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AN INVESTIGATION OF UNITARILY INVARIANT NORM INEQUALITIES OF LÖWNER–HEINZ TYPE

M. Fujii[1], M.S. Moslehian[2], R. Nakamoto[3] and M. Tominaga[4]

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Abstract. We utilize some 2×2 matrix tricks to obtain several unitarily invariant norm inequalities corresponding to the Löwner–Heinz inequality, the arithmetic–geometric mean inequality and the Corach–Porta–Recht inequality. Among others, we establish some norm inequalities for unitarily invariant norms implying an extended Löwner–Heinz inequality.

1 Introduction. Let $\mathbb{B}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ and let I be its identity. We write $A \geq 0$ if A is a positive operator in the sense that $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$. Further, $A \geq B$ if A and B are self-adjoint and $A - B \geq 0$. By a strictly positive operator A, denoted by $A > 0$, we mean a positive operator being invertible. If A, B are operators in $\mathbb{B}(\mathscr{H})$, we write the direct sum $A \oplus B$ for the 2×2 operator matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, regarded as an operator on $\mathscr{H} \oplus \mathscr{H}$. Let $\mathbb{K}(\mathscr{H})$ denote the ideal of compact operators on $\mathscr{H}$. For any operator $A \in \mathbb{K}(\mathscr{H})$, let $s_1(A), s_2(A), \cdots$ be the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$ in decreasing order and repeated according to multiplicity. If $A \in \mathcal{M}_n$, we take $s_k(A) = 0$ for $k > n$. A unitarily invariant norm in $\mathbb{K}(\mathscr{H})$ is a map $|||\cdot||| : \mathbb{K}(\mathscr{H}) \to [0, \infty]$ given by $|||A||| = g(s(A))$, $A \in \mathbb{K}(\mathscr{H})$, where g is a symmetric gauge function; cf. [12]. The set $\mathcal{I} = \mathcal{I}_{|||\cdot|||} = \{A \in \mathbb{K}(\mathscr{H}) : |||A||| < \infty\}$ is a (two-sided) ideal of $\mathbb{B}(\mathscr{H})$ by the basic property (1) in the below. An operator $A \in \mathbb{K}(\mathscr{H})$ is said to be in the Schatten p-class $\mathcal{C}_p$ $(1 \leq p < \infty)$, if $\sum_j s_j(A)^p < \infty$. The Schatten p-norm of A is defined by $\|A\|_p = \left(\sum_j s_j(A)^p\right)^{\frac{1}{p}}$, which is a typical example of a unitarily invariant norm. Other examples of unitarily invariant norms are the operator norm and the Ky Fan norms $\|A\|_{(k)} := \sum_{j=1}^k s_j(A)$, $k \in \mathbb{N}$ under decreasingly arranged on j. Some of basic properties are as follow:

(1) If $B \in \mathcal{I}$, then $|||B||| = |||\,|B|\,||| = |||B^*|||$ and $|||ABC||| \leq \|A\|\, |||B|||\, \|C\|$ for any $A, C \in \mathbb{B}(\mathscr{H})$.

(2) It follows from the Fan dominance principle (see e.g. [1]) that $|||A||| \leq |||B|||$ for all unitarily invariant norms if and only if $|||A \oplus 0||| \leq |||B \oplus 0|||$ for all unitarily invariant norms.

Let $\Lambda^k\mathscr{H}$ be the subspace of the k-fold tensor product $\otimes^k\mathscr{H}$ spanned by antisymmetric tensors. Then the k-fold product $\otimes^k A$ of an operator A on $\mathscr{H}$ leaves this space invariant and the restriction of $\otimes^k A$ to it, denoted by $\Lambda^k A$, is called the exterior power of A. Λ^k is multiplicative, $*$-preserving and unital. We denote the weak-log majorization and the weak majorization , $\prec_{w-log}$ and $\prec_w$, respectively. The following relations among them are known; cf. [1]. Let $X, Y \in \mathbb{K}(\mathscr{H})$. Then

$$\begin{aligned} &|X| \prec_{w-log} |Y| \quad (\text{i.e., } \left\|\Lambda^k X\right\| \leq \left\|\Lambda^k Y\right\| \text{ for any } k \leq n) \\ &\Rightarrow |X| \prec_w |Y| \quad (\text{i.e., } \|X\|_{(k)} \leq \|Y\|_{(k)} \text{ for any } k \leq n). \end{aligned}$$

So the Fan Dominance theorem is rephrased as

$$|X| \prec_{w-log} |Y| \quad \Longrightarrow \quad |||X||| \leq |||Y|||\,.$$

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Now, let us pay attention to some literature reviews. The Heinz inequality states that for $A, B, X \in \mathbb{B}(\mathscr{H})$ with $A, B \geq 0$,

$$\|AX + XB\| \geq \left\|A^\alpha XB^{1-\alpha} + A^{1-\alpha}XB^\alpha\right\|$$

for $0 \leq \alpha \leq 1$, which is one of essential inequalities in operator theory.

McIntosh [11] proved that for all $A, B, X \in \mathbb{B}(\mathscr{H})$,

$$\|A^*AX + XB^*B\| \geq 2\,\|AXB^*\|\,, \tag{1.1}$$

which is called the arithmetic-geometric mean inequality; see also [7]. Bhatia and Kittaneh [4] proved that (1.1) holds for any unitarily invariant norm.

If $A \in \mathbb{B}(\mathscr{H})$ is invertible and self-adjoint, Corach et al. [6] proved that

$$\left\|A^{-1}XA + AXA^{-1}\right\| \geq 2\,\|X\|$$

for every $X \in \mathbb{B}(\mathscr{H})$. It plays a key role in the study of differential geometry of self-adjoint operators, and it has been investigated in [8] as well as [5]. On the other hand, it is known that the Löwner–Heinz inequality

$$A \geq B \geq 0 \quad \text{implies} \quad A^p \geq B^p \quad \text{for all } 0 \leq p \leq 1$$

is equivalent to the Araki–Cordes inequality (see [1], [8])

$$\|AB\|^p \geq \|A^pB^p\| \qquad \text{for all} \quad A, B \geq 0 \quad \text{and} \quad 0 \leq p \leq 1. \tag{1.2}$$

In particular, the case $p = \frac{1}{2}$ in (1.2), i.e.

$$\left\|A^2B^2\right\| \geq \left\|AB^2A\right\| \quad \text{for all} \quad A, B \geq 0, \tag{1.3}$$

is essential, which is implied by the Heinz inequality; see [7] and [8].

In this paper, we investigate several unitarily invariant norm inequalities corresponding to the Löwner–Heinz inequality, the arithmetic–geometric mean inequality and the Corach–Porta–Recht inequality. Among others, we propose some norm inequalities for unitarily invariant norms implying an extended Löwner–Heinz inequality.

2 Löwner–Heinz type inequalities. As stated in [1], a Heinz type inequality can be regarded as the arithmetic–geometric mean inequality as follows: Let $A \geq 0$ be a matrix and X a self-adjoint matrix. Then

$$|\!|\!|\mathrm{Re}\;(\alpha AX + (1-\alpha)XA)|\!|\!| \geq |\!|\!|\mathrm{Re}\;(A^\alpha XA^{1-\alpha})|\!|\!| \quad \text{for } \alpha \in [0, 1].$$

We note that the equivalence among Heinz type inequalities for matrices is discussed by Furuta [9]. Now we recall some relations among the Heinz inequality, the Löwner–Heinz inequality and corresponding norm inequalities for the operator norm $\|\cdot\|$; see [7]:

$$\text{Heinz inequality} \Longleftrightarrow \|\mathrm{Re}\,AX\| \geq \|XA\| \text{ if } A \geq 0 \text{ and } XA \text{ is self-adjoint,}$$
$$\text{Löwner–Heinz inequality} \Longleftrightarrow \|AX\| \geq \|XA\| \text{ if } A \geq 0 \text{ and } XA \text{ is self-adjoint.}$$

In the above inequality, if we take $X = AY$ for any $Y = Y^*$, then we have the inequality $\left\|A^2Y\right\| \geq \|AYA\|$ for $A \geq 0$. In other word, we have

$$\|AX\| \geq \left\|A^{1/2}XA^{1/2}\right\| \quad \text{for } A \geq 0 \text{ and } X = X^*. \tag{2.1}$$

Conversely, if we assume that (2.1) holds for $A \geq 0$ and $X = X^*$, then it implies

$$\|AB\| \geq \left\|A^{1/2}B^{1/2}\right\|^2,$$

that is, (1.3) is obtained and so it ensures the Löwner–Heinz inequality. Namely it is proved that (2.1) is equivalent to the Löwner–Heinz inequality.

