dimension and extensions*, North-Holland Publishing Co., Amsterdam, (1992).

[4] R. Z. Buzyakova, *On multivalued fixed-point free maps on* $\mathbb{R}^n$, Proc. Amer. Math. Soc., 140 (2012), 2929-2936.

[5] R. Z. Buzyakova and A. Chigogidze, *Fixed-point free maps of Euclidean spaces*, Fund. Math., 212 (2011), 1-16.

[6] E. K. van Douwen, *βX and fixed-point free maps*, Topology Appl., 51 (1993), 191-195.

[7] R. Engelking, *General Topology*, PWN, Warszawa, (1977).

[8] O. Frink, *Topology in lattices*, Trans. Amer. Math. Soc., 51 (1941), 569-582.

[9] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc., 7 (1951), 152-182.

[10] J. van Mill, *The Infinite-Dimensional Topology of Function Spaces*, North-Holland Publishing Co., Amsterdam, (2001).

Communicated by *Yasunao Hattori*

Atsushi Ebisawa
Institute of Mathematics
University of Tsukuba
Ibaraki 305-8571, Japan
E-mail adress: ebisawa554@gmail.com

Hisao Kato
Institute of Mathematics
University of Tsukuba
Ibaraki 305-8571, Japan
E-mail adress: hkato@math.tsukuba.ac.jp

WEIGHTED VARIABLE MODULATION SPACES

TAKAHIRO NOI AND YOSHIHIRO SAWANO

Received October 10,2014 ; Revised January 15, 2015

ABSTRACT. The aim of this paper is to develop a theory of weighted modulation spaces with variable exponent. All we assume on the exponent is that the essential infimum of the exponent is positive. We shall show that the auxiliary parameter can be removed assuming, in addition, that the weight belongs to the variable Muckenhoupt class and that the exponents satisfy the log-Hölder condition and the log-decay condition. Under these assumptions, we prove the molecular decomposition theorem and boundedness of pseudo-differential operators with symbol S^0_{00}.

1 Introduction The aim of this paper is two-fold: one is to develop a theory for modulation spaces with variable exponents; the other is to establish that the results carry over to the weighted setting to a large extent by introducing an auxiliary parameter $a > 0$.

Let us start by recalling the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ proposed by Nakano in 1951 [10, 11], where Nakano actually worked on $[0,1]$; see [8] for a detailed account. Let $L^0(\mathbb{R}^n)$ be the set of all complex-valued measurable functions defined on $\mathbb{R}^n$. Let also $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a measurable function throughout this paper, which is sometimes referred to as an exponent. Define the Lebesgue space $f \in L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent by:

$$L^{p(\cdot)}(R^n) \equiv \left\{ f \in L^0(R^n) : \int_{R^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx < \infty \text{ for some } \lambda > 0 \right\}.$$

Equip $L^{p(\cdot)}(\mathbb{R}^n)$ with the norm given by

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \equiv \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}$$

for $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

Now we move on to the weighted setting. By a "weight function", we mean a measurable function w defined on $\mathbb{R}^n$ such that $0 < w(x) < \infty$ for almost every $x \in \mathbb{R}^n$. Let $p(\cdot)$ be an exponent such that

$$0 < p_- \equiv \operatorname*{essinf}_{x\in\mathbb{R}^n} p(x) \le p_+ \equiv \operatorname*{esssup}_{x\in\mathbb{R}^n} p(x) < \infty$$

and w be a weight function. One defines the weighted variable exponent Lebesgue space $L^{p(\cdot)}(w)$ by

$$L^{p(\cdot)}(w) \equiv \left\{ f \in L^0(\mathbb{R}^n) \, : \, \int_{\mathbb{R}^n} |f(x)|^{p(x)} w(x) \, dx < \infty \right\},$$

as a linear space, and the norm is given by:

$$\|f\|_{L^{p(\cdot)}(w)} \equiv \inf \left\{ \lambda > 0 \, : \, \int_{\mathbb{R}^n} \left(\frac{|f(x)| w(x)^{1/p(x)}}{\lambda} \right)^{p(x)} dx \le 1 \right\}.$$

We define the weighted vector-valued Lebesgue space $\ell^{q(\cdot)}(L^{p(\cdot)}(w))$ with variable exponents based on the above definition.

2000 *Mathematics Subject Classification.* 42B35, 41A17.

Definition 1.1. Let $p(\cdot)$ and $q(\cdot)$ be exponents satisfying

$$0 < p_- \le p_+ < \infty, \quad 0 < q_- \le q_+ < \infty, \tag{1.1}$$

and let w be a weight. One defines the weighted vector-valued function space $\ell^{q(\cdot)}(L^{p(\cdot)}(w))$ by

$$\ell^{q(\cdot)}(L^{p(\cdot)}(w)) \equiv \left\{ \{f_m\}_{m\in\mathbb{Z}^n} \subset L^0(\mathbb{R}^n) \,:\, \sum_{m\in\mathbb{Z}^n} \left\| |f_m|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(w)} < \infty \right\}$$

as a linear space, and the norm is given by:

$$\|\{f_m\}_{m\in\mathbb{Z}^n}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} = \inf \left\{ \lambda > 0 \,:\, \sum_{m\in\mathbb{Z}^n} \left\| \left| \frac{f_m}{\lambda} \right|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(w)} \le 1 \right\}$$

for $\{f_m\}_{m\in\mathbb{Z}^n} \subset L^0(\mathbb{R}^n)$.

Now we define the weighted modulation space $M_{p(\cdot),q(\cdot),a}(w)$ with variable exponents by using the following standard operators in time frequency analysis:

- For a measurable function f on $\mathbb{R}^n$ and $m, l \in \mathbb{Z}^n$, define $M_m f$ and $T_l f$ by $M_m f(x) \equiv \exp(im\cdot x)f(x)$ and $T_l f(x) \equiv f(x-l)$, respectively.
- Define the Fourier transform and its inverse by

$$\mathcal{F}f(\xi) \equiv \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x)\exp(-ix\cdot\xi)\,dx, \ \mathcal{F}^{-1}f(x) \equiv \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(\xi)\exp(ix\cdot\xi)\,d\xi.$$

- Let $Q(r) = \{x \in \mathbb{R}^n \,:\, \max\{|x_1|, |x_2|, \ldots, |x_n|\} \le r\}$.

With these definitions in mind, we present the definition of the weighted modulation space $M_{p(\cdot),q(\cdot),a}(w)$ with variable exponents.

Definition 1.2. Suppose that $p(\cdot)$ and $q(\cdot)$ satisfies (1.1) and $a > 0$. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfying

$$\chi_{Q(1/4)} \le \mathcal{F}\phi \le \chi_{Q(2)} \tag{1.2}$$

and

$$\sum_{m\in\mathbb{Z}^n} T_m[\mathcal{F}\phi](x) > 0 \tag{1.3}$$

for all $x \in \mathbb{R}^n$. Then the space $M_{p(\cdot),q(\cdot),a}(w)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quasi-norm

$$\|f\|_{M_{p(\cdot),q(\cdot),a}(w)} \equiv \|\{(M_m\phi * f)_a\}_{m\in\mathbb{Z}^n}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}$$

is finite, where

$$(M_m\phi * f)_a(x) = \sup_{y\in\mathbb{R}^n} \frac{|M_m\phi * f(y)|}{(1+|x-y|)^a}. \tag{1.4}$$

The next result justifies the notation $M_{p(\cdot),q(\cdot),a}(w)$.

Theorem 1.1. *Let $p(\cdot), q(\cdot) : \mathbb{R}^n \to (0,\infty)$ be variable exponents satisfying* (1.1). *Then the definition of the set $M_{p(\cdot),q(\cdot),a}(w)$ is independent of the choice of ϕ; different choices of admissible functions yield equivalent norms.*

With the new parameter a, our assumption on w can be minimized in order that $M_{p(\cdot),q(\cdot),a}(w) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. More quantitatively, we have the following assertion:

Theorem 1.2. *Let w be a weight such that*

$$\int_{[0,1]^n} w(x)\,dx < \infty. \tag{1.5}$$

Let $f \in M_{p(\cdot),q(\cdot),a}(w)$. Then there exists $C > 0$ such that

$$\sup_{m\in\mathbb{Z}^n} \|(1+|\cdot|)^{-a}(M_m\phi * f)_a\|_{L^\infty} \le C\|f\|_{M_{p(\cdot),q(\cdot),a}(w)}.$$

In particular, $M_{p(\cdot),q(\cdot),a}(w) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$.

Theorem 1.2 shows that (1.5) is sufficient to guarantee that our new space $M_{p(\cdot),q(\cdot),a}(w)$ is a subset of $\mathcal{S}'(\mathbb{R}^n)$.

We impose on $p(\cdot)$ the log-Hölder continuity condition:

$$|p(x)-p(y)| \le \frac{c_{\log}(p)}{\log(\mathrm{e}+|x-y|^{-1})} \quad \text{for} \quad x, y \in \mathbb{R}^n, \tag{1.6}$$

and the log decay condition;

$$|p(x)-p_\infty| \le \frac{c^*}{\log(e+|x|)} \quad \text{for} \quad x \in \mathbb{R}^n, \tag{1.7}$$

where p_∞ is a real number, $c_{\log}(p)$ and c^* are positive constants independent of x and y. We say that $p(\cdot)$ satisfies the globally log-Hölder condition if $p(\cdot)$ satisfies both (1.6) and (1.7).

We also consider the sequence space $m_{p(\cdot),q(\cdot),a}(w)$ to prove the molecular decomposition theorem.

Definition 1.3. Let $p(\cdot)$ and $q(\cdot)$ be exponents satisfying (1.1) and $a > 0$. One defines a space $m_{p(\cdot),q(\cdot),a}(w)$ as the set of all complex sequences $\lambda = \{\lambda_{ml}\}_{m,l\in\mathbb{Z}^n}$ such that

$$\sum_{m\in\mathbb{Z}^n} \left\| \left| \sup_{y\in\mathbb{R}^n} \left(\sum_{l\in\mathbb{R}^n} \frac{|\lambda_{ml}|\chi_{l+[0,1)^n}(\cdot-y)}{(1+|y|)^a} \right) \right|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(w)} < \infty.$$

For such a sequence λ, define the quasi-norm by

$$\|\lambda\|_{m_{p(\cdot),q(\cdot),a}(w)} \equiv \inf\left\{ T>0 : \sum_{m\in\mathbb{Z}^n} \left\| \left| \sup_{y\in\mathbb{R}^n} \left(\sum_{l\in\mathbb{R}^n} \frac{|\lambda_{ml}|\chi_{l+[0,1)^n}(\cdot-y)}{T(1+|y|)^a} \right) \right|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(w)} \le 1 \right\}.$$

We consider the following condition on weights:

$$(1+|\cdot|)^{-a} \in L^{p(\cdot)}(w) < \infty. \tag{1.8}$$

As the following theorem shows, (1.8) is a natural and minimal condition.

Theorem 1.3. *Let $w : \mathbb{R}^n \to [0,\infty)$ be a measurable function. Assume that $p(\cdot)$ and $q(\cdot)$ satisfy* (1.1) *and $a > 0$. Then assumption* (1.8) *is necessary and sufficient for $m_{p(\cdot),q(\cdot),a}(w)$ to contain an element other than* 0.

We may ask ourselves whether the parameter a is essential. If the weight is good enough, then we can show that a is not essential as long as $a \gg 0$. We invoke the following definition from [2, 3, 4].

For a variable exponent $p(\cdot) : \mathbb{R}^n \to [1,\infty)$, a measurable function w is said to be an $A_{p(\cdot)}$ weight if $0 < w(x) < \infty$ for almost every $x \in \mathbb{R}^n$ and

$$\sup_Q \left(\frac{1}{|Q|} \|w^{1/p(\cdot)}\chi_Q\|_{L^{p(\cdot)}} \|w^{-1/p(\cdot)}\chi_Q\|_{L^{p'(\cdot)}} \right) < \infty \tag{1.9}$$

holds, where the supremum is taken over all open cubes $Q \subset \mathbb{R}^n$ whose sides are parallel to the coordinate axes and $p'(x)$ is the conjugate exponent of $p(x)$, that is, $1/p(x)+1/p'(x) = 1$.

In the above definition, when $a \gg 0$ and $w \in A_{p(\cdot)}$, the space $m_{p(\cdot),q(\cdot),a}(w)$ does not depend on a, as the following theorem shows.

Theorem 1.4. *Assume that $p(\cdot)$ and $q(\cdot)$ satisfy* (1.1)*, $p_- > 1$, $w \in A_{p(\cdot)}$ and $a \gg 0$. Assume, in addition, that $p(\cdot)$ and $q(\cdot)$ are globally* log*-Hölder continuous. Then $\lambda \in m_{p(\cdot),q(\cdot),a}(w)$ if and only if*

$$\left\| \left\{ \sum_{l\in\mathbb{Z}^n} |\lambda_{ml}|\chi_{l+[0,1)^n}(\cdot) \right\}_{m\in\mathbb{Z}^n} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} < \infty.$$

We also consider the molecular decomposition. For $x \in \mathbb{R}^n$, we write $\langle x\rangle \equiv \sqrt{1+|x|^2}$. Suppose that $K, N \in \mathbb{N}$ are large enough and fixed. A C^K-function $\tau : \mathbb{R}^n \to \mathbb{C}$ is said to be an (m,l)-molecule if it satisfies $|\partial^\alpha(e^{-im\cdot x}\tau(x))| \le \langle x-l\rangle^{-N},\ x \in \mathbb{R}^n$ for $|\alpha| \le K$. Set

$$\mathcal{M} \equiv \{M = \{\mathrm{mol}_{ml}\}_{m,l\in\mathbb{Z}^n} \subset C^K \ :\ \mathrm{mol}_{ml} \text{ is an } (m,l)\text{-molecule for every } m,l \in \mathbb{Z}^n\}.$$

We shall develop a theory of decomposition based on the above definition.

Theorem 1.5. *Let $a \gg N+n$. Assume, in addition, that $p(\cdot)$ and $q(\cdot)$ satisfy* (1.1).

(i) *Let $\phi, \kappa \in \mathcal{S}(\mathbb{R}^n)$ satisfy*

$$\chi_{Q(1/4)} \le \mathcal{F}\phi \le \chi_{Q(2)}, \quad \sum_{l\in\mathbb{Z}^n} T_l[\mathcal{F}\phi] \equiv 1 \tag{1.10}$$

and

$$0 \le \kappa \le \chi_{Q(2)}, \quad \sum_{l\in\mathbb{Z}^n} T_l\kappa \equiv 1. \tag{1.11}$$

The decomposition, called Gabor decomposition, holds for $M_{p(\cdot),q(\cdot),a}(w)$. More precisely, we have $\{T_lM_m[\mathcal{F}^{-1}\kappa]\}_{m,l\in\mathbb{Z}^n} \in \mathcal{M}$ and the mapping

$$f \in M_{p(\cdot),q(\cdot),a}(w) \mapsto \lambda = \{M_m\phi * f(l)\}_{m,l\in\mathbb{Z}^n} \in m_{p(\cdot),q(\cdot),a}(w)$$

is bounded. Furthermore, any $f \in M_{p(\cdot),q(\cdot),a}(w)$ admits the following Gabor decomposition

$$f = \sum_{m,l\in\mathbb{Z}^n} \lambda_{ml} \cdot T_lM_m[\mathcal{F}^{-1}\kappa],$$

$$\lambda = \{\lambda_{ml}\}_{m,l\in\mathbb{Z}^n} = \{M_m\phi * f(l)\}_{m,l\in\mathbb{Z}^n} \in m_{p(\cdot),q(\cdot),a}(w). \tag{1.12}$$

(ii) *Suppose we are given* $M = \{\mathrm{mol}_{ml}\}_{m,l \in \mathbb{Z}^n} \in \mathcal{M}$ *and* $\lambda = \{\lambda_{ml}\}_{m,l \in \mathbb{Z}^n} \in m_{p(\cdot),q(\cdot),a}(w)$. *Then*

$$f \equiv \sum_{m,l \in \mathbb{Z}^n} \lambda_{ml} \cdot \mathrm{mol}_{ml} \tag{1.13}$$

converges unconditionally in $\mathcal{S}'(\mathbb{R}^n)$. *Furthermore,* f *belongs to* $M_{p(\cdot),q(\cdot),a}(w)$ *and satisfies the quasi-norm estimate* $\|f\|_{M_{p(\cdot),q(\cdot),a}(w)} \le C\,\|\lambda\|_{m_{p(\cdot),q(\cdot),a}(w)}$. *In particular, the convergence of* (1.13) *takes place in* $M_{p(\cdot),q(\cdot),a}(w)$.

Corollary 1.6. *Under assumption* (1.8), $\mathcal{S}(\mathbb{R}^n) \subset M_{p(\cdot),q(\cdot),a}(w)$.

As an application, we shall show that the pseudo-differential operator with symbol S^0_{00} is bounded on $M_{p(\cdot),q(\cdot),a}(w)$. Recall that $a \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ is an S^0_{00}-symbol if

$$\partial^\beta_x \partial^\alpha_\xi a \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$$

for all multi-indices α and β. The pseudo-differential operator $a(X,D)$ is defined by

$$a(X,D)f(x) \equiv \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} a(x,\xi)\mathcal{F}f(\xi) e^{ix\cdot\xi}\,d\xi.$$

In [9, Lemma 3.2], the authors showed that the set $\mathcal{M}$ is preserved by $a(X,D)$. Thus, we have the following result, which is a direct corollary of Theorem 1.5:

Theorem 1.7. *Let* $a \in S^0_{00}$. *Then* $a(X,D)$ *is a bounded linear operator on* $M_{p(\cdot),q(\cdot),a}(w)$.

Remark 1.1. When $p(\cdot) \equiv q(\cdot) \equiv 2$, $a > n/2$ and $w \equiv 1$, we have $M_{p(\cdot),q(\cdot),a}(w) = L^2(\mathbb{R}^n)$. It may be interesting to note that Sjöstrand proved this result when $M_{p(\cdot),q(\cdot),a}(w) = L^2(\mathbb{R}^n)$ by using the so-called T^*T-method, while our method is beyond the reach of this method employed in [12].

We organize the remaining part of this paper as follows: The proofs of Theorem 1.1 through Theorem 1.4 can be found in Section 2. In Section 3, we shall develop a theory of decomposition and we prove Theorem 1.5.

2 Fundamental structure of $M_{p(\cdot),q(\cdot),a}(w)$

2.1 Proof of Theorem 1.1 Let ϕ, ψ be functions in $\mathcal{S}(\mathbb{R}^n)$ satisfying (1.2) and (1.3). Let us choose a smooth function $\Phi \in \mathcal{S}(\mathbb{R}^n)$ so that

$$\mathcal{F}\Phi(\xi) \sum_{m \in \{-2,-1,0,1,2\}^n} \mathcal{F}\psi(\xi - m) = \mathcal{F}\phi(\xi).$$

Then we have

$$\phi = (2\pi)^{n/2}\Phi * \sum_{m \in \{-2,-1,0,1,2\}^n} M_m\psi$$

and hence

$$M_l\phi = (2\pi)^{n/2} M_l\Phi * \sum_{m \in \{-2,-1,0,1,2\}^n} M_{l+m}\psi,$$

which implies

$$
\begin{aligned}
&|M_l\phi * f(x-y)| \\
&\le C \sum_{m\in\{-2,-1,0,1,2\}^n} \int_{\mathbb{R}^n} |\Phi(z)|\cdot|M_{l+m}\psi * f(x-y-z)|\,dz \\
&\le C\left(\int_{\mathbb{R}^n}|\Phi(z)|(1+|z+y|)^a\,dz\right)\sum_{m\in\{-2,-1,0,1,2\}^n}\sup_{w\in\mathbb{R}^n}\frac{|M_{l+m}\psi * f(x-w)|}{(1+|w|)^a} \\
&\le C(1+|y|)^a \sum_{m\in\{-2,-1,0,1,2\}^n}\sup_{w\in\mathbb{R}^n}\frac{|M_{l+m}\psi * f(x-w)|}{(1+|w|)^a}
\end{aligned}
$$

for all $x, y \in \mathbb{R}^n$. This in turn implies

$$(M_l\phi * f)_a(x) \le C \sum_{m\in\{-2,-1,0,1,2\}^n} (M_{l+m}\psi * f)_a(x).$$

Due to symmetry, we see that different choices of admissible functions yield equivalent norms.

2.2 Proof of Theorem 1.2 Let $m \in \mathbb{Z}^n$ be fixed and take $x \in m + [0,1]^n$. Then we have

$$
\begin{aligned}
(1+|x|)^{-a}(M_m\phi * f)_a(x) &= \sup_{y\in\mathbb{R}^n}\frac{|M_m\phi * f(y)|}{(1+|x-y|)^a(1+|x|)^a} \\
&\le C\sup_{y\in\mathbb{R}^n}\frac{|M_m\phi * f(y)|}{(1+|y|)^a} \\
&\le C\inf_{z\in[0,1]^n}\sup_{y\in\mathbb{R}^n}\frac{|M_m\phi * f(y)|}{(1+|y-z|)^a}.
\end{aligned}
$$

If we use (1.5), then we obtain

$$(1+|x|)^{-a}(M_m\phi * f)_a(x) \le C\left\|\sup_{y\in\mathbb{R}^n}\frac{|M_m\phi * f(y)|}{(1+|y-\cdot|)^a}\right\|_{L^{p(\cdot)}(w)} = C\|(M_m\phi * f)_a\|_{L^{p(\cdot)}(w)}.$$

This then yields

$$(1+|x|)^{-a}(M_m\phi * f)_a(x) \le C\|f\|_{M_{p(\cdot),q(\cdot),a}(w)}.$$

Thus, the proof is complete.

2.3 Proof of Theorem 1.3 We justify the condition (1.8); we prove Theorem 1.3.

Proof of Theorem 1.3. Let $\lambda = \{\lambda_{ml}\}_{m,l\in\mathbb{Z}^n} \in m_{p(\cdot),q(\cdot),a}(w)\setminus\{0\}$. Then there exist $m_0, l_0 \in \mathbb{Z}^n$ such that $\lambda_{m_0 l_0} \ne 0$. Set

$$\rho_{ml} \equiv \begin{cases}\lambda_{m_0l_0} & (m,l) = (m_0,l_0),\\ 0 & \text{otherwise.}\end{cases}$$

Then $\rho \equiv \{\rho_{ml}\}_{m,l\in\mathbb{Z}^n}$ belongs to $m_{p(\cdot),q(\cdot),a}(w)\setminus\{0\}$. This implies

$$0 < \|\rho\|_{m_{p(\cdot),q(\cdot),a}(w)} = \left\|\sup_{y\in\mathbb{R}^n}\left(|\lambda_{m_0l_0}|\frac{\chi_{l_0+[0,1)^n}(\cdot-y)}{(1+|y|)^a}\right)^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(w)} < \infty.$$

Hence, we have

$$\left\|\sup_{y\in\mathbb{R}^n}\left(\frac{\chi_{[0,1)^n}(\cdot-y)}{(1+|y|)^a}\right)^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(w)} \le (1+|l_0|)^{aq_+}\left\|\sup_{y\in\mathbb{R}^n}\left(\frac{\chi_{l_0+[0,1)^n}(\cdot-y)}{(1+|y|)^a}\right)^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(w)} < \infty.$$

Therefore, we obtain

$$\begin{aligned}&\left\|\left|\sup_{y\in\mathbb{R}^n}\left(\frac{\chi_{[0,1)^n}(\cdot-y)}{(1+|y|)^a}\right)\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(w)}\\ &\le \max\left\{\left\|\sup_{y\in\mathbb{R}^n}\left(\frac{\chi_{[0,1)^n}(\cdot-y)}{(1+|y|)^a}\right)\right\|_{L^{p(\cdot)}(w)}^{q_-},\left\|\sup_{y\in\mathbb{R}^n}\left(\frac{\chi_{[0,1)^n}(\cdot-y)}{(1+|y|)^a}\right)\right\|_{L^{p(\cdot)}(w)}^{q_+}\right\}<\infty\end{aligned}$$

and hence (1.8). □

2.4 Proof of Theorem 1.4 Firstly, we prove the following lemma to prove Lemma 2.2.

Lemma 2.1. *Let $p(\cdot)$ satisfy the* log*-Hölder conditions. Define $\eta_a(x) \equiv \frac{1}{(1+|x|)^a}$ for $x \in \mathbb{R}^n$. If $a > c_{\log}(p)$, then there exists a constant $C > 0$ such that*

$$b^{p(x)}\eta_{2a}(x-y) \le Cb^{p(y)}\eta_a(x-y) \tag{2.1}$$

holds for any $1 \le b < \infty$ and $x, y \in \mathbb{R}^n$.

Proof. We use a similar argument to the proof of [5, Lemma 6.1]. We may assume $|x-y| \ge b$ due to the log Hölder continuity of $p(\cdot)$. We fix the smallest natural number $k \ge 2$ such that $|x-y| \le b^{-1+k}$. Then, for such x, y and k, $1 + |x-y| \sim b^k$ holds and we have

$$\frac{\eta_{2a}(x-y)}{\eta_a(x-y)} \le c(1+b^k)^{-a} \le cb^{-ka}. \tag{2.2}$$

Furthermore, by the Hölder continuity of $p(\cdot)$ and $a > c_{\log}(p)$, we see that

$$b^{p(y)-p(x)} \ge b^{-c_{\log}(p)/\log(e+|x-y|^{-1})} \ge b^{-c_{\log}(p)} \ge b^{-(k-1)a}. \tag{2.3}$$

Hence, the desired inequality (2.1) holds thanks to (2.2) and (2.3) as well as the fact that $a > c_{\log}(p)$. □

We need the following auxiliary estimate akin to the one in [1].

Lemma 2.2. *Let $p(\cdot), q(\cdot)$ satisfy the log-Hölder condition as well as the log decay condition. Let $1 \le p_- \le p_+ < \infty$ and let also $w \in A_{p(\cdot)}$. Let $a > 2\max\{n, c_{\log}(q)\}$. Set $\eta_a(x) \equiv (1+|x|)^{-a}$. Then, for any $\{f_m\}_{m\in\mathbb{Z}^n} \in \ell^{q(\cdot)}(L^{p(\cdot)}(w))$,*

$$\|\{\eta_a * f_m\}_{m\in\mathbb{Z}^n}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} \le C\|\{f_m\}_{m\in\mathbb{Z}^n}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}.$$

Proof. We follow the idea in the work by Almeida and Hästö, which is listed above. Without loss of generality, we can assume $\|\{f_m\}_{m\in\mathbb{Z}^n}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} = 1$. Then it is easy to see that $\|f_m\|_{L^{p(\cdot)}(w)} \le 1$ hold for any $m \in \mathbb{Z}^n$. Let $m \in \mathbb{Z}^n$ be fixed and $\delta = \||f_m|^{q(\cdot)}\|_{L^{p(\cdot)/q(\cdot)}(w)}$. By the argument of [1, Proof of Lemma 4.7] with Lemma 2.1 which takes the place of [1, Lemma 4.3], we have $\|\delta^{-1/q(\cdot)}(\eta_a * f_m)\|_{L^{p(\cdot)}(w)} \le C\|\eta_{a/2} * [\delta^{-1/q(\cdot)} f_m]\|_{L^{p(\cdot)}(w)}$.

Denote by M the Hardy-Littlewood maximal operator; for a measurable function f define

$$Mf(x) = \sup_{r>0} \frac{1}{(2r)^n} \int_{[-r,r]^n} |f(x-y)|\,dy \quad (x \in \mathbb{R}^n).$$

Since $a > 2n$, we have $\|\eta_{a/2}\|_{L^1} < \infty$ and hence

$$|\eta_{a/2} * F(x)| \le CMF(x)$$

for all positive measurable functions F. Since $w \in A_{p(\cdot)}$, we have

$$\|\eta_{a/2} * F\|_{L^{p(\cdot)}(w)} \le C\|F\|_{L^{p(\cdot)}(w)}. \tag{2.4}$$

Note that

$$\|\delta^{-1/q(\cdot)}(\eta_a * f_m)\|_{L^{p(\cdot)}(w)} \le C\|\eta_{a/2} * [\delta^{-1/q(\cdot)} f_m]\|_{L^{p(\cdot)}(w)} \le C\|\delta^{-1/q(\cdot)} f_m\|_{L^{p(\cdot)}(w)},$$

where for the second inequality we used (2.4). Note that

$$\begin{aligned}
&\min\{\|h^{q(\cdot)}\|^{1/q_-}_{L^{p(\cdot)/q(\cdot)}(w)}, \|h^{q(\cdot)}\|^{1/q_+}_{L^{p(\cdot)/q(\cdot)}(w)}\} \\
&\le \|h\|_{L^{p(\cdot)}} \le \max\{\|h^{q(\cdot)}\|^{1/q_-}_{L^{p(\cdot)/q(\cdot)}(w)}, \|h^{q(\cdot)}\|^{1/q_+}_{L^{p(\cdot)/q(\cdot)}(w)}\}
\end{aligned}$$

for any non-negative measurable function h. Therefore,

$$\begin{aligned}
&\min\{\delta^{-1/q_-}\||\eta_a * f_m|^{q(\cdot)}\|^{1/q_-}_{L^{p(\cdot)/q(\cdot)}(w)}, \delta^{-1/q_+}\||\eta_a * f_m|^{q(\cdot)}\|^{1/q_+}_{L^{p(\cdot)/q(\cdot)}(w)}\} \\
&\le \max\{\delta^{-1/q_-}\||f_m|^{q(\cdot)}\|^{1/q_-}_{L^{p(\cdot)/q(\cdot)}(w)}, \delta^{-1/q_+}\||f_m|^{q(\cdot)}\|^{1/q_+}_{L^{p(\cdot)/q(\cdot)}(w)}\} = 1.
\end{aligned}$$

This implies that either

$$\delta^{-1/q_-}\||\eta_a * f_m|^{q(\cdot)}\|^{1/q_-}_{L^{p(\cdot)/q(\cdot)}(w)} \le 1$$

or

$$\delta^{-1/q_+}\||\eta_a * f_m|^{q(\cdot)}\|^{1/q_+}_{L^{p(\cdot)/q(\cdot)}(w)} \le 1.$$

Hence,

$$\|\,|\eta_a * f_m|^{q(\cdot)}\,\|_{L^{p(\cdot)/q(\cdot)}(w)} \le C\|\,|f_m|^{q(\cdot)}\,\|_{L^{p(\cdot)/q(\cdot)}(w)}.$$

□

Now we prove Theorem 1.4. Let $\eta_a(x) \equiv (1+|x|)^{-a}$ as before. Then we have

$$\begin{aligned}
\sup_{y\in\mathbb{R}^n}\left(\sum_{l\in\mathbb{Z}^n}\frac{|\lambda_{ml}|\chi_{l+[0,1)^n}(x-y)}{(1+|y|)^a}\right) &\le C\sum_{k\in\mathbb{Z}^n}\left(\sum_{l\in\mathbb{Z}^n}\frac{|\lambda_{ml}|\chi_{l+[0,1)^n}(x-k)}{(1+|k|)^a}\right) \\
&\le C\int_{\mathbb{R}^n}\left(\sum_{l\in\mathbb{Z}^n}\frac{|\lambda_{ml}|\chi_{l+[0,1)^n}(x-z)}{(1+|z|)^a}\right)dz \\
&= C\eta_a * \left[\sum_{l\in\mathbb{Z}^n}|\lambda_{ml}|\chi_{l+[0,1)^n}\right](x).
\end{aligned}$$

Taking the $\ell^{q(\cdot)}(L^{p(\cdot)}(w))$-norm, we obtain

$$\begin{aligned}&\left\|\left\{\sup_{y\in\mathbb{R}^n}\left(\sum_{l\in\mathbb{Z}^n}\frac{|\lambda_{ml}|\chi_{l+[0,1)^n}(\cdot-y)}{(1+|y|)^a}\right)\right\}_{m\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}\\ &\le C\left\|\left\{\eta_a*\left[\sum_{l\in\mathbb{Z}^n}|\lambda_{ml}|\chi_{l+[0,1)^n}\right]\right\}_{m\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}.\end{aligned}$$

Hence, we invoke Lemma 2.2 to see that

$$\begin{aligned}&\left\|\left\{\sup_{y\in\mathbb{R}^n}\left(\sum_{l\in\mathbb{Z}^n}\frac{|\lambda_{ml}|\chi_{l+[0,1)^n}(\cdot-y)}{(1+|y|)^a}\right)\right\}_{m\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}\\ &\le C\left\|\left\{\sum_{l\in\mathbb{Z}^n}|\lambda_{ml}|\chi_{l+[0,1)^n}\right\}_{m\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))},\end{aligned}$$

as was to be shown.