We here remark that (2.1) does not hold for nonselfadjoint X in general. As a matter of fact, we have a counterexample as follows: Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } X = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}.$$

Then $\|A^2X\| = 4$ and $\|AXA\| = 6$.

In succession, we consider the convexity of the function

$$h(t) = \left\|A^t X A^{1-t}\right\| \quad \text{for } t \in [0,1],$$

where $A \geq 0$ and $X = X^*$.

Theorem 2.1. *The function $h(t)$ defined above is convex.*

Proof. For $\mu < \nu$, we take $t = (\mu+\nu)/2$ and $p = t - \mu = \nu - t > 0$. Then

$$\begin{aligned} h(t)^2 &= \left\|A^t X A^{2(1-t)} X A^t\right\| \\ &= r(A^t X A^{2(1-t)} X A^t) = r(A^\mu X A^{1-\mu} A^{1-\nu} X A^\nu) \\ &\leq \left\|A^\mu X A^{1-\mu} A^{1-\nu} X A^\nu\right\| \\ &\leq \left\|A^\mu X A^{1-\mu}\right\| \left\|A^\nu X A^{1-\nu}\right\| = h(\mu) \cdot h(\nu), \end{aligned}$$

where $r(C)$ denotes the spectral radius of the operator $C \in \mathbb{B}(\mathscr{H})$. Therefore

$$h(t) \leq h(\mu)^{1/2} h(\nu)^{1/2} \leq \frac{1}{2}(h(\mu) + h(\nu))$$

so that the continuous function $h(t)$ is convex. $\square$

Next we consider the function

$$g(t) := \left|\!\left|\!\left|A^t X A^{1-t}\right|\!\right|\!\right| \quad \text{for } t \in [0,1], \tag{2.2}$$

where $A \geq 0$ and $X \in \mathcal{I}$ with $X = X^*$. Here we remark that every normalized unitarily invariant norm is submultiplicative (see [1, p.94]):

$$|||AB||| \leq |||A||| \cdot |||B||| \quad \text{for all } A, B \in \mathbb{K}(\mathscr{H}). \tag{2.3}$$

Corollary 2.2. *If $|||\cdot|||$ is normalized and $X = X^*$, then the function $g(t)$ defined in* (2.2) *is* log-*convex on* $[0,1]$ *and is symmetric at* $\frac{1}{2}$. *Consequently, $g(t)$ is convex for arbitrary unitarily invariant norm and so $g(t) \geq g(\frac{1}{2})$.*

Proof. As in the proof of Theorem 2.1, we have, under the same notation,

$$\begin{aligned} \left\|\Lambda^k(A^t X A^{1-t})\right\|^2 &= \left\|(\Lambda^k A^t)(\Lambda^k X)(\Lambda^k A^{1-t})\right\|^2 \\ &\leq \left\|(\Lambda^k A^\mu)(\Lambda^k X)(\Lambda^k A^{1-\mu})(\Lambda^k A^{1-\nu})(\Lambda^k X)(\Lambda^k A^\nu)\right\| \\ &= \left\|\Lambda^k(A^\mu X A^{1-\mu} \cdot A^{1-\nu} X A^\nu)\right\|, \end{aligned}$$

whence

$$\left|\!\left|\!\left|A^t X A^{1-t}\right|\!\right|\!\right|^2 \leq \left|\!\left|\!\left|A^\mu X A^{1-\mu} \cdot A^{1-\nu} X A^\nu\right|\!\right|\!\right|.$$

Moreover, since every normalized unitarily invariant norm is submultiplicative, we get

$$\left|\!\left|\!\left|A^t X A^{1-t}\right|\!\right|\!\right|^2 \leq \left|\!\left|\!\left|A^\mu X A^{1-\mu}\right|\!\right|\!\right| \left|\!\left|\!\left|A^{1-\nu} X A^\nu\right|\!\right|\!\right|,$$

that is, $g(t)^2 \leq g(\mu)g(\nu)$. Therefore $g(t)$ is log-convex and so

$$g(t) \leq \frac{1}{2}(g(\mu) + g(\nu)).$$

Hence the continuous function $g(t)$ is convex. In addition, since the convexity is invariant under positive scalar multiple, $g(t)$ is convex for any arbitrary unitarily invariant norm. $\square$

As a result, the following inequalities are obtained:

Corollary 2.3. (1) *The following inequality holds:*

$$\|AX\| \geq \left\| A^\alpha X A^{1-\alpha} \right\| \quad \text{for } A \geq 0,\ X = X^* \in \mathcal{I} \text{ and } 0 \leq \alpha \leq 1. \tag{2.4}$$

(2) *The function $g(t)$ defined in* (2.2) *is monotone decreasing on* $[0, \frac{1}{2}]$ *and monotone increasing on* $[\frac{1}{2}, 1]$ *and consequently*

$$\left\| A^t X A^{1-t} \right\| \geq \left\| A^{\frac{1}{2}} X A^{\frac{1}{2}} \right\| \ (0 \leq t \leq 1) \quad \text{and} \quad \|AX\| \geq \left\| A^{\frac{1}{2}} X A^{\frac{1}{2}} \right\|.$$

Remark 2.4. *We should mention that inequality* (2.4) *follows from [2, Theorem 2], and (2) of Corollary 2.3 follows from the generalized Heinz inequality proved by Bhatia and Davis in [3], but our both approaches are rather different.*

Under these preparations, we have several Löwner–Heinz type inequalities as follows:

Theorem 2.5. *The following mutually equivalent inequalities hold:*

$$\left\| AXA^{-1} \right\| \geq \|X\| \quad \text{for any invertible } A \text{ and } X = X^* \in \mathcal{I}; \tag{2.5}$$

$$\|AX\| \geq \|XA\| \quad \text{for any invertible } A \text{ and } X \in \mathcal{I} \text{ such that } XA \text{ is selfadjoint}; \tag{2.6}$$

$$\|AA^*X\| \geq \|A^*XA\| \quad \text{for any invertible } A \text{ and } X = X^* \in \mathcal{I}. \tag{2.7}$$

Proof. First of all, by putting $\alpha = \frac{1}{2}$ and replacing A by $AA^* = |A^*|^2$ in (2.4), (2.7) is obtained:

$$\|AA^*X\| \geq \||A^*|X|A^*|\| = \|A^*XA\|$$

because $A^* = U|A^*|$ with unitary U.

Next we show that (2.5) $\Rightarrow$ (2.6) $\Rightarrow$ (2.7) $\Rightarrow$ (2.5).

(2.5) $\Rightarrow$ (2.6): Since XA is selfadjoint, it follows from (2.5) that

$$\|XA\| \leq \left\| A(XA)A^{-1} \right\| = \|AX\|.$$

(2.6) $\Rightarrow$ (2.7): Since a given X is selfadjoint, so is A^*XA. Hence (2.7) is obtained by replacing X by $X_1 = A^*X$ in (2.6), that is,

$$\|AA^*X\| = \|AX_1\| \geq \|X_1A\| = \|A^*XA\|.$$

(2.7) $\Rightarrow$ (2.5): It is obtained by replacing X by $A^{*-1}XA^{-1}$ in (2.7). □

Theorem 2.6. *For $A, B \geq 0$ and $X \in \mathcal{I}$ it holds that*

$$\|AX \oplus BX^*\| \geq \left\| A^\alpha X B^{1-\alpha} \oplus B^\alpha X^* A^{1-\alpha} \right\| \quad \text{for } 0 \leq \alpha \leq 1. \tag{2.8}$$

Consequently,

$$\left\| A^{2m+n} X B^{-n} \oplus B^{2m+n} X^* A^{-n} \right\| \geq \left\| A^{2m} X \oplus B^{2m} X^* \right\|, \tag{2.9}$$

where m, n are arbitrary nonnegative integers.