3 Molecular decomposition Assumption (1.5) is also appropriate to develop a theory for the decomposition of weighted modulation spaces with variable exponent.

The following well-known lemma is used to prove Theorem 1.5. For example, we refer for the proof to the paper [6, Lemma 2.1] due to M. Frazier and B. Jawerth, who took full advantage of this equality in [7, Lemma 2.1].

Lemma 3.1. [6, Lemma 2.1], [7, Lemma 2.1] *Let $f\in\mathcal{S}'(\mathbb{R}^n)$ with frequency support contained in $Q(2)$;*

$$\operatorname{supp}(\mathcal{F}f)\subset Q(2). \tag{3.1}$$

Assume, in addition, that $\kappa\in\mathcal{S}(\mathbb{R}^n)$ is supported on $Q(2)$ and that

$$\sum_{l\in\mathbb{Z}^n}T_l\kappa\equiv 1.$$

Then we have

$$f=(2\pi)^{-\frac{n}{2}}\sum_{l\in\mathbb{Z}^n}f(l)\cdot T_l[\mathcal{F}^{-1}\kappa]. \tag{3.2}$$

Remark 3.1. In the original version of [6, Lemma 2.1], Frazier and Jawerth did not consider condition (3.1). Instead, they decomposed f according the size of frequency support; see [6, (2.5)]. Apart from the mollification done in [6, (2.7)], their key idea of the proof is to expand a function into Fourier series; see [6, (2.8)]. This technique will be used to prove Lemma 3.1. Despite the fact that Frazier and Jawerth dealt with Besov spaces and Triebel-Lizorkin spaces in [6, 7] and that we deal with (weighted) modulation spaces, we can say that Lemma 3.1 is essentially due to Frazier and Jawerth because of the important contribution to the theory of decompositions obtained in [6, 7].

Proof of Theorem 1.5. Define $M_{p(\cdot),q(\cdot),a}(w)$ according to Definition 1.2 by using ϕ satisfying (1.10).

(i) Let $f \in M_{p(\cdot),q(\cdot),a}(w)$. Then by using (1.10) and (1.11) we expand f according to Lemma 3.1:

$$f = \sum_{m\in Z^n} M_m\phi * f = (2\pi)^{-\frac{n}{2}} \sum_{m\in Z^n} \left(\sum_{l\in Z^n} M_m\phi * f(l) \cdot T_l M_m[\mathcal{F}^{-1}\kappa] \right) .$$

Thus, if we set $\lambda_{ml} \equiv (2\pi)^{-\frac{n}{2}} M_m\phi * f(l)$, then we obtain a decomposition of f as follows:

$$f = \sum_{m\in\mathbb{Z}^n} \left(\sum_{l\in\mathbb{Z}^n} \lambda_{ml} \cdot T_l M_m[\mathcal{F}^{-1}\kappa] \right).$$

Let us check that this decomposition fulfills the desired property in Theorem 1.5(i).

Let $x, y \in \mathbb{R}^n$. Denote by $l_{x,y}$ an element in $\mathbb{Z}^n$ such that $x-y \in l_{x,y}+[0,1)^n$. Observe that

$$\begin{aligned}
(2\pi)^{\frac{n}{2}} \sup_{y\in\mathbb{R}^n} \left(\sum_{l\in\mathbb{R}^n} \frac{|\lambda_{ml}|\chi_{l+[0,1)^n}(x-y)}{(1+|y|)^a} \right) &= \sup_{y\in\mathbb{R}^n} \left(\sum_{l\in\mathbb{R}^n} \frac{|M_m\phi * f(l)|\chi_{l+[0,1)^n}(x-y)}{(1+|y|)^a} \right) \\
&= \sup_{y\in\mathbb{R}^n} \frac{|M_m\phi * f(l_{x,y})|}{(1+|y|)^a} \\
&\le \sup_{y\in\mathbb{R}^n} \frac{|M_m\phi * f(l_{x,y})|}{(1+|x-l_{x,y}|)^a}(1+|y-x+l_{x,y}|)^a \\
&\le 2^a \sup_{y\in\mathbb{R}^n} \frac{|M_m\phi * f(l_{x,y})|}{(1+|x-l_{x,y}|)^a} \\
&= 2^a (M_m\phi * f)_a(x).
\end{aligned}$$

Therefore, we obtain

$$\|\lambda\|_{m_{p(\cdot),q(\cdot),a}(w)} \le C\|f\|_{M_{p(\cdot),q(\cdot),a}(w)},$$

as was to be shown.

(ii) Let $m' \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$. Then we have

$$\begin{aligned}
|M_{m'}\phi * f(x)| &\le \sum_{m,l\in\mathbb{Z}^n} |\lambda_{ml}| \cdot |M_{m'}\phi * \mathrm{mol}_{ml}(x)| \\
&= \sum_{m,l\in\mathbb{Z}^n} |\lambda_{ml}| \cdot |M_{m'}\phi * [M_m[M_{-m}\mathrm{mol}_{ml}]](x)| \\
&= \sum_{m,l\in\mathbb{Z}^n} |\lambda_{ml}| \cdot |M_{m'-m}\phi * [M_{-m}\mathrm{mol}_{ml}](x)|.
\end{aligned}$$

Note that

$$M_{m'-m}\phi * [M_{-m}\mathrm{mol}_{ml}](x) = (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{i(m'-m)y}\phi(y)(M_{-m}\mathrm{mol}_{ml})(x-y)\,dy$$

satisfies

$$|M_{m'-m}\phi * [M_{-m}\mathrm{mol}_{ml}](x)| \le C\langle m'-m\rangle^{-N}\langle x-l\rangle^{-N}.$$

Thus, it follows that

$$\frac{|M_{m'-m}\phi * [M_{-m}\mathrm{mol}_{ml}](y)|}{(1+|x-y|)^N} \le C\langle m'-m\rangle^{-N}\langle x-l\rangle^{-N}$$

for all $y \in \mathbb{R}^n$. Consequently,

$$
\begin{aligned}
&\sup_{y\in\mathbb{R}^n} \frac{|M_{m'}\phi * f(y)|}{(1+|x-y|)^N} \\
&\le C \sum_{m\in\mathbb{Z}^n} \left(\sum_{l\in\mathbb{Z}^n} |\lambda_{ml}| \langle m'-m\rangle^{-N} \langle x-l\rangle^{-N} \right) \\
&= C \sum_{m\in\mathbb{Z}^n} \langle m'-m\rangle^{-N} \left(\sum_{l\in\mathbb{Z}^n} |\lambda_{ml}| \langle x-l\rangle^{-a} \langle x-l\rangle^{-N+a} \right) \\
&\le C \sum_{m\in\mathbb{Z}^n} \langle m'-m\rangle^{-N} \left(\sum_{l\in\mathbb{Z}^n} \sup_{z\in\mathbb{R}^n} \left(\sum_{l_1\in\mathbb{Z}^n} \frac{|\lambda_{ml_1}|\chi_{l_1+[0,1)^n}(x-z)}{(1+|z|)^a} \right) \langle x-l\rangle^{-N+a} \right) \\
&\le C \sum_{m\in\mathbb{Z}^n} \langle m'-m\rangle^{-N} \left(\sup_{z\in\mathbb{R}^n} \left(\sum_{l_1\in\mathbb{Z}^n} \frac{|\lambda_{ml_1}|\chi_{l_1+[0,1)^n}(x-z)}{(1+|z|)^a} \right) \right)
\end{aligned}
$$

as long as $N > a+n$. Thus, $f \in M_{p(\cdot),q(\cdot),a}(w)$.

□

Acknowledgment Yoshihiro Sawano was partially supported by Grant-in-Aid for Young Scientists (B), No. 24740085, Japan Society for the Promotion of Science. Both authors wish to express his thank to the anonymous referee and Mr. Stefan Horocholyn for their careful reading of the paper.

REFERENCES

[1] A. Almeida and P. Hästö, Besov spaces with variable smoothness and integrability, J. Funct. Anal. **258** (2010), 1628–1655.

[2] D. Cruz-Uribe, L. Diening and P. Hästö, The maximal operator on weighted variable Lebesgue spaces, Fract. Calc. Appl. Anal. **14** (2011), 361–374.

[3] D. Cruz-Uribe, A. Fiorenza and C.J. Neugebauer, Weighted norm inequalities for the maximal operator on variable Lebesgue spaces, J. Math. Anal. Appl. **394** (2012), no. 2, 744–760.

[4] L. Diening and P. Hästö, Muckenhoupt weights in variable exponent spaces, preprint, 2010.

[5] L. Diening, P. Hästö and R. Roudenko, Function spaces of variable smoothness and integrability, J. Funct. Anal. **256** (2009), 1731–1768.

[6] M. Frazier and B. Jawerth, Decomposition of Besov spaces, Indiana Univ. Math. J. **34** (1985), no. 4, 777–799.

[7] M. Frazier and B. Jawerth, A discrete transform and decompositions of distribution spaces, J. Funct. Anal. **93** (1990), no. 1, 34–170.

[8] M. Izuki, E. Nakai and Y. Sawano, Function spaces with variable exponents – an introduction –, Scientiae Mathematicae Japonicae, Volume 77, No. 2 (2014 August), 187–315.

[9] M. Kobayashi and Y. Sawano, Molecular decomposition of the modulation spaces, Osaka J. Math. **47** (2010), no. 4, 1029–1053.

[10] H. Nakano, Modulared Semi-Ordered Linear Spaces, Maruzen Co., Ltd., Tokyo, 1950.

[11] H. Nakano, Topology of Linear Topological Spaces, Maruzen Co. Ltd., Tokyo, 1951.

[12] J. Sjöstrand, An algebra of pseudodifferential operators, Math. Res. Lett. **1** (1994), no. 2, 185–192.

Communicated by *Eiichi Nakai*

Takahiro NOI
Department of Mathematics and Information Science
Tokyo Metropolitan University
1-1 Minami-Ohsawa, Hachioji, Tokyo 192-0397, Japan
e-mail: taka.noi.hiro@gmail.com

Yoshihiro SAWANO
Department of Mathematics and Information Science
Tokyo Metropolitan University
1-1 Minami-Ohsawa, Hachioji, Tokyo 192-0397, Japan
e-mail: ysawano@tmu.ac.jp, yoshihiro-sawano@celery.ocn.ne.jp

BERTRAND VERSUS COURNOT COMPETITION IN A VERTICAL DUOPOLY

DongJoon Lee * SangHeon Han † Joonghwa Oh ‡

Received 6 October 2014; Revised 15 December 2014, 28 December 2014, 25 January 2015

Abstract. This paper examines whether firms prefer to choose prices or quantities with a manufacturing duopoly in which each upstream firm sells its product to its own downstream firm. The degree of product differentiation plays an important role in whether firms set prices or quantities. We show that price competition performs better than quantity competition, from the upstream and downstream firms' point of view, regardless of the product differentiation. We also show that pay-offs are larger in Bertrand (price) competition than in Cournot (quantitiy) competition if both products are differentiated to a certain extent.

1 Introduction As we well know, two classical models in oligopoly theory are Cournot and Bertrand. In a non-cooperative profit maximization environment, one may wonder whether firms prefer to choose prices (Bertland) or quantities (Cournot). Singh and Vives (1984) first analyzed the issue of whether firms prefer to set prices or quantities. They show that consumer and total surplus in Bertrand competition are larger than those in Cournot competition regardless of the nature of goods.[1] They also show that Cournot equilibrium profits are higher than Bertrand equilibrium profits when the goods are substitutes, and vice versa when the goods are complements.[2]

During the past 30 years, many literatures have produced an array of extensions and generalizations of the analysis in Singh and Vives (1984). Previous literature on the issue has followed two separate streams. One stream focuses on extensions and generalizations of Singh and Vives (1984). For example, Dastidar (1997), Qiu (1997), Lambertini (1997), Häckner (2000), and Amir and Jin (2001), among others, have analyzed counter-examples based on the framework of Singh and Vives (1984) by allowing for a wider range of cost and demand asymmetries.[3] The other stream of the literature focuses on expanding the Bertrand-Cournot competition with vertically related duopoly. Correa-Lopez (2007) examines the Bertrand-Cournot profits ranking in a vertically related duopoly model focusing on substitutes and vertical product differentiation. They show that Bertrand profits may exceed Cournot profits when decentralized bargaining over the labor cost is introduced.[4]

2010 *Mathematics Subject Classification.* 91A80, 47N10.

Key words and phrases. Price and Quantity Competition, Imperfect Downstream Competition.

*Faculty of Management Administration Nagoya University of Commerce & Business 4-4 Sagamine, Komenoki-cho, Nisshin-shi, Aichi-ken, Japan 470-0193, Phone: +81-(0)561-73-2111, E-mail: dongjoon@nucba.ac.jp

†Faculty of Management Administration Nagoya University of Commerce & Business 4-4 Sagamine, Komenoki-cho, Nisshin-shi, Aichi-ken, Japan 470-0193, Phone: +81-(0)561-73-2111, E-mail: han-sh@nucba.ac.jp

‡Faculty of Commerce Administration Nagoya University of Commerce & Business 4-4 Sagamine, Komenoki-cho, Nisshin-shi, Aichi-ken, Japan 470-0193, Phone: +81-(0)561-73-2111, E-mail: jhoh@nucba.ac.jp, This research was supported by JSPS KAKENHI Grant Number 26780262.

[1] When the goods are independent, they are equal.

[2] See Cheng (1985) for a graphical description of Singh and Vives' analysis.

[3] In particular, Zanchettin (2006) found that Singh and Vives's (1984) result that firms always make larger profits under Cournot competition than under Bertrand competition fails to hold.

[4] Symeonidis (2003, 2008) also analyzes the effects of downstream competition when there is bargaining between downstream firms and upstream agents (firms or unions).

Arya et al. (2008) explore the standard conclusions about duopoly competition when the production of key input is outsourced to a vertically integrated retail competitor with upstream market power. They show that prices and industry profits can be larger in Bertrand competition than in Cournot, while consumer and total surplus can be smaller in Bertrand than in Cournot. Mukherjee et al. (2012) compare Cournot with Bertrand competition in a vertical structure in which a monopoly upstream firm sells its product to two downstream firms, assuming there are asymmetric costs between downstream firms and homogeneous final goods. They demonstrate that the technology differences among the downstream firms and the pricing strategy (i.e., uniform pricing or price discrimination) of the upstream firm play an important role in the ranking of profit and social welfare. We revisit the profit ranking under Bertrand and Cournot competition in a vertically related duopoly in which each upstream firm sells its product to its own downstream firm. Our paper differs from the existing literature in at least two important aspects. First, previous studies consider Bertrand and Cournot competitions under wage bargaining and input prices negotiation. Our study examines them without negotiation. Second, previous ones produced the counter-results of Signs and Vives (1984) under costs and demand asymmetry. However, this paper analyzes the issue under symmetric conditions. This paper is organized as follows; in Section 2, we set up the model. Section 3 examines the Cournot competition, and then, Section 4 analyzes the Bertrand competition. Section 5 deals with comparative analysis. Finally, Section 6 contains concluding remarks.

Consider a manufacturing duopoly in which each upstream firm sells its product to its own downstream firm. There is a continuum of consumers of the same type with a utility function separable and linear in numeraire goods. Therefore, there are no income effects. The representative consumer maximizes $U(q_i, q_j) - \Sigma p_i q_i; i = 1, 2; i \neq j$, where q_i is the quantity of good i and p_i its price. U is assumed to be quadratic and strictly concave $U(q_i, q_j) = q_i + q_j - (q_i^2 + 2bq_iq_j + q_j^2)/2; i = 1, 2; i \neq j$. This utility function gives rise to a linear demand structure. Inverse demands are given by

$$p_i = 1 - q_i - bq_j,\ 0 \leq b \leq 1,\ i, j = 1, 2, i \neq j. \tag{1}$$

where p_i is the retail price for product i, and q_i and q_j are the amount of goods produced by channel i and j, respectively. Each unit of retail output requires exactly one unit of the input. The products are differentiated $(0 \leq b \leq 1)$. Upstream firms and downstream firms are risk-neutral and there are no production or retailing costs.

We posit a two-stage game. At stage one, each upstream firm sets an wholesale price. At stage two, each downstream firm also sets the retail price or quantity.

2 Cournot Competition We first consider Cournot competition in which each downstream firm sets a quantity. In this case the equilibrium concept is the sub-game perfect Nash equilibrium.

Stage Two (Quantity): At stage two, downstream firm i sets a quantity, q_i, so as to maximize its profit for a given input price, w_i. Downstream firm i's maximization problem is as follows:

$$\max \pi_i = (p_i - w_i)q_i,\ w.r.t.\ q_i.$$

where w_i is the input price. Therefore, downstream firm i sets the quantity, q_i, as the function of input prices as follows:

$$q_i(w_i, w_j) = \frac{2(1 - w_i) - b(1 - w_j)}{4 - b^2}. \tag{2}$$

Stage one (Wholesale Price): At stage one, upstream firm i sets wholesale, w_i, to maximize its profit for a given w_j. Upstream firm i's maximization problem is as follows:

$$\max \Pi_i = w_i q_i(w_i, w_j) = \frac{w_i[(2-w_i) - b(1-w_j)]}{4-b^2}, \ w.r.t.\ w_i.$$

The equilibrium wholesale price for upstream firm i is derived as follows:

$$w_i = \frac{2-b}{4-b}. \tag{3.1}$$

Substituting the wholesale price into Eq. (1) and Eq. (2), we obtain the retail price, p_i, the quantity, q_i, the upstream firm i's payoff, Π_i, and downstream firm i's payoff, π_i,

$$p_i^C = \frac{6-b^2}{(2+b)(4-b)}, \tag{3.2}$$

$$q_i^C = \frac{2}{(2+b)(4-b)}, \tag{3.3}$$

$$\Pi_i^C = \frac{2(2-b)}{(2+b)(4-b)^2}, \text{ and} \tag{3.4}$$

$$\pi_i^C = \frac{4}{(2+b)^2(4-b)^2}. \tag{3.5}$$

where superscripts C denote Cournot equilibrium.

3 Bertrand Competition We now turn to Bertrand competition in which each downstream firm sets a retail price. From Eq. (1), the following direct demand function can be derived as follows:

$$q_i = \frac{1-b-p_i+bp_j}{1-b^2}, \ 0 \le b \le 1, \ i,j = 1,2, i \ne j. \tag{4}$$

Stage Two (Retail Price): At stage two, downstream firm i sets retail price, p_i, so as to maximize its profit for a given wholesale price, w_i. Downstream firm i's maximization problem is as follows:

$$\max \pi_i = (p_i - w_i) q_i = \frac{(p_i - w_i)(1-b-p_i+bp_j)}{1-b^2}, \ w.r.t.\ p_i.$$

Therefore, downstream firm i sets the retail price, p_i, as the function of wholesale prices as follows:

$$p_i(w_i, w_j) = \frac{2(1-w_i) - b(1-w_j) - b^2}{4-b^2}. \tag{5}$$

Stage One (Wholesale Price): At stage one, upstream firm i sets a wholesale price, w_i, to maximize its profit for a given wholesale price, w_j. Upstream firm i's maximization problem is as follows:

$$\max \Pi_i = w_i q_i(w_i, w_j) = \frac{w_i[(2-b^2)(1-w_i) - b(1-w_j)]}{(4-b^2)(1-b^2)}, \ w.r.t.\ w_i.$$

The equilibrium wholesale price for upstream firm i is derived as follows:

$$w_i = \frac{2-b-b^2}{4-b-2b^2}. \tag{6.1}$$

Substituting the wholesale price into Eq. (4) and Eq. (5), we obtain the retail price, p_i, the quantity, q_i, the upstream firm i's payoff, Π_i, and downstream firm i's payoff, π_i,

$$p_i^B = \frac{2(1-b)(3-b^2)}{(2-b)(4-b-2b^2)}, \tag{6.2}$$

$$q_i^B = \frac{(2-b^2)}{(2-b)(4-b-2b^2)}, \tag{6.3}$$

$$\Pi_i^B = \frac{(1-b)(2+b)(2-b^2)}{(1+b)(2-b)(4-b-2b^2)^2}, \text{ and} \tag{6.4}$$

$$\pi_i^B = \frac{(1-b)(2-b^2)^2}{(1+b)(2-b)2(4-b-2b^2)^2}. \tag{6.5}$$

4 Comparative Analysis We turn now to compare the equilibrium under Bertrand and Cournot competition. Firstly, we compare wholesale prices between two types of contracts. From Eq. (3.1) and Eq. (6.1), we obtain the following results:

$$w_i^C - w_i^B = \frac{b^3}{(4-b)(4-b-2b^2)} \geq 0.$$

where superscripts B and C denote Bertrand and Cournot, respectively.

Lemma 1. Under Eq. (1) and Eq. (4), if $0 < b \leq 1$, the equilibrium wholesale prices are higher in Cournot than in Bertrand competition. If $b = 0$, both have the same wholesale prices.

Secondly, the equilibrium levels of retail prices and quantities are shown in Table 1.

Table 1: Equilibrium Levels of Retail Price and Quantity

	Retail Price	Quantity
Bertrand	$\frac{2(1-b)(3-b^2)}{(2-b)(4-b-2b^2)}$	$\frac{2-b^2}{(2-b)(4-b-2b^2)}$
Cournot	$\frac{(6-b^2)}{(2+b)(4-b)}$	$\frac{2}{(2+b)(4-b)}$

Lemma 2. Under Eq. (1) and Eq. (4), if $0 < b \leq 1$, the equilibrium prices for both downstream firms are higher in Cournot than in Bertrand competition. If $b = 0$, both have the same prices.

Lemma 3. Under Eq. (1) and Eq. (4), if $0 < b \leq 1$, the equilibrium outputs for both downstream firms are larger in Bertrand than in Cournot competition. If $b = 0$, both have the same input prices.

Quantities are larger and prices lower in Bertrand than in Cournot competition regardless of the nature of goods.[5] Lower prices and higher quantities are always better in welfare terms. Consumer and total surplus are decreasing as a function of prices. Therefore, in terms of consumer surplus and total surplus, the Bertrand equilibrium dominates the Cournot one. Proposition 1 summarizes the results thus far.

[5] When $b = 0$, they are equal.

Proposition 1. Under Eq. (1) and Eq. (4), if $0 < b \leq 1$, consumer surplus and total surplus are larger in Bertrand than in Cournot competition. If $b = 0$, they are equal.
For proof, see Appendix.

Thirdly, we turn to the equilibrium profits for Bertrand and Cournot competition. From Eq. (3.4) and Eq. (6.4), when $0 \leq b \leq 1$, notice that the following results are satisfied:

$$\Pi_i^B - \Pi_i^C = \frac{b^2(4+b-b^2)(16-b(2-b)(10+7b))}{(1+b)(2-b)(2+b)(4-b)^2(4-b-2b^2)^2},$$
$$\Pi_i^B > \Pi_i^C \Leftrightarrow 0 < b < 0.8868 \equiv \bar{b}.$$

Proposition 2. Under Eq. (1) and Eq. (4), if $0 < b \leq \bar{b}$, the Bertrand strategy is dominant for upstream firms. If $\bar{b} < b \leq 1$, the Cournot strategy is dominant for upstream firms. If $b = 0$, payoffs for both upstream firm are equal.

Proposition 2 can be explained as follows. If $0 < b < \bar{b}$, pay-offs in Bertrand competition are higher than those in Cournot, and vise versa. The degree of product differentiation plays an important role in equilibrium. As the degree of product differentiation decreases, the product market competition is more intense under Bertrand compared with Cournot competition. Therefore, pay-offs of Cournot competition are higher than those of Bertrand competition because of monopolistic effect. On the other hand, as the degree of product differentiation decreases, even if the wholesale price is lower in Bertrand competition than in Cournot competition, a more intense competition in the former helps to create a larger wholesale demand than in the latter. As a result, the upstream firm obtains higher pay-offs in Bertrand competition than in Cournot competition.

5 Concluding Remarks We may summarize the results derived from the model as follows:
(1) With linear demand function, if $0 < b \leq 1$, consumer and total surplus are larger in Bertrand than in Cournot competition.
(2) Pay-offs of both upstream firms are larger, equal, or smaller in Bertrand competition than in Cournot competition, according to whether $0 < b < \bar{b}$, or $\bar{b} < b \leq 1$.

We can also extend our analysis for each upstream firm and each downstream firm to make a precommitment to quantity or price contract in a vertically related market. In such a situation, we are wondering the results are the same as Singh and Vives (1984).

Appendix

Proof of Proposition 1. Consumer Surplus ranking of Bertrand and Cournot equilibria. In view of Lemma 2, consumer surplus is clearly higher under Bertrand than under Cournot competition. From the utility function, we get

$$CS = U(q_i, q_j) - (p_i q_i + p_j q_j) = q_i + q_j - \frac{(q_i^2 + 2bq_iq_j + q_j^2)}{2} - (p_i q_i + p_j q_j)$$
$$= q_i + q_j - \frac{(q_i+q_j)^2}{2} + (1-b)q_iq_j - (p_iq_i + p_jq_j) = (1-p_i)\frac{q_i}{2} + (1-p_j)\frac{q_j}{2}.$$

For $0 \leq b \leq 1$, inequality $CS^B > CS^C$ reduces to

$$CS^B - CS^C = \frac{b^2(8-3b^2)(32+8b-28b^2-4b^3+5b^4)}{(1+b)(2-b)^2(2+b)^2(4-b)^2(4-b-2b^2)^2} > 0.$$

This inequality holds for any $0 < b \leq 1$. For $b = 0$, consumer surplus is equal. From the utility function, we get

$$\begin{aligned}
TS &= CS + \Pi_i + \Pi_j + \pi_i + \pi_j \\
&= U(q_i, q_j) - (p_i q_i + p_j q_j) + (w_i q_i + w_j q_j) + (p_i - w_i)q_i + (p_j - w_j)q_j \\
&= q_i + q_j - \frac{(q_i^2 + 2bq_iq_j + q_j^2)}{2} \\
&= q_i + q_j - \frac{(q_i + q_j)^2}{2} + (1-b)q_iq_j \\
&= \frac{(1-p_i)q_i}{2} + p_iq_i + \frac{(1-p_j)q_j}{2} + p_jq_j \\
&= \frac{(1+p_i)q_i}{2} + \frac{(1+p_j)q_j}{2}.
\end{aligned}$$

For $0 \leq b \leq 1$, inequality $TS^B > TS^C$ reduces to

$$TS^B - TS^C = \frac{b^2(8-3b^2)(96-72b-60b^2+36b^3+9b^4-4b^5)}{(1+b)(2-b)^2(2+b)^2(4-b)^2(4-b-2b^2)^2} > 0$$

This inequality holds for any $0 < b \leq 1$. For $b = 0$, total surplus is equal.

References

[1] Amir, R. and J.Y. Jin. 2001. Cournot and Bertrand Equilibria Compared: Substitutability, Complementarity and Concavity, *International Journal of Industrial Organization*, 19; 303-317.

[2] Arya, A., B. Mittendorf, and D. E. M. Sappington. 2006. Outsourcing, Vertical Integration, and Price vs. Quantity Competition, *International Journal of Industrial Organization*, 26; 1-16.

[3] Belleflamme, P. and Peitz, M. 2010. *Industrial Organization Markets and Strategies*, Cambridge University Press.

[4] Dastidar, K.G. 1997. Comparing Cournot and Bertrand in a Homogenous Product Market, *Journal of Economic Theory*, 75; 205-212.

[5] Dixit, A. K. 1979. A Model of Duopoly Suggesting a Theory of Entry Barriers, *Bell Journal of Economics*, 10; 20-32.

[6] Häckner, J. 2000. A Note on Price and Quantity Competition in Differentiated Oligopolies, *Journal of Economic Theory*, 93; 233-239.

[7] Horn, H. and A. Wolinsky. 1988 Bilateral Monopolies and Incentives for Merger, *RAND Journal of Economics*, 19; 408-419.

[8] Lambertini, L. 1997. Prisoners' Dilemma in Duopoly (Super) Games, *Journal of Economic Theory*, 77; 181-191.

[9] Lopez, M. 2007. Price and Quantity Competition in a Differentiated Duopoly with Upstream Suppliers, *Journal of Economics Management Strategy*, 16; 469-505.

[10] Qiu, L. D. 1997. On the Dynamic Efficiency of Bertrand and Cournot Equilibria, *Journal of Economic Theory*, 75; 213-229.

[11] Singh, N. and X, Vives. 1984. Price and Quantity Competition in a Differentiated Duopoly, *Rand Journal of Economics*, 15; 546-554.

[12] Vives, X. 1984. Duopoly Information Equilibrium: Cournot and Bertrand, *Journal of Economic Theory*, 34; 71-94.

[13] Vives, X. 1985. On the Efficiency of Bertrand and Cournot Equilibria with Product Differentiation, *Journal of Economic Theory*, 36; 166-175.

Communicated by *Yosio Tabata*

COMBINATION OF OPTIMAL STOPPING ALGORITHMS

Marek Skarupski[1]

Received October 15, 2014

Abstract. In this paper we investigate the possibility of combination two optimal stopping algorithms: Odds algorithm and Elimination algorithm. We show how reduce a problem to monotone problem and after this step find the optimal strategy which will be valid also in the original problem.

1 Introduction Bruss (2000) in [3] developed Odds algorithm which is very simple tool used to solve optimal stopping problems. In this model observe sequence of independent indicators and want to stop on the last (if any) success. Extension of this idea was presented in [4] and [9]. Different approaches are presented in work of Dendievel [6]. The result of Bruss' can be obtain in another way if we focus on monotonicity of a problem of selecting last success in sequence of events. However there are some problems which are not monotone and therefore Odds algorithm can give us strategy that is not optimal. Sonin (1999) in [13] presented so called Elimination Algorithm (EA) for solving optimal stopping problems (OSP). The idea is to combine this two algorithms by reducing original problem to monotone problem using EA and then find the optimal strategy by One-Step-Look-Ahead (1-SLA) method. Similar work was done by Ferguson [8]. This problem was also considered by Ano [1].

2 Optimal stopping for unobservable event Let a probability space $(\Omega, \mathcal{G}, P)$ be given and let $\{X_k\}_{k=1}^{\infty}$ be a sequence of random variables whose joint distribution is known. Let $\mathcal{F}_k = \sigma(X_1, ..., X_k)$ be a sigma field generated by $X_1, ..., X_k$ (natural filtration). In many cases we deal with Markov chain. We assume that we have finite horizon n. Define function $g_k((X_1, ...X_k))$ and call it reward function. g_k is $\mathcal{F}_k$ measurable. Further we will denote $g_k((X_1, ...X_k))$ as G_k. We observe X_k sequentially. The goal is to stop observation on index i for which reward function reach the maximum value. The triplet (space, filtration, function) we will call an optimal stopping problem (OSP).

Definition 1. *Let A_k denote a set $\{G_k \geq E[G_{k+1}|\mathcal{F}_k]\}$. We say that the stopping rule problem is monotone if*

$$A_0 \subset A_1 \subset A_2 \subset ... \ a.s. \tag{1}$$

One of the simplest stopping rule is known as One-Step-Look-Ahead (OSLA or 1-SLA). The 1-SLA is the rule which calls for stopping on the first k for which the return for stopping is greater or equal as the expected return of continuing one step and then stopping.

Definition 2. *1-SLA is described by the stopping time*

$$\nu_1 = \min\{k \geq 0 : G_k \geq E[G_{k+1}|\mathcal{F}_k]\}.$$

2010 *Mathematics Subject Classification.* 60G40.
Key words and phrases. odds algorithm, elimination algorithm, optimal stopping, monotone problem, Markov chain .

[1]Faculty of Pure and Applied Mathmatics Wrocław University of Science and Technology Wyb. Wyspiańskiego 27 50-370 Wrocław, Poland

Theorem 1. *In a finite horizon monotone stopping rule problem, the 1-SLA rule is optimal.*

The proof of this fact is here omitted. It can be found in [7]

3 Odds theorem Idea is that we consider n independent indicators $I_k, 1 \leq k \leq n$ observed sequentially. If the indicator on place k has value 1 we say that the success occur. If 0 then we say that the failure occur. The aim is to stop on last 1.

Let $(\Omega, \mathcal{G}, P)$ be a probability space. On this space we define sequence of independent events $\{A_k\}_{k=1}^n$. We observe sequence of indicators of this events $\{I_k\}_{k=1}^n$. Let us denote by $\mathcal{F}_k = \sigma(I_1, ... I_k)$ sequence of sigma fields generated by indicators and let $\mathcal{T}$ be the set of all stopping moments τ wrt σ -fields $\mathcal{F}_k, k = 1, ..., n$. We want to stop on such time τ^* that will maximize $P(I_t = 1, I_{t+1} = ... = I_n = 0)$ over all $t \in \mathcal{T}$.

Theorem 2. *(Bruss 2000)*
Let $I_1, I_2, ..., I_n$ be a sequence of independent indicator functions with $p_j = E[I_j]$. Let $q_j = 1 - p_j$ and $r_j = \frac{p_j}{q_j}$. Then an optimal rule τ_n for stopping on the last success exists and is to stop on the first index (if any) k with $I_k = 1$ and $k \geq s$ where

$$s = \sup\{1, \sup\{1 \leq k \leq n : \sum_{j=k}^{n} r_j \geq 1\}\}$$

with $\sup\{\emptyset\} = -\infty$. The optimal reward (win probability) is given by

$$V(n) = \prod_{j=s}^{n} q_j \sum_{j=s}^{n} r_j.$$

Proof presented by Bruss in [3] is based on probability generating function. We present different approach.

Proof. Define a process ξ_t in the following way

$$\xi_t = \inf\{k \geq \xi_{t-1} : I_k = 1\}$$

with initial point $\xi_0 = 1$. Calculate transition probabilities

$$\begin{aligned} p_{i,s} &= P(\xi_{k+1} = s | \xi_k = i) = \frac{P(\xi_{k+1} = s, \xi_k = i)}{P(\xi_k = i)} = \\ &= \frac{P(I_i = 1, I_{i+1} = ... = I_{s-1} = 0, I_s = 1)}{P(I_i = 1)} = p_s \prod_{j=i+1}^{s-1} q_j. \end{aligned} \tag{2}$$

Define a gain function g in the following way

$$g(i) = P(I_{i+1} = ... = I_n = 0) = \prod_{j=i+1}^{n} q_j. \tag{3}$$

Definition 3. *An operator $T(\cdot)$ defined as follows*

$$Tf(x) = \sum_y p(x, y) f(y)$$

is called the averaging operator.

Using averaging operator calculate the expected pay-off in next step.

$$
\begin{aligned}
Tg(i) = \sum_{s=i+1}^{n} p_{i,s} g(s) &= \sum_{s=i+1}^{n} p_s \prod_{j=i+1}^{s-1} q_j \prod_{j=s+1}^{n} q_j = \\
&= \sum_{s=i+1}^{n} p_s \prod_{j=i+1}^{s-1} q_j \prod_{j=s+1}^{n} q_j \frac{q_s}{q_s} = \\
&= \prod_{j=i+1}^{n} q_j \sum_{s=i+1}^{n} r_s.
\end{aligned} \tag{4}
$$

To find an optimal stopping rule we check when $Tg \leq g$, i.e. when the expected value of doing one step more is less or equal to pay-off in current state. We get condition that stopping rule is

$$
s = \min\{1 \leq k \leq n : \sum_{j=k}^{n} r_j \leq 1\}. \tag{5}
$$

We show that it is optimal. In Bruss' theorem we can see that problem is monotone, because sets $A_k = \{Tg(k) \leq g(k)\}$ satisfies condition (1). Therefore we know that method 1-SLA is optimal. In this case, because we deal with independent events 1-SLA is described as follows

$$
\nu_0 = \min\{1 \leq k \leq n : \sum_{j=k}^{n} r_j \leq 1\}. \tag{6}
$$

So it is exactly the same rule as in (5). Therefore we get the thesis. Win probability is calculated as follows

$$
V(n) = Eg(\nu_0) = \prod_{j=\nu_0}^{n} q_j \sum_{s=\nu_0}^{n} r_s. \tag{7}
$$

□

3.1 Extension of Bruss' theorem ¿From Odds theorem we can find the moment of last success in n trials. The obvious question is how to find the moment of last l-th success in n independent trials. Idea is to find such a stopping time τ_l^* that will maximize $P(I_t = 1, I_{t+1} + ... + I_n = l)$ and its value. The following theorem gives us the answer of this question.

Theorem 3. *(Bruss, Paindaveine 2000)*
Let $I_1, I_2, ..., I_n$ be a sequence of independent indicator functions with $p_j = E[I_j]$. Let $q_j = 1 - p_j$ and $r_j = \frac{p_j}{q_j}$. Then an optimal rule τ_n for stopping on the l-th last success exists and is to stop on the first index (if any) k with $I_k = 1$ and $k \geq s_l$ where

$$
s_l = \sup\{1, \sup\{1 \leq k \leq n - l + 1 : R_{l,k} \geq l R_{l-1,k} \text{ and } \pi_k \geq l\}\}
$$

where

$$
R_{l,k} = \sum_{j_1,...j_l = k, all \neq}^{n} r_{j_1} ... r_{j_l}
$$

$$
\pi_k = \#\{j \geq k | r_j > 0\}
$$

with $\sup\{\emptyset\} = -\infty$. *The optimal reward (win probability) is given by:*

$$V(l,n) = \prod_{j=s_l}^{n} q_j \frac{R_{l,s_l}}{l!}.$$

The proof of this fact can be found in [4].
Another similar problem is to stop on any of last l-th success. The following theorem gives the solution of it.

Theorem 4. *(Tamaki 2010)*
Let $I_1, I_2, ..., I_n$ *be a sequence of independent indicator functions with* $p_j = E[I_j]$. *Let* $q_j = 1 - p_j$ *and* $r_j = \frac{p_j}{q_j}$. *Then an optimal rule* τ_n *for stopping on any of the* l*-th last success exists and is to stop on the first index (if any)* k *with* $I_k = 1$ *and* $k \geq s_l$ *where*

$$s_l = \sup\{1, \sup\{1 \leq k \leq n : \widehat{R}_{l,k+1} \geq 1\}\}$$

where

$$\widehat{R}_{l,k} = \sum_{k \leq j_1 < ... < j_l \leq n} r_{j_1} ... r_{j_l}$$

with $\sup\{\emptyset\} = -\infty$. *The optimal reward (win probability) is given by*

$$V(l,n) = \prod_{j=s_l}^{n} q_j \left(\sum_{j=1}^{l} \widehat{R}_{j,s_l}\right).$$

The proof of this fact can be found in [16].

4 Eliminate *and* Stop. Theorem 2 provides a simple rule for stopping on problems which can be described via simple indicator functions. As an example we consider Classical Secretary problem:

4.1 Example 1 - Selecting the best object. Consider the classical secretary problem. Let X_k be the absolute rank of the k-th candidate. We define

$$Y_k = \#\{1 \leq i \leq k : X_i \leq X_k\}.$$

The random variable Y_k is called the relative rank of k-th candidate.
Let $(\Omega, \mathcal{F}, P)$ be the probability space, where elementary events are permutations of the elements from $\{1, ..., n\}$ and the probability measure P is the uniform distribution on Ω. For $k = 1, ..., n$ let $\mathcal{F}_k = \sigma\{Y_1, ..., Y_k\}$ be a sequence of σ -fields. It can be proved that Y_k are independent and $P(Y_k = i) = \frac{1}{k}, i = 1, ..., k$. Set a function

$$I_k := I_{\{Y_k = 1\}}.$$

Then we get that $p_k = E[I_k] = P(Y_k = 1) = \frac{1}{k}$ and $q_k = \frac{k-1}{k}, r_k = \frac{1}{k-1}$. The optimal stopping rule is therefore

$$s = \min\{1 \leq k \leq n : \sum_{i=k}^{n} \frac{1}{i-1} \leq 1\}.$$

The gain is

$$V(n) = \frac{s-1}{n} \sum_{i=s}^{n} \frac{1}{i-1}.$$

4.2 Example 2A - Selecting the second best object. There are problems that can be described similarly as in Odds theorem: we want to maximize the probability of unobservable event describing them via Indicator functions. But because of non-monotonicity of the problem there does not exist a simple rule as the above. As an example consider secretary problem with choosing the second best applicant.
Let $A = \{X_k = 2\}$ denote an event that k-th absolute rank is equal to 2.

$$\{X_k = 2\} =$$
$$= \bigcup_{s=k+1}^{n} \{Y_k = 1, Y_{k+1} > 1, ..., Y_s = 1, Y_{s+1} > 2, ..., Y_n > 2\} \cup \{Y_k = 2, Y_{k+1} > 2, ..., Y_n > 2\} :=$$
$$:= \bigcup_{s=k+1}^{n} B_1^{(s)} \cup B_2.$$

The sets $B_1^{(s)}, B_2$ for all indexes s are disjoint. We have that

$$P(A) = P(X_k = 2) = P(\bigcup_{s=k+1}^{n} B_1^{(s)} \cup B_2) = \sum_{s=k+1}^{n} P(B_1^{(s)}) + P(B_2). \tag{8}$$

First calculate $P(B_2)$. Let us introduce function G

$$G(Y_i) = \begin{cases} I_{\{Y_i=2\}} & \text{for } i = k \\ I_{\{Y_i\in\{1,2\}\}} & \text{for } k+1 \leq i \leq n. \end{cases} \tag{9}$$

$$P(B_2) = P(G(Y_k) = 1, G(Y_{k+1}) = 0, ..., G(Y_n) = 0) = P(\sum_{i=k}^{n} G(Y_i) = 1). \tag{10}$$

Now we calculate $P(B_1^{(s)})$. Let us introduce function $F_{(s)}$

$$F_{(s)}(Y_i) = \begin{cases} I_{\{Y_i=1\}} & \text{for } k \leq i \leq s \\ I_{\{Y_i\in\{1,2\}\}} & \text{for } s < i \leq n \end{cases} \tag{11}$$

$$P(\bigcup_{s=k+1}^{n} B_1^{(s)}) = \sum_{s=k+1}^{n} P(B_1^{(s)}) =$$

$$= \sum_{s=k+1}^{n} P(F_s(Y_k) = 1, F_s(Y_{k+1}) = 0, ..., F_s(Y_s) = 1, F_s(Y_{s+1}) = 0, ..., F_s(Y_n) = 0) =$$

$$= \sum_{s=k+1}^{n} P(\sum_{i=k}^{n} F_s(Y_i) = 2). \tag{12}$$

From 8, 10 and 12 we get that

$$\sum_{s=k+1}^{n} P(\sum_{i=k}^{n} F_s(Y_i) = 2) + P(\sum_{i=k}^{n} G(Y_i) = 1). \tag{13}$$

We are looking for such $\tau^* \in \mathcal{T}$ that $P(A)$ is the greatest, i.e.

$$\tau^* = \arg\sup_{\tau\in\mathcal{T}} P(A).$$

From Theorem 2 we can find a stopping time $\tau_2 \in \mathcal{T}$ that:

$$\tau_2 = \arg\sup_{\tau\in\mathcal{T}} P(\sum_{i=\tau}^{n} G(Y_i) = 1).$$

We have

$$p_i = P(G(Y_i) = 1) = \begin{cases} P(Y_i = 2) = \frac{1}{k} & \text{for } i = k \\ P(Y_i \in \{1,2\}) = \frac{2}{i} & \text{for } k+1 \le i \le n \end{cases}$$

and

$$q_i = \begin{cases} \frac{k-1}{k} & \text{for } i = k \\ \frac{i-2}{i} & \text{for } k+1 \le i \le n \end{cases}$$

$$r_i = \begin{cases} \frac{1}{k-1} & \text{for } i = k \\ \frac{2}{i-2} & \text{for } k+1 \le i \le n \end{cases}$$

For $i = 1, p_1 = 0, q_1 = 1, r_1 = 1$. We get that

$$\tau_2 = \sup\{1, \sup\{1 \le k \le n : \frac{1}{k-1} + \sum_{i=k+1}^{n} \frac{2}{i-2} \ge 1\}\}.$$

$$\tau_2 = \sup\{1, \sup\{1 \le k \le n : \sum_{i=k}^{n-1} \frac{1}{i-1} \ge \frac{k-2}{2k-2}\}\}.$$

The win probability is $V(n) = \frac{(k-1)^2}{n(n-1)}(\frac{1}{k-1} + \sum_{i=k+1}^{n} \frac{2}{i-2})$.

From Theorem 3 we can find a stopping time $\tau\ _1^{(s)} \in \mathcal{T}$ that

$$\tau_1^{(s)} = \arg\sup_{\tau\in\mathcal{T}} P(\sum_{i=\tau}^{n} F_s(Y_i) = 2).$$

We have

$$p_i = P(F_s(Y_i) = 1) = \begin{cases} P(Y_i = 1) = \frac{1}{i} & \text{for } k \le i \le s \\ P(Y_i \in \{1,2\}) = \frac{2}{i} & \text{for } s < i \le n \end{cases}$$

and

$$q_i = \begin{cases} \frac{i-1}{i} & \text{for } k \le i \le s \\ \frac{i-2}{i} & \text{for } s < i \le n \end{cases}$$

$$r_i = \begin{cases} \frac{1}{i-1} & \text{for } k \le i \le s \\ \frac{2}{i-2} & \text{for } s < i \le n \end{cases}$$

Let us consider the following inequality

$$\sum_{i,j=k, i\neq j}^{n} r_i r_j \ge 2 \sum_{j=k}^{n} r_j.$$

$$
\begin{aligned}
LHS = \sum_{i,j=k,i\neq j}^{n} r_i r_j = ((\sum_{i=k}^{n} r_i)^2 - \sum_{i=k}^{n} r_i^2) = \\
=((\sum_{i=k}^{s}\frac{1}{i-1} + \sum_{i=s+1}^{n}\frac{2}{i-2})^2 - \sum_{i=k}^{s}\frac{1}{(i-1)^2} - \sum_{i=s+1}^{n}\frac{4}{(i-2)^2}).\\
RHS =2\sum_{i=k}^{n} r_j = 2(\sum_{i=k}^{s}\frac{1}{i-1} + \sum_{i=s+1}^{n}\frac{2}{i-2}).
\end{aligned}
$$

We get that

$$
\begin{aligned}
\tau_1^{(s)} = \sup\{1, \sup\{1 \le k \le n-1 : (\sum_{i=k}^{s}\frac{1}{i-1} + \sum_{i=s+1}^{n}\frac{2}{i-2})^2 - \sum_{i=k}^{s}\frac{1}{(i-1)^2} - \sum_{i=s+1}^{n}\frac{4}{(i-2)^2}\\
\ge 2(\sum_{i=k}^{s}\frac{1}{i-1} + \sum_{i=s+1}^{n}\frac{2}{i-2}) \quad \text{and} \quad \pi_k \ge 2\}\}.
\end{aligned}
$$

Which after some simplifications gives us

$$
\begin{aligned}
\tau_1^{(s)} = \sup\{1, \sup\{1 \le k \le n-1 : \sum_{i=k}^{s}(\frac{i}{i-1})^2 + \sum_{i=s+1}^{n}(\frac{i}{i-2})^2 - (\sum_{i=k}^{s}\frac{1}{i-1} + \sum_{i=s+1}^{n}\frac{2}{i-2})^2\\
\le n-k+1 \quad \text{and} \quad \pi_k \ge 2\}\}.
\end{aligned}
$$

The value of the problem is (according to Theorem 3)

$$
V(n) = \frac{(k-1)(s-1)}{n(n-1)}((\sum_{i=k}^{s}\frac{1}{i-1} + \sum_{i=s+1}^{n}\frac{2}{i-2})^2 - \sum_{i=k}^{s}\frac{1}{(i-1)^2} - \sum_{i=s+1}^{n}\frac{4}{(i-2)^2}).
$$

Remark 5. *Exact results for stopping on second best object can be found in [11]. The above probabilities are conditional probabilities that selected relatively best object is the second one from the end. Denote as k^* the first moment after k when relatively first occurs and let $S_j := I_j + ... + I_n$. Then we have the following approximation*

$$
\begin{aligned}
&P(X_{k^*} = 2|S_k = 2) = \\
&= \sum_{s=k+1}^{n}\frac{k}{s(s-1)}(\sum_{l=s+1}^{n}\frac{s}{l(l-1)}(1 - 2\sum_{j=l+1}^{n}\frac{l(l-1)}{j(j-1)(j-2)})) \to \\
&\to x\int_x^1 \frac{1}{t^2}(t\int_t^1 \frac{1}{u^2}(1 - 2\int_u^1 \frac{u^2}{z^3}dz)du)dt = \\
&= x\int_x^1 \frac{t(1-t)}{t^2}dt = x(x - 1 - \log(x)) := v(x).
\end{aligned} \tag{14}
$$

We have that

$$
k^* = s_2^*, \quad x^* := \frac{s_2^*}{n} \to e^{-2} \approx 0.13534 \text{ as } n \to \infty. \tag{15}
$$

Approximated reward (probability of stopping on relative rank 1, such that $S_k = 2$) is

$$
V(2, n) \to \frac{2^2}{2!e^2} = \frac{2}{e^2} \approx 0.27067 \text{ as } n \to \infty. \tag{16}
$$

But approximating win probability of $P(X_{k^*} = 2)$ we get that $P(X_{k^*} = 2) = v(x)$. Substituting $x^* = e^{-2}$ to this formula we get

$$
v(e^{-2}) = e^{-2} + e^{-4} \approx 0.15361. \tag{17}
$$

4.3 Reduction of states We want to consider the above example as a stopping problem of some Markov chain. It is obvious that the problem is not monotone. Thus we can not use 1-SLA method. In similar problems we would like to find the most simple optimal stopping rule. But the simplest rule is provided by monotone problems. Idea is to eliminate those states that spoils monotonicity and afterwards use 1-SLA.

State reduction approach (SRA). Let us assume that the model $(\mathbb{X}_1, P_1)$, where $\mathbb{X}_1$ is a state space and P_1 is a transition matrix is given. Let Z_n be a Markov chain in this model and let $\tau_1, ..., \tau_n$ be the sequence of the moments of first,..., n-th exit of Z_n from set $D \subset \mathbb{X}_1$. Consider the chain $Z'_n = Z_{\tau_n}$. Denote by $\mathbb{X}_2 = \mathbb{X}_1 \setminus D$. Let us denote by $u_1(z, \mathbb{X}_2, \cdot)$ the distribution of the Markov chain Z_n for the initial model at the moment τ_1 of first exit from D starting at $z, z \in D$.

The sequence Z'_n is a Markov chain in model $(\mathbb{X}_2, P_2)$, where the transition matrix is given by the formula

$$p_2(x, y) = p_1(x, y) + \sum_{z \in D} p_1(x, z) u_1(z, \mathbb{X}_2, y), \quad x, y \in \mathbb{X}_2. \tag{18}$$

In case when $D = \{\widehat{z}\}$ and it is not absorbing point we get simpler formula

$$p_2(x, y) = p_1(x, y) + \frac{p_1(x, \widehat{z}) p_1(\widehat{z}, y)}{1 - p_1(\widehat{z}, \widehat{z})}. \tag{19}$$

New model is called D-reduced model. Z_n and Z'_n are different chains, with different state spaces and transition probabilities, but there are some characteristics that are common for them. We formulate one result that will be used later.

Lemma 1. *Let us assume that we have two models* $(\mathbb{X}_1, P_1)$ *and* $(\mathbb{X}_2, P_2)$ *defined as above,* $U \subset \mathbb{X}_2$ *and* $\tau_U, (\tau'_U)$ *be the moment of first visit to* U *in the first (second) model. Then*

$$\forall x \in \mathbb{X}_2 \quad u_1(x, U, y) = u_2(x, U, y), \quad (x \in \mathbb{X}_2, y \in U).$$

Proof of this lemma can be found in [13]. In a finite model we can use procedure of eliminating states recursively by eliminating on each step one state. This is very simple implication from the Lemma 1.

Elimination theorem. Let us assume that we have Markov model $M = (\mathbb{X}, P_1, g)$, where $\mathbb{X}$ is a state space, and P_1 is a transition matrix and g is reward function. Let Z_n be a Markov chain specified on this model with initial point z. We denote by P_z, E_z the probability measure and expectation of the Markov chain with the initial point z. We introduce natural filtration and with respect to it we define stopping times. Denote by $\mathcal{T}$ the set of all stopping times.

Let v be the value function, i.e. $v(z) = \sup_{\tau \in \mathcal{T}} E_z g(Z_\tau)$. Let T be an averaging operator. By D let us denote a subset of $\mathbb{X}$ and by τ_D we denote moment of first visit of the chain in set D, i.e: $\tau_D = \min\{k \geq 1 : Z_k \in D\}$.

Definition 4. *We call a set* S *an optimal stopping set if*

$$S = \{x : v(x) = g(x)\} \text{ and } P(\tau_S < \infty) = 1.$$

The idea of state elimination approach is to eliminate states where is not optimal to stop. We want to eliminate those states, where doing one step more is optimal. In this case we want to satisfy the condition

$$Tg(x) > g(x). \tag{20}$$

Theorem 6. *(Sonin 1995)*
Let $M_1 = (\mathbb{X}_1, P_1, g)$ be an OSP,$D \subseteq \{z \in \mathbb{X}_1 : T_1 g(z) > g(z)\}$ and $P_{1,x}(\tau_{\mathbb{X}_1 \setminus D} < \infty) = 1$ for all $x \in D$. Consider an OSP $M_2 = (\mathbb{X}_2, P_2, g)$ with $\mathbb{X}_2 = \mathbb{X}_1 \setminus D, p_2(x, y)$ defined by (18). Let S be the optimal stopping set in M_2. Then S is the optimal stopping set in the problem M_1 also and $v_1(x) = v_2(x),\ \forall x \in \mathbb{X}_2$.

Second theorem from [13] deals with situation when the problem can be divided into disjoint classes with two properties:

- for any class the transition probability from each state in one class to another class are the same for all states in first class

- the reward function is a constant inside of each of these classes.

Theorem 7. *Let $M_1 = (\mathbb{X}_1, P_1, g)$ and $M_2 = (\mathbb{X}_2, P_2, g)$ be two optimal stopping problems and let $f : \mathbb{X}_1 \to \mathbb{M}_2$ be surjection such that*

- $P_1(x, f^{-1}(y)) = p_2(f(x), y)\ \forall x \in \mathbb{X}_1, y \in \mathbb{X}_2$

- $g(x) = g(f(x))\ \forall x \in \mathbb{X}_1$.

Then

1. $v_1(x) = v_2(f(x)),\ \forall x \in \mathbb{X}_1$

2. *if S_2 is an optimal stopping set for the problem $\mathbb{X}_2$ then $S_1 = \{f^{-1}(S_2)\}$ is an optimal stopping set for the problem $\mathbb{M}_1$.*

Proof. 1. Denote $f(z) = y$. Then

$$\begin{aligned} Tg_1(x) &= \sum_z p_1(x, z) g_1(z) = \\ &= \sum_{f^{-1}(y)} p_1(x, f^{-1}(y)) g_1(f^{-1}(y)) = \\ &= \sum_y p_2(f(x), y) g_2(f(f^{-1}(y))) = Tg_2(f(x)). \end{aligned}$$

Thus

$$v_1(x) = \max\{g_1(x), Tv_1(x)\} = \max\{g_2(f(x)), Tv_2(f(x))\} = v_2(f(x)).$$

2.

$$S_2 = \{y : g_2(y) = v_2(y)\}.$$

$$\begin{aligned} f^{-1} S_2 = f^{-1}\{y : g_2(y) = v_2(y)\} &= \{x : g_2(f(x)) = v_2(f(x))\} = \\ &= \{x : g_1(x) = v_1(x)\} = S_1. \end{aligned}$$

$\square$

4.4 The monotonicity of the model after the state reduction Consider a Markov model $(\mathbb{X}_1, P_1, g)$, where $\mathbb{X}_1$ is a state space and P_1 is a transition matrix. Let Z_n be a Markov chain in this model with special absorbing state 0. Denote $G_k = g_k(Z_1, .., Z_k)$ Consider sets

$$D^{(1)} = \{z_k \in \mathbb{X}_1 : G_k < E[G_{k+1}|\mathcal{F}_k]\}.$$

We denote by T_i an averaging operator in model $\mathbb{X}_i$.
Idea is to eliminate all states from set D. We do it sequentially till we get such a model $(\mathbb{X}_j, P_j, g)$, that $T_j g(z) \leq g(z)$. It means that

$$D^{(j)} = \{z \in \mathbb{X}_j : G_k < E[G_{k+1}|\mathcal{F}_k]\} = \emptyset$$

and therefore

$$\forall\ z \in \mathbb{X}_j : T_j g(z) \leq g(z). \tag{21}$$

We get that new Markov chain $Z_k^{(j)}$. For every index k we have that

$$G_k^{(j)} \geq E[G_{k+1}^{(j)}|\mathcal{F}_k^{(j)}].$$

Denote this set by A_k^j. It is easy to see that in this model condition (1) is satisfied. Thus we get a monotone stopping problem.
In this new problem we want to find an optimal stopping rule. But according to Theorem 1 1-SLA is optimal for this problem.

Lemma 2. *Suppose that we have Markov model* $(\mathbb{X}_1, P_1, g)$ *and reduced model* $(\mathbb{X}_2, P_2, g)$ *such that condition (21) is satisfied. Then 1-SLA stopping rule optimal in model* $\mathbb{X}_2$ *is also optimal is* $\mathbb{X}_1$.

Proof. Suppose that in reduced model $\mathbb{X}_1$. From SRA we can reduce this model to $\mathbb{X}_2$. We do it sequentially till condition (21) is satisfied. Therefore stopping set is

$$\mathbb{X}_2 = \{z : g_k(z'_1, ..., z'_k) \geq E[g_k(z'_1, ..., z'_k, Z'_{k+1})|z'_1, ..., z'_k])\}$$

where Z'_i is a Markov chain in reduced model. Consider set $A'_k = \{G_k \geq E[G_{k+1}|\mathcal{F}'_k]\}$, where $\mathcal{F}'_k$ is sigma-field generated by $Z'_1, ..., Z'_k$. We show that $A'_k \subset A'_{k+1}$.
Take an arbitrary elementary event $\omega \in A'_k$. Then we have

$$G_{k+1} = g_{k+1}(Z'_1(\omega), ..., Z'_k(\omega), Z'_{k+1}(\omega)) \quad (*)$$

Since $Z'_{k+1}(\omega) \in \mathbb{X}_2$ thus we have:

$$(*) \geq E[g_{k+1}(Z'_1(\omega), ..., Z'_{k+1}(\omega), Z'_{k+2})|Z'_1(\omega), ..., Z'_{k+1}(\omega)])$$

Therefore $\omega \in A'_{k+1}$. Because ω and k was arbitrary we have that

$$\omega \in A'_k \Rightarrow \omega \in A'_{k+1},$$

$$A'_k \subset A'_{k+1}.$$

So we have that 1-SLA is optimal in model $\mathbb{X}_2$. From Theorem 6 we have the that the same stopping rule is valid in model $\mathbb{X}_1$. $\square$

4.5 General model for monotone problems One of the most important modifications of Odds theorem provided in [8] was finding the connection between Bruss' result and 1-SLA method. Let $Z_1, Z_2, ...$ be a stochastic process on an arbitrary space with special absorbing state which will be denoted as 0. Z_k denote the set of random variables observed after $k-1$ success up to and including success k. If there are less than k successes then $Z_k = 0$. Assume that the process will be absorbed with probability one. We want to predict when the process will first hit state 0. If we predict correctly then we win 1, if we predict incorrectly we win nothing, if the process hits 0 before our prediction then we win $\omega < 1$. Therefore the pay-off function is given by

$$\begin{aligned} G_n &= \omega I(Z_n = 0) + I(Z_n \neq 0) P(Z_{n+1} = 0 | \mathcal{G}_n) \\ G_\infty &= \omega. \end{aligned} \tag{22}$$

where $\mathcal{G}_n = \sigma(Z_1, ..., Z_n)$.
This problem is solved by 1-SLA described in Definition 2. The optimal stopping rule is given by

$$\nu_1 = \min\{k \geq 1 : Z_k = 0 \text{ or } (Z_k \neq 0 \text{ and } \frac{W_k}{V_k} \leq 1 - \omega)\} \tag{23}$$

where

$$\begin{aligned} V_k &= P(Z_{k+1} = 0 | \mathcal{G}_k) \\ W_k &= P(Z_{k+1} \neq 0, Z_{k+2} = 0 | \mathcal{G}_k). \end{aligned}$$

¿From the condition in Definition 1 it is easy to see that the sufficient condition for the problem to be monotone is

$$\frac{W_k}{V_k} \text{ is } a.s \text{ non-increasing in } k. \tag{24}$$

Theorem 8. *(Ferguson 2008)*
Suppose that process $Z_1, Z_2, ...$ has an absorbing state 0 such that probability that the process is absorbed is 1 and that the stopping problem with reward sequence (22) satisfies the condition (24). Then the 1-SLA is optimal.

The problem for the Bruss' theorem deals with situation where we observe independent indicators and natural filtration generated by this indicators. Nevertheless this method can be also applied to possibly dependent indicators. Then we have that

$$\begin{aligned} V_k &= P(I_{k+1} = I_{k+2} = ... = 0 | \mathcal{G}_k) \\ W_k &= \sum_{j=k+1}^{\infty} P(I_{k+1} = I_{k+2} = ... = I_{j-1} = 0, I_j = 1, I_{j+1} = I_{j+2} = ... = 0 | \mathcal{G}_k). \end{aligned}$$

In Bruss' result we have also $\omega = 0$. From Theorem 8 we get the following corollary.