Proof. We note that

$$\|AX \oplus BX^*\| = \left\| \begin{bmatrix} AX & 0 \\ 0 & BX^* \end{bmatrix} \right\|$$

$$= \left|\left|\left| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix} \right|\right|\right|$$
$$= \left|\left|\left| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \right|\right|\right|$$
$$= \left|\left|\left| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \right|\right|\right|$$

and

$$\begin{aligned} |||A^\alpha XB^{1-\alpha} \oplus B^\alpha X^* A^{1-\alpha}||| &= \left|\left|\left| \begin{bmatrix} A^\alpha & 0 \\ 0 & B^\alpha \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix} \begin{bmatrix} B^{1-\alpha} & 0 \\ 0 & A^{1-\alpha} \end{bmatrix} \right|\right|\right| \\ &= \left|\left|\left| \begin{bmatrix} A^\alpha & 0 \\ 0 & B^\alpha \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} B^{1-\alpha} & 0 \\ 0 & A^{1-\alpha} \end{bmatrix} \right|\right|\right| \\ &= \left|\left|\left| \begin{bmatrix} A^\alpha & 0 \\ 0 & B^\alpha \end{bmatrix} \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \begin{bmatrix} A^{1-\alpha} & 0 \\ 0 & B^{1-\alpha} \end{bmatrix} \right|\right|\right| \\ &= \left|\left|\left| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^\alpha \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^{1-\alpha} \right|\right|\right|. \end{aligned}$$

Hence the desired inequality (2.8) is ensured by (2.4). Inequality (2.9) can be obtained from (2.8) if we replace A, B, X by A^{2m+2n}, B^{2m+2n}, $A^{-n}XB^{-n}$, respectively and put $\alpha = (2m+n)(2m+2n)^{-1}$. □

Remark 2.7. *Related to inequality* (2.8)*, we have the following complementary inequality:*

$$|||A^\beta XB^{1-\beta} \oplus B^\beta X^* A^{1-\beta}||| \geq |||AX \oplus BX^*||| \quad (\beta \notin (0,1)).$$

Indeed, it can be shown by replacing A, B, X and α by $A^{2\beta-1}$, $B^{2\beta-1}$, $A^{1-\beta}XB^{1-\beta}$ and $\frac{\beta}{2\beta-1}$, respectively in (2.8)*.*

If A and B are positive invertible, then (2.7) *holds for $\beta \notin (0,1)$.*

Corollary 2.8. *The following inequalities hold and equivalent:*

(2.10) $|||A^*AX \oplus B^*BX^*||| \geq |||AXB^* \oplus BX^*A^*|||$ *for $A, B \in B(\mathscr{H})$ and $X \in \mathcal{I}$;*

(2.11) $|||AXB^{-1} \oplus BX^*A^{-1}||| \geq |||X \oplus X^*|||$ *for any invertible A, B and $X \in \mathcal{I}$.*

Proof. First of all, we show (2.10) by utilizing (2.8). Let $A = U|A|$ and $B = V|B|$ be the polar decompositions of A and B, respectively. We replace A and B by A^*A and B^*B, respectively, in (2.8) and put $\alpha = \frac{1}{2}$. Then we have

$$\begin{aligned} |||A^*AX \oplus B^*BX^*||| &\geq |||\,|A|X|B| \oplus |B|X^*|A|\,||| \\ &= \|U \oplus V\|\, |||\,|A|X|B| \oplus |B|X^*|A|\,|||\, \|V^* \oplus U^*\| \\ &\geq |||(U \oplus V)(|A|X|B| \oplus |B|X^*|A|)(V^* \oplus U^*)||| \\ &= |||AXB^* \oplus BX^*A^*|||\,. \end{aligned} \tag{2.12}$$

Next (2.10) ⇒ (2.11) has been mentioned in [10]. We state its proof for the sake of completeness. Replacing X by $A^{-1}XB^{*-1}$ in (2.10), we have

$$|||A^*XB^{*-1} \oplus B^*X^*A^{*-1}||| \geq |||X \oplus X^*|||\,,$$

so that (2.11) is obtained by replacing A^* and B^* by A and B, respectively.

Finally we show (2.11) ⇒ (2.10). Let $A = U|A|$ and $B = V|B|$ be the polar decompositions of A and B. We may assume that $|A|, |B|$ are invertible. It follows from (2.11) that

$$\begin{aligned} |||A^*AX \oplus B^*BX^*||| &= |||\,|A|(|A|X|B|)|B|^{-1} \oplus |B|(|B|X^*|A|)|A|^{-1}\,||| \\ &\geq |||\,|A|X|B| \oplus |B|X^*|A|\,||| \\ &\geq |||AXB^* \oplus BX^*A^*||| \end{aligned}$$

as we observed in (2.12). □

Remark 2.9. *We comment that* (2.11) *is implied by* (2.7)*. Put* $C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ *and* $Y = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}$*. It follows from* (2.7) *that*

$$\begin{aligned} \left|\!\left|\!\left| AXB^{-1} \oplus BX^*A^{-1} \right|\!\right|\!\right| &= \left|\!\left|\!\left| \begin{bmatrix} 0 & AXB^{-1} \\ BX^*A^{-1} & 0 \end{bmatrix} \right|\!\right|\!\right| \\ &= \left|\!\left|\!\left| CYC^{-1} \right|\!\right|\!\right| = \left|\!\left|\!\left| C^2(C^{-1}YC^{-1}) \right|\!\right|\!\right| \\ &\geq \left|\!\left|\!\left| C(C^{-1}YC^{-1})C \right|\!\right|\!\right| = \left|\!\left|\!\left| Y \right|\!\right|\!\right| = \left|\!\left|\!\left| X \oplus X^* \right|\!\right|\!\right|. \end{aligned}$$

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Communicated by *Junichi Fujii*

[1] Department of Mathematics, OsakaKyoiku University, Kashiwara, Osaka 582-8582, Japan.

E-mail address, mfujii@cc.osaka-kyoiku.ac.jp

[2] Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran.

E-mail address, moslehian@um.ac.ir and moslehian@member.ams.org

[3] 1-4-13, Daihara-cho, Hitachi, Ibaraki 316-0021, Japan.

E-mail address, r-naka@net1.jway.ne.jp

[4] Practical School Education, Osaka Kyoiku University, 4-88 Minamikawahori-cho, Tennoji-ku, Osaka 543-0054, Japan.

E-mail address, tommy@cc.osaka-kyoiku.ac.jp

THE EDGEWORTH EXPANSION FOR THE NUMBER OF DISTINCT OBSERVATIONS WITH THE MIXTURE OF DIRICHLET PROCESSES

HAJIME YAMATO AND MASAO KONDO

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ABSTRACT. Let a random distribution $\mathcal{P}$ on the real line $\mathbb{R}$ have the mixture of Dirichlet processes. Let $S^{(n)} = (S_1, \cdots, S_n)$ be the random partition of the positive integer n based on a sample of size n from $\mathcal{P}$. For the number $K_n = S_1 + \cdots + S_n$ of distinct observations among the sample, Yamato (2012) gives the asymptotic distribution of K_n and the rate $O(1/\log^{1/3} n)$ of its convergence. In this pager we give the Edgeworth expansion for K_n with the rate $O(1/\log^{2/5} n)$ and the rate $O(1/\log^{3/7} n)$.

1 Introduction. Let H_0 be a continuous distribution on the real line $\mathbb{R}$ and $\mathcal{B}$ be the σ-field which consists of the subsets of $\mathbb{R}$. Let θ be a positive random variable having the distribution γ. Given θ, let the random distribution $\mathcal{P}$ have the Dirichlet process $\mathcal{D}(\theta H_0)$ on ($\mathbb{R}$, $\mathcal{B}$) with parameters θ and H_0. Then this random distribution $\mathcal{P}$ has the mixture of Dirichlet processes $\mathcal{D}(\theta H_0)$ with the mixing distribution γ (Antoniak (1974)). For a sample of size n from the random distribution $\mathcal{P}$, S_1 denotes the number of observations which occur only once, S_2 the number of observations which occur exactly twice, ... and so on. Then $K_n = S_1 + \cdots + S_n$ denotes the number of distinct observations among the sample. For the convergence of K_n, Yamato (2012) gives

$$\sup_{-\infty<x<\infty} \left| P\left(\frac{K_n}{\log n} \le x \right) - \gamma(x) \right| = O\left(\frac{1}{\log^{1/3} n} \right).$$

In case the distribution γ is degenerate at θ_0, that is the θ equals to a positive constant θ_0, the random distribution $\mathcal{P}$ has the Dirichlet process $\mathcal{D}(\theta_0 H_0)$. Then, K_n has the well-known Ewens sampling formula and the asymptotic normality, whose Edgeworth expansion is given by

$$P\left(\frac{K_n - \theta_0 \log n}{\sqrt{\theta_0 \log n}} \le x \right) = \Phi(x) - \frac{1}{6\sqrt{\theta_0 \log n}} \phi(x) \left[x^2 - 1 - 6\theta_0 \psi(\theta_0) \right] + O\left(\frac{1}{\log n} \right),$$

which holds uniformly in $x \in \mathbb{R}$ (Yamato (2013)). Here Φ and ϕ are the distribution function and the density function of the standard normal distribution, respectively, and ψ is the digamma function defined by $\psi(x) = \Gamma'(x)/\Gamma(x)$, where $\Gamma(x)$ is the gamma function. The purpose of this paper is to give the Edgeworth expansion for K_n, in case $\mathcal{P}$ has the mixture of Dirichlet processes $\mathcal{D}(\theta H_0)$ with the mixing distribution γ which is not degenerate. We denote the distribution function (d.f) of the distribution γ by $G(x)$. Let g be the bounded density function of the d.f. G.