Corollary 1. *Suppose the Bernoulli variables $I_1, I_2, ...$ satisfy the condition that there are finite number of successes with probability one. Let $\mathcal{G}_1, \mathcal{G}_2, ...$ be an increasing sequence of sigma-fields such that $\{I_k = 1\}$ is in $\mathcal{G}_k$ for any $k = 1, 2, ...$. Then among stopping rules adapted to the sequence $\{\mathcal{G}_k\}$, the rule (23) is an optimal stopping rule provided condition (24) is satisfied.*

It is easy to see that this corollary implies the Bruss' theorem. In the theorem of Bruss indicators I_k are independent so the ratio $\frac{W_k}{V_k}$ in (23) may be written as $\frac{W_k}{V_k} = \sum_{j=k+1}^{\infty} \frac{p_j}{1-p_j}$. All conditions for monotonicity of the problem are satisfied. Thus problem is monotone and 1-SLA is optimal. This also proves the Bruss' result in the infinite horizon case. Using this approach we can easily find 1-SLA rule in reduced model from Lemma 2. Therefore it is also optimal stopping rule in non-reduced model.

4.6 Example 2B - Selecting the second best object We want to find optimal stopping set for event $\{X_k = 2\}$. Gain function is given by:

$$g((n,k)) = E[I_{\{X_n=2\}}|Y_n = k], \qquad n = 1,...,N; k = 1,...,n.$$

Because absolute rank 2 we can obtain only if we focus on relative ranks 1 or 2 then we get that

$$g((n,l)) = 0, \ \forall\, l \geq 3.$$

$$\begin{aligned} g((n,1)) &= E[I_{\{X_n=2\}}|Y_n = 1] = \\ &= P(X_n = 2|Y_n = 1) = \frac{\binom{1}{0}\binom{N-2}{n-1}}{\binom{N}{n}} = \\ &= \frac{(N-2)!}{(n-1)!(N-n-1)!} \cdot \frac{n!(N-n)!}{N!} = \frac{n(N-n)}{N(N-1)}. \end{aligned} \tag{25}$$

$$\begin{aligned} g((n,2)) &= E[I_{\{X_n=2\}}|Y_n = 2] = \\ &= P(X_n = 2|Y_n = 2) = \frac{\binom{1}{1}\binom{N-2}{n-2}}{\binom{N}{n}} = \\ &= \frac{(N-2)!}{(n-2)!(N-n)!} \cdot \frac{n!(N-n)!}{N!} = \frac{n(n-1)}{N(N-1)}. \end{aligned} \tag{26}$$

Define mapping

$$f((Y_1,...,Y_k)) = \begin{cases} (k,2) & \text{for } Y_k = 2 \\ (k,1) & \text{for } Y_k = 1 \\ (k,0) & \text{otherwise} \end{cases}$$

New transition probabilities are given by $p_2((k-1,j),(k,1)) = p_2((k-1,j),(k,2)) = \frac{1}{k}$ and $p_2((k-1,j),(k,0)) = \frac{k-2}{k}$. We want to create a simpler model M_3 and eliminate states in which is not optimal to stop. First notice that all states (n,l) where $l \geq 3$ are eliminated, because

$$Tg(n,l) > 0 = g(n,l).$$

Thus we get new model M_3:

1. $\mathbb{X}_3$ is set of all pairs (n,k), where $1 \leq n \leq N$ and $k = 1,2$
2. transition matrix is defines as

$$p_3((n,k),(m,j)) = \frac{n(n-1)}{m(m-1)(m-2)}, \quad 2 \leq n < m \leq N,$$

$$p_3((1,1),(2,j)) = \frac{1}{2}, \quad j = 1,2$$

   and satisfies monotonicity property, i.e. for $m \leq n$, $p_3((n,k),(m,j)) = 0$.

3. Z_n be a Markov chain with initial point $z = (1,1)$.

There are also some states with relative ranks 1 and 2 that should be eliminated. We will find condition for that. First calculate $Tg(n,j), j = 1,2$.

$$\begin{aligned}
Tg(n,1) &= \sum p((n,1),(m,k))g((m,k)) = \\
&= \sum_{m=n+1}^{N} p((n,1),(m,1))g((m,1)) + p((n,1),(m,2))g((m,2)) = \\
&= \sum_{m=n+1}^{N} \frac{n(n-1)}{m(m-1)(m-2)}\frac{m(N-m)}{N(N-1)} + \frac{n(n-1)}{m(m-1)(m-2)}\frac{m(m-1)}{N(N-1)} = \\
&= \sum_{m=n+1}^{N} \frac{n(n-1)}{N(N-1)(m-2)}\left(\frac{N-m}{m-1}+1\right) = \\
&= \frac{n(n-1)}{N(N-1)}\sum_{m=n+1}^{N}\frac{1}{m-2}\left(\frac{N-m+m-1}{m-1}\right) = \\
&= \frac{n(n-1)}{N}\sum_{m=n+1}^{N}\frac{1}{(m-1)(m-2)}.
\end{aligned} \tag{27}$$

Similarly

$$\begin{aligned}
Tg(n,2) &= \sum p((n,2),(m,k))g((m,k)) = \\
&= \sum_{m=n+1}^{N} p((n,2),(m,1))g((m,1)) + p((n,2),(m,2))g((m,2)) = \\
&= \sum_{m=n+1}^{N} \frac{n(n-1)}{m(m-1)(m-2)}\frac{m(N-m)}{N(N-1)} + \frac{n(n-1)}{m(m-1)(m-2)}\frac{m(m-1)}{N(N-1)} = \\
&= \sum_{m=n+1}^{N} \frac{n(n-1)}{N(N-1)(m-2)}\left(\frac{N-m}{m-1}+1\right) = \\
&= \frac{n(n-1)}{N(N-1)}\sum_{m=n+1}^{N}\frac{1}{m-2}\left(\frac{N-m+m-1}{m-1}\right) = \\
&= \frac{n(n-1)}{N}\sum_{m=n+1}^{N}\frac{1}{(m-1)(m-2)}.
\end{aligned} \tag{28}$$

We see that $Tg((n,1)) = Tg((n,2))$. From (20), (25) and (27) we get

$$\sum_{m=n+1}^{N}\frac{1}{(m-1)(m-2)} > \frac{N-n}{(n-1)(N-1)} \tag{29}$$

and from (20), (26) and (28)

$$\sum_{m=n+1}^{N}\frac{1}{(m-1)(m-2)} > \frac{1}{N-1}. \tag{30}$$

Then we eliminate states for which conditions (29) and (30) are satisfied and recalculate transition probabilities using (29). We get simpler model M_4 and from Theorem 6 we know that optimal stopping set in M_4 is also optimal stopping set in M_1.

From calculus we know that

$$\sum_{m=n+1}^{N} \frac{1}{(m-1)(m-2)} = \frac{N-n}{(n-1)(N-1)}. \tag{31}$$

It means that we do not eliminate any state $(n, 1)$ and eliminate states $(n, 2)$ such that

$$\begin{aligned} \frac{N-n}{(n-1)(N-1)} &> \frac{1}{N-1} \\ \frac{N-n}{n-1} &> 1 \\ n &< \frac{N+1}{2}. \end{aligned} \tag{32}$$

Denote: $K = \lfloor \frac{N}{2} \rfloor$. According to the Lemma 1 we can eliminate the states recursively using formula (18). Therefore the new transition probabilities are

$$\begin{aligned} p_4((n,1),(m,1)) &= \frac{n}{m(m-1)}, \quad 1 \le n < m \le K \\ p_4((n,1),(m,j)) &= \frac{n(K-1)}{m(m-1)(m-2)}, \quad n \le K < m \\ p_4((n,k),(m,j)) &= \frac{n(n-1)}{m(m-1)(m-2)}, \quad K < n < m \end{aligned} \tag{33}$$

Continuing this procedure of course should give us the minimal optimal stopping set and transition probabilities. Once again calculate $Tg(n,j), j = 1, 2$.
For $n < K$

(34)

$$\begin{aligned} Tg(n,1) &= \\ &= \sum_{m=n+1}^{K} \frac{n}{m(m-1)} \cdot \frac{m(N-m)}{N(N-1)} + \sum_{m=K+1}^{N} \frac{n(K-1)}{m(m-1)(m-2)} \frac{m(N-m)+m(m-1)}{N(N-1)} = \\ &= \frac{n}{N(N-1)} \Big((N-1) \sum_{m=n+1}^{K} \frac{1}{m-1} - K + n + N - K \Big) = \\ &= \frac{n}{N} \sum_{m=n+1}^{K} \frac{1}{m-1} + \frac{n}{N(N-1)} (N + n - 2K). \end{aligned}$$

Using 20 we get

$$\sum_{m=n+1}^{K} \frac{1}{m-1} > \frac{2(K-1)}{N-1}. \tag{35}$$

From this we find an index k^* such that the above condition is satisfied. Of course neither for $n \ge K$ states $(n, 1)$ and states $(n, 2)$ are eliminated.

It is easy to check, that there are no more states that can be eliminated. Thus the optimal stopping rule is

$$N^* = \min\{1 \le n \le N : (Y_n = 1 \text{ and } \sum_{m=n+1}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{m-1} \le \frac{2(\lfloor \frac{N}{2} \rfloor - 1)}{N-1})$$
$$\text{or } (Y_n \in \{1,2\} \text{ and } n > \lfloor \frac{N}{2} \rfloor)\}.$$

Now from Lemma 2 we know that the same optimal stopping rule holds for initial model.

5 Conclusion We have shown two important results: one is that Odds Theorem comes from problem of optimal stopping of Markov chains. Second is that optimal stopping problem of Markov chain can be reduced to monotone stopping problem. The procedure is the following: eliminate those states which is not optimal to stop on, apply 1-SLA method to find the optimal stopping rule and calculate the expected reward. This explains why the procedure was called 'Eliminate and stop'. This algorithm can be used to solve many problems. One of them is 'secretary problem'.

6 Acknowledgements The author thank the referee for making several suggestions which improved the quality of this paper. The author would like to thank professor Krzysztof Szajowski for discussions and comments.

References

[1] K. Ano, *Monotone problem in optimal stopping via discrete version of Dynkin formula and Bruss' odds theorem*, RIMS Kokyuroku, No.1802, 281-287, 2012

[2] P. Billingsley *Probability and measure*, 2nd edition, John Wiley & Sons, Inc., New York 1986

[3] F.T.Bruss *Sum the odds to one and stop*, The Annals of Probability, Vol. 28, No.3, 1384-1391, 2000

[4] F.T. Bruss and D. Paindaveine. *Selecting a Sequence of Last Successes in Independent Trials*, Journal of Applied Probability, Vol 37, 389-399, 2000

[5] Y.S. Chow, H. Robbins, D. Siegmund, *Great Expectations: The Theory of Optimal Stopping*, Houghton Mifflin Co., Boston, 1971

[6] R. Dendievel *New developments of the odds theorem*, http://arxiv.org/abs/1212.1391v1, 2012

[7] T.S. Ferguson, *Optimal stopping and applications*, http://www.math.ucla.edu/∼tom/Stopping/Contents.html, 2000

[8] T.S.Ferguson *The sum-the-odds theorem with application to a stopping game of Sakaguchi*,http://www.math.ucla.edu/∼tom/papers/oddsThm.pdf ,2008

[9] S. Hsiau, J. Yang *Selecting the last success in Markov-dependent trials*, Journal of Applied Probability, Vol 39, 271-281, 2002

[10] J. Jakubowski, R. Sztencel *Wstep do teorii prawdopodobienstwa*, (english title: *Introductin to probability theory*), wyd. II, SCRIPT, 2001, [in polish]

[11] M.P. Quine, J.S. Law *Exact Results for a Secretary Problem*, Journal of Applied Probability, Vol 33, 630-639, 1996

[12] A.N. Shiryayev *Optimal Stopping Rules*, Springer, New York, 1978

[13] I. Sonin *The Elimination Algorithm for the problem of optimal stopping*, Mathematical Methods of Operations Research, 49: 111-123, 1999

[14] K. Szajowski, *An apartment problem*, Mathematica Applicanda, vol. 42, nr 2, p. 1-14, 2014.

[15] M. Tamaki *Markov version of Bruss' odds-theorem*, RIMS Kokyuroku, 1504, 184-187, 2006

[16] M. Tamaki *Sum the multiplicative odds to one and stop*, Journal of Applied Probability, Vol 47, 761-777, 2010

Communicated by *Vladimir. V. Mazalov*

LOGICAL ANALYSIS OF RATIO INFERENCE BY CHILDREN

TAKESHI SAKAI, AND TADASHI TAKAHASHI

Received January 30, 2015

Abstract. In this study, we applied propositional and predicate logic for mathematical explication of the processes of inference by children. This facilitated extraction and comparison of children-specific inference processes, which are difficult to derive from a child's protocol itself, and elucidation of the structure of children's ratio-related conceptual and procedural knowledge.

1 Introduction In arithmetic education, ascertaining the concepts of given domains in terms of conceptual and procedural knowledge is essential as a mechanism of knowledge change during knowledge acquisition. Conceptual knowledge consists of an implicit or explicit system of interlinked pieces of knowledge for a given domain, and procedural knowledge comprises systems of multiple execution series for problem solution [1], [2].

The concept of ratio is applied in ascertaining the relation between two quantities and in comparing the relative quantities of two sets. It differs in meaning from simple multiplication and is active in the sense of comparing the relative sizes (multiples) of given quantities and base quantities rather than directly comparing quantities [3]. In the present study, we therefore focus on comparison of the relations between quantities in two different sets. It has been noted that the concept of ratio can be investigated in a fairly pure form as a logical mathematical recognition [4], and in this light we treat this comparison as a probabilistic comparison task. Ratio and probability are different concepts, but for children unschooled in probability, the ratio concept can be utilized as an approach for probability settings. Studies that have utilized probability comparison tasks include A. Nakagaki [4], [5], N. Fujimura [6], G. Noelting [7], [8], J. Piaget and B. Inhelder [9], and R. S. Siegler [10], [11], which in relation to quantification of probability all share the view that recognition of equivalence based on recognition of multiple relationships provides the foundation for the intensive quantity concept, and formation of that concept begins at the age of 11 or 12 years [6]. A. Nakagaki [5] identifies the psychological stage of development of the ratio concept as a process of balancing in which the ability to compensate affirmation with negation becomes complete. Moreover, children inherently possess and apply the concept of "half" 1/2) as an intuitive approach for quantification of probability [12], and the "half" benchmark strategy [13] is of key significance during the stage in which children recognize and develop ratio inference leading to "part-whole" comparison in probability comparison problems.

Previous studies have not included integrated analyses of children's recognition in the three situational contexts of ratio, comparative quantity, and base quantity, and are generally protocol-based analyses of children's recognition of ratio rather than mathematical representations of children's thought processes, using test problems that include numbers, and thereby make it difficult to determine the relationship between children's conceptual and procedural knowledge in their ratio recognition.

2010 *Mathematics Subject Classification.* Mathematics Education .
Key words and phrases. ratio, propositional logic, predicate logic.

The present study was undertaken to develop test problems that distinguish between conceptual and procedural knowledge relating to ratios, express the thought processes of children mathematically, and elucidate the structures of ratio-related conceptual and procedural knowledge.

2 Development of test problems

(1) Symbolization of inference process by propositional and predicate logic In the development of each test problem, it is necessary to prove that a given inference process can derive the correct conclusion from the perspective of probability with the conditions given in the problem statement as assumptions. In analysis of the test results, moreover, it is essential to explain the children-specific logic used in the inference process mathematically. In the present study, we perform these proofs and analyses by using propositional logic and predicate logic with reference to the views of S. Tamura, K. Aragane, and T. Hirai [14] and K. Todayama [15]. The symbols and the rules and laws of inference as used in the present study are essentially as follows. Note that we express $A \Rightarrow B$, i.e., if $A \equiv \top$ then $B \equiv \top$, as inference schemata with a horizontal line of the form as below.

$$\boxed{A \Rightarrow B \qquad \frac{A}{B}}$$

1) Inference rules and laws We let x, y, z, a, b, c, and d be nonnegative variables, and let $f(x)$ be $x = y, x > y$, or $x < y$. We refer to $f(x)$ containing variable x as the expression. The focus is on the thought processes of children, and we accordingly allow the use of operations on the variables and take the operation rules to be applicable to inference rules. Tables 1 through 3 show the unit element, zero element, and reflective, symmetric, and transitive laws, the inference rules, and the inference laws, respectively, for operations on the variables. The proofs of the inference laws are not shown.

$Unit\ Element(UE)$	If $x \times y = y \times x = x$, take y as a unit element and write $y = 1$.
$Zero\ Element(ZE)$	If $x + y = y + x = x$, take y as a zero element and write $y = 0$.
$Reflective\ Law(RL)$	$x = x$
$Symmetric\ Law(SL)$	$\dfrac{x = y}{y = x}$
$Transitive\ Law(TL)$	$\dfrac{x = y \quad y = z}{x = z}$ $\quad \dfrac{x > y \quad y > z}{x > z}$ $\quad \dfrac{x < y \quad y < z}{x < z}$ $\dfrac{x > y \quad y = z}{x > z}$ $\quad \dfrac{x = y \quad y > z}{x > z}$ $\quad \dfrac{x < y \quad y = z}{x < z}$ $\quad \dfrac{x = y \quad y < z}{x < z}$

Table 1: Unit element, zero element, and reflective, symmetric, and transitive laws for operations on the variables

Rule name	Inference rule
$Operation-Inference(OI)$	Where $a \circ b = c$, allow $\dfrac{f(a \circ b)}{f(c)}$ and $\dfrac{f(c)}{f(a \circ b)}$
$==$	$\dfrac{a = b \quad c = d}{a \circ c = b \circ d}$ $\quad \circ : +, -, \times,$ or $\div$

>= 1	$\dfrac{a > b \quad c = d}{a \circ c > b \circ d}$	$\dfrac{a < b \quad c = d}{a \circ c < b \circ d}$	$\circ : +, -, \times,$ or $\div$
	$\dfrac{a = b \quad c > d}{a \circ c > b \circ d}$	$\dfrac{a = b \quad c < d}{a \circ c < b \circ d}$	$\circ : +$ or $\times$
>= 2	$\dfrac{a = b \quad c > d}{a \circ c < b \circ d}$	$\dfrac{a = b \quad c < d}{a \circ c > b \circ d}$	$\circ : -$ or $\div$
>> 1	$\dfrac{a > b \quad c > d}{a \circ c > b \circ d}$	$\dfrac{a < b \quad c < d}{a \circ c < b \circ d}$	$\circ : +$ or $\times$
>> 2	$\dfrac{a > b \quad c > d}{a \circ d > b \circ c}$	$\dfrac{a < b \quad c < d}{a \circ d < b \circ c}$	$\circ : -$ or $\div$
<>	$\dfrac{a > b}{b < a}$	$\dfrac{a < b}{b > a}$	

Table 2: Rules of inference for operations on variables

In all of the above operations, $\div$ is applicable so long as $c \neq 0$ and $d \neq 0$.

The following rules are allowed as operation-inference rules for $a \circ b = c$.
(1) $x \times 1 = 1 \times x = x$
(2) $x \times 1/x = 1/x \times x = x \div x = x/x = 1$
(3) $x + 0 = 0 + x = x$
(4) $x - x = 0$
(5) $x \circ y = y \circ x \quad (\circ : +$ or $\times)$ [Commutative Law]
(6) $(x \circ y) \circ z = x \circ (y \circ z) \quad (\circ : +$ or $\times)$ [Associative Law]
(7) $x \times (y \circ z) = x \times y \circ x \times z \quad (\circ : +$ or $-)$ [Distributive Law]
(8) $(y \circ z) \div x = y \div x \circ z \div x \quad (\circ : +$ or $-)$ [Distributive Law]

The following calculations are allowed as operation-inference rules for $a \circ b = c$.
(1) $x \times 1/y = x \div y = x/y$
(2) $a \div b = (a \times c) \div (b \times c)$
(3) $(a/b \times bd) \div (c/d \times bd) = (a \times d) \div (b \times c)$

Law name	Inference law
$= Substitution (= Sub)$	$\dfrac{f(a_1, a_2, \cdots, a_n) \quad a_1 = b_1, a_2 = b_2, \cdots, a_n = b_n}{f(b_1, b_2, \cdots, b_n)}$

Table 3: Laws of inference for operations on variables

The next four tables show the inference rules (Table 4) and inference laws (Table 5) for propositional logic, and the inference rules (Table 6) and inference law (Table 7) for predicate logic. $F(X)$ is a logical expression containing propositional variable X. The proofs of the inference laws are not shown.

Rule name	Inference rule	Rule name	Inference rule
$\to Introduction(\to Int)$	$\dfrac{\begin{matrix}(k)\\ [A]\\ B\end{matrix}}{A\to B}\,(k)$	$\lor Removal(\lor Rem)$	$\dfrac{A\lor B \quad \begin{matrix}(k)\\ [A]\\ C\end{matrix} \quad \begin{matrix}(k)\\ [B]\\ C\end{matrix}}{C}\,(k)$
$\to Removal(\to Rem)$	$\dfrac{A \quad A\to B}{B}$	$\lor Introduction(\lor Int)$	$\dfrac{A}{A\lor B}$
$Transition(Trn)$	$\dfrac{A\to B \quad B\to C}{A\to B}$	$\neg Removal(\neg Rem)$	$\dfrac{A \quad \neg\neg A}{\perp}$
$\land Introduction(\land Int)$	$\dfrac{A \quad B}{A\land B}$	$\neg Introduction(\neg Int)$	$\dfrac{\begin{matrix}[A]\\ \perp\end{matrix}}{\neg A}$
$\land Removal(\land Rem)$	$\dfrac{A\land B}{A} \qquad \dfrac{A\land B}{B}$	$\neg\neg Removal(\neg\neg Rem)$	$\dfrac{\neg\neg A}{A}$

Table 4: Rules of inference for propositional logic

Law name	Inference law	Law name	Inference law
$\equiv Removal(\equiv Rem)$	$\dfrac{A\equiv B}{A\to B} \qquad \dfrac{A\equiv B}{B\to A}$	$\equiv Substiution(\equiv Sub)$	$\dfrac{F(A) \quad A\equiv B}{F(B)}$
$\land\land Introduction(\land\land Int)$	$\dfrac{A_1 \quad A_2 \quad A_3\cdots A_n}{A_1\land A_2\land A_3\land\cdots\land A_n}$	$\veebar\to\lor$	$\dfrac{A\veebar B}{A\lor B}$
$\neg\neg Introduction(\neg\neg Int)$	$\dfrac{A}{\neg\neg A}$	$Importation(Imp)$	$\dfrac{A\to(B\to C)}{A\land B\to C}$
$Contraposition(Cont)$	$\dfrac{A\to B}{\neg B\to\neg A}$		

Table 5: Laws of inference for propositional logic

Rule name	Inference rule
$\forall Removal(\forall Rem)$	$\dfrac{\forall x[P(x)]}{P(a_i)}$
$\exists Introduction(\exists Int)$	$\dfrac{P(a_i)}{\exists x[P(x)]}$
$\forall Introduction(\forall Int)$	$\dfrac{P(a_1) \quad P(a_2)\cdots P(a_n)}{\forall x[P(x)]}$
$\exists Removal(\exists Rem)$	$\dfrac{\exists x[P(x)] \quad \dfrac{P(a_1)}{C} \quad \dfrac{P(a_2)}{C}\cdots\dfrac{P(a_n)}{C}}{C}$

Table 6: Rules of inference for predicate logic

Law name	Inference law
$\exists\exists Introduction(\exists\exists Int)$	$\dfrac{P_1(a_{1_{i_1}})\quad P_2(a_{2_{i_2}})\cdots P_n(a_{n_{i_n}})}{\exists x_1\exists x_2\cdots\exists x_n[P_1(x_1)\wedge P_2(x_2)\wedge\cdots\wedge P_n(x_n)]}$

Table 7: Law of inference for predicate logic

2) Symbolization for single-lot drawing trials Table 8 shows the symbolization for the number of events, elementary events, and probabilities in a trial drawing of one lot from a set containing winning and losing lots and in a trial drawing of one lot each from sets A and B (thus an " A lot " and a " B lot ", respectively) with both sets containing winning and losing lots. Variable x may be $n(X), n(Y), n(S), P(X)$, or $P(Y)$, either alone or in combination.

X	Event: Drawing of winning lot	$n(S)$	Total number of lots
X_A	Event: Drawing of winning A lot	$n(S_A)$	Total number of A lots
X_B	Event: Drawing of winning B lot	$n(S_B)$	Total number of B lots
Y	Event: Drawing of losing lot	$P(X)$	Probability of drawing winning lot
Y_A	Event: Drawing of losing A lot	$P(X_A)$	Probability of drawing winning A lot
Y_B	Event: Drawing of losing B lot	$P(X_B)$	Probability of drawing winning B lot
S	All events	$P(Y)$	Probability of drawing losing lot
S_A	All A-lot events	$P(Y_A)$	Probability of drawing losing A lot
S_B	All B-lot events	$P(Y_B)$	Probability of drawing losing B lot
$n(X)$	Number of winning lots	$P(S)$	Probability of all events
$n(X_A)$	Number of winning A lots	$P(S_A)$	Probability of all events for A lots
$n(X_B)$	Number of winning B lots	$P(S_B)$	Probability of all events for B lots
$n(Y)$	Number of losing lots		
$n(Y_A)$	Number of losing A lots		
$n(Y_B)$	Number of losing B lots		

Table 8: Number and probability of events and elementary events in single-lot drawing trials

Table 9 shows the symbolization of comparative conditions in the settings, with the total number of lots, number of winning lots, number of losing lots, probability of winning, and probability of losing as the objects of comparison. The expression $(A\wedge\neg B)\vee(\neg A\wedge B)$ is abbreviated $A\veebar B$, and exclusive disjunction is symbolized as $\veebar$.

Condition	Symbolization
Equal total numbers of A and B lots	A_1: $n(S_A)=n(S_B)$
Larger total number of A lots	A_2: $n(S_A)>n(S_B)$
Larger total number of B lots	A_3: $n(S_A)<n(S_B)$
Different total numbers of A and B lots	$\neg A_1$: $\neg(n(S_A)=n(S_B))$
Equal numbers of winning A and B lots	B_1: $n(X_A)=n(X_B)$
Larger number of winning A lots	B_2: $n(X_A)>n(X_B)$
Larger number of winning B lots	B_3: $n(X_A)<n(X_B)$
Different numbers of winning A and B lots	$\neg B_1$: $\neg(n(X_A)=n(X_B))$
Equal numbers of losing A and B lots	C_1: $n(Y_A)=n(Y_B)$
Larger number of losing A lots	C_2: $n(Y_A)>n(Y_B)$

Larger number of losing B lots	C_3: $n(Y_A) < n(Y_B)$
Different numbers of losing A and B lots	$\neg C_1$: $\neg(n(Y_A) = n(Y_B))$
Equal chances of winning with A and B lots	D_1: $P(X_A) = P(X_B)$
Greater chance of winning with A lots	D_2: $P(X_A) > P(X_B)$
Greater chance of winning with B lots	D_3: $P(X_A) < P(X_B)$
Different chances of winning with A and B lots	$\neg D_1$: $\neg(P(X_A) = P(X_B))$
Equal chances of losing with A and B lots	E_1: $P(Y_A) = P(Y_B)$
Greater chance of losing with A lots	E_2: $P(Y_A) > P(Y_B)$
Greater chance of losing with B lots	E_3: $P(Y_A) < P(Y_B)$
Different chances of losing with A and B lots	$\neg E_1$: $\neg(P(Y_A) = P(Y_B))$

Table 9: Comparative setting conditions relating to probabilities

From $A_1 \veebar A_2 \veebar A_3$, $\neg A_1 \equiv A_2 \veebar A_3$; from $B_1 \veebar B_2 \veebar B_3$, $\neg B_1 \equiv B_2 \veebar B_3$; from $C_1 \veebar C_2 \veebar C_3$, $\neg C_1 \equiv C_2 \veebar C_3$; from $D_1 \veebar D_2 \veebar D_3$, $\neg D_1 \equiv D_2 \veebar D_3$; and from $E_1 \veebar E_2 \veebar E_3$, $\neg E_1 \equiv E_2 \veebar E_3$

3) Axioms, definitions, and theorems for single-lot drawing trials Table 10 shows the axioms, definitions, and theorems for the trials in which a single lot is drawn. The theorem proofs are not shown.

$Axiom1(Ax1)$	$P(S) = 1, P(\phi) = 0$
$Axiom2(Ax2)$	$P(S) = P(X) + P(Y)$
$Axiom3(Ax3)$	$0 \leqq P(X) \leqq 1,\ 0 \leqq P(Y) \leqq 1 \quad (X \subseteqq S, Y \subseteqq S)$
$Definition(Def)$	$P(Z) = n(Z) \div n(S) \quad (Z : X, Y)$
$Theorem1(Thm1)$	$P(Y) = 1 - P(X)$
$Theorem2(Thm2)$	$P(X) = 1 - P(Y)$
$Theorem3(Thm3)$	$n(Z) = n(S) \times P(Z) \quad (Z : X, Y)$
$Theorem4(Thm4)$	$n(S) = n(Z) \div P(Z) \quad (Z : X, Y)$
$Theorem5(Thm5)$	$n(S) = n(X) + n(Y)$
$Theorem6(Thm6)$	$n(Y) = n(S) - n(X)$
$Theorem7(Thm7)$	$n(X) = n(S) - n(Y)$

Table 10: Axioms, definitions, and theorems for single-lot drawing trials

(2) Test problems The test problems in the probability comparison tasks are in the two categories of ratio-related conceptual and ratio-related procedural knowledge. Each of the two categories includes the three contextual categories of ratio, comparative-quantity, and base-quantity. The conceptual-knowledge problems are those that contain no numbers and thus require approaches based primarily on concepts. The procedural-knowledge problems are those that contain numbers and thus allow approaches based primarily on procedures. In the following, we provide examples of ratio-context test problems that pertain to ratio-related conceptual and procedural knowledge. Tables 11 and 12 show the supposition and conclusion of each of these test problems. Please refer to Supplements 1 through 4 for test problems in the comparative-quantity and base-quantity contexts pertaining to ratio-related conceptual and procedural knowledge.

Example test problem for ratio-related conceptual knowledge in the ratio context

Sample question

In this lot drawing, some of the lots are winning lots and some of them are losing lots. There are two groups of lots. Lots from one group are called " A lots " and lots from the other group are called " B lots ". Both groups include winning lots and losing lots. The " total number of lots " in one group means all the winning and losing lots in that group. If a winning lot is easy to draw, we call the group an " easy winner ".

The total number of A lots is the same as the total number of B lots.
There are more winning A lots than winning B lots.
There are more losing B lots than losing A lots.

(Supposition)

If just one lot is drawn, will it be easier to win with an A lot or a B lot, or will it be the same for an A lot and a B lot? Draw a circle in the box above any of the following answers that you think may be correct. Note that in some questions, a circle can be drawn in all of the boxes.

☐	☐	☐
It is easier to win with an A lot.	No difference between an A lot and a B lot.	It is easier to win with a B lot.

(Conclusion)

	Supposition	Correct conclusion
Question 1	A_1, B_2, C_3	D_2
Question 2	A_1, B_1, C_1	D_1
Question 3	$\neg A_1, B_2, C_3$	D_2
Question 4	$\neg A_1, B_2, C_2$	D_1, D_2, D_3
Question 5	$\neg A_1, B_2, C_1$	D_2
Question 6	$\neg A_1, B_1, C_2$	D_3

Table 11: Test problem suppositions and correct conclusions for ratio-related conceptual knowledge in the ratio context

Example test problem for ratio-related procedural knowledge in the ratio context

Sample question

In this lot drawing, some of the lots are winning lots and some of them are losing lots. There are two groups of lots. Lots from one group are called " A lots " and lots from the other group are called " B lots ". Both groups include winning lots and losing lots. The " total number of lots " in one group means all the winning and losing lots in that group. We call how easy it is to draw a winning lot " chance of winning ". If chance of winning is high, we call the group an " easy winner ".

The total number of A lots is 5, and 3 of them are winning lots.
The total number of B lots is 5, and 1 of them is a winning lot.

(Supposition)

If just one lot is drawn, will it be easier to win with an A lot or a B lot, or will it be the same for an A lot and a B lot? Draw a circle in the box above any of the following answers that you think may be correct.

☐	☐	☐	
It is easier to win with an A lot.	No difference between an A lot and a B lot.	It is easier to win with a B lot.	(Conclusion)

	Supposition	Correct conclusion
Question 1	$n(X_A)=3, n(X_B)=1, n(S_A)=5, n(S_B)=5$	D_2
Question 2	$n(X_A)=1, n(X_B)=3, n(S_A)=2, n(S_B)=6$	D_1
Question 3	$n(X_A)=3, n(X_B)=3, n(S_A)=4, n(S_B)=5$	D_2
Question 4	$n(X_A)=1, n(X_B)=3, n(S_A)=4, n(S_B)=4$	D_3
Question 5	$n(X_A)=3, n(X_B)=6, n(S_A)=4, n(S_B)=8$	D_1
Question 6	$n(X_A)=2, n(X_B)=2, n(S_A)=4, n(S_B)=5$	D_2
Question 7	$n(X_A)=1, n(X_B)=4, n(S_A)=2, n(S_B)=5$	D_3
Question 8	$n(X_A)=1, n(X_B)=3, n(S_A)=4, n(S_B)=6$	D_3
Question 9	$n(X_A)=2, n(X_B)=3, n(S_A)=4, n(S_B)=5$	D_3
Question 10	$n(X_A)=2, n(X_B)=3, n(S_A)=8, n(S_B)=10$	D_3
Question 11	$n(X_A)=3, n(X_B)=4, n(S_A)=4, n(S_B)=5$	D_3
Question 12	$n(X_A)=4, n(X_B)=3, n(S_A)=10, n(S_B)=6$	D_3

Table 12: Test problem suppositions and correct conclusions for ratio-related procedural knowledge in the ratio context

(3) Test-problem proofs We proved the validity of the correct conclusions given the problem descriptions and suppositions, by propositional logic in cases resulting in one correct answer and by predicate logic in cases not resulting in one correct answer. The following two examples are typical of the proof process. One is for a problem involving ratio-related conceptual knowledge in the ratio context and the other is for a problem involving ratio-related procedural knowledge in the ratio context.

1) Ratio-related conceptual knowledge in the ratio context
Case resulting in one correct answer: Question 1

Supposition	A_1, B_2, C_3
Correct conclusion	D_2

$$\frac{B_2: n(X_A) > n(X_B) \quad A_1: n(S_A) = n(S_B)}{n(X_A) \div n(S_A) > n(X_B) \div n(S_B)}(>=1)$$

$$\cfrac{Def: P(Z) = n(Z) \div n(S) \qquad n(Z) = n(X_A) \;\; n(S) = n(S_A) \;\; P(Z) = P(X_A)}{\cfrac{P(X_A) = n(X_A) \div n(S_A)}{n(X_A) \div n(S_A) = P(X_A)}(SL)}(=Sub)$$

$$\cfrac{Def: P(Z) = n(Z) \div n(S) \qquad n(Z) = n(X_B) \;\; n(S) = n(S_B) \;\; P(Z) = P(X_B)}{\cfrac{P(X_B) = n(X_B) \div n(S_B)}{n(X_B) \div n(S_B) = P(X_B)}(SL)}(=Sub)$$

$$\frac{n(X_A)\div n(S_A)>n(X_B)\div n(S_B) \quad n(X_A)\div n(S_A)=P(X_A) \quad n(X_B)\div n(S_B)=P(X_B)}{D_2 : P(X_A) > P(X_B)} \ (=Sub)$$

This proves that D_2 is the correct conclusion, given supposition A_1 and B_2.

Case not resulting in one correct answer: Question 4

Supposition	$\neg A_1, B_2, C_2$
Correct conclusion	D_1, D_2, D_3

$$\frac{n(X_A)=2 \quad n(X_B)=1 \quad n(S_A)=4 \quad n(S_B)=2 \quad 2>1 \quad 4>2 \quad 4-2>2-1 \quad 2/4=1/2}{\begin{array}{l}\exists x\exists y\exists z\exists w[n(X_A)=x\wedge n(X_B)=y\wedge x>y\\ \quad\wedge n(S_A)=z\wedge n(S_B)=w\wedge z>w\\ \quad\wedge n(Y_A)=z-x\wedge n(Y_B)=w-y\wedge z-x>w-y\\ \quad\wedge P(X_A)=x/z\wedge P(X_B)=y/w\wedge x/z=y/w]\cdots(a)\end{array}} \ (\exists\exists Int)$$

$$\frac{n(X_A)=3 \quad n(X_B)=1 \quad n(S_A)=5 \quad n(S_B)=2 \quad 3>1 \quad 5>2 \quad 5-3>2-1 \quad 3/5>1/2}{\begin{array}{l}\exists x'\exists y'\exists z'\exists w'[n(X_A)=x'\wedge n(X_B)=y'\wedge x'>y'\\ \quad\wedge n(S_A)=z'\wedge n(S_B)=w'\wedge z'>w'\\ \quad\wedge n(Y_A)=z'-x'\wedge n(Y_B)=w'-y'\wedge z'-x'>w'-y'\\ \quad\wedge P(X_A)=x'/z'\wedge P(X_B)=y'/w'\wedge x'/z'>y'/w']\cdots(b)\end{array}} \ (\exists\exists Int)$$

$$\frac{n(X_A)=2 \quad n(X_B)=1 \quad n(S_A)=5 \quad n(S_B)=2 \quad 2>1 \quad 5>2 \quad 5-2>2-1 \quad 2/5<1/2}{\begin{array}{l}\exists x''\exists y''\exists z''\exists w''[n(X_A)=x''\wedge n(X_B)=y''\wedge x''>y''\\ \quad\wedge n(S_A)=z''\wedge n(S_B)=w''\wedge z''>w''\\ \quad\wedge n(Y_A)=z''-x''\wedge n(Y_B)=w''-y''\wedge z''-x''>w''-y''\\ \quad\wedge P(X_A)=x''/z''\wedge P(X_B)=y''/w''\wedge x''/z''<y''/w'']\cdots(c)\end{array}} \ (\exists\exists Int)$$

$$\frac{(a) \qquad (b) \qquad (c)}{\begin{array}{l}(\exists x\exists y\exists z\exists w[n(X_A)=x\wedge n(X_B)=y\wedge x>y\\ \quad\wedge n(S_A)=z\wedge n(S_B)=w\wedge z>w\\ \quad\wedge n(Y_A)=z-x\wedge n(Y_B)=w-y\wedge z-x>w-y\\ \quad\wedge P(X_A)=x/z\wedge P(X_B)=y/w\wedge x/z=y/w])\\ \wedge(\exists x'\exists y'\exists z'\exists w'[n(X_A)=x'\wedge n(X_B)=y'\wedge x'>y'\\ \quad\wedge n(S_A)=z'\wedge n(S_B)=w'\wedge z'>w'\\ \quad\wedge n(Y_A)=z'-x'\wedge n(Y_B)=w'-y'\wedge z'-x'>w'-y'\\ \quad\wedge P(X_A)=x'/z'\wedge P(X_B)=y'/w'\wedge x'/z'>y'/w'])\\ \wedge(\exists x''\exists y''\exists z''\exists w''[n(X_A)=x''\wedge n(X_B)=y''\wedge x''>y''\\ \quad\wedge n(S_A)=z''\wedge n(S_B)=w''\wedge z''>w''\\ \quad\wedge n(Y_A)=z''-x''\wedge n(Y_B)=w''-y''\wedge z''-x''>w''-y''\\ \quad\wedge P(X_A)=x''/z''\wedge P(X_B)=y''/w''\wedge x''/z''<y''/w''])\end{array}} \ (\wedge\wedge\, Int)$$

This proves that there exist $n(X_A), n(X_B), n(S_A)$, and $n(S_B)$ that satisfy $\neg A_1, B_2, C_2$, and D_1; $\neg A_1, B_2, C_2$, and D_2; and $\neg A_1, B_2, C_2$, and D_3, respectively.

2) Ratio-related procedural knowledge in the ratio context
Question 1

Supposition	$n(X_A) = 3, n(X_B) = 1, n(S_A) = 5, n(S_B) = 5$
Correct conclusion	D_2

$$\dfrac{Def : P(Z) = n(Z) \div n(S) \quad n(Z) = n(X_A) \;\; n(S) = n(S_A) \;\; P(Z) = P(X_A)}{P(X_A) = n(X_A) \div n(S_A)}\,(= Sub)$$

$$\dfrac{\dfrac{P(X_A) = n(X_A) \div n(S_A) \quad n(X_A) = 3 \;\; n(S_A) = 5}{P(X_A) = 3 \div 5}\,(= Sub)}{P(X_A) = 3/5}\,(OI)$$

$$\dfrac{Def : P(Z) = n(Z) \div n(S) \quad n(Z) = n(X_B) \;\; n(S) = n(S_B) \;\; P(Z) = P(X_B)}{P(X_B) = n(X_B) \div n(S_B)}\,(= Sub)$$

$$\dfrac{\dfrac{P(X_B) = n(X_B) \div n(S_B) \quad n(X_B) = 1 \;\; n(S_B) = 5}{P(X_B) = 1 \div 5}\,(= Sub)}{P(X_B) = 1/5}\,(OI)$$

$$\dfrac{\dfrac{P(X_A) = 3/5 \quad 3/5 > 1/5}{P(X_A) > 1/5}\,(TL) \qquad \dfrac{P(X_B) = 1/5}{1/5 = P(X_B)}\,(SL)}{D_2 : P(X_A) > P(X_B)}\,(TL)$$

This proves D_2 as the correct conclusion.

We similarly proved all of the test problems by mathematically deriving the correct answers from the suppositions, using propositional or predicate logic. The results showed all of the test problems to be free from contradiction and demonstrated their correct inference processes. By similarly representing the inference processes performed by the children, it was then possible to obtain a clear comparison between the correct reasoning based on probability definitions and the children's reasoning based on theorems of their own making.

(4) Children tested The tests were administered to children in the fifth and sixth grades of elementary schools. The sixth graders had been schooled in unit-element ratios and the fifth graders had not. The number of children in each test category was as follows.

Ratio-related conceptual knowledge in the ratio context
125 5^{th} graders, 129 6^{th} graders, 254 total
Ratio-related conceptual knowledge in the comparative-quantity context
117 5^{th} graders, 114 6^{th} graders, 231 total
Ratio-related conceptual knowledge in the base-quantity context
144 5^{th} graders, 139 6^{th} graders, 283 total
Ratio-related procedural knowledge in the ratio context
214 5^{th} graders, 229 6^{th} graders, 443 tota
Ratio-related procedural knowledge in the comparative-quantity context
188 5^{th} graders, 203 6^{th} grader, 391 total
Ratio-related procedural knowledge in the base-quantity context
207 5^{th} graders, 220 6^{th} graders, 427 total

3 Analysis of test results

(1) Mathematical explication of children's inference processes We listed the test problems in order from high to low correct-answer rate and analyzed the children's protocols. As a result, we found that the children's manner of reasoning was characteristic and that because they consistently used the same manner of reasoning it tended to be applicable only to specific problems. As shown in Table 13, we therefore added symbols relating to determinations based on half (1/2) as the basis/standard and then performed the symbolization of inferences seen in classic child protocols to obtain a mathematical explication of the children's manner of reasoning. We also performed level and stage categorization, with structural and qualitative changes in the children's manner of reasoning taken as changes of level and changes of stages within levels, respectively. For integrated analysis relating to the two types of ratio-related knowledge and the three contexts, we extracted the children-specific manner of reasoning as reasoning that is central to the reasoning of children.

In the following, we show typical examples of our symbolization of inferences made by the children and the related level and stage categories for several test problems on ratio-related conceptual knowledge in the ratio context. In these examples, we refer to correct conclusions derived from the suppositions as "correct answers" and answers derived by the children simply as "conclusions", and highlight the children-specific reasoning in inference schemata.

$W(z)$	$P(z) > 1/2$
$L(z)$	$P(z) < 1/2$
$H(z)$	$P(z) = 1/2$

Table 13: Determinations from base 1/2

1) Level 0

Question 2	
Supposition	A_1, B_1, C_1
Correct answer	D_1
Conclusion	D_3
Correct or Incorrect	Incorrect

2) Level 1, Stage 1A

Question 2	
Supposition	A_1, B_1, C_1
Correct answer	D_1
Conclusion	D_1
Correct or Incorrect	correct
$\dfrac{B_1 : n(X_A) = n(X_B) \quad \boldsymbol{n(X_A) = n(X_B) \to P(X_A) = P(X_B)}}{D_1 : P(X_A) = P(X_B)} \; (\to Rem)$	
Question 6	
Supposition	$\neg A_1, B_1, C_2$
Correct answer	D_3

Conclusion	D_1
Correct or Incorrect	Incorrect

$$\frac{B_1 : n(X_A) = n(X_B) \qquad \boldsymbol{n(X_A) = n(X_B) \to P(X_A) = P(X_B)}}{D_1 : P(X_A) = P(X_B)}(\to Rem)$$

3) Level 1, Stage 1B

Question 6	
Supposition	$\neg A_1, B_1, C_2$
Correct answer	D_3
Conclusion	D_3
Correct or Incorrect	correct

$$\frac{\dfrac{\dfrac{B_1 \quad C_2}{B_1 \wedge C_2}(\wedge Int)}{C_2 : n(Y_A) > n(Y_B)}(\wedge Rem) \qquad \boldsymbol{n(Y_A) > n(Y_B) \to P(X_A) < P(X_B)}}{D_3 : P(X_A) < P(X_B)}(\to Rem)$$

Question 4	
Supposition	$\neg A_1, B_2, C_2$
Correct answer	D_1, D_2, D_3
Conclusion	D_2
Correct or Incorrect	Incorrect

$$\frac{\dfrac{\dfrac{B_2 \quad C_2}{B_2 \wedge C_2}(\wedge Int)}{B_2 : n(X_A) > n(X_B)}(\wedge Rem) \qquad \boldsymbol{n(X_A) > n(X_B) \to P(X_A) > P(X_B)}}{D_2 : P(X_A) > P(X_B)}(\to Rem)$$

4) Level 2

Question 4	
Supposition	$\neg A_1, B_2, C_2$
Correct answer	D_1, D_2, D_3
Conclusion	D_1, D_2, D_3
Correct or Incorrect	correct

$n(X_A) = 3, n(X_B) = 2, n(Y_A) = 3, n(Y_B) = 2 \cdots (1)$

In the following inference schemata, [1] should be replaced with (1), excluding the commas.

$$\frac{[1] \qquad 3 > 2 \qquad 3 > 2 \qquad 3/(3+3) = 2/(2+2)}{\begin{aligned}\exists x \exists y \exists z \exists w[&n(X_A) = x \wedge n(X_B) = y \wedge x > y \\ &\wedge n(Y_A) = z \wedge n(Y_B) = w \wedge z > w \\ &\wedge P(X_A) = x/(x+z) \wedge P(X_B) = y/(y+w) \wedge x/(x+z) = y/(y+w)] \cdots (a)\end{aligned}}(\exists\exists Int)$$

$n(X_A) = 6, n(X_B) = 2, n(Y_A) = 3, n(Y_B) = 2 \cdots (2)$

In the following inference schemata, [2] should be replaced with (2), excluding the commas.

$$\frac{[2] \quad 6>2 \quad 3>2 \quad 6/(6+3)>2/(2+2)}{\begin{array}{l}\exists x'\exists y'\exists z'\exists w'[n(X_A)=x' \wedge n(X_B)=y' \wedge x'>y' \\ \quad \wedge n(Y_A)=z' \wedge n(Y_B)=w' \wedge z'>w' \\ \quad \wedge P(X_A)=x'/(x'+z') \wedge P(X_B)=y'/(y'+w') \wedge x'/(x'+z')>y'/(y'+w')] \cdots (b)\end{array}} (\exists\exists Int)$$

$n(X_A)=6, n(X_B)=4, n(Y_A)=6, n(Y_B)=2 \cdots (3)$

In the following inference schemata, [3] should be replaced with (3), excluding the commas.

$$\frac{[3] \quad 6>4 \quad 6>2 \quad 6/(6+6)<4/(4+2)}{\begin{array}{l}\exists x''\exists y''\exists z''\exists w''[n(X_A)=x'' \wedge n(X_B)=y'' \wedge x''>y'' \\ \quad \wedge n(Y_A)=z'' \wedge n(Y_B)=w'' \wedge z''>w'' \\ \quad \wedge P(X_A)=x''/(x''+z'') \wedge P(X_B)=y''/(y''+w'') \wedge x''/(x''+z'')<y''/(y''+w'')] \cdots (c)\end{array}} (\exists\exists Int)$$

$$\frac{(a) \quad (b) \quad (c)}{\begin{array}{l}(\exists x\exists y\exists z\exists w[n(X_A)=x \wedge n(X_B)=y \wedge x>y \\ \quad \wedge n(Y_A)=z \wedge n(Y_B)=w \wedge z>w \\ \quad \wedge P(X_A)=x/(x+z) \wedge P(X_B)=y/(y+w) \wedge x/(x+z)=y/(y+w)]) \\ \wedge(\exists x'\exists y'\exists z'\exists w'[n(X_A)=x' \wedge n(X_B)=y' \wedge x'>y' \\ \quad \wedge n(Y_A)=z' \wedge n(Y_B)=w' \wedge z'>w' \\ \quad \wedge P(X_A)=x'/(x'+z') \wedge P(X_B)=y'/(y'+w') \wedge x'/(x'+z')>y'/(y'+w')]) \\ \wedge(\exists x''\exists y''\exists z''\exists w''[n(X_A)=x'' \wedge n(X_B)=y'' \wedge x''>y'' \\ \quad \wedge n(Y_A)=z'' \wedge n(Y_B)=w'' \wedge z''>w'' \\ \quad \wedge P(X_A)=x''/(x''+z'') \wedge P(X_B)=y''/(y''+w'') \wedge x''/(x''+z'')<y''/(y''+w'')])\end{array}} (\wedge\wedge\, Int)$$

The processes of inference in children unschooled in probability are not based on an explicit definition of probability. In their inference processes, leaps therefore tend to occur due to children-specific reasoning. The children's reasoning sequences $\boldsymbol{n(X_A)=n(X_B)\rightarrow P(X_A)=P(X_B)}$ in Level 1 Stage 1A and $\boldsymbol{n(Y_A)>n(Y_B)\rightarrow P(X_A)<P(X_B)}$ in Level 1 Stage 1B generally hold in cases where $n(S_A)=n(S_B)$, but it appears that they were also excessively applied in cases where $n(S_A)\neq n(S_B)$. The children at Level 2 apparently focused on $n(X_A), n(X_B), n(Y_A)$, and $n(Y_B)$, and derived inference schema conclusion (a) based on the following manner of reasoning.

$n(X_A)=3, n(X_B)=2, n(Y_A)=3, n(Y_B)=2$

$$\frac{\dfrac{n(X_A)=3 \quad 3>2}{n(X_A)>2}(TL) \qquad \dfrac{n(X_B)=2}{2=n(X_B)}(SL)}{B_2: n(X_A)>n(X_B)}(TL)$$

$$\frac{\dfrac{n(Y_A)=3 \quad 3>2}{n(Y_A)>2}(TL) \qquad \dfrac{n(Y_B)=2}{2=n(Y_B)}(SL)}{C_2: n(Y_A)>n(Y_B)}(TL)$$

$$\frac{\dfrac{\dfrac{n(X_A)=3 \quad n(Y_A)=3}{n(X_A)\div n(Y_A)=3\div 3}(==)}{n(X_A)\div n(Y_A)=1}(OI) \qquad \dfrac{\dfrac{\dfrac{n(X_B)=2 \quad n(Y_B)=2}{n(X_B)\div n(Y_B)=2\div 2}(==)}{n(X_B)\div n(Y_B)=1}(OI)}{1=n(X_B)\div n(Y_B)}(SL)}{n(X_A)\div n(Y_A)=n(X_B)\div n(Y_B)}(TL)$$

$$\dfrac{n(X_A)\div n(Y_A)=n(X_B)\div n(Y_B) \qquad \boldsymbol{n(X_A)\div n(Y_A)=n(X_B)\div n(Y_B)\rightarrow P(X_A)=P(X_B)}}{D_1 : P(X_A)=P(X_B)}\ (\rightarrow Rem)$$

The Level-2 children's reasoning, $\boldsymbol{n(X_A)\div n(Y_A)=n(X_B)\div n(Y_B)\rightarrow P(X_A)=P(X_B)}$, is not correct in terms of probability. It does have a certain generality, as in this reasoning the ratio $n(X_A)$ to $n(Y_A)$ extended to the ratio $n(X_A)$ to $n(S_A)$ and the ratio $n(X_B)$ to $n(Y_B)$ extended to the ratio $n(X_B)$ to $n(S_B)$. It is accordingly a mathematically correct concept in special cases, but its generality is not guaranteed. In the following, we show in terms of propositional logic the process of obtaining $\boldsymbol{n(X_A)\div n(Y_A)=n(X_B)\div n(Y_B)\rightarrow P(X_A)=P(X_B)}$.

$(OI)^{*}1$	$1\div(n(X_A)\div n(Y_A))=1\times n(Y_A)\div(n(X_A)\div n(Y_A)\times n(Y_A))$ $=n(Y_A)\div n(X_A)$

$(OI)^{*}2$	$1+n(Y_A)\div n(X_A)=n(X_A)\div n(X_A)+n(Y_A)\div n(X_A)$ $=(n(X_A)+n(Y_A))\div n(X_A)$

$(OI)^{*}3$	$n(X_A)\times(n(X_A)+n(Y_A))\div n(X_A)=(n(X_A)+n(Y_A))\times n(X_A)\div n(X_A)$ $=(n(X_A)+n(Y_A))\times 1$ $=n(X_A)+n(Y_A)$

$$\dfrac{\dfrac{\dfrac{1=1 \qquad \dfrac{\dfrac{n(X_A)=3 \quad n(Y_A)=3}{n(X_A)\div n(Y_A)=3\div 3}\,(==)}{n(X_A)\div n(Y_A)=1}\,(OI)}{1\div(n(X_A)\div n(Y_A))=1\div 1}\,(==)}{1\div(n(X_A)\div n(Y_A))=1}\,(OI)}{n(Y_A)\div n(X_A)=1}\,(OI)^{*}1$$

$$\dfrac{\dfrac{1=1 \quad n(Y_A)\div n(X_A)=1}{1+n(Y_A)\div n(X_A)=1+1}\,(==)}{(n(X_A)+n(Y_A))\div n(X_A)=1+1}\,(OI)^{*}2$$

$$\dfrac{\dfrac{\dfrac{n(X_A)=3 \quad (n(X_A)+n(Y_A))\div n(X_A)=1+1}{n(X_A)\times(n(X_A)+n(Y_A))\div n(X_A)=3\times(1+1)}\,(==)}{n(X_A)\times(n(X_A)+n(Y_A))\div n(X_A)=3+3}\,(OI)}{n(X_A)+n(Y_A)=3+3}\,(OI)^{*}3$$

$$\dfrac{Thm5 : n(S)=n(X)+n(Y) \quad n(X)=n(X_A) \quad n(Y)=n(Y_A) \quad n(S)=n(S_A)}{n(S_A)=n(X_A)+n(Y_A)}\,(=Sub)$$

$$\dfrac{n(X_A)=3 \qquad \dfrac{n(X_A)+n(Y_A)=3+3 \qquad \dfrac{n(S_A)=n(X_A)+n(Y_A)}{n(X_A)+n(Y_A)=n(S_A)}\,(SL)}{n(S_A)=3+3}\,(=Sub)}{n(X_A)\div n(S_A)=3\div(3+3)}\,(==)$$

$$\dfrac{Def : P(Z)=n(Z)\div n(S) \quad n(Z)=n(X_A) \quad n(S)=n(S_A) \quad P(Z)=P(X_A)}{P(X_A)=n(X_A)\div n(S_A)}\,(=Sub)$$

$$\dfrac{\dfrac{n(X_A)\div n(S_A)=3\div(3+3) \qquad \dfrac{P(X_A)=n(X_A)\div n(S_A)}{n(X_A)\div n(S_A)=P(X_A)}\,(SL)}{P(X_A)=3\div(3+3)}\,(=Sub)}{P(X_A)=1/2}\,(OI)$$

$$\dfrac{\dfrac{\dfrac{1=1 \qquad \dfrac{\dfrac{n(X_B)=2 \quad n(Y_B)=2}{n(X_B)\div n(Y_B)=2\div 2}\,(==)}{n(X_B)\div n(Y_B)=1}\,(OI)}{1\div(n(X_B)\div n(Y_B))=1\div 1}\,(==)}{1\div(n(X_B)\div n(Y_B))=1}\,(OI)}{n(Y_B)\div n(X_B)=1}\,(OI)^{*}1$$

$$\dfrac{\dfrac{1=1 \quad n(Y_B)\div n(X_B)=1}{1+n(Y_B)\div n(X_B)=1+1}\,(==)}{(n(X_B)+n(Y_B))\div n(X_B)=1+1}\,(OI)^{*}2$$

$$\dfrac{\dfrac{\dfrac{n(X_B)=2 \quad (n(X_B)+n(Y_B))\div n(X_B)=1+1}{n(X_B)\times(n(X_B)+n(Y_B))\div n(X_B)=2\times(1+1)}\,(==)}{n(X_B)\times(n(X_B)+n(Y_B))\div n(X_B)=2+2}\,(OI)}{n(X_B)+n(Y_B)=2+2}\,(OI)^{*}3$$

$$\dfrac{Thm5: n(S)=n(X)+n(Y) \quad n(X)=n(X_B) \;\; n(Y)=n(Y_B) \;\; n(S)=n(S_B)}{n(S_B)=n(X_B)+n(Y_B)}\,(=Sub)$$

$$\dfrac{n(X_B)=2 \qquad \dfrac{n(X_B)+n(Y_B)=2+2 \quad \dfrac{n(S_B)=n(X_B)+n(Y_B)}{n(X_B)+n(Y_B)=n(S_B)}\,(SL)}{n(S_B)=2+2}\,(=Sub)}{n(X_B)\div n(S_B)=2\div(2+2)}\,(==)$$

$$\dfrac{Def: P(Z)=n(Z)\div n(S) \quad n(Z)=n(X_B) \;\; n(S)=n(S_B) \;\; P(Z)=P(X_B)}{P(X_B)=n(X_B)\div n(S_B)}\,(=Sub)$$

$$\dfrac{\dfrac{n(X_B)\div n(S_B)=2\div(2+2) \qquad \dfrac{P(X_B)=n(X_B)\div n(S_B)}{n(X_B)\div n(S_B)=P(X_B)}\,(SL)}{P(X_B)=2\div(2+2)}\,(=Sub)}{P(X_B)=1/2}\,(OI)$$

$$\dfrac{P(X_A)=1/2 \quad \dfrac{P(X_B)=1/2}{1/2=P(X_B)}\,(SL)}{D1: P(X_A)=P(X_B)}\,(TL)$$

(2) Children-specific reasoning As a result of the symbolization of the inferences performed by the children for all of the test problems and their level and stage classification as shown in Tables 14 and 15, we found children-specific reasoning to be present in all levels and stages. Tables 16 and 17 provide a summary of the children-specific reasoning extracted from the children's manner of reasoning at each level and stage, and the correct answers based on probabilistic definitions.