In the section 2 , we give the Edgeworth expansion for K_n with the rate $O(1/\log^{2/5} n)$. In the section 3, we give it with the rate $O(1/\log^{3/7} n)$. In the section 4, we show numerical examples.

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2 The Edgeworth expansion with the rate $1/\log^{2/5} n$. We first note the random variable P_n^*, which has the Poisson distribution with the mean $\theta(\log n - \psi(\theta))$, given θ. By Lemma 2.1 of Yamato (2013), we have:

Lemma 2.1 *Under the condition that $E_\gamma\theta$ and $E_\gamma\theta^{-1}$ are finite ,*

$$\sup_{B\subset\mathbb{Z}_+}\big|P(K_n\in B)-P(P_n^*\in B)\big| = O\Big(\frac{1}{\log n}\Big),\quad n\to\infty, \tag{2.1}$$

where where $\mathbb{Z}_+=\{0,1,2,\cdots\}$ and E_γ denotes the expectation with respect to the distribution γ.

In this section 2, we suppose that $E_\gamma(\theta^{-1})$, $E_\gamma(\theta^2)$, and $E_\gamma[\theta^2\psi(\theta)^2]$ exist. The following conditions are necessary for the proof of the proposition 2.2 using the smoothing lemma (see, for example, Petrov (1995; Theorem 5.2)); (i) $g(x)$ is twice differentiable, (ii) $x\psi(x+1)g(x)$, $g(x)$ and $xg'(x)$ are the functions of bounded variation, (iii) $g'(x)$, $xg''(x)$ and $[x\psi(x+1)g(x)]'$ are bounded, and (iv) $g(x)\to 0$, $xg'(x)\to 0$ as $x\to 0$, and $x\psi(x+1)g(x)\to 0$, $xg'(x)\to 0$ as $x\to+\infty$. Note that for $x\geq 0$, $\psi(x+1)$ is monotone increasing and $\psi(x+1)\geq\psi(1)$, where $-\psi(1)$ equals Euler's constant $(=0.57721\cdots)$. Then we have:

Proposition 2.2 *For $n>3$, we have*

$$\sup_{-\infty<x<\infty}\left|P\Big(\frac{K_n}{\log n}\leq x\Big)-\Big[G(x)+\frac{1}{2\log n}\Big\{\big[2x\psi(x+1)-1\big]g(x)+xg'(x)\Big\}\Big]\right| = O\Big(\frac{1}{\log^{2/5} n}\Big). \tag{2.2}$$

In the following proof, we use the well-known relations for any complex number z such that

$$e^z = 1+z+\frac{c_1}{2}\mid z\mid^2, \tag{2.3}$$

$$= 1+z+\frac{1}{2}z^2+\frac{c_2}{6}\mid z\mid^3, \tag{2.4}$$

where for $i=1,2$ c_i is a complex number satisfying $\mid c_i\mid\leq 1$.

Proof of Proposition 2.2. Given θ, the characteristic function of $P_n^*/\log n$ is given by the following; For $-\infty<t<\infty$,

$$E\Big[\exp\Big\{it\frac{P_n^*}{\log n}\Big\}\;\Big|\;\theta\Big] = \exp\big\{\theta[\log n-\psi(\theta)]\big[e^{it/\log n}-1\big]\big\}, \tag{2.5}$$

which is written as

$$= \exp\theta\Big\{\big[\log n-\psi(\theta)\big]\Big[\frac{it}{\log n}-\frac{t^2}{2\log^2 n}+\frac{c_{1n}}{6}\frac{\mid t\mid^3}{\log^3 n}\Big]\Big\}$$

by (2.4), where c_{1n} is a complex number such that $\mid c_{1n}\mid\leq 1$. Thus we have

$$E\Big[\exp\Big\{it\frac{P_n^*}{\log n}\Big\}\;\Big|\;\theta\Big] = \exp\theta\{it+A_1\} = e^{it\theta}\times e^{\theta A_1} \tag{2.6}$$

where

$$A_1 = -\psi(\theta)\frac{it}{\log n}-\frac{t^2}{2\log n}+\psi(\theta)\frac{t^2}{2\log^2 n}+\frac{c_{1n}}{6}\frac{\mid t\mid^3}{\log^2 n}-\psi(\theta)\frac{c_{1n}}{6}\frac{\mid t\mid^3}{\log^3 n}.$$

By using (2.3) to the term $e^{\theta A_1}$ of the right-hand side of (2.6), we have

$$E\left[\exp\left\{it\frac{P_n^*}{\log n}\right\} \mid \theta\right] = e^{it\theta}\left\{1+\theta A_1 + \frac{c_{2n}(\theta)}{2}\theta^2 \mid A_1 \mid^2\right\},$$

where $c_{2n}(\theta)$ is a complex number such that $\mid c_{2n}(\theta) \mid\leq 1$. This is written as

$$E\left[\exp\left\{it\frac{P_n^*}{\log n}\right\} \mid \theta\right] = e^{it\theta}\left\{1-\theta\psi(\theta)\frac{it}{\log n} - \frac{\theta t^2}{2\log n} + B_1\right\},$$

where

$$B_1 = \theta B_{10} + \frac{c_{2n}(\theta)}{2}\theta^2 \mid A_1 \mid^2, \quad B_{10} = \left[\psi(\theta)\frac{t^2}{2\log^2 n} + \frac{c_{1n}}{6}\frac{\mid t\mid^3}{\log^2 n} - \psi(\theta)\frac{c_{1n}}{6}\frac{\mid t \mid^3}{\log^3 n}\right]. \tag{2.7}$$

Thus we get

$$\left|E\left[\exp\left\{it\frac{P_n^*}{\log n}\right\} \mid \theta\right] - e^{it\theta}\left\{1-\theta\psi(\theta)\frac{it}{\log n} - \frac{\theta t^2}{2\log n}\right\}\right| \leq \mid B_1 \mid . \tag{2.8}$$

About B_{10}, for $\mid t \mid < \log^{2/5} n$ $(n>3)$ we have

$$\mid B_{10} \mid \leq \frac{\mid t \mid}{\log^{4/5} n}\left[\frac{1}{6} + \frac{2}{3}\mid \psi(\theta) \mid\right], \tag{2.9}$$

because of the following relations,

$$\frac{\mid t \mid}{\log^2 n} < \frac{1}{\log^{8/5} n} < \frac{1}{\log^{4/5} n}, \quad \frac{t^2}{\log^2 n} < \frac{1}{\log^{6/5} n} < \frac{1}{\log^{4/5} n} \quad \text{and} \quad \frac{t^2}{\log^3 n} < \frac{1}{\log^{4/5} n}.$$

The similar inequalities to these are used, hereafter. About $|A_1|$, for $|t| < \log^{2/5} n$ $(n>3)$, by $t^2 < \log n$ and $\log n > 1$ we have

$$\begin{aligned}
\mid A_1 \mid^2 &\leq \left\{\mid \psi(\theta) \mid \frac{\mid t \mid}{\log n} + \frac{t^2}{2\log n} + \mid \psi(\theta) \mid \frac{t^2}{2\log n} + \frac{\mid t \mid}{6\log n} + \mid \psi(\theta) \mid \frac{\mid t \mid}{6 \log n}\right\}^2 \\
&= \left\{\left(\frac{7}{6}\mid \psi(\theta) \mid + \frac{1}{6}\right)\frac{\mid t \mid}{\log n} + \left(\mid \psi(\theta) \mid + 1\right)\frac{t^2}{2\log n}\right\}^2 \\
&\leq 2\left\{\left(\frac{7}{6}\mid \psi(\theta) \mid + \frac{1}{6}\right)^2\frac{t^2}{\log^2 n} + \left(\mid \psi(\theta) \mid + 1\right)^2\frac{t^4}{4\log^2 n}\right\} \\
&\leq 2\left\{\left(\frac{7}{6}\mid \psi(\theta) \mid + \frac{1}{6}\right)^2 + \frac{1}{4}\left(\mid \psi(\theta) \mid + 1\right)^2\right\}\frac{\mid t \mid}{\log^{4/5} n}. \qquad (2.10)
\end{aligned}$$