Level	Stage	Ratio	Comparative quantity	Base quantity
0				
1	1A	Question 2 Question 1 Question 3 Question 5	Question 2 Question 1	Question 2 Question 1
	1B	Question 6	Question 6 Question 4	Question 6 Question 3
2		Question 4	Question 5 Question 3	Question 5 Question 4

Table 14: Levels and stages of ratio-related conceptual knowledge

Level	Stage	Ratio	Comparative quantity	Base quantity
0				
1	1A	Question 1 Question 4	Question 1 Question 4 Question 11 Question 9 Question 10 Question 8 Question 7	Question 2 Question 5 Question 7 Question 11 Question 8 Question 9 Question 10 Question 12
	1B	Question 6 Question 3 Question 12	Question 2 Question 5	Question 3 Question 6
	1C	Question 2 Question 8 Question 9 Question 7		
2		Question 5 Question 10 Question 11	Question 6 Question 12 Question 3	Question 1 Question 4

Table 15: Levels and stages of ratio-related procedural knowledge

Level	Stage	Ratio	Comparative quantity	Base quantity
0				
1	1A	$\cdot n(X_A) = n(X_B) \to$ $P(X_A) = P(X_B)$ $\cdot n(X_A) > n(X_B) \to$ $P(X_A) > P(X_B)$	$\cdot n(Y_A) = n(Y_B) \to$ $n(X_A) = n(X_B)$ $\cdot n(Y_A) < n(Y_B) \to$ $n(X_A) > n(X_B)$	$\cdot n(X_A) = n(X_B) \to$ $n(Y_A) = n(Y_B)$ $\cdot n(X_A) > n(X_B) \to$ $n(Y_A) < n(Y_B)$
	1B	$\cdot n(X_A) = n(X_B) \to$ $n(Y_A) > n(Y_B) \to$ $P(X_A) < P(X_B)$	$\cdot n(Y_A) = n(Y_B) \to$ $P(X_A) > P(X_B) \to$ $n(X_A) > n(X_B)$	$\cdot n(X_A) = n(X_B) \to$ $P(X_A) < P(X_B) \to$ $n(Y_A) > n(Y_B)$

2		$\cdot n(X) = n(Y) \to H(X)$ $\cdot n(X) > n(Y) \to W(X)$ $\cdot n(X) < n(Y) \to L(X)$ $\cdot H(X_A) \wedge H(X_B) \to$ $P(X_A) = P(X_B)$ $\cdot W(X_A) \wedge L(X_B) \to$ $P(X_A) > P(X_B)$ $\cdot L(X_A) \wedge W(X_B) \to$ $P(X_A) < P(X_B)$	$\cdot n(X) = n(Y) \to H(X)$ $\cdot n(X) > n(Y) \to W(X)$ $\cdot n(X) < n(Y) \to L(X)$ $\cdot H(X_A) \wedge H(X_B) \to$ $P(X_A) = P(X_B)$ $\cdot L(X_A) \wedge W(X_B) \to$ $P(X_A) < P(X_B)$	$\cdot n(X) = n(Y) \to H(X)$ $\cdot n(X) > n(Y) \to W(X)$ $\cdot n(X) < n(Y) \to L(X)$ $\cdot H(X_A) \wedge H(X_B) \to$ $P(X_A) = P(X_B)$ $\cdot W(X_A) \wedge L(X_B) \to$ $P(X_A) > P(X_B)$
		$\cdot n(X_A) \div n(Y_A) =$ $n(X_B) \div n(Y_B) \to$ $P(X_A) = P(X_B)$ $\cdot n(X_A) \div n(Y_A) >$ $n(X_B) \div n(Y_B) \to$ $P(X_A) > P(X_B)$ $\cdot n(X_A) \div n(Y_A) <$ $n(X_B) \div n(Y_B) \to$ $P(X_A) < P(X_B)$ $\cdot n(X_A) \div n(S_A) =$ $n(X_B) \div n(S_B) \to$ $P(X_A) = P(X_B)$ $\cdot n(X_A) \div n(S_A) >$ $n(X_B) \div n(S_B) \to$ $P(X_A) > P(X_B)$ $\cdot n(X_A) \div n(S_A) <$ $n(X_B) \div n(S_B) \to$ $P(X_A) < P(X_B)$	$\cdot n(X_A) \div n(Y_A) =$ $n(X_B) \div n(Y_B) \to$ $P(X_A) = P(X_B)$ $\cdot n(X_A) \div n(Y_A) <$ $n(X_B) \div n(Y_B) \to$ $P(X_A) < P(X_B)$ $\cdot n(S_A) \times P(X_A) >$ $n(S_B) \times P(X_B) \to$ $n(X_A) > n(X_B)$	$\cdot n(X_A) \div n(Y_A) =$ $n(X_B) \div n(Y_B) \to$ $P(X_A) = P(X_B)$ $\cdot n(X_A) \div n(Y_A) >$ $n(X_B) \div n(Y_B) \to$ $P(X_A) > P(X_B)$ $\cdot n(S_A) \times P(Y_A) >$ $n(S_B) \times P(Y_B) \to$ $n(Y_A) > n(Y_B)$

Table 16: Children's reasoning related to conceptual knowledge

Level	Stage	Ratio	Comparative quantity	Base quantity
0		$\cdot n(S_A) = n(S_B) \to$ $P(X_A) = P(X_B)$	$\cdot n(S_A) = n(S_B) \to$ $n(X_A) = n(X_B)$	$\cdot P(X_A) = P(X_B) \to$ $n(S_A) = n(S_B)$
1	1A	$\cdot n(X_A) = (nX_B) \to$ $P(X_A) = P(X_B)$ $\cdot n(X_A) > n(X_B) \to$ $P(X_A) > P(X_B)$	$\cdot P(X_A) = P(X_B) \to$ $n(X_A) = n(X_B)$ $\cdot P(X_A) > P(X_B) \to$ $n(X_A) > n(X_B)$ $\cdot P(X_A) < P(X_B) \to$ $n(X_A) < n(X_B)$	$\cdot n(X_A) = n(X_B) \to$ $n(S_A) = n(S_B)$ $\cdot n(X_A) > n(X_B) \to$ $n(S_A) > n(S_B)$ $\cdot n(X_A) < n(X_B) \to$ $n(S_A) < n(S_B)$
	1B	$\cdot n(X_A) = (nX_B) \to$ $n(S_A) > n(S_B) \to$ $P(X_A) < P(X_B)$ $\cdot n(X_A) = (nX_B) \to$ $n(S_A) < n(S_B) \to$ $P(X_A) > P(X_B)$	$\cdot P(X_A) = P(X_B) \to$ $n(S_A) < n(S_B) \to$ $n(X_A) < n(X_B)$	$\cdot n(X_A) = n(X_B) \to$ $P(X_A) > P(X_B) \to$ $n(S_A) < n(S_B)$

	1B	$\cdot n(Y_A) > n(Y_B) \rightarrow$ $P(X_A) < P(X_B)$ $\cdot n(Y_A) < n(Y_B) \rightarrow$ $P(X_A) > P(X_B)$		
	1C	$\cdot n(X) = n(Y) \rightarrow H(X)$ $\cdot n(X) > n(Y) \rightarrow W(X)$ $\cdot n(X) < n(Y) \rightarrow L(X)$ $\cdot H(X_A) \wedge H(X_B) \rightarrow$ $P(X_A) = P(X_B)$ $\cdot L(X_A) \wedge H(X_B) \rightarrow$ $P(X_A) < P(X_B)$ $\cdot H(X_A) \wedge W(X_B) \rightarrow$ $P(X_A) < P(X_B)$		
		$\cdot n(X_A) - n(Y_A) <$ $\cdot n(X_B) - n(Y_B) \rightarrow$ $P(X_A) < P(X_B)$		
2		$\cdot n(S_A) \times a = n(S_B) \times b$ $\rightarrow n(X_A) \times a = n(X_B) \times b$ $\rightarrow P(X_A) = P(X_B)$ $\cdot n(S_A) \times a = n(S_B) \times b$ $\rightarrow n(X_A) \times a < n(X_B) \times b$ $\rightarrow P(X_A) < P(X_B)$		
		$\cdot n(X_A) \div n(S_A) =$ $n(X_B) \div n(S_B) \rightarrow$ $P(X_A) = P(X_B)$ $\cdot n(X_A) \div n(S_A) <$ $n(X_B) \div n(S_B) \rightarrow$ $P(X_A) < P(X_B)$	$\cdot n(S_A) \times P(X_A) =$ $n(S_B) \times P(X_B) \rightarrow$ $n(X_A) = n(X_B)$ $\cdot n(S_A) \times P(X_A) >$ $n(S_B) \times P(X_B) \rightarrow$ $n(X_A) > n(X_B)$	$\cdot n(X_A) \div P(X_A) =$ $n(X_B) \div P(X_B) \rightarrow$ $n(S_A) = n(S_B)$

Table 17: Children's reasoning related to procedural knowledge

4 Discussion In reasoning, children consider relations between two sets and relations within a set, which we refer to here as "Between" and "Within" relations, respectively. For Between relations, such as that of $n(X_A)$ and $n(X_B)$, they consider the relation between two quantities with each occurring in a different set. For Within relations, such as that of $n(X_A)$ and $n(Y_A)$, they consider the relation between two quantities occurring in the same set.

In comparing the children's reasoning processes, as shown in Tables 16 and 17, we found that additive reasoning (including size comparison) for Between relations and multiplicative reasoning for Within relations occur in relation to both conceptual knowledge and procedural knowledge in all three of the contexts, and that additive reasoning for the Between relation precedes multiplicative reasoning for the Within relation. We found additive reasoning (including size comparison) for the Within relation to occur consistently in relation to ratio-related conceptual knowledge in all three contexts. In relation to ratio-related procedural knowledge in the ratio context, we found additive reasoning (including size comparison) for the Within relation and multiplicative reasoning for the Between relation, again with additive reasoning for the Within relation preceding multiplicative reasoning for the Between relation. These findings indicate that the Between relation is easier for children to recognize than the Within relation, and that additive reasoning is easier than multiplicative reasoning. They also indicate that the transitions in reasoning proceed from additive reasoning for the Between relation to additive reasoning for the Within relation, to multiplicative reasoning for the Between relation, and finally to multiplicative reasoning for the Within relation. Additive reasoning for the Within relation was found to involve the use of half as a basis strategy. This is in accord with the findings in studies made to present on the stages of children's knowledge and development in proportional reasoning.

In cases where the number of winning lots and total number of lots in two sets were in a double-half (1/2) relation, some of the children considered the related numbers and performed inferences based on multiplicative reasoning for the Between relation. Even in problems containing no explicit numbers, some of the children on their own initiative set up actual numbers that were in the double-half (1/2) relation, e.g., (4, 2), (6, 3), (8, 4), and (10, 5), for the number of winning and losing lots and performed their inferences based on multiplicative reasoning for the Within relation. In these cases, they used half as a ratio rather than as a basis strategy. Their unprompted introduction of the half concept, in any case, clearly suggests that it holds a key role as a prime mover in the transition from additive reasoning in the Within relation to multiplicative reasoning in the Between relation and to multiplicative reasoning in the Within relation.

The occurrence of additive reasoning relating to ratio-related conceptual and procedural knowledge for Between relations and multiplicative reasoning for Within relations in all three contexts indicates that in each of the contexts an association is formed between ratio-related conceptual and procedural knowledge under additive reasoning and the structural change in the manner of thinking then leads to a formation of a new association under multiplicative reasoning. The structural change is a basic change from an additive to a multiplicative algebraic structure that is the foundation of the children's manner of reasoning and corresponds to a structural change in level. The emergence of Additive reasoning for Within relations can also be regarded as a qualitative change from the Between relation to the Within relation in additive reasoning, and the emergence of multiplicative reasoning for Between relations can be regarded as a qualitative change from the Between relation to the Within relation in multiplicative reasoning.

We also found an increase from one to two in the number of events considered in additive reasoning for the Between relation, with the proviso that although two events were considered in all three contexts for ratio-related conceptual and procedural knowledge, in those cases where equality was established for one event there was a tendency to perform the determination based only on the other event.

These qualitative changes signify a change in the children's mode of consideration from one event to two and from the Between relation to the Within relation, and correspond to a change in stage. Until the structural change from additive to multiplicative reasoning occurs, children consistently perform inferences based on additive reasoning. In summary, the findings indicate that the three contexts do not become integrated in terms of additive reasoning until after ratio-related conceptual and procedural knowledge become linked in additive reasoning in each of the three.

Additional note

This work was supported by JSPS KAKENHI Grant Number 25381204.

References

[1] J. Hiebert and P. Lefevre, " Conceptual and Procedural Knowledge in Mathematics: An Introductory Analysis " , in J. Hiebert (ed.), Conceptual and Procedural Knowledge: The Case of Mathematics, Hillsdale, N. J.: Lawrence Erlbaum Associates, pp.1-27, 1986.

[2] B. Rittle-Johnson, R. S. Siegler, and N. W. Alibali, " Developing Conceptual Understanding and Procedural Skill in Mathematics: An Iterative Process ", Journal of Educational Psychology, Vol.93, No.2, pp.346-362, 2001.

[3] R. Maeda, "Concept of the Ratio", New Arithmetic Education Lecture Vol. 3: Mathematical Relations, in C. Akahane (ed.), Yoshino-shobo Publishing Ltd, pp.239-247, 1960 (in Japanese).

[4] A. Nakagaki, " How do Children Judge the Size of the Ratio? -The Developmental Study ", Research report of the National Institute for Educational Research, Vol.13, pp.35-55, 1986 (in Japanese).

[5] A. Nakagaki, "Dynamism of the Cognitive System in a Ratio-Comparison Problem", Research report of the National Institute for Educational Research, Vol.34, pp.31-51, 1997 (in Japanese).

[6] N. Fujimura, " Formation Process of Intensive Quantity Concepts in School Age Children ", Japanese Journal of Educational Psychology, Vol.38, No.3, pp.277-286, 1990 (in Japanese).

[7] G. Noelting, " The Development of Proportional Reasoning and the Ratio Concept, Part I -Differentiation of Stages ", Educational Studies in Mathematics, Vol.11, pp.217-253, 1980a.

[8] G. Noelting, " The Development of Proportional Reasoning and the Ratio Concept, Part II -Problem-structure at Successive Stages: Problem-solving Strategies and the Mechanism of Adaptive Restructuring ", Educational Studies in Mathematics, Vol.11, pp.331-363, 1980b.

[9] J. Piaget and B. Inhelder, The Origin of the Idea of Chance in Children, Translated by L. Leake, Jr., P. Burrell, and H. D. Fishbein, New York: Norton, 1975.

[10] R. S. Siegler, "Three Aspects of Cognitive Development", Cognitive Psychology, Vol.8, pp.481-520, 1976.

[11] R. S. Siegler, "The Origins of Scientific Reasoning", in R. S. Siegler (ed.), Children's Thinking: What Develop?, Lawrence Erlbaum Associates, Hillsdale, N. J., pp.109-149, 1978.

[12] A. G. Spinillo, " Children's Use of Part-part Comparisons to Estimate Probability ", Journal of Mathematical Behavior, Vol.21, pp.357-369, 2002.

[13] L. O. Lembke and B. J. Reys, " The Development of, and Interaction Between, Intuitive and School-Taught Ideas about Percent", Journal for Research in Mathematics Education, Vol.25, No.3, pp.237-259, 1994.

[14] S. Tamura, K. Aragane, and T. Hirai, The Logic and Thought: Introduction to Symbolic Logic, Osaka-kyoiku-tosho Publishing Ltd, 1997 (in Japanese).

[15] K. Todayama, Learning Logic through Building it, Nagoya University Publishing Ltd, 2000 (in Japanese).

Appendix 1

Example test problem for ratio-related conceptual knowledge in the comparative quantity context

Sample question

In this lot drawing, some of the lots are winning lots and some of them are losing lots. There are two groups of lots. Lots from one group are called " A lots " and lots from the other group are called " B lots ". Both groups include winning lots and losing lots. The " total number of lots " in one group means all the winning and losing lots in that group. If a winning lot is easy to draw, we call the group an " easy winner ".

The total number of A lots is the same as the total number of B lots.
There are more losing B lots than losing A lots.
If just one lot is drawn, it is easier to win with an A lot than with a B lot. (Supposition)

Which of the A lots or the B lots have a larger number of winning lots, or is it the same for the A lots and B lots? Draw a circle in the box above any of the following answers that you think may be correct. Note that in some questions, a circle can be drawn in all of the boxes.

☐	☐	☐
There are more winning A lots.	No difference between the A lots and B lots.	There are more winning B lots.

(Conclusion)

Test problem suppositions and correct conclusions for ratio-related conceptual knowledge in the comparative quantity context

	Supposition	Correct conclusion
Question 1	A_1, C_3, D_2	B_2
Question 2	A_1, C_1, D_1	B_1
Question 3	$\neg A_1, C_2, D_3$	B_1, B_2, B_3
Question 4	$\neg A_1, C_2, D_2$	B_2
Question 5	$\neg A_1, C_2, D_1$	B_2
Question 6	$\neg A_1, C_1, D_2$	B_2

Appendix 2
Example test problem for ratio-related conceptual knowledge in the base quantity context

Sample question

In this lot drawing, some of the lots are winning lots and some of them are losing lots. There are two groups of lots. Lots from one group are called “ A lots ” and lots from the other group are called “ B lots ”. Both groups include winning lots and losing lots. The “ total number of lots ” in one group means all the winning and losing lots in that group. If a winning lot is easy to draw, we call the group an “ easy winner ”.

The total number of A lots is the same as the total number of B lots.
There are more winning A lots than winning B lots.
If just one lot is drawn, it is easier to win with an A lot than with a B lot. (Supposition)

Which of the A lots or the B lots have a larger number of losing lots, or is it the same for the A lots and B lots? Draw a circle in the box above any of the following answers that you think may be correct. Note that in some questions, a circle can be drawn in all of the boxes.

☐	☐	☐
There are more losing A lots.	No difference between the A lots and B lots.	There are more losing B lots.

(Conclusion)

Test problem suppositions and correct conclusions for ratio-related conceptual knowledge in the base quantity context

	Supposition	Correct conclusion
Question 1	A_1, B_2, D_2	C_3
Question 2	A_1, B_1, D_1	C_1
Question 3	$\neg A_1, B_2, D_3$	C_2
Question 4	$\neg A_1, B_2, D_2$	C_1, C_2, C_3
Question 5	$\neg A_1, B_2, D_1$	C_2
Question 6	$\neg A_1, B_1, D_3$	C_2

Appendix 3
Example problem for ratio-related procedural knowledge in the comparative quantity context

Sample question

In this lot drawing, some of the lots are winning lots and some of them are losing lots. There are two groups of lots. Lots from one group are called “ A lots ” and lots from the other group are called “ B lots ”. Both groups include winning lots and losing lots. The “ total number of lots ” in one group means all the winning and losing lots in that group. We call how easy it is to draw a winning lot “ chance of winning ”. If chance of winning is high, we call the group an “ easy winner ”.

The total number of A lots is 5, and chance of winning is 0.6.
The total number of B lots is 5, and chance of winning is 0.2.

(Supposition)

Which of the A lots or the B lots have a larger number of winning lots, or is it the same for the A lots and B lots? Draw a circle in the box above any of the following answers that you think may be correct.

☐	☐	☐
There are more winning A lots.	No difference between the A lots and B lots.	There are more winning B lots.

(Conclusion)

Test problem suppositions and correct conclusions for ratio-related procedural knowledge in the comparative quantity context

	Supposition	Correct conclusion
Question 1	$n(S_A) = 5, n(S_B) = 5, P(X_A) = 0.6, P(X_B) = 0.2$	B_2
Question 2	$n(S_A) = 2, n(S_B) = 6, P(X_A) = 0.5, P(X_B) = 0.5$	B_3
Question 3	$n(S_A) = 4, n(S_B) = 5, P(X_A) = 0.75, P(X_B) = 0.6$	B_1
Question 4	$n(S_A) = 4, n(S_B) = 4, P(X_A) = 0.25, P(X_B) = 0.75$	B_3
Question 5	$n(S_A) = 4, n(S_B) = 8, P(X_A) = 0.75, P(X_B) = 0.75$	B_3
Question 6	$n(S_A) = 4, n(S_B) = 5, P(X_A) = 0.5, P(X_B) = 0.4$	B_1
Question 7	$n(S_A) = 2, n(S_B) = 5, P(X_A) = 0.5, P(X_B) = 0.8$	B_3
Question 8	$n(S_A) = 4, n(S_B) = 6, P(X_A) = 0.25, P(X_B) = 0.5$	B_3
Question 9	$n(S_A) = 4, n(S_B) = 5, P(X_A) = 0.5, P(X_B) = 0.6$	B_3
Question 10	$n(S_A) = 8, n(S_B) = 10, P(X_A) = 0.25, P(X_B) = 0.3$	B_3
Question 11	$n(S_A) = 4, n(S_B) = 5, P(X_A) = 0.75, P(X_B) = 0.8$	B_3
Question 12	$n(S_A) = 10, n(S_B) = 6, P(X_A) = 0.4, P(X_B) = 0.5$	B_2

Appendix 4
Example problem for ratio-related procedural knowledge in the base quantity context

Sample question

In this lot drawing, some of the lots are winning lots and some of them are losing lots. There are two groups of lots. Lots from one group are called "A lots" and lots from the other group are called "B lots". Both groups include winning lots and losing lots. The "total number of lots" in one group means all the winning and losing lots in that group. We call how easy it is to draw a winning lot "chance of winning". If chance of winning is high, we call the group an "easy winner".

Chance of winning of an A lot is 0.6, and the number of winning lots is 3.
Chance of winning of a B lot is 0.2, and the number of winning lots is 1.

(Supposition)

Which of the A lots or the B lots have a larger total number of lots, or is it the same for the A lots and B lots? Draw a circle in the box above any of the following answers that you think may be correct.

☐	☐	☐
The total number of A lots is larger.	No difference between the A lots and B lots.	The total number of B lots is larger.

(Conclusion)

Test problem suppositions and correct conclusions for ratio-related procedural knowledge in the base quantity context

	Supposition	Correct conclusion
Question 1	$P(X_A) = 0.6, P(X_B) = 0.2, n(X_A) = 3, n(X_B) = 1$	A_1
Question 2	$P(X_A) = 0.5, P(X_B) = 0.5, n(X_A) = 1, n(X_B) = 3$	A_3
Question 3	$P(X_A) = 0.75, P(X_B) = 0.6, n(X_A) = 3, n(X_B) = 3$	A_3
Question 4	$P(X_A) = 0.25, P(X_B) = 0.75, n(X_A) = 1, n(X_B) = 3$	A_1
Question 5	$P(X_A) = 0.75, P(X_B) = 0.75, n(X_A) = 3, n(X_B) = 6$	A_3
Question 6	$P(X_A) = 0.5, P(X_B) = 0.4, n(X_A) = 2, n(X_B) = 2$	A_3
Question 7	$P(X_A) = 0.5, P(X_B) = 0.8, n(X_A) = 1, n(X_B) = 4$	A_3
Question 8	$P(X_A) = 0.25, P(X_B) = 0.5, n(X_A) = 1, n(X_B) = 3$	A_3
Question 9	$P(X_A) = 0.5, P(X_B) = 0.6, n(X_A) = 2, n(X_B) = 3$	A_3
Question 10	$P(X_A) = 0.25, P(X_B) = 0.3, n(X_A) = 2, n(X_B) = 3$	A_3
Question 11	$P(X_A) = 0.75, P(X_B) = 0.8, n(X_A) = 3, n(X_B) = 4$	A_3
Question 12	$P(X_A) = 0.4, P(X_B) = 0.5, n(X_A) = 4, n(X_B) = 3$	A_2

Communicated by *Tadashi Takahash*

Takeshi Sakai
tsakai@naruto-u.ac.jp
Naruto University of Education
Aza-Nakashima, Takashima, Naruto-cho, Naruto, Tokushima 772-8502, Japan

Tadashi Takahashi
takahasi@konan-u.ac.jp
Department of Intelligence and Informatics, Konan University,
8-9-1 Okamoto,Higashinada, Kobe, Hyogo 658-8501, Japan

THE STRUCTURE OF PROJECTION METHODS FOR VARIATIONAL INEQUALITY PROBLEMS AND WEAK CONVERGENCE THEOREMS

RIEKO KUBOTA, WATARU TAKAHASHI, AND YUKIO TAKEUCHI

Received December 24, 2013

Abstract. In this paper, we study the structure of projection methods for variational inequality problems and then prove weak convergence theorems which generalize Takahashi and Toyoda [W. Takahashi and M. Toyoda, Weak convergence theorems for nonepxansive mappings and monotone mappings, J. Optim. Theory Appl. 118 (2003), 417–428] and Nadezhkina and Takahashi [N. Nadezhkina and W. Takahashi, Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. 128 (2006), 191-201]. Our proofs are different from them. Furthermore, using these weak convergence theorems, we obtain some new results.

1. Introduction

Throughout this paper, we denote by R the set of real numbers and by N the set of positive integers. Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C be a non-empty subset of H. Let T be a mapping of C into H. We denote by $F(T)$ the set of fixed points of T and by $A(T)$ the set of attractive points [23] of T, i.e.,

$$F(T) = \{u \in C : Tu = u\},$$
$$A(T) = \{u \in H : \|Tx - u\| \leq \|x - u\|, \ \forall x \in C\}.$$

A mapping $T : C \to H$ is said to be k-Lipschitz continuous if there exists $k > 0$ such that $\|Tx - Ty\| \leq k\|x - y\|$ for all $x, y \in C$. If a mapping $T : C \to H$ is 1-Lipschitz continuous, it is said to be nonexpansive, i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to H$ is called quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - v\| \leq \|x - v\|$ for all $x \in C$ and $v \in F(T)$. We note that the condition $F(T) \subset A(T)$ always holds if T is quasi-nonexpansive. We denote by I the identity mapping on H. A mapping $A : C \to H$ is said to be monotone if $\langle x-y, Ax-Ay \rangle \geq 0$ for all $x, y \in C$. Let $\alpha > 0$. A mapping $A : C \to H$ is said to be α-inverse strongly monotone if $\langle x-y, Ax-Ay \rangle \geq \alpha\|Ax-Ay\|^2$ for all $x, y \in C$. It is obvious that if A is α-inverse strongly monotone, then A is monotone and $1/\alpha$-Lipschitz continuous. In the case $a \in (0, 2\alpha]$, it is known that $I - aA$ is nonexpansive. In fact, we have that for any $x, y \in C$

$$\|(I - aA)x - (I - aA)y\|^2 \leq \|x - y\|^2 - a(2\alpha - a)\|Ax - Ay\|^2;$$

2010 *Mathematics Subject Classification.* 47H10, 47H09.

Key words and phrases. Variational inequality problem, extragradient method, attractive point, fixed point, monotone mapping, generalized hybrid mapping.

see, for instance, [21]. Assume that C is non-empty, closed and convex. In this case, for each $x \in H$, there exists a unique $x_0 \in C$ such that $\|x - x_0\| = \min\{\|x - y\| : y \in C\}$. The mapping P_C defined by $P_C x = x_0$ for $x \in H$ is called the metric projection of H onto C. Let C be a subset of a Hilbert space H and let A be a mapping of C into H. We denote by $VI(C, A)$ the set of solutions of the variational inequality for A, i.e.,

$$VI(C, A) = \{x \in C : \langle y - x, Ax \rangle \geq 0, \ \forall y \in C\}.$$

Let C be a closed and convex subset of a n-dimensional Euclidean space R^n. Let A be a monotone and k-Lipschitz continuous mapping of C into R^n with $VI(C, A) \neq \emptyset$. For $a \in (0, 1/k)$, let V_a and U_a be a self–mappings on C defined by

$$V_a x = P_C(I - aA)x, \quad U_a x = P_C(I - aAV_a)x, \quad \forall x \in C.$$

Let $x_1 \in C$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in C such that $y_n = V_a x_n$ and $x_{n+1} = U_a x_n$ for all $n \in N$. This iterative procedure called the extragradient method was introduced by Korplevich [8]. Under these conditions, he proved that both sequences $\{x_n\}$ and $\{y_n\}$ converge to the same point in $VI(C, A)$. In 2003, Takahashi and Toyoda [24] proved the following theorem; also see [7].

Theorem 1.1. *Let C be a closed and convex subset of a Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H. Let $\{a_n\}$ be a sequence in $[c_1, d_1]$ as $0 < c_1 \leq d_1 < 2\alpha$. For each $n \in N$, let V_{a_n} be a mapping of C into itself defined by $V_{a_n} x = P_C(I - a_n A)x$ for all $x \in C$. Let S be a nonexpansive mapping of C into itself. Assume that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[c_2, d_2]$ as $0 < c_2 \leq d_2 < 1$. Let $x_1 \in C$ and let $\{x_n\}$ and $\{y_n\}$ be sequences in C defined by*

$$y_n = V_{a_n} x_n, \qquad x_{n+1} = \alpha_n S V_{a_n} x_n + (1 - \alpha_n) x_n, \quad \forall n \in N.$$

Then $\{x_n\}$ and $\{y_n\}$ converge weakly to a point $u \in F(S) \cap VI(C, A)$.

In 2006, Nadezhkina and Takahashi [17] also proved the following theorem.

Theorem 1.2. *Let C be a closed and convex subset of a Hilbert space H and A be a monotone and k-Lipschitz continuous mapping of C into H. Let $\{a_n\}$ be a sequence in $[c_1, d_1]$ as $0 < c_1 \leq d_1 < 1/k$. For each $n \in N$, let V_{a_n} and U_{a_n} be mappings of C into itself defined by*

$$V_{a_n} x = P_C(I - a_n A)x, \quad U_{a_n} x = P_C(I - a_n A V_{a_n})x, \quad \forall x \in C.$$

Let S be a nonexpansive mapping of C into itself. Assume that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[c_2, d_2]$ as $0 < c_2 \leq d_2 < 1$. Let $x_1 \in C$ and let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences in C defined by

$$y_n = V_{a_n} x_n, \quad z_n = U_{a_n} x_n, \quad x_{n+1} = \alpha_n S U_{a_n} x_n + (1 - \alpha_n) x_n, \quad \forall n \in N.$$

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge weakly to a point $u \in F(S) \cap VI(C, A)$.

Motivated by Takahashi and Toyoda [24] and Nadezhkina and Takahashi [17], we study properties of projection methods for variatinal inequality problems and then prove weak convergence theorems which generalize Theorems 1.1 and 1.2. Though almost all techniques in this paper are in Takahashi and Toyoda [24] and Nadezhkina and Takahashi [16, 17], our proofs are different from them. Our techniques depend on the structure of projection methods for variatinal inequality problems and our class of nonlinear mappings S in Theorems 1.1 and 1.2 is a broad class including nonexpansive mappings. Furthermore, using these weak convergence theorems, we obtain some new results.

2. Preliminaries

Let H be a Hilbert space. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. From [21] we have that for $x, y \in H$ and $\lambda \in R$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.1}$$

We also know that for $x, y, u, v \in H$

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.2}$$

A Hilbert space satisfies Opial's condition [18], that is,

$$\liminf_{n\to\infty} \|x_n - u\| < \liminf_{n\to\infty} \|x_n - v\|$$

if $x_n \rightharpoonup u$ and $u \neq v$; see [18]. Let C be a non-empty subset of H. A mapping $T : C \to H$ is called firmly nonexpansive if $\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle$ for all $x, y \in C$. If a mapping T is firmly nonexpansive, then it is nonexpansive. If $T : C \to H$ is nonexpansive, then $F(T)$ is closed and convex; see [21]. We also know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$$

for all $x, y \in H$. Furthermore, $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$. This inequality is equivalent to

$$\|x - P_C x\|^2 + \|y - P_C x\|^2 \leq \|x - y\|^2 \tag{2.3}$$

for all $x \in H$ and $y \in C$; see, for instance, [20]. Recently, many researchers considered broad classes of nonlinear mappings which contain nonexpansive mappings. Kocourek, Takahashi and Yao [9] introduced a class of mappings called generalized hybrid. Let C be a non-empty subset of a Hilbert space H. Then a mapping $T : C \to H$ is called generalized hybrid if there exist $\alpha, \beta \in R$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x-y\|^2$$

for all $x, y \in C$; see also [1]. Such a mapping T is also called (α, β)-generalized hybrid. A (1,0)-generalized hybrid mapping is nonexpansive. A (2,1)-generalized hybrid mapping is nonspread; see [10, 11]. It is also hybrid in the sense of [22] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. Suzuki [19] also introduced a new class of nonlinear mappings. A mapping T of C into itself is said to satisfy Condition (C) if for any $x, y \in C$

$$\tfrac{1}{2}\|x - Tx\| \leq \|x - y\| \quad \Rightarrow \quad \|Tx - Ty\| \leq \|x - y\|.$$

It is obvious that if T is nonexpansive, then T satisfies Condition (C). Motivated by these mappings, Takahashi and Takeuchi [23] considered a class of mappings which satisfies the following condition:

$$F(T) \subset A(T). \tag{2.4}$$

Falset, Fuster and Suzuki [6] also considered the following class of mappings: There exists $s \in [0, \infty)$ such that

$$\|x - Ty\| \leq s\|x - Tx\| + \|x - y\|, \quad \forall x, y \in C. \tag{2.5}$$

We note that a nonexpansive mapping and a mapping satisfying Condition (C) satisfy (2.5) as $s = 1$ and $s = 3$, respectively. We also note that (2.5) is stronger than (2.4). In fact, if (2.5) holds and $u \in F(T)$, then we have that $\|u-Ty\| \leq \|u-y\|$ for all $y \in C$. A mapping T is quasi-nonexpansive if T satisfies $F(T) \neq \emptyset$ and (2.4). We finally note that a generalized hybrid mapping satisfies (2.4). Let C be a non-empty subset of H and let S be a mapping of C into H. $I - S$ is called demiclosed at 0 if a sequence $\{x_n\}$ in C converges weakly to $u \in C$ and $\lim_n \|Sx_n - x_n\| = 0$, then $u \in F(S)$. The following lemma was proved by Takahashi, Wong and Yao [25].

Lemma 2.1 ([25].). *Let C be a non-empty subset of a Hilbert space H and let S be a generalized hybrid mapping of C into itself. Let $\{x_n\}$ be a sequence in C which converges weakly to $u \in H$ and satisfies $\lim_n \|Sx_n - x_n\| = 0$. Then $u \in A(S)$. In addition, if C is closed and convex, then $u \in F(S)$.*

The following lemma was essentially proved in [19].

Lemma 2.2. *Let C be a closed and convex subset of a Hilbert space H and let S be a mapping of C into itself which satisfies (2.5). Let $\{x_n\}$ be a sequence in C which converges weakly to $u \in C$ and satisfies $\lim_n \|Sx_n - x_n\| = 0$. Then $u \in F(S)$.*

Proof. Assume $u \neq Su$. Since $\{x_n\}$ converges weakly to u, from the Opial property we have $\liminf_n \|x_n - u\| < \liminf_n \|x_n - Su\|$. We also have that there exists $s \in [0, \infty)$ such that

$$\|x_n - Su\| \leq s\|x_n - Sx_n\| + \|x_n - u\|, \quad \forall n \in N.$$

By $\lim_n \|Sx_n - x_n\| = 0$, this implies that $\liminf_n \|x_n - Su\| \leq \liminf_n \|x_n - u\|$. We have a contradiction. This completes the proof. □

Let C be a non-empty subset of a Hilbert space H. For a mapping A of C into H, we define the set $vi(C, A)$ as follows:

$$vi(C, A) = \{v \in C : \langle z - v, Az \rangle \geq 0, \ \forall z \in C\}.$$

From [20, Lemma 7.1.7] we have the following:

Lemma 2.3. *Let C be a convex subset of a Hilbert space H. Let A be a mapping of C into H. Then the following hold:*

1. *If A is continuous, then $vi(C, A) \subset VI(C, A)$.*
2. *If A is monotone then $\langle y - u, Ay \rangle \geq \langle y - u, Au \rangle \geq 0$ for $u \in VI(C, A)$ and $y \in C$. That is, if A is monotone then $VI(C, A) \subset vi(C, A)$.*
3. *If A is monotone and continuous, then $VI(C, A) = vi(C, A)$.*

3. Lemmas

In this section, we present some lemmas which are connected with properties of projection methods. The following lemma is well-known. For the sake of completeness, we give the proof.

Lemma 3.1. *Let C be a non-empty, closed and convex subset of a Hilbert space H. Let A be a mapping of C into H. Let $a \in (0, \infty)$ and let V_a be a mapping of C into itself defined by $V_a x = P_C(I - aA)x$ for all $x \in C$. Then $F(V_a) = VI(C, A)$.*

Proof. Let $u \in F(V_a)$. Then $u = P_C(I - aA)u$. From the property of P_C we have that for any $y \in C$

$$0 \leq \langle y - u, u - (u - aAu) \rangle = \langle y - u, aAu \rangle = a \langle y - u, Au \rangle .$$

From $a > 0$ we have that $\langle y - u, Au \rangle \geq 0$ for all $y \in C$. This implies $u \in VI(C, A)$. The reverse is similar. □

Lemma 3.2. *Let $c, k > 0$ and $\{a_n\} \subset [c, \infty)$. Let C be a non-empty, closed and convex subset of a Hilbert space H and let A be a monotone and k-Lipschitz continuous mapping of C into H with $VI(C, A) \neq \emptyset$. Let $\{V_{a_n}\}$ be a sequence of mappings on C defined by $V_{a_n}x = P_C(I - a_nA)x$ for all $x \in C$ and $n \in N$. Let $\{x_n\}$ be a bounded sequence in C. If $\lim_n \|V_{a_n}x_n - x_n\| = 0$, then the weak limit of any weakly convergent subsequence of $\{x_n\}$ is in $VI(C, A)$.*

Proof. Let $y_n = V_{a_n}x_n$ for all $n \in N$. Since $\{x_n\}$ is bounded, $\{x_n\}$ has a weakly convergent subsequence. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ which converges weakly to some $u \in C$. By $\lim_n \|V_{a_n}x_n - x_n\| = 0$, we also have that $\{y_{n_j}\}$ converges weakly to u. We first show $\langle z - u, Az \rangle \geq 0$ for all $z \in C$. Take $z \in C$. Since A is monotone, we have that $\left\langle z - y_{n_j}, Az - Ay_{n_j} \right\rangle \geq 0$ for all $j \in N$, that is,

$$\left\langle z - y_{n_j}, Az \right\rangle \geq \left\langle z - y_{n_j}, Ay_{n_j} \right\rangle . \tag{3.1}$$

Using $y_{n_j} = P_C(x_{n_j} - a_{n_j}Ax_{n_j})$ and $z \in C$, we also have from the property of P_C that

$$0 \geq \left\langle z - y_{n_j}, (x_{n_j} - a_{n_j}Ax_{n_j}) - y_{n_j} \right\rangle .$$

From $a_{n_j} > 0$ we have that

$$0 \geq \frac{1}{a_{n_j}} \left\langle z - y_{n_j}, x_{n_j} - y_{n_j} \right\rangle - \left\langle z - y_{n_j}, Ax_{n_j} \right\rangle . \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\left\langle z - y_{n_j}, Az \right\rangle \geq \frac{1}{a_{n_j}} \left\langle z - y_{n_j}, x_{n_j} - y_{n_j} \right\rangle + \left\langle z - y_{n_j}, Ay_{n_j} - Ax_{n_j} \right\rangle .$$

Since $1/a_{n_j} \leq 1/c$ and A is k-Lipschitz continuous, we have that

$$\left\langle z - y_{n_j}, Az \right\rangle \geq -\frac{1}{c} \|z - y_{n_j}\| \, \|x_{n_j} - y_{n_j}\| - k\|z - y_{n_j}\| \, \|y_{n_j} - x_{n_j}\|. \tag{3.3}$$

Since $\{y_{n_j}\}$ converges weakly to u, we have that $\langle z - u, Az \rangle \geq 0$. Since $z \in C$ is arbitrary, we have that $\langle z - u, Az \rangle \geq 0$ for all $z \in C$. By the continuity of A and Lemma 2.3 (1), we have $u \in VI(C, A)$. □

Remark 1. The inequality (3.3) is essential in the proof of Lemma 3.2. In the case $\lim_j a_{n_j} = 0$, we cannot prove the result. This problem appears when we deal with Halpern's type iterations with extragradient methods. We really know that there are some articles which have mathematical errors for this problem.

The following lemma plays crucial roll in the proof of Theorem 4.1.

Lemma 3.3. *Let C be a non-empty, closed and convex subset of a Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H with $VI(C, A) \neq \emptyset$. Let $\{a_n\}$ be a sequence in $[c, d]$ as $0 < c \leq d < 2\alpha$. Let $\{V_{a_n}\}$ be a sequence of mappings on C defined by $V_{a_n}x = P_C(I - a_nA)x$ for $x \in C$. If $\{x_n\}$ is a sequence in C such that $\lim_n \|x_n - u\| = \lim_n \|V_{a_n}x_n - u\|$ for some $u \in VI(C, A)$, then $\lim_n \|V_{a_n}x_n - x_n\| = 0$.*

Proof. Set $y_n = V_{a_n} x_n = P_C(I - a_n A)x_n$ for all $n \in N$. By Lemma 3.1, we have that $F(V_{a_n}) = VI(C, A)$ for $n \in N$. By our assumptions, $\{x_n\}$ and $\{y_n\}$ are bounded. Since $u \in VI(C, A)$ and A is α-inverse strongly monotone, we have

$$\begin{aligned}\|y_n - u\|^2 &= \|P_C(I - a_n A)x_n - P_C(I - a_n A)u\|^2 \\ &\le \|(I - a_n A)x_n - (I - a_n A)u\|^2 \\ &\le \|x_n - u\|^2 - a_n(2\alpha - a_n)\|Ax_n - Au\|^2\end{aligned}$$

for $n \in N$. From $a_n \in [c, d] \subset (0, 2\alpha)$, it follows that for $n \in N$

$$c(2\alpha - d)\|Ax_n - Au\|^2 \le a_n(2\alpha - a_n)\|Ax_n - Au\|^2 \le \|x_n - u\|^2 - \|y_n - u\|^2.$$

By $c(2\alpha - d) > 0$ and $\lim_n \|x_n - u\| = \lim_n \|y_n - u\|$, we have $\lim_n \|Ax_n - Au\| = 0$. Since P_C is firmly nonexpansive and $I - a_n A$ is nonexpansive, we have

$$\begin{aligned}2\|y_n - u\|^2 &= 2\|P_C(I - a_n A)x_n - P_C(I - a_n A)u\|^2 \\ &\le 2\langle P_C(I - a_n A)x_n - P_C(I - a_n A)u, (I - a_n A)x_n - (I - a_n A)u\rangle \\ &= 2\langle y_n - u, (I - a_n A)x_n - (I - a_n A)u\rangle \\ &= \|y_n - u\|^2 + \|(I - a_n A)x_n - (I - a_n A)u\|^2 \\ &\qquad - \|(y_n - u) - ((I - a_n A)x_n - (I - a_n A)u)\|^2 \\ &\le \|y_n - u\|^2 + \|x_n - u\|^2 \\ &\qquad - \|(y_n - x_n) + a_n(Ax_n - Au)\|^2 \\ &= \|y_n - u\|^2 + \|x_n - u\|^2 \\ &\qquad - \|y_n - x_n\|^2 - 2a_n\langle y_n - x_n, Ax_n - Au\rangle - a_n^2\|Ax_n - Au\|^2\end{aligned}$$

for all $n \in N$. Thus it follows that for $n \in N$

$$\begin{aligned}\|y_n - x_n\|^2 \le \|x_n - u\|^2 &- \|y_n - u\|^2 \\ &- 2a_n\langle y_n - x_n, Ax_n - Au\rangle - a_n^2\|Ax_n - Au\|^2.\end{aligned}$$

By $\lim_n \|x_n - u\| = \lim_n \|y_n - u\|$ and $\lim_n \|Ax_n - Au\| = 0$, we have

$$\lim_n \|y_n - x_n\| = \lim_n \|V_{a_n} x_n - x_n\| = 0.$$

This completes the proof. □

Let $\{a_n\}$ be a sequence in $(0, \infty)$. Let C be a non-empty, closed and convex subset of a Hilbert space H. Let A be a mapping of C into H such that $VI(C, A) \ne \emptyset$. Let $\{V_{a_n}\}$ be a sequence of mappings on C defined by $V_{a_n} x = P_C(I - a_n A)x$ for all $x \in C$ and let $\{W_n\}$ be a sequence of mappings on C such that $F(W_n) \subset A(W_n)$ for all $n \in N$. Then $\{W_n\}$ said to satisfy Condition (E) with $\{V_{a_n}\}$ if there exist $M_1, M_2 > 0$ such that for any $n \in N$

(E_1) $\|W_n x - x\| \le M_1\|V_{a_n} x - x\|, \quad \forall x \in C;$

(E_2) $\|x - V_{a_n} x\|^2 \le M_2(\|x - u\|^2 - \|W_n x - u\|^2), \quad \forall x \in C,\ u \in VI(C, A).$

We note that $F(W_n) \subset A(W_n)$ and $F(W_n) \ne \emptyset$ if and only if W_n is quasi-nonexpansive.

Lemma 3.4. *Let $\{a_n\}$ be a sequence in $(0, \infty)$. Let C be a non-empty, closed and convex subset of a Hilbert space H. Let A be a mapping of C into H with $VI(C, A) \ne \emptyset$. Let $\{V_{a_n}\}$ be a sequence of mappings on C defined by $V_{a_n} x =$*

$P_C(I - a_n A)x$ for $x \in C$. If $\{W_n\}$ is a sequence of mappings on C which satisfies Condition (E) with $\{V_{a_n}\}$, then for each $n \in N$

$$F(V_{a_n}) = F(W_n) = VI(C, A).$$

Proof. Fix $n \in N$ arbitrarily. We already know that $F(V_{a_n}) = VI(C, A)$. Let $v \in F(V_{a_n}) = VI(C, A)$. From (E_1) we have

$$\|W_n v - v\| \leq M_1 \|V_{a_n} v - v\| = 0.$$

Then $\emptyset \neq F(V_{a_n}) \subset F(W_n)$. Let $u \in VI(C, A)$ and $w \in F(W_n)$. From (E_2) we have

$$\|w - V_{a_n} w\|^2 \leq M_2(\|w - u\|^2 - \|W_n w - u\|^2) = M_2(\|w - u\|^2 - \|w - u\|^2) = 0.$$

Then $F(W_n) \subset F(V_{a_n})$. Thus $F(V_{a_n}) = F(W_n) = VI(C, A)$ for all $n \in N$. □

The following lemma is a result to simplify the proof of Lemma 3.6.

Lemma 3.5. *Let C be a non-empty, closed and convex subset of a Hilbert space H. Let $k > 0$ and let A be a monotone and k-Lipschitz continuous mapping of C into H such that $VI(C, A) \neq \emptyset$. Let $a \in (0, 1/k]$. Let $x \in C$, $y = P_C(x - aAx)$, $z = P_C(x - aAy)$ and $u \in VI(C, A)$. Then the following hold:*

(1) $\langle y - z, aAy \rangle \geq \langle u - z, aAy \rangle$;
(2) $\|x - z\|^2 + 2\langle z - y, aAy \rangle \geq (1 - a^2k^2)\|x - y\|^2 + (ak\|x - y\| - \|y - z\|)^2 \geq 0$;
(3) $\|z - u\|^2 \leq \|x - u\|^2 - (1 - a^2k^2)\|x - y\|^2 \leq \|x - u\|^2$.

Proof. We prove (1). Let $u \in VI(C, A)$. Since A is monotone, we have

$$\langle y - u, Ay \rangle \geq \langle y - u, Au \rangle \geq 0.$$

From $a > 0$ we have that

$$\langle y - z, aAy \rangle - \langle u - z, aAy \rangle = a\langle y - u, Ay \rangle \geq a\langle y - u, Au \rangle \geq 0$$

and hence $\langle y - z, aAy \rangle \geq \langle u - z, aAy \rangle$. We prove (2). By $y = P_C(x - aAx)$ and $z \in C$, we have

$$\langle z - y, (x - aAx) - y \rangle \leq 0.$$

Then the following inequality holds:

$$\begin{aligned}\langle z - y, x - y \rangle - \langle z - y, aAy \rangle &= \langle z - y, (x - aAx) - y \rangle + a\langle z - y, Ax - Ay \rangle \\ &\leq a\langle z - y, Ax - Ay \rangle.\end{aligned}$$

Since A is k-Lipschitz continuous and $ak \leq 1$, it follows that

$$\begin{aligned}&\|x - z\|^2 + 2\langle z - y, aAy \rangle \\ &\quad = \big(\|x - y\|^2 + \|z - y\|^2 - 2\langle z - y, x - y \rangle\big) + 2\langle z - y, aAy \rangle \\ &\quad \geq \|x - y\|^2 + \|z - y\|^2 - 2a\langle z - y, Ax - Ay \rangle \\ &\quad \geq \|x - y\|^2 + \|y - z\|^2 - 2ak\|z - y\|\,\|x - y\| \\ &\quad = (1 - a^2k^2)\|x - y\|^2 + (ak\|x - y\| - \|y - z\|)^2 \geq 0.\end{aligned}$$

We prove (3). Using $z = P_C(x - aAy)$, (1), (2) and properties of P_C, we have

$$\begin{aligned}\|z-u\|^2 &\le \|(x-aAy)-u\|^2 - \|(x-aAy)-z\|^2\\ &= (\|x-u\|^2 + \|aAy\|^2 - 2\langle x-u, aAy\rangle)\\ &\qquad - (\|x-z\|^2 + \|aAy\|^2 - 2\langle x-z, aAy\rangle)\\ &= \|x-u\|^2 - \|x-z\|^2 - 2\langle z-u, aAy\rangle\\ &\le \|x-u\|^2 - \|x-z\|^2 - 2\langle z-y, aAy\rangle\\ &\le \|x-u\|^2 - (1-a^2k^2)\|x-y\|^2 - (ak\|x-y\| - \|z-y\|)^2\\ &\le \|x-u\|^2 - (1-a^2k^2)\|x-y\|^2 \le \|x-u\|^2.\end{aligned}$$

This completes the proof. $\square$

Lemma 3.6. *Let C be a non-empty, closed and convex subset of a Hilbert space H. Let $k > 0$ and let A be a monotone and k-Lipschitz continuous mapping of C into H such that $VI(C, A) \neq \emptyset$. Let $0 < d < 1/k$ and $\{a_n\}$ be a sequence in $(0, d]$. Let $\{V_{a_n}\}$ be a sequence of mappings on C defined by $V_{a_n}x = P_C(I - a_nA)x$ for $x \in C$ and let $\{U_{a_n}\}$ be a sequence of mappings on C defined by*

$$U_{a_n}x = P_C(I - a_nAV_{a_n})x$$

for $x \in C$. Then each U_{a_n} is a quasi-nonexpansive mapping such that $F(V_{a_n}) = F(U_{a_n}) = VI(C, A)$ and $\{U_{a_n}\}$ satisfies Condition (E) with $\{V_{a_n}\}$.

Proof. We show that $\{U_{a_n}\}$ satisfies Condition (E_1). Fix $n \in N$ arbitrarily. Since $0 < a_nk \le dk < 1$, P_C is nonexpansive and A is k-Lipschitz continuous, we have that for all $x \in C$

$$\begin{aligned}\|U_{a_n}x - V_{a_n}x\| &= \|P_C(x - a_nAV_{a_n}x) - P_C(x - a_nAx)\|\\ &\le \|(x-x) - a_n(AV_{a_n}x - Ax)\| \le a_nk\|V_{a_n}x - x\|\end{aligned}$$

and hence

$$\begin{aligned}\|U_{a_n}x - x\| &\le \|U_{a_n}x - V_{a_n}x\| + \|V_{a_n}x - x\|\\ &\le a_nk\|V_{a_n}x - x\| + \|V_{a_n}x - x\|\\ &\le (1 + a_nk)\|V_{a_n}x - x\| \le 2\|V_{a_n}x - x\|.\end{aligned}$$

This implies that $\{U_{a_n}\}$ satisfies Condition (E_1) as $M_1 = 2$. We show that $\{U_{a_n}\}$ satisfies Condition (E_2). Fix $n \in N$ arbitrarily. Let $x \in C$, $u \in VI(C, A)$ and set $y = V_{a_n}x$. By $U_{a_n}x = P_C(x - a_nAy)$ and Lemma 3.5 (3), we have

$$\|U_{a_n}x - u\|^2 \le \|x-u\|^2 - (1 - a_n^2k^2)\|x-y\|^2 \le \|x-u\|^2.$$

Thus we have that for $x \in C$ and $u \in VI(C, A)$

(*a*) $\|U_{a_n}x - u\| \le \|x - u\|$;

(*b*) $(1 - d^2k^2)\|x - V_{a_n}x\|^2 \le (1 - a_n^2k^2)\|x - V_{a_n}x\|^2 \le \|x-u\|^2 - \|U_{a_n}x - u\|^2$.

From (*b*), it follows that $\{U_{a_n}\}$ satisfies Condition (E_2) as $M_2 = 1/(1 - d^2k^2)$. We have from Lemma 3.4 that $F(V_{a_n}) = F(U_{a_n}) = VI(C, A)$ for each $n \in N$. By (*a*), each U_{a_n} is a quasi–nonexpansive mapping. This completes the proof. $\square$

4. Main Results

We present our main results.

Theorem 4.1. *Let C be a closed and convex subset of a Hilbert space H and let $\alpha > 0$. Let A be an α-inverse strongly monotone mapping of C into H. Let $\{a_n\}$ be a sequence in $[c, d]$ as $0 < c \leq d < 2\alpha$. For each $n \in N$, let V_{a_n} be a mapping of C into itself defined by $V_{a_n}x = P_C(I - a_nA)x$ for all $x \in C$. Let S be a mapping of C into itself such that $F(S) \subset A(S)$ and $I - S$ is demiclosed at 0. Assume $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[a, b]$ as $0 < a \leq b < 1$. Let $x_1 \in C$ and let $\{x_n\}$ and $\{y_n\}$ be sequences in C defined by*

$$y_n = V_{a_n}x_n, \quad x_{n+1} = \alpha_n SV_{a_n}x_n + (1 - \alpha_n)x_n, \quad \forall n \in N.$$

Then $\{x_n\}$ and $\{y_n\}$ converge weakly to a point $u \in F(S) \cap VI(C, A)$.

Proof. Under our assumptions, it follows that each V_{a_n} is a nonexpansive mapping such that $F(V_{a_n}) = VI(C, A) \neq \emptyset$. Since $F(S) \subset A(S)$ and $F(S) \neq \emptyset$, S is also quasi-nonexpansive. Let $w \in F(S) \cap VI(C, A)$. We have that

$$\begin{aligned}\|x_{n+1} - w\| &\leq \alpha_n\|SV_{a_n}x_n - w\| + (1 - \alpha_n)\|x_n - w\| \\ &\leq \alpha_n\|x_n - w\| + (1 - \alpha_n)\|x_n - w\| = \|x_n - w\|\end{aligned}$$

for all $n \in N$. Then $\{\|x_n - w\|\}$ is non-increasing and converges to some $s \in [0, \infty)$. It follows that $\{x_n\}$ are bounded. We also have that

$$\begin{aligned}&\alpha_n\|x_{n+1} - w\| + (1 - \alpha_n)(\|x_{n+1} - w\| - \|x_n - w\|) \\ &\qquad \leq \alpha_n\|SV_{a_n}x_n - w\| \leq \alpha_n\|V_{a_n}x_n - w\| \leq \alpha_n\|x_n - w\|.\end{aligned}$$

Since $\alpha_n \in [a, b]$ and $\|x_{n+1} - w\| - \|x_n - w\| \leq 0$, we have that

$$\|x_{n+1} - w\| + \tfrac{1}{a}(\|x_{n+1} - w\| - \|x_n - w\|) \leq \|V_{a_n}x_n - w\| \leq \|x_n - w\|$$

for all $n \in N$. This implies $\lim_n \|V_{a_n}x_n - w\| = \lim_n \|x_n - w\| = s$. We have from Lemma 3.3 that $\lim_n \|V_{a_n}x_n - x_n\| = 0$. On the other hand, we have from (2.1) that for any $x, y \in H$ and $\alpha \in R$

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2.$$

Setting $\alpha = \alpha_n$, $x = SV_{a_n}x_n - w$, $y = x_n - w$, we have that for any $n \in N$

$$\begin{aligned}&\alpha_n(1 - \alpha_n)\|SV_{a_n}x_n - x_n\|^2 \\ &\qquad = \alpha_n\|SV_{a_n}x_n - w\|^2 + (1 - \alpha_n)\|x_n - w\|^2 - \|x_{n+1} - w\|^2 \\ &\qquad \leq \alpha_n\|x_n - w\|^2 + (1 - \alpha_n)\|x_n - w\|^2 - \|x_{n+1} - w\|^2 \\ &\qquad = \|x_n - w\|^2 - \|x_{n+1} - w\|^2.\end{aligned}$$

Since $\{\|x_n - w\|\}$ is a convergent sequence and $\alpha_n \in [a, b]$ for all $n \in N$, we have that $\lim_n \|SV_{a_n}x_n - x_n\| = 0$. Moreover, since

$$\|SV_{a_n}x_n - V_{a_n}x_n\| \leq \|SV_{a_n}x_n - x_n\| + \|V_{a_n}x_n - x_n\|.$$

for all $n \in N$, we have that

$$\lim{}_n \|Sy_n - y_n\| = \lim{}_n \|SV_{a_n}x_n - V_{a_n}x_n\| = 0.$$

Since $\{x_n\}$ is bounded, there exists a weakly convergent subsequence. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ which converges weakly to some $u \in C$. From $\lim{}_n \|V_{a_n}x_n - x_n\| = 0$, $\{y_{n_j}\}$ also converges weakly to u. Since A is monotone and

$1/\alpha$-Lipschitz continuous, from $\lim_j \|V_{a_{n_j}} x_{n_j} - x_{n_j}\| = 0$ and Lemma 3.2, we have $u \in VI(C, A)$. Since $I - S$ is demi–closed at 0 and $\lim_n \|SV_{a_n} x_n - V_{a_n} x_n\| = 0$, we also have $u \in F(S)$. Thus $u \in VI(C, A) \cap F(S)$.

Finally, let us show that $\{x_n\}$ converges weakly to $u \in VI(C, A) \cap F(S)$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be subsequences of $\{x_n\}$ which converge weakly to $u, v \in VI(C, A) \cap F(S)$, respectively. To have the result, it is sufficient to show $u = v$. Assume $u \neq v$. By the Opial property, we have that

$$\begin{aligned}\lim_i \|x_{n_i} - u\| &< \lim_i \|x_{n_i} - v\| = \lim_j \|x_{n_j} - v\| \\ &< \lim_j \|x_{n_j} - u\| = \lim_i \|x_{n_i} - u\|.\end{aligned}$$

This is a contradiction. Then we have $u = v$. Therefore we have the desired result. □

Theorem 4.2. *Let C be a closed and convex subset of a Hilbert space H and let $k > 0$. Let A be a monotone and k-Lipschitz continuous mapping of C into H. Let $\{a_n\}$ be a sequence in $[c, \infty)$ as $c \in (0, \infty)$. For each $n \in N$, let V_{a_n} be a mapping of C into itself defined by $V_{a_n} x = P_C(I - a_n A)x$ for all $x \in C$. Let $\{W_n\}$ be a sequence of mappings on C with $F(W_n) \subset A(W_n)$ such that $\{W_n\}$ satisfies Condition (E) with $\{V_{a_n}\}$. Let S be a mapping of C into itself such that $F(S) \subset A(S)$ and $I - S$ is demiclosed at 0. Assume $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[a, b]$ as $0 < a \leq b < 1$. Let $x_1 \in C$ and let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be sequences defined by*

$$y_n = V_{a_n} x_n, \; z_n = W_n x_n, \quad x_{n+1} = \alpha_n S W_n x_n + (1 - \alpha_n) x_n, \quad \forall n \in N.$$

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge weakly to a point $u \in F(S) \cap VI(C, A)$.

Proof. By Lemma 3.4, we know that W_n is quasi-nonexpansive and $F(W_n) = VI(C, A)$ for all $n \in N$. Since $F(S) \subset A(S)$ and $F(S) \cap VI(C, A) \neq \emptyset$, S is also quasi-nonexpansive. Let $w \in F(S) \cap VI(C, A)$. We have that

$$\begin{aligned}\|x_{n+1} - w\| &\leq \alpha_n \|SW_n x_n - w\| + (1 - \alpha_n)\|x_n - w\| \\ &\leq \alpha_n \|x_n - w\| + (1 - \alpha_n)\|x_n - w\| = \|x_n - w\|\end{aligned}$$

for all $n \in N$. Then $\{\|x_n - w\|\}$ is non–increasing and converges to some $s \in [0, \infty)$. Thus we have that $\{x_n\}$ are bounded. As in the proof of Theorem 4.1, we also have that

$$\begin{aligned}&\alpha_n \|x_{n+1} - w\| + (1 - \alpha_n)(\|x_{n+1} - w\| - \|x_n - w\|) \\ &\qquad \leq \alpha_n \|SW_n x_n - w\| \leq \alpha_n \|W_n x_n - w\| \leq \alpha_n \|x_n - w\|.\end{aligned}$$

Since $\alpha_n \in [a, b]$ and $\|x_{n+1} - w\| - \|x_n - w\| \leq 0$, we have

$$\|x_{n+1} - w\| + \tfrac{1}{a}(\|x_{n+1} - w\| - \|x_n - w\|) \leq \|W_n x_n - w\| \leq \|x_n - w\|$$

for all $n \in N$. This implies $\lim_n \|W_n x_n - w\| = s$. By (E_2) of Condition (E), there is $M_2 > 0$ such that

$$\|V_{a_n} x_n - x_n\|^2 \leq M_2(\|x_n - w\|^2 - \|W_n x_n - w\|^2)$$

for all $n \in N$. Since $\lim_n \|x_n - w\| = \lim_n \|W_n x_n - w\| = s$, we have that $\lim_n \|V_{a_n} x_n - x_n\| = 0$. By (E_1) of Condition (E), we also have that $\lim_n \|W_n x_n -$

$x_n\| = 0$. On the other hand, using (2.1), we have that for any $n \in N$

$$\begin{aligned}
\alpha_n(1-\alpha_n)&\|SW_nx_n - x_n\|^2 \\
&= \alpha_n\|SW_nx_n - w\|^2 + (1-\alpha_n)\|x_n - w\|^2 - \|x_{n+1} - w\|^2 \\
&\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|x_n - w\|^2 - \|x_{n+1} - w\|^2 \\
&= \|x_n - w\|^2 - \|x_{n+1} - w\|^2.
\end{aligned}$$

Since $\{\|x_n - w\|\}$ converges and $\alpha_n \in [a,b]$ for all $n \in \mathbb{N}$, we have $\lim_n \|SW_nx_n - x_n\| = 0$. Moreover, since

$$\|SW_nx_n - W_nx_n\| \leq \|SW_nx_n - x_n\| + \|W_nx_n - x_n\|.$$

for all $n \in N$, we have that

$$\lim{}_n \|Sz_n - z_n\| = \lim{}_n \|SW_nx_n - W_nx_n\| = 0.$$

Since $\{x_n\}$ is bounded, there exists a weakly convergent subsequence. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ which converges weakly to some $u \in C$. By $\lim_n \|V_{a_n}x_n - x_n\| = 0$ and $\lim_n \|W_nx_n - x_n\| = 0$, we also have that $\{y_{n_j}\}$ and $\{z_{n_j}\}$ converge weakly to u. Since A is monotone and k-Lipschitz continuous, from $\lim_j \|V_{a_{n_j}}x_{n_j} - x_{n_j}\| = 0$ and Lemma 3.2, we have that $u \in VI(C,A)$. Since $I - S$ is demi–closed at 0 and $\lim_j \|SW_{n_j}x_{n_j} - W_{n_j}x_{n_j}\| = 0$, we also have $u \in F(S)$. Thus $u \in VI(C,A) \cap F(S)$. To show that $\{x_n\}$ converges weakly to a point of $VI(C,A) \cap F(S)$, let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be subsequences of $\{x_n\}$ which converge weakly to $u, v \in VI(C,A) \cap F(S)$, respectively. To have the result, it is sufficient to show $u = v$. Assume $u \neq v$. As in the proof of Theorem 4.1, we have that

$$\begin{aligned}
\lim{}_i \|x_{n_i} - u\| &< \lim{}_i \|x_{n_i} - v\| = \lim{}_j \|x_{n_j} - v\| \\
&< \lim{}_j \|x_{n_j} - u\| = \lim{}_i \|x_{n_i} - u\|.
\end{aligned}$$

This is a contradiction. Then we have the desired result. □

5. Applications

Using Theorems 4.1 and 4.2, we present some new results. The following are extensions of Theorem 1.1.

Theorem 5.1. *Let C be a closed and convex subset of a Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H. Let $\{a_n\}$ be a sequence in $[c,d]$ as $0 < c \leq d < 2\alpha$. For each $n \in N$, let V_{a_n} be a mapping of C into itself defined by $V_{a_n}x = P_C(I - a_nA)x$ for all $x \in C$. Let S be a generalized hybrid mapping of C into itself. Assume that $F(S) \cap VI(C,A) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[a,b]$ as $0 < a \leq b < 1$. Let $x_1 \in C$ and let $\{x_n\}$ and $\{y_n\}$ be sequences in C defined by*

$$y_n = V_{a_n}x_n, \quad x_{n+1} = \alpha_nSV_{a_n}x_n + (1-\alpha_n)x_n, \quad \forall n \in N.$$

Then $\{x_n\}$ and $\{y_n\}$ converge weakly to a point $u \in F(S) \cap VI(C,A)$.

Proof. Since $S : C \to C$ is generalized hybrid, S satisfies $F(S) \subset A(S)$. By Lemma 2.1 we have that $I - S$ is demiclosed at 0. Then, by Theorem 4.1, we have the desired result. □

Theorem 5.2. *Let C be a closed and convex subset of a Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H. Let $\{a_n\}$ be a sequence in $[c,d]$ as $0 < c \le d < 2\alpha$. For each $n \in N$, let V_{a_n} be a mapping of C into itself defined by $V_{a_n}x = P_C(I - a_nA)x$ for all $x \in C$. Let $S : C \to C$ be a mapping such that, for some $s \in [0,\infty)$,*

$$\|x - Ty\| \le s\|x - Tx\| + \|x - y\|, \quad \forall x, y \in C. \tag{5.1}$$

Assume that $F(S) \cap VI(C,A) \ne \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[a,b]$ as $0 < a \le b < 1$. Let $x_1 \in C$ and let $\{x_n\}$ and $\{y_n\}$ be sequences in C defined by

$$y_n = V_{a_n}x_n, \qquad x_{n+1} = \alpha_n SV_{a_n}x_n + (1-\alpha_n)x_n, \quad \forall n \in N.$$

Then $\{x_n\}$ and $\{y_n\}$ converge weakly to a point $u \in F(S) \cap VI(C,A)$.