By applying (2.9) and (2.10) to (2.7), for $|t| < \log^{2/5} n$ $(n>3)$,

$$\mid B_1 \mid \leq \frac{\mid t \mid}{\log^{4/5} n}\left[\frac{1}{6}\theta + \frac{2}{3}\theta \mid \psi(\theta) \mid + \left\{\left(\frac{7}{6}\theta \mid \psi(\theta) \mid + \frac{1}{6}\theta\right)^2 + \frac{1}{4}\left(\theta \mid \psi(\theta) \mid + \theta\right)^2\right\}\right]. \tag{2.11}$$

Therefore, for $|t| < \log^{2/5} n$ $(n>3)$, by $E_\gamma(\theta^2)$ and $E_\gamma[\theta^2\psi(\theta)^2] < \infty$, (2.8) and (2.11) yield

(2.12)
$$\left|E\left[\exp\left\{it\frac{P_n^*}{\log n}\right\}\right] - \left\{E_\gamma e^{it\theta} - \frac{it}{\log n}E_\gamma\left[\theta\psi(\theta)e^{it\theta}\right] - \frac{t^2}{2\log n}E_\gamma\left[\theta e^{it\theta}\right]\right\}\right| \leq d_{11}\frac{\mid t \mid}{\log^{4/5} n},$$

where d_{11} is a positive constant. Since $E_\gamma e^{it\theta}$ is the characteristic function of the distribution function G, that is, it is the Fourier transform of the distribution function $G(x)$. Similarly, $-itE_\gamma[\theta\psi(\theta)e^{it\theta}]$ is the Fourier transform of the function $x\psi(x)g(x)$, and $-t^2E_\gamma[\theta e^{it\theta}]$ is the Fourier transform of the function $\{xg(x)\}' = g(x) + xg'(x)$. Therefore, by applying the smoothing lemma to (2.12), we have the following.

$$(2.13)\quad \sup_x \left| P\left(\frac{P_n^*}{\log n} \le x\right) - \left[G(x) + \frac{1}{\log n} x\psi(x)g(x) + \frac{1}{2\log n}\{g(x) + xg'(x)\}\right]\right|$$
$$\le \frac{d_{11}}{\log^{4/5} n}\int_0^{\log^{2/5} n} dt \;+\; \frac{d_{12}}{\log^{2/5} n} = O\left(\frac{1}{\log^{2/5} n}\right),$$

where d_{12} is positive constant depending only on d_{11}. We get (2.2) by (2.1) and (2.13), using the relation $x\psi(x) = x\psi(x+1) - 1$.

3 The Edgeworth expansion with the rate $1/\log^{3/7} n$. In addition to the assumption of the section 2, we assume that $g(x)$ is differentiable four times, $\{x^2g(x)\}^3$ is the function of bounded variation and $\{x^2g(x)\}^4$ is bounded. Besides, we suppose $E_\gamma(\theta^3)$ and $E[\theta^3|\psi(\theta)|^3]$ exist. Then we have:

Proposition 3.1 *For $n > 3$, we have*

$$(3.1)\quad \sup_{-\infty<x<\infty} \left| P\left(\frac{K_n}{\log n} \le x\right) - \left[G(x) + \frac{1}{2\log n}\Big\{\big[2x\psi(x+1) - 1\big]g(x) + xg'(x)\Big\}\right.\right.$$
$$\left.\left. + \frac{1}{8\log^2 n}\{x^2g(x)\}^{(3)}\right]\right| = O\left(\frac{1}{\log^{3/7} n}\right).$$

In the following proof, in addition to (2.4) we use the well-known relation for any complex number z such that

$$(3.2)\qquad e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{c_3}{24} \mid z \mid^4,$$

where c_3 is a complex number satisfying $\mid c_3 \mid \le 1$.

Proof of Proposition 3.1. Given θ, the characteristic function (2.5) of $P_n^*/\log n$ is written as

$$E\left[\exp\left\{it\frac{P_n^*}{\log n}\right\} \;\Big|\; \theta\right] = \exp\left\{\theta[\log n - \psi(\theta)]\left[\frac{it}{\log n} - \frac{t^2}{2\log^2 n} - \frac{it^3}{6\log^3 n} + \frac{c_{3n}}{24}\frac{t^4}{\log^4 n}\right]\right\},$$

by (3.2), where $-\infty < t < \infty$ and c_{3n} is a complex number satisfying $\mid c_{3n} \mid \le 1$. Thus we can write

$$(3.3)\qquad E\left[\exp\left\{it\frac{P_n^*}{\log n}\right\} \;\Big|\; \theta\right] = \exp\theta\{it + A_2\} = e^{it\theta} \times e^{\theta A_2},$$

where

$$A_2 = -\psi(\theta)\frac{it}{\log n} - \frac{t^2}{2\log n} + \psi(\theta)\frac{t^2}{2\log^2 n} - \frac{it^3}{6\log^2 n} + \psi(\theta)\frac{it^3}{6\log^3 n} + \frac{c_{3n}}{24}\frac{t^4}{\log^3 n} - \frac{c_{3n}}{24}\psi(\theta)\frac{t^4}{\log^4 n}.$$

By using (2.4) to the term $e^{\theta A_2}$ of the right-hand side of (3.3), we have

$$E\left[\exp\left\{it\frac{P_n^*}{\log n}\right\}\Bigm|\theta\right] = e^{it\theta}\left\{1+\theta A_2+\frac{1}{2}\theta^2 A_2^2+\frac{c_{4n}(\theta)}{6}\theta^3\mid A_2\mid^3\right\},$$

where $c_{4n}(\theta)$ is a complex number such that $\mid c_{4n}(\theta)\mid\leq 1$. This is written as

$$E\left[\exp\left\{it\frac{P_n^*}{\log n}\right\}\Bigm|\theta\right] = e^{it\theta}\left\{1-\theta\psi(\theta)\frac{it}{\log n}-\frac{\theta t^2}{2\log n}+\frac{\theta^2 t^4}{8\log^2 n}+B_2\right\}, \tag{3.4}$$

where

$$\begin{aligned}
B_2 &= \theta B_{21}+\frac{\theta^2}{2}B_{22}+\frac{c_{4n}(\theta)}{6}\theta^3\mid A_2\mid^3,\\
B_{21} &= \psi(\theta)\frac{t^2}{2\log^2 n}-\frac{it^3}{6\log^2 n}+\psi(\theta)\frac{it^3}{6\log^3 n}+\frac{c_{3n}}{24}\frac{t^4}{\log^3 n}-\frac{c_{3n}}{24}\psi(\theta)\frac{t^4}{\log^4 n},\\
B_{22} &= A_2^2-\frac{t^4}{4\log^2 n}.
\end{aligned}$$

For $|t|<\log^{3/7} n$ $(n>3)$, by $|t|<\log^{1/2} n$ we have

$$\mid B_{21}\mid\leq\left(\frac{5}{24}+\frac{17}{24}\mid\psi(\theta)\mid\right)\frac{\mid t\mid}{\log n} < \left(\frac{5}{24}+\frac{17}{24}\mid\psi(\theta)\mid\right)\frac{\mid t\mid}{\log^{6/7} n}. \tag{3.5}$$

About $\mid A_2\mid$, at first for $|t|<\log^{3/7} n$ $(n>3)$, by $|t|<\log^{1/2} n$ we have

$$\begin{aligned}
\mid A_2\mid &\leq\mid\psi(\theta)\mid\frac{\mid t\mid}{\log n}+\frac{t^2}{2\log n}+\mid\psi(\theta)\mid\frac{t^2}{2\log^2 n}+\frac{\mid t\mid^3}{6\log^2 n}\\
&\qquad+\mid\psi(\theta)\mid\frac{\mid t\mid^3}{6\log^3 n}+\frac{t^4}{24\log^3 n}+\mid\psi(\theta)\mid\frac{t^4}{24\log^4 n}\\
&\leq\mid\psi(\theta)\mid\frac{\mid t\mid}{\log n}+\frac{t^2}{2\log n}+\mid\psi(\theta)\mid\frac{\mid t\mid}{2\log^{3/2} n}+\frac{\mid t\mid}{6\log n}\\
&\qquad+\mid\psi(\theta)\mid\frac{\mid t\mid}{6\log^2 n}+\frac{\mid t\mid}{24\log^{3/2} n}+\mid\psi(\theta)\mid\frac{\mid t\mid}{24\log^{5/2} n}.
\end{aligned}$$

Thus, for $|t|<\log^{3/7} n$ $(n>3)$, we have

$$\mid A_2\mid\leq\frac{\mid t\mid}{\log n}\eta(\theta)+\frac{t^2}{2\log n}\quad\text{where}\quad\eta(\theta)=\frac{41}{24}\mid\psi(\theta)\mid+\frac{5}{24}. \tag{3.6}$$