Proof. Since S is a mapping satisfying (5.1), S satisfies $F(S) \subset A(S)$. By Lemma 2.2 we have that $I - S$ is demiclosed at 0. Then, by Theorem 4.1, we have the desired result. □

Using Theorem 5.2, we have the following result.

Theorem 5.3. *Let C be a closed and convex subset of a Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H. Let $\{a_n\}$ be a sequence in $[c,d]$ as $0 < c \le d < 2\alpha$. For each $n \in N$, let V_{a_n} be a mapping of C into itself defined by $V_{a_n}x = P_C(I - a_nA)x$ for $x \in C$. Let $S : C \to C$ be a mapping which satisfies Condition (C). Assume that $F(S) \cap VI(C,A) \ne \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[a,b]$ as $0 < a \le b < 1$. Let $x_1 \in C$ and let $\{x_n\}$ and $\{y_n\}$ be sequences in C defined by*

$$y_n = V_{a_n}x_n, \qquad x_{n+1} = \alpha_n SV_{a_n}x_n + (1-\alpha_n)x_n, \quad \forall n \in N.$$

Then $\{x_n\}$ and $\{y_n\}$ converge weakly to a point $u \in F(S) \cap VI(C,A)$.

Proof. If a mapping S satisfies Condition (C), then we know that S satisfies (5.1). Thus we obtain the desired result from Theorem 5.2. □

As in the proofs of Theorems 5.1 and 5.2 we have the following extensions of Theorem 1.2 from Lemma 3.6 and Theorem 4.2.

Theorem 5.4. *Let C be a closed and convex subset of a Hilbert space H and A be a monotone and k-Lipschitz continuous mapping of C into H. Let $\{a_n\}$ be a sequence in $[c,d]$ as $0 < c \le d < 1/k$. For each $n \in N$, let V_{a_n} and U_{a_n} be mappings of C into itself defined by*

$$V_{a_n}x = P_C(I - a_nA)x, \quad U_{a_n}x = P_C(I - a_nAV_{a_n})x, \quad \forall x \in C,$$

respectively. Let $S : C \to C$ be a generalized hybrid mapping. Assume that $F(S) \cap VI(C,A) \ne \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[a,b]$ as $0 < a \le b < 1$. Let $x_1 \in C$ and let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be sequences defined by

$$y_n = V_{a_n}x_n, \quad z_n = U_{a_n}x_n, \quad x_{n+1} = \alpha_n SU_{a_n}x_n + (1-\alpha_n)x_n, \quad \forall n \in N.$$

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge weakly to a point $u \in F(S) \cap VI(C,A)$.

Theorem 5.5. *Let C be a closed and convex subset of a Hilbert space H and A be a monotone and k-Lipschitz continuous mapping of C into H. Let $\{a_n\}$ be a* *sequence in* $[c, d]$ *as* $0 < c \leq d < 1/k$. *For each* $n \in N$, *let* V_{a_n} *and* U_{a_n} *be mappings of* C *into itself defined by*

$$V_{a_n}x = P_C(I - a_nA)x, \quad U_{a_n}x = P_C(I - a_nAV_{a_n})x, \quad \forall x \in C,$$

respectively. Let $S : C \to C$ *be a mapping such that, for some* $s \in [0, \infty)$,

$$\|x - Ty\| \leq s\|x - Tx\| + \|x - y\|, \quad \forall x, y \in C.$$

Assume that $F(S) \cap VI(C, A) \neq \emptyset$. *Let* $\{\alpha_n\}$ *be a sequence in* $[a, b]$ *as* $0 < a \leq b < 1$. *Let* $x_1 \in C$ *and let* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ *be sequences defined by*

$$y_n = V_{a_n}x_n, \quad z_n = U_{a_n}x_n, \quad x_{n+1} = \alpha_n SU_{a_n}x_n + (1 - \alpha_n)x_n, \quad \forall n \in N.$$

Then $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge weakly to a point* $u \in F(S) \cap VI(C, A)$.

References

[1] K. Aoyama, S. Iemoto, F. Kohsaka, and W. Takahashi, *"Fixed point and ergodic theorems for -hybrid mappings in Hilbert spaces"*, J. Nonlinear Convex Anal. **11** (2010), 335–343.

[2] S. Banach, gSur les opérations dans les ensembles abstraits et leur application atix équations intégrales h , Fund. Math. **3** (1922), 133–181.

[3] F. E. Browder, *"Nonlinear operators and nonlinear equations of evolution in Banach spaces"*, Nonlinear functional analysis, Proc. Sympos. Pure Math., Vol XVIII, Part2, Chicago, III., (1968), Amer. Math. Soc., Providence, R.I., (1973), 251–262.

[4] R. E. Bruck, *"A simple proof of the mean ergodic theorems for nonlinear contractions in Banach spaces"*, Israel J. Math. **32** (1974), 107–116.

[5] P. E. Combettes and S. A. Hirstoaga, *"Equilibrium programing in Hilbert spaces"*, J. Nonlinear Convex Anal. **6** (2005), 117–136.

[6] J. G. Falset, E. L. Fuster, and T. Suzuki, *" Fixed point theory for a class of generalized nonexpansive mappings"*, J. Math. Anal. Appl. **375** (2011) 185–195.

[7] H. Iiduka, W. Talahashi, and M. Toyodai, *" Approximation of solutions of variational inequalities for monotone mappings"*, Panamer. Math. J. **14** (2004), 49–61.

[8] G. M. Korpelevich, *"The extragradient method for finding saddle points and other problems"*, Matecon 12 (1976), 747–756.

[9] P. Kocourek, W. Takahashi, and J.-C. Yao, *"Fixed point theorems and weak convergence theorems for genelalized hybrid mappings in Hilbert spaces"*, Taiwanese J. Math. **14** (2010), 2497–2511.

[10] F. Kosaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*, SIAM. J.Optim. **19** (2008), 824-835.

[11] F. Kosaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces.*, Arch. Math. (Basel) **91** (2008), 166-177.

[12] M. A. Krasnoselskii, *"Two remarks on the method of successive approximations"*, Uspehi Mat. Nauk 10 (1955), 123–127 (Russian).

[13] R. Kubota and Y. Takeuchi, *"On Ishikawa's strong convergence theorem"*, to apear.

[14] F. Liu and M. Z. Nashed, *"Regularization of nonlinear ill-posed variational inequalities and convergence rates"*, Set-valued Anal. **6** (1998), 313-344.

[15] W. R. Mann, *"Mean value methods in iteration"*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[16] N. Nadezhkina and W. Takahashi, *"Strong convergence theorem by the hybrid and extragradient method for nonexpansive mappings and monotone mappings"*, Kyoto University Research Information Repository, 1396 (2004), 42–48.

[17] N. Nadezhkina and W. Takahashi, *"Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings"*, J. Optim. Theory Appl. **128** (2006), 191-201.

[18] Z. Opial, *"Weak convergence of the sequence of successive approximations for nonexpansive mappings"*, Bull. Amer. Math. Soc. 73 (1967), 591–597.

[19] T. Suzuki, *"Fixed point theorems and convergence theorems for some generalized nonexpansive mappings"*, J. Math. Anal. Appl. **340**, (2008) 1088-1095.

[20] W. Takahashi, *"Nonlinear Functional Analysis"*, Yokohama Publishers, Yokohama, 2000.

[21] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2009.
[22] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79–88.
[23] W. Takahashi and Y. Takeuchi, *"Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space"*, J. Nonlinear Convex Anal. **12 No 2** (2011), 399–406.
[24] W. Takahashi and M. Toyoda, *"Weak convergence theorems for nonepxansive mappings and monotone mappings"*, J. Optim. Theory Appl. **118** (2003), 417–428.
[25] W. Takahashi, N.-C. Wong, and J.-C. Yao, *"Attractive point and weak convergence theorems for new generalized hybrid mappings in Hilbert spaces"*, J. Nonlinear Convex Anal. **13** (2012), 745–757.

Communicated by *Wataru Takahashi*

(Rieko Kubota) Yokohama So-gakukan High School, 1-43-1 Mutsuura-Higashi, Kanazawa, Yokohama 236-0037, Japan
E-mail address: lri2e3ko_9@yahoo.co.jp

(Wataru Takahashi) Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan; Keio Research and Education Center for Natural Sciences, Keio University, Japan; and Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Ookayama, Meguro-ku, Tokyo 152-8552, Japan
E-mail address: wataru@is.titech.ac.jp

(Yukio Takeuchi) Takahashi Institute for Nonlinear Analysis, 1-11-11 Nakazato, Minami, Yokohama 232-0063, Japan
E-mail address: aho31415@yahoo.co.jp

INTERNATIONAL ADVISORS

BOARD OF EDITORS

CZECH REPUBLIC

(a) **Milan Vlach**
(b) Charles University, Faculty of Mathematics and Physics
Malostranske namesti 25, 118 00 Prague 1, Czech Republic
(b') milan.vlach@mff.cuni.cz and mvlach@ksi.ms.mff.cuni.cz
(c) Game theory, Fair division, Optimization

(a) **Vaclav Koubek**
(b) Department of Theoretical Computer Science and Mathematical Logic (KTIML), The Faculty of Mathematics and Physics (MFF), Malostranske nam. 25, 118 00 Praha 1, Czech Republic
(b') koubek@ktiml.mff.cuni.cz
(c) Theory of lattices. Theory of semigroups. Theory of graphs and their complexity. Complexity theory. Universal algebra

(a) **Vera Trnková**
(b) Mathematical Institute of Charles University, 186 75 Praha 8, Sokolovska 83, Czech Republik
(b') trnkova@karlin.mff.cz
(c) Universal algebra, Category theory, General topology

FINLAND

(a) **Arto Salomaa**
(b) Jaanintie 34 A 26, 20540 Turku, Finland
(b') asalomaa@utu.fi
(c) Formal languages, automata, computability, cryptography

(a) **Jarkko Kari**
(b) Department of Mathematics and Statistics, FI-20014 University of Turku, Finland
(b') jkari@utu.fi
(c) automata theory, cellular automata, tilings, symbolic dynamics

GERMANY

(a)**Klaus Denecke**
(b) 14542 Werder (Havel), Germany, Zanderweg 3
(b') klausdenecke@hotmail.com
(c) General Algebra, Discrete Mathematics, Multiple-valued Logic, Ordered Sets and Lattices, Theory of Semigroups

GREECE

(a) **Maria Fragoulopoulou**
(b) Department of Mathematics, University of Athens, Panepistimiopolis, Athens 157 84, Greece
(b') fragoulop@math.uoa.gr
(c) Non-normed Topological Algebras, Topological Algebras with an Involution, Unbounded Operator Theory, Tensor products of Topological Algebras and Topological *-Algebras.

HUNGARY

(a) **Gyula Maksa**
(b) Institute of Mathematics, University of Debrecen, H-4010 Debrecen, Pf. 12, Hungary
(b') maksa@science.unideb.hu
(c) Functional equations

UNITED STATES OF AMERICA

(a) **Andreas Blass**
(b) Mathematics Department, University of Michigan, Ann Arbor, MI 48109-1043, USA
(b') ablass@umich.edu
(c) Mathematical logic, set theory, category theory

(a) **John B Conway**
(b) Professor Emeritus, George Washington University, 2115 G Street NW; Room 240; Washington, DC 20052, U.S.A
(b') Conway@gwu.edu
(c) Functional Analysis and Operator Theory

(a) **Paul Cull**
(b) Computer Science, Kelley Engineering Center, Oregon State University, Corvallis, OR 97331, USA
(b') pc@cs.orst.edu
(c) Difference Equations and Dynamical Systems, Computer Science (Theory, Algorithms, Networks), Mathematical Biology (Population Models, Neural Nets)

(a) **W. Wistar Comfort**
(b) Department of Mathematics, Wesleyan University, Wesleyan Station, Middletown, CT USA 06459
(b') wcomfort@wesleyan.edu
(c) Topological theory of topological groups, General (set-theoretic) topology

JAPAN

(a) **Mariko Yasugi**
(b) 28-21 Misasagiikezutsumicho, Yamashinaku, Kyoto 607-8433, Japan
(b') yasugi@cc.kyoto-su.ac.jp
(c) Logic Oriented Mathematics

(a) **Haruo Maki**
(b) 2-10-13 Wakagi-dai, Fukutsu-shi, Fukuoka 811-3221, Japan
(b') makih@pop12.odn.ne.jp
(c) (Topological) digital n-spaces (n>0), Generalized closed sets (after Levine), Operation theory in topology (in the sense of Kasahara and Ogata)

(a) **Kohzo Yamada**
(b) Faculty of Education, Shizuoka Univ., 836 Ohya, Shizuoka 422-8529, Japan
(b') eckyama@ipc.shizuoka.ac.jp
(c) General Topology

(a) **Yasunao Hattori**
(b) Dept. of Mathematics and Computer Sciences, Shimane Univ., Matsue, Shimane 690-8504, Japan
(b') hattori@riko.shimane-u.ac.jp
(c) General Topology

(a) **Yoshikazu Yasui**
(b) Department of Modern Education, Faculty of Education, Kio University, 4-2-2, Umami-naka, Koryo-cho, Kitakaturagi-gun, Nara, 635-0832, Japan
(b') y.yasui@kio.ac.jp
(c) General Topology

(a) **Eiichi Nakai**
(b) Department of Mathematics, Ibaraki University, Mito, Ibaraki 310-8512, Japan
(b') enakai@mx.ibaraki.ac.jp
(c) Real analysis, harmonic analysis, Fourier analysis, function spaces, singular and fractional integrals

(a) **Jun Kawabe**
(b) Department of Mathematical Sciences, Shinshu University, 4-17-1 Wakasato, Nagano 380-8553, Japan
(b') jkawabe@shinshu-u.ac.jp
(c) Measure and integration, Vector measure, Fuzzy measure

(a) **Shizu Nakanishi**
(b) 4-15 Jonanteramachi, Tennoji-Ku, Osaka 543-0017, Japan
(b') shizu.nakanishi@nifty.ne.jp
(c) measures and integrations

(a) **Jun Ichi Fujii**
(b) Department of Arts and Sciences (Information Science), Osaka Kyoiku
University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan
(b') fujii@cc.osaka-kyoiku.ac.jp
(c) Operator Theory

(a) **Masaru Nagisa**
(b) Dept. of Mathematics and Informatics, Faculty of Science, Chiba Univ., Yayoi-cho,
Chiba, 263-8522, Japan
(b') nagisa@math.s.chiba-u.ac.jp
(c) operator algebra, operator theory

(a) **Hiroyuki Osaka**
(b) Noji-Higashi 1-1-1, Kusatsu,525-8577, Japan
(b') osaka@se.ritsumei.ac.jp
(c) Operator Theory and Operator Algebras

(a) **Masatoshi Fujii**
(b) 46 Taka, Kashiba, Nara 639-0241, Japan
(b') mfujii@cc.osaka-kyoiku.ac.jp
(c) Operator Theory

(a) **Moto O'uchi**
(b) Dept. of Mathematics and Information Sciences, Graduate School of Science, Osaka Prefecture University, Sakai City,
Osaka 599-8531, Japan
(b') ouchi@mi.s.osakafu-u.ac.jp
(c) C*-algebras, C*-modules, groupoid C*-algebras

(a) **Wataru Takahashi**
(b) Shiratoridai 52-27, Aoba-ku, Yokohama 227-0054, Japan
(b') wataru@is.titech.ac.jp, wataru@a00.itscom.net
(c) Nonlinear Functional Analysis

(a) **Shigeo Akashi**
(b) Department of Information Sciences, Faculty of Science and Technology, Tokyo University of Science,
2641, Yamazaki, Noda-City, Chiba-Prefecture, 278-8510, Japan
(b') akashi@is.noda.tus.ac.jp
(c) Information Theory, Entropy Analysis, Applied Mathematics, Functional Analysis

(a) **Yoshitsugu Kabeya**
(b) Department of Mathematical Sciences, Osaka Prefecture University, 1-1, Gakuen-cho, Naka-ku, Sakai, Osaka 599-8531, Japan
(b') kabeya@ms.osakafu-u.ac.jp
(c) Partial Differential Equations, Ordinary Differential Equations

(a) **Atsushi Yagi**
(b) Dept. of Applied Physics, Graduate School of Engineering, Osaka Univ., 2-1 Yamadaoka, Suita, Osaka 565-0871, Japan
(b') yagi@ap.eng.osaka-u.ac.jp
(c) Nonlinear partial differential equations, Infinite-dimensional dynamical systems

(a) **Yoshimasa Nakamura**
(b) Graduate School of Informatics, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan
(b)' ynaka@i.kyoto-u.ac.jp
(c) integrable systems, numerical linear algebra, special functions

(a) **Yasumasa Fujisaki**
(b) Department of Information and Physical Sciences, Graduate School of Information Science and Technology, Osaka University, 1-5 Yamadaoka, Suita, Osaka 565-0871, Japan
(b') fujisaki@ist.osaka-u.ac.jp
(c) Control Systems Theory

(a) **Naruhiko Aizawa**
(b) Department of Mathematics and Information Sciences,Graduate School of Science,Osaka Prefecture University,Sakai,Osaka 599-8531,Japan
(b')aizawa@mi.s.osakafu-u.ac.jp
(c) representation theory

(a) **Hisao Nagao**
(b) 5-1-25-506, Senrioka, Settu, Osaka 566-0001, Japan
(b') nagao@sb.dcns.ne.jp
(c) Multivariate Analysis, Sequential Analysis, Jackknife Statistics and Bootstrap Method

(a) **Masamori Ihara**
(b) 18-8 Neayagawa, Osaka, 572-8530, Japan
(b') ihara@isc.osakac.ac.jp
(c) Structural equations modeling, Statistical Quality Managment, Factor analysis, Multivariate Analysis

(a) **Masanobu Taniguchi**
(b) Dept. of Applied Mathematics, School of Fundamental Science & Engineering, Waseda University, 3-4-1, Okubo, Shinjuku-ku, Tokyo,169-8555, Japan, Tel & Fax: 03-5286-8095
(b') taniguchi@waseda.jp
(c) Statistical Inference for Stochastic Processes

(a) **Masao Kondo**
(b) Dept. of Mathematics and Computer Science, Faculty of Science, Kagoshima University, 1-21-35 Korimoto, Kagoshima 890-0065, Japan
(b') kondo@sci.kagoshima-u.ac.jp
(c) Time Series Analysis

(a) **Masao Fukushima**
(b) Dept. of Systems and Mathematical Science, Faculty of Science and Engineering, Nanzan University, Nagoya, Aichi 466-8673, Japan
(b)' fuku@nanzan-u.ac.jp
(c) Mathematical Programming, Nonlinear Optimization

(a) **Ryusuke Hohzaki**
(b) Department of Computer Science, National Defense Academy, 1-10-20, Hashirimizu, Yokosuka, 239-8686, Japan
(b’) hozaki@cc.nda.ac.jp
(c) Reviewable area: Operations Research, Search theory, Game theory

(a) **Hiroaki Ishii**
(b) Department of Mathematical Sciences, School of Science and Technology, Kwansei Gakuin University
2-1 Gakuen, Sanda, Hyogo 669-1337, Japan
(b’) ishiihiroaki@kwansei.ac.jp
(c) Operations Research and Fuzzy Theory, especially Mathematical Programming (Stochastic Programming, Combinatorial Optimization, Fuzzy Programming), Scheduling Theory, Graph and Network Theory, Inventory control, Mathematical evaluation method

(a) **Junzo Watada**
(b) Graduate School of Information, Production and Systems (IPS) Waseda Univ., 2-7 Hibikino, Wakamatsuku, Kitakyushu, Fukuoka 808-0135, Japan
(b’) junzow@osb.att.ne.jp
(c) Fuzzy systems, Management Engineering

(a) **Kensaku Kikuta**
(b) School of Business Administration, University of Hyogo, 8-2-1 Gakuen-nishi-machi, Nishi-ku, Kobe City 651-2197 JAPAN
(b’) kikuta@biz.u-hyogo.ac.jp
(c) Game Theory, Operations Research,

(a) **Wuyi Yue**
(b) Dept. of Intelligence and Informatics, Faculty of Intelligence and Informatics, Konan University, 8-9-1 Okamoto, Higashinada-ku , Kobe 658-8501, JAPAN
(b’) yue@konan-u.ac.jp
(c) Queueing Networks, Performance Analysis and Modeling, Communications Networks, Operations Research, Markov Processes, Probabilistic Methods, Systems Engineering

(a) **Hiroaki Sandoh**
(b) Graduate School of Economics, Osaka University, 1-7, Machikaneyama, Toyonaka, Osaka 560-0043, Japan
(b’)sandoh@econ.osaka-u.ac.jp
(c) Operations Research and Management Science, Stochastic modeling

(a) **Yoshio Tabata**
(b) Takigi Ogaki 21, Kyotanabe, Kyoto 610-0341, Japan
(b’) tabata@econ.osaka-u.ac.jp
(c) Mathematical Finance, Sequential Decision Theory

(a) Katsunori Ano
(b) Department of Mathematical Sciences, Shibaura Institute of Technology, 307 Fukasaku Minuma-ku Saitama-city, 337-8570, Japan
(b)’ k-ano@shibaura-it.ac.jp
(c) Optimal Stopping, Mathematical Finance, Applied Probability

(a) **Koyu Uematsu**
(b)Dept.of Economics and Finance,Faculty of Business Administration,Osaka International University
3-50-1 Sugi Hirakata Osaka,573-0192,Japan
(b’) uematsu@oiu.jp
(c)Stochastic Process and its Applications,Reliability Analysis,and Game Theory

(a) **Yoshiki Kinoshita**
(b) Dept. of Information Sciences , Faculty of Science, Kanagawa University, Tsuchiya 2946, Hiratsuka-shi, Kanagawa 259-1293, Japan
(b') yoshiki@kanagawa-u.ac.jp
(c) Software Science, Programming language semantics

(a) **Shunsuke Sato**
(b)2-6-20 Hanayashiki-Soen,Takarazuka,Hyogo 665-0808, Japan
(b')ss_22362@nifty.com
(c) Mathematical biology in general, Neural networks, application of stochastic process

(a)**Tadashi Takahashi**
(b)Department of Intelligence and Informatics, Konan University, 8-9-1 Okamoto, Higashinada, Kobe, Hyogo 658-8501, Japan
(b') takahasi@konan-u.ac.jp
(c)Mathematics Education

(a) **Benoit Collins**
(b) Department of Mathematics, Faculty of Science, Kyoto University
(b') collins@math.kyoto-u.ac.jp
(c) Random Matrix Theory, Free Probability, Quantum Information Theory
Quantum Groups (operator algebra side), Operator Algebra

****

Managing Editor
Koyu Uematsu (Professor of Osaka International University)
International Society for Mathematical Sciences
1-5-12-202 Kaorigaoka-cho, Sakai-ku, Sakai-city, 590-0011,Japan
uematsu@jams.jp

Submission to the SCMJ

In September 2012, the way of submission to Scientiae Mathematicae Japonicae (SCMJ) was changed. Submissions should be sent electronically (in PDF file) to the editorial office of International Society for Mathematical Sciences (ISMS).

(1) Preparation of files and Submission

a. Authors who would like to submit their papers to the SCMJ should make source files of their papers in LaTeX2e using the ISMS style file (scmjlt2e.sty) Submissions should be in PDF file compiled from the source files. Send the PDF file to s1bmt@jams.jp .

b. Prepare a Submission Form and send it to the ISMS. The required items to be contained in the form are:

1. Editor's name whom the author chooses from the Editorial Board (http://www.jams.or.jp/hp/submission_f.html)and would like to take in charge of the paper for refereeing.
2. Title of the paper.
3. Authors' names.
4. Corresponding author's name, e-mail address and postal address (affiliation).
5. Membership number in case the author is an ISMS member.

Japanese authors should write 3 and 4 both in English and in Japanese.

At http://www.jams.or.jp/hp/submission_f.html, the author can find the Submission Form. Fulfill the Form and sent it to the editorial office by pushing the button "transmission". Or, without using the Form, the author may send an e-mail containing the items 1-5 to s1bmt@jams.jp

(2) Registration of Papers

When the editorial office receives both a PDF file of a submitted paper and a Submission Form, we register the paper. We inform the author of the registration number and the received date. At the same time, we send the PDF file to the editor whom the author chooses in the Submission Form and request him/her to begin the process of refereeing. (Authors need not send their papers to the editor they choose.)

(3) Reviewing Process

a. The editor who receives, from the editorial office, the PDF file and the request of starting the reviewing process, he/she will find an appropriate referee for the paper.

b. The referee sends a report to the editor. When revision of the paper is necessary, the editor informs the author of the referee's opinion.

c. Based on the referee report, the editor sends his/her decision (acceptance of rejection) to the editorial office.

(4) a. Managing Editor of the SCMJ makes the final decision to the paper valuing the editor's decision, and informs it to the author.

b. When the paper is accepted, we ask the author to send us a source file and a PDF file of the final manuscript.

c. The publication charges for the ISMS members are free if the membership dues have been paid without delay. If the authors of the accepted papers are not the ISMS members, they should become ISMS members and pay ¥6,000 (US$75, Euro55) as the membership dues for a year, or should just pay the same amount without becoming the members.

Items required in Submission Form

1. Editor's name who the authors wish will take in charge of the paper
2. Title of the paper
3. Authors' names

3'. 3. in Japanese for Japanese authors

4. Corresponding author's name and postal address (affiliation)

4'. 4. in Japanese for Japanese authors

5. ISMS membership number
6. E-mail address

Call for ISMS Members

Call for Academic and Institutional Members

Discounted subscription price: When organizations become the Academic and Institutional Members of the ISMS, they can subscribe our journal Scientiae Mathematicae Japonicae at the yearly price of US$225. At this price, they can add the subscription of the online version upon their request.

Invitation of two associate members: We would like to invite two persons from the organizations to the associate members with no membership fees. The two persons will enjoy almost the same privileges as the individual members. Although the associate members cannot have their own ID Name and Password to read the online version of SCMJ, they can read the online version of SCMJ at their organization.

To apply for the Academic and Institutional Member of the ISMS, please use the following application form.

Application for Academic and Institutional Member of ISMS

Subscription of SCMJ Check one of the two.	☐Print (US$225)	☐Print ＋ Online (US$225)
University (Institution)		
Department		
Postal Address where SCMJ should be sent		
E-mail address		
Person in charge	Name: Signature:	
Payment Check one of the two.	☐Bank transfer	☐Credit Card (Visa, Master)
Name of Associate Membership	1.	
	2.	

Call for Individual Members

We call for individual members. The privileges to them and the membership dues are shown in "**Join ISMS !**" on the inside of the back cover.

Items required in Membership Application Form

1. Name
2. Birth date
3. Academic background
4. Affiliation
5. 4's address
6. Doctorate
7. Contact address
8. E-mail address
9. Special fields
10. Membership category (See **Table 1** in "**Join ISMS !**")

Individual Membership Application Form

1. **Name**	
2. **Birth date**	
3. **Academic background**	
4. **Affiliation**	
5. **4's address**	
6. **Doctorate**	
7. **Contact address**	
8. **E-mail address**	
9. **Special fields**	
10. **Membership category**	

Contributions (Gift to the ISMS)

We deeply appreciate your generous contributions to support the activities of our society.

The donation are used (1) to make medals for the new prizes (Kitagawa Prize, Kunugi Prize, and ISMS Prize), (2) to support the IVMS at Osaka University Nakanoshima Center, and (3) for a special fund designated by the contributors.

Your remittance to the following accounts of ours will be very much appreciated.

(1) Through a post office, remit to our giro account (in Yen only):
No. 00930-1-11872, Japanese Association of Mathematical Sciences (JAMS)
or send International Postal Money Order (in US Dollar or in Yen) to our address:
International Society for Mathematical Sciences
2-1-18 Minami Hanadaguchi, Sakai-ku, Sakai, Osaka 590-0075, Japan

(2) A/C 94103518, ISMS
CITIBANK, Japan Ltd., Shinsaibashi Branch
Midosuji Diamond Building
2-1-2 Nishi Shinsaibashi, Chuo-ku, Osaka 542-0086, Japan

****

Payment Instructions:

Payment can be made through a post office or a bank, or by credit card. Members may choose the most convenient way of remittance. Please note that we do not accept payment by bank drafts (checks). For more information, please refer to an invoice.

Methods of Overseas Payment:

Payment can be made through (1) a post office, (2) a bank, (3) by credit card, or (4) UNESCO Coupons.

Authors or members may choose the most convenient way of remittance as are shown below. Please note that **we do not accept payment by bank drafts (checks)**.

(1) Remittance through a post office to our giro account No. 00930-1-11872 or send International Postal Money Order to our postal address (2) Remittance through a bank to our account No. 94103518 at Shinsaibashi Branch of CITIBANK (3) **Payment by credit cards** (AMEX, VISA, MASTER or NICOS), or (4) Payment by UNESCO Coupons.

Methods of Domestic Payment:

Make remittance to:

(1) Post Office Transfer Account - 00930-3-73982 or

(2) Account No.7726251 at Sakai Branch, SUMITOMO MITSUI BANKING CORPORATION, Sakai, Osaka, Japan.

All of the correspondences concerning subscriptions, back numbers, individual and institutional memberships, should be addressed to the Publications Department, International Society for Mathematical Sciences.

Join ISMS !

ISMS Publications: We published **Mathematica Japonica (M.J.)** in print, which was first published in 1948 and has gained an international reputation in about sixty years, and its offshoot **Scientiae Mathematicae (SCM)** both online and in print. In January 2001, the two publications were unified and changed to **Scientiae Mathematicae Japonicae (SCMJ)**, which is the "21st Century New Unified Series of Mathematica Japonica and Scientiae Mathematicae" and published both online and in print. Ahead of this, the online version of SCMJ was first published in September 2000. The whole number of SCMJ exceeds 270, which is the largest amount in the publications of mathematical sciences in Japan. The features of SCMJ are:

1) About 80 eminent professors and researchers of not only Japan but also 20 foreign countries join the Editorial Board. The accepted papers are published both online and in print. SCMJ is reviewed by Mathematical Review and Zentralblatt from cover to cover.
2) SCMJ is distributed to many libraries of the world. The papers in SCMJ are introduced to the relevant research groups for the positive exchanges between researchers.
3) **ISMS Annual Meeting**: Many researchers of ISMS members and non-members gather and take time to make presentations and discussions in their research groups every year.

The privileges to the individual ISMS Members:

(1) No publication charges
(2) Free access (**including printing out**) to the online version of SCMJ
(3) Free copy of each printed issue

The privileges to the Institutional Members:

Two associate members can be registered, free of charge, from an institution.

Table 1: Membership Dues for 2015

Categories	**Domestic**	**Overseas**	**Developing countries**
1-year Regular member	￥8,000	US$80 , Euro75	US$50, Euro47
1-year Students member	￥4,000	US$50 , Euro47	US$30 , Euro28
Life member*	Calculated as below*	US$750 , Euro710	US$440, Euro416
Honorary member	Free	Free	Free

(Regarding submitted papers,we apply above presented new fee after April 15 in 2015 on registoration date.) * Regular member between 63 - 73 years old can apply the category.

(73 − age) × ￥3,000

Regular member over 73 years old can maintain the qualification and the privileges of the ISMS members, if they wish.

Categories of 3-year members were abolished.

CONTENTS

Notices from the ISMS