Therefore, for $|t|<\log^{3/7} n$ $(n>3)$ we have

$$\mid A_2\mid^3\leq 4\left\{\frac{\mid t\mid^3}{\log^3 n}\eta(\theta)^3+\frac{t^6}{8\log^3 n}\right\}\leq 4\left\{\eta(\theta)^3+\frac{1}{8}\right\}\frac{\mid t\mid}{\log^{6/7} n}. \tag{3.7}$$

For the evaluation of B_{22}, at first we write A_2 as

$$A_2=-\frac{t^2}{2\log n}+A_{21}$$

where

$$A_{21} = -\psi(\theta)\frac{it}{\log n} + \psi(\theta)\frac{t^2}{2\log^2 n} - \frac{it^3}{6\log^2 n} + \psi(\theta)\frac{it^3}{6\log^3 n} + \frac{c_{3n}}{24}\frac{t^4}{\log^3 n} - \frac{c_{3n}}{24}\psi(\theta)\frac{t^4}{\log^4 n}.$$

Then we

$$| B_{22} |\leq \frac{t^2}{\log n} | A_{21} | + | A_{21} |^2 . \tag{3.8}$$

We note that A_{21} is obtained by deleting $-t^2/(2\log n)$ from A_2. Similarly to (3.6), for $|t| < \log^{3/7} n$ $(n > 3)$, we have

$$| A_{21} |\leq \frac{| t |}{\log n}\eta(\theta). \tag{3.9}$$

Applying (3.9) to (3.8), for $|t| < \log^{3/7} n$ $(n > 3)$, we have

$$| B_{22} |\leq \frac{| t |^3}{\log^2 n}\eta(\theta) + \frac{t^2}{\log^2 n}\eta(\theta)^2 \leq \{\eta(\theta) + \eta(\theta)^2\}\frac{| t |}{\log n}. \tag{3.10}$$

From (3.4) we get

$$\left|E\left[\exp\left\{it\frac{P_n^*}{\log n}\right\} \,\Big|\, \theta\right] - e^{it\theta}\left\{1 - \theta\psi(\theta)\frac{it}{\log n} - \frac{\theta t^2}{2\log n} + \frac{\theta^2 t^4}{8\log^2 n}\right\}\right| \leq B_2, \tag{3.11}$$

and from (3.5), (3.7) and (3.10) we have

$$\begin{aligned} | B_2 | \leq{}& \theta\left(\frac{5}{24} + \frac{17}{24} | \psi(\theta) | \right)\frac{| t |}{\log^{6/7} n} + \frac{1}{2}\theta^2\{\eta(\theta) + \eta(\theta)^2\}\frac{| t |}{\log n} \\ &+ \frac{2}{3}\theta^3\left\{\eta(\theta)^3 + \frac{1}{8}\right\}\frac{| t |}{\log^{6/7} n}. \end{aligned} \tag{3.12}$$

Therefore, for $|t| < \log^{3/7} n$ $(n > 3)$, under the condition $E_\gamma(\theta^3),\ E_\gamma[\theta^3|\psi(\theta)|^3] < \infty$, (3.11) and (3.12) give

$$\begin{aligned} \Bigg|E\left[\exp\left\{it\frac{P_n^*}{\log n}\right\}\right] - \Bigg\{&E_\gamma e^{it\theta} - \frac{it}{\log n}E_\gamma\left[\theta\psi(\theta)e^{it\theta}\right] \\ &- \frac{t^2}{2\log n}E_\gamma\left[\theta e^{it\theta}\right] + \frac{t^4}{8\log^2 n}E_\gamma\left[\theta^2 e^{it\theta}\right]\Bigg\}\Bigg| \leq d_{21}\frac{|t|}{\log^{6/7} n}, \end{aligned}$$

where d_{21} is a positive constant. Since $t^4 E_\gamma[\theta^2 e^{it\theta}]$ is the Fourier transform of the function $\{xg(x)\}^{(3)}$, by the reason similar to (2.13) we have

$$\begin{aligned} \sup_x \Bigg|P\left(\frac{P_n^*}{\log n} \leq x\right) - \Bigg[&G(x) + \frac{1}{\log n}x\psi(x)g(x) + \frac{1}{2\log n}\{g(x) + xg'(x)\} \\ &+ \frac{1}{8\log^2 n}\{x^2 g(x)\}^{(3)}\Bigg]\Bigg| = O\left(\frac{1}{\log^{3/7} n}\right). \end{aligned} \tag{3.13}$$

Therefore we get (3.1) by (2.1) and (3.13), using the relation $x\psi(x) = x\psi(x+1) - 1$.

4 numerical examples. We examine the Propositions 2.2 and 3.1 graphically by using the gamma distribution as γ whose density is given by $g_c(x) = x^{c-1}e^{-x}/\Gamma(c)$. The distribution function of $K_n/\log n$ is obtained approximately by using the random numbers of R and described by the step function.

At first, for the examination of (2.2) by taking $c > 1$. Then, the conditions of the Propositions 2.2 are satisfied. The approroximate function $G_1(x) = G(x) + \big\{[2x\psi(x+1) - 1]g(x) + xg'(x)\big\}/(2\log n)$ is described by the broken curve. The distribution function G_c of $g_c(x)$ is described by the dotted curve. For $n = 50$, the Figure's 1, 2, 3 and 4 give the cases of $c = 1.1$, $c = 1.5$, $c = 2$ and $c = 3$. If c is small and near 1, then the function $G_1(x)$ is good approximation to the distribution function of $K_n/\log n$. Even if c increases, the function $G_1(x)$ is better than $G_c(x)$ as the approximation to the distribution function of $K_n/\log n$. But, the tail is not good approximation, similar to the usual Edgeworth expansion.

Figure 1: c=1.1, n=50

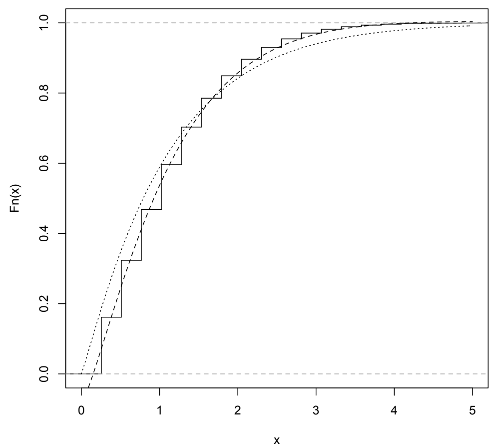


Figure 2: c=1.5, n=50

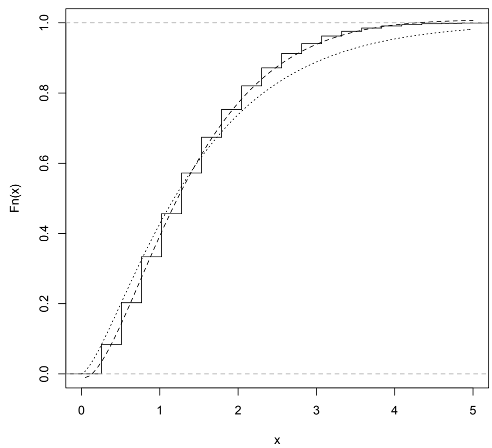


Figure 3: c=2.0, n=50

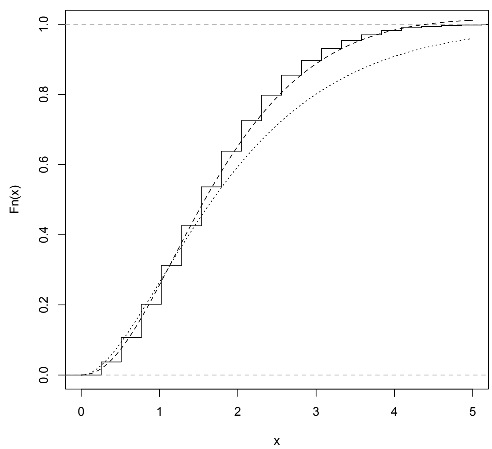
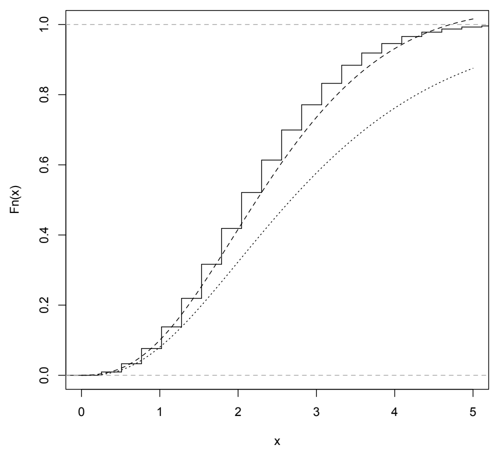


Figure 4: c=3.0, n=50



Next, we examine the relation (3.5) by taking $c = 4$. Then, the conditions of the Propositions 3.1 are satisfied.

The approximate distribution G_1 is described by the broken curve. The approximate function $G_2(x) = G(x) + \big\{[2x\psi(x+1) - 1]g(x) + xg'(x)\big\}/(2\log n) + \{x^2 g(x)\}^{(3)}/(8\log^2 n)$

is described by the dot-broken curve. The distribution function G_c of $g_c(x)$ is described by the dotted curves. For $c = 4$, the Figure's 5 and 6 give the cases of $n = 50$ and $n = 100$, respectively. Both the functions G_1 and G_2 give a little good approximate to the distribution function of $K_n/\log n$. But there are no obvious difference between G_1 and G_2, becuase the value of $G_2(x) - G_1(x) = \{x^2 g(x)\}^{(3)}/(8\log^2 n)$ is small. The little difference may be seen at the left tail.

Figure 5 : c=4, n=50

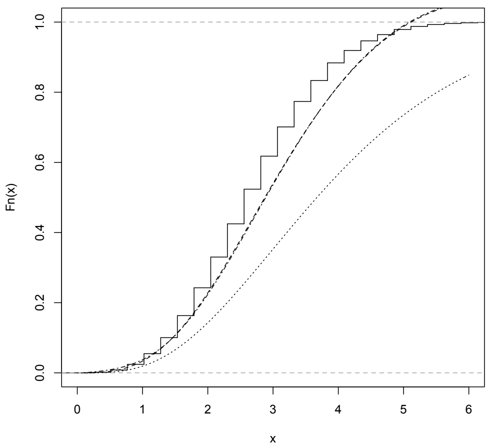
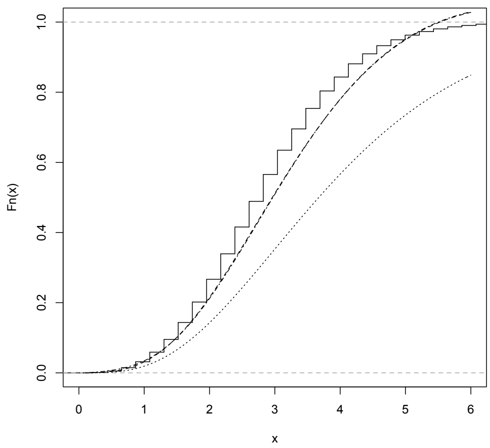


Figure 6 : c=4, n=100



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First author:
Emeritus of Kagoshima University, Take 3-32-1-708 Kagoshima 890–0045, Japan
Second author:
Department of Mathematics and Computer Science, Kagoshima University, Kagosima 890-0065, Japan

QUASI DROP PROPERTIES, (α)-PROPERTIES AND THE STRICT MACKEY CONVERGENCE

ARMANDO GARCÍA

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Abstract. The Mackey (α)-property and quasi-weak (α)-property are defined. For disks in locally convex spaces with the strict Mackey convergence condition, the quasi drop property (resp. quasi weak drop property) implies Mackey (α)-property (resp. quasi weak (α)-property). In Frechet spaces, the quasi drop property and Mackey (α)-property are equivalent. Other equivalences are given.

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space and B_X its closed unit ball. By the drop $D(x, B_X)$ defined by an element $x \in X \setminus B_X$ we mean the set $conv(\{x\} \cup B_X)$. Danes [2] proved that, for any Banach space $(X, \|\cdot\|)$ and every non-empty closed set $A \subset X$ at positive distance from B_X, there exists an $x_0 \in A$ such that $D(x_0, B_X) \cap A = \{x_0\}$. Motivated by Danes theorem, Rolewicz [21] introduced the notion of drop property for the norm of a Banach space: the norm $\|\cdot\|$ in X has the drop property if for every non-empty closed set A disjoint from B_X there exists $x_0 \in A$ such that $D(x_0, B_X) \cap A = \{x_0\}$. He proved that if the norm has the drop property then $(X, \|\cdot\|)$ is reflexive (see [21] Theorem 5). Later, Montesinos (see [14] Theorem 4) proved that a Banach space is reflexive if and only if it can be renormed to have the drop property.

Let B be a subset of a Banach space $(X, \|\cdot\|)$. The Kuratowski index of noncompactness of B, $\alpha(B)$, is the infimum of all positive numbers r such that B can be covered by a finite number of sets of diameter less than r. Given $f \in X^*$ such that $\|f\| = 1$ and $0 < \delta \leq 2$, consider the slice $S(f, B_X, \delta) = \{x \in B_X : f(x) \geq 1 - \delta\}$. The norm $\|\cdot\|$ in a Banach space X has property (α), if $\lim_{\delta\to 0} \alpha(S(f, B_X, \delta)) = 0$ for every $f \in X^*$, $\|f\| = 1$. Also, Rolewicz ([21] Theorem 4), proved that if the norm has the drop property then it has property (α), and Montesinos ([14] Theorem 3) established that these two properties are equivalent.

Giles, Sims and Yorke [3] said that the norm has the weak drop property if for every non-empty weakly sequentially closed set A disjoint from B_X, there exists an $x_0 \in A$ such that $D(x_0, B_X) \cap A = \{x_0\}$, and they proved that this property is equivalent to $(X, \|\cdot\|)$ being reflexive. Kutzarova [8] and Giles and Kutzarova [4] extended the discussion of these drop properties to closed bounded convex sets in Banach spaces. Cheng, Zhou and Zang [1], Zheng [23] and other authors studied those drop properties in locally convex spaces: a bounded, convex and closed subset

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B of a locally convex space (E,τ) is said to have the drop property if it is non-empty and for every non-empty sequentially closed subset $A \subset E$ disjoint from B there exists $a \in A$ such that $D(a,B) \cap A = \{a\}$.

Qiu, in [17] and Monterde and Montesinos in [13], introduced another drop properties in locally convex spaces: a non-empty closed bounded convex subset B of a locally convex space (E,τ) is said to have the quasi weak drop (resp. quasi drop) property if for every non-empty weakly closed (resp. closed) subset $A \subset E$ disjoint from B, there exists an $x_0 \in A$ such that $D(x_0,B) \cap A = \{x_0\}$. In [17] and [18], Qiu established a number of equivalences for the quasi-weak drop property in Frèchet spaces and in quasi-complete locally convex spaces. He characterized reflexivity of those spaces by the condition that every closed bounded convex subset of the space must satisfy the quasi weak drop property. Concerning drop properties and their applications, see for example [1], [2], [9], [10], [11],[12], [16], [19] and [20].

Also, in [17], Qiu proved that for every non-empty bounded, closed and convex subset of a Frechet space $B \subset (E,\tau)$ quasi-weak drop property and weak compactness are equivalent, and he asked if this and other properties can be extended to locally complete spaces. In [18] he answered these questions in negative. In order to extend some results on quasi drop properties to a bigger family of locally convex spaces, based on techniques of Qiu and Rolewickz, we consider locally convex spaces satisfying the strict Mackey convergence condition (sMc), and finally, based on a theorem of A. Martellotti we characterize quasi drop property for Frechet spaces.

2. PRELIMINARIES

Throughout this paper, (E,τ) is a Hausdorff locally convex space over $\mathbb{R}$. A closed, bounded and absolutely convex subset will be called a disk. If D is a disk in the space (E,τ) then we let E_D denote the linear span of D, equipped with the topology given by ρ_D the gauge (Minkowski's functional) of D. This topology has a base of zero neighborhoods of the form $\{aD : a > 0\}$, and makes E_D into a normed space such that $\tau|_{E_D} \leq \rho_D|_{E_D}$, for τ the original topology of E. And (E,τ) is said to be locally complete if every disk $D \subset E$, is a Banach disk, that is (E_D, ρ_D) is a Banach space. Note that for metrizable spaces, completeness and local completeness are equivalent. For local completeness, see [7] and [15].

According to Grothendieck (see [6]), we have that a space (E,τ) satisfies the strict Mackey convergence condition (sMc) if for every bounded subset $B \subset (E,\tau)$, there exists a disk $D \subset E$ containing B such that the topologies of E and E_D agree on B, i.e. $\tau|_B = \rho_D|_B$. Note that every metrizable space satisfies the sMc (see [15], 5.1.27(ii)). And following Gilsdorf [5], every space with a boundedly compatible web satisfies the sMc (see [5] and [7] for webs). In particular, every strictly barrelled space satisfies the sMc (see [22] for strictly barrelled spaces).

Let $B \subset (E,\tau)$ be a disk, for $f \in (E,\tau)' \setminus \{0\}$ let $M_f = \sup\{f(x) : x \in B\}$, and for $\delta > 0$ consider the slice $S(f,B,\delta) = \{x \in B : f(x) \geq M_f - \delta\}$. The disk B is said to have the (α)-property if for every $f \in (E,\tau)' \setminus \{0\}$ and for every neighborhood U of 0 in τ, there exists $\delta > 0$ such that $S(f,B,\delta)$ can be covered by a finite number of translates of U.

Suppose now that (E,τ) is a space that satisfies sMc and $B \subset E$ is a disk. So, there exists a disk $D \subset E$ containing B such that $\tau|_B = \rho_D|_B$. The Kuratowski index of noncompactness of $S(f,B,\delta)$ associated to the disk D is $\alpha_D(S(f,B,\delta))$ the infimum of all positive numbers r such that $S(f,B,\delta)$ is covered by a finite number of sets of ρ_D-diameter less than r. The disk $B \subset E$ is said to have the Mackey (α)-property if for D as above $\lim\limits_{\delta\to 0} \alpha_D(S(f,B,\delta) = 0$ for every $f \in (E,\tau)' \setminus \{0\}$. In this case, due to the fact that ρ_D and τ induce the same topology on B, we get that B has the (α)-property with respect to τ. Obviously, if $(E,\|\cdot\|)$ is a normed space both (α)-properties coincide.

3. RESULTS

Let (E,τ) be a locally convex space and $B \subset E$ a disk.

a) Suppose there exists a disk $D \subset E$ such that $B \subset D$ and B has the quasi drop (resp. quasi weak drop) property in (E_D,ρ_D). Note that for every non-empty subset $A \subset E$, τ-closed (resp. $\sigma(E,E')$-closed), we have $A_D := A \cap E_D$ is ρ_D-closed (resp. $\sigma(E_D,E'_D)$-closed, where $E'_D = (E_D,\rho_D)')$. Then B has the quasi drop (resp. quasi-weak drop) property in (E,τ).

b) Now, for $f \in (E,\tau)' \setminus \{0\}$, find $x_0 \in E$ such that $f(x_0) > M_f$; where $M_f := \sup\{f(x) : x \in B\}$. Suppose that for the disk $C = \overline{abconv}\{B \cup \{x_0\}\}$ in E, there exists a disk $D \subset E$ containing C such that $\tau|_C = \rho_D|_C$; so, in particular, $\tau|_B = \rho_D|_B$. If $\inf\limits_{\varepsilon>0} \alpha_D(S(f,B,\varepsilon)) > 2\delta_0$ for some $\delta_0 > 0$, then (see [21], Theorem 4) for every finite dimensional subspace $L \subset E_D$ we have:

$$\sup_{x\in S(f,B,\varepsilon)} (\inf_{y\in L} \rho_D(x-y)) \geq \frac{1}{2}\inf_{\varepsilon>0} \alpha_D(S(f,B,\varepsilon)) > \delta_0 \qquad \cdots (1)$$

Take $\varepsilon_1 < f(x_0) - M_f$. And choose $\overline{x_1} \in S(f,B,\varepsilon_1)$ such that

$$\inf\{\rho_D(\overline{x_1} - z) : z \in span\{x_0\}\} > \delta_0.$$

Let $x_1 = \frac{x_0+\overline{x_1}}{2}$, then

$$f(x_1) = f(\frac{x_0+\overline{x_1}}{2}) = \frac{f(x_0)}{2} + \frac{f(\overline{x_1})}{2} > \frac{M_f+\varepsilon_1}{2} + \frac{M_f-\varepsilon_1}{2} = M_f.$$

Moreover

$$\inf\{\rho_D(x_1 - z) : z \in span\{x_0\}\} = \frac{1}{2}\inf\{\rho_D(\overline{x_1} - z) : z \in span\{x_0\}\} > \frac{\delta_0}{2}$$

Now, suppose we have $\{x_0, x_1, ..., x_n\}$, such that $x_i \neq x_j$ if $i \neq j \leq n$, and
i) $f(x_i) > M_f$
ii) $\inf\{\rho_D(x_i - z) : z \in span\{x_0, ..., x_{i-1}\}\} > \frac{\delta_0}{2}$
iii) $x_i \in D(x_{i-1}, B)$
for every $i \leq n$. Take $\varepsilon_{n+1} < f(x_n) - M_f$ and by (1) find $\overline{x_{n+1}} \in S(f,B,\varepsilon_{n+1})$ such that

$$\inf\{\rho_D(\overline{x_{n+1}} - z) : z \in span\{x_0, x_1, ..., x_n\}\} > \delta_0.$$

Let $x_{n+1} = \frac{x_n+\overline{x_{n+1}}}{2}$ then, in an analogous way to x_1, $f(x_{n+1}) > M_f$ and

$$\begin{aligned} &\inf\{\rho_D(x_{n+1} - z) : z \in span\{x_0, ..., x_n\}\} \\ = \; &\frac{1}{2}\inf\{\rho_D(\overline{x_{n+1}} - z) : z \in span\{x_0, ..., x_n\}\} > \frac{\delta_0}{2}. \end{aligned}$$

Then the sequence $(x_n)_n$ satisfies (i,ii,iii) and the set $A = \{x_0, x_1, ..., x_n, ...\} \subset C$ is ρ_D-closed. Since the topologies τ and ρ_D agree on C, A is τ-closed and $A \cap B = \emptyset$. Hence B, does not have the quasi drop property. So, we conclude

Proposition 1. *Let (E,τ) be a locally convex space satisfying the sMc and $B \subset E$ a disk. Suppose that B has the quasi drop property. Then there exists a disk $D \subset E$ containing B such that $\tau|_B = \rho_D|_B$ and B has the Mackey (α)-property associated to the disk D. B has (α)-property respect to τ, too.*

For the next proposition, we will suppose that the disk $B \subset E$ has the Mackey (α)-property associated to a Banach disk D. So, if we take $x_n \in S(f,B,\frac{1}{n})$ for every $n \in \mathbb{N}$, there exists a subsequence $(x_{n_k})_k \subset (x_n)_n$ convergent to some $x_0 \in B$ respect to ρ_D, and hence $x_{n_k} \to x_0 \in B$ respect to τ.

Proposition 2. *Let (E,τ) be a locally complete space satisfying the sMc. Let $B, D \subset E$ be disks such that $B \subset D$, $\tau|_B = \rho_D|_B$ and for every $f \in (E,\tau)' \setminus \{0\}$, $\alpha_D(S(f,B,\delta)) \to 0$; as $\delta \to 0$. Then B has the quasi drop property on (E,τ).*

Proof. Suppose that B does not have the quasi drop property on (E,τ). It implies, there exists a τ-closed set $A \subset E$, $A \cap B = \emptyset$, such that for every $x \in A$, $D(x,B) \cap A \neq \{x\}$. Let $A_D := A \cap E_D$. So A_D is ρ_D-closed in E_D and $A_D \cap B = \emptyset$. Note that $D(x,B) \cap A_D \neq \{x\}$, for every $x \in A_D$. By [17] lemma 2.2, $\rho_D(D(z,B) \cap A_D, B) = 0$, for every $z \in A_D$. Take $x_1 \in A_D$ fixed and find $f \in (E,\tau)'$ such that $f(x_1) = M_f + 1$, for $M_f = \sup\{f(x) : x \in B\}$. We may suppose $M_f > 1$. Since $\rho_D(D(x_1,B) \cap A_D, B) = 0$, take $y_1 \in S(f,B,1)$ and let $R = \rho_D(x_1 - y_1)$. Take $x_2 \in D(x_1,B) \cap A_D \setminus \{x_1\}$, then $\rho_D(D(x_2,B) \cap A_D, B) = 0$. And find $y_2 \in S(f,B,\frac{1}{2}) \setminus \{y_1\}$ such that $\rho_D(x_2 - y_2) < \frac{R}{2}$. Continue this process inductively, to construct sequences $(x_n)_n \in A_D \subset (E \setminus B)$ and $(y_n)_n \in B$, $y_n \in S(f,B,\frac{1}{n})$ such that $x_n \neq x_m$ and $y_n \neq y_m$ if $n \neq m$, and $\rho_D(x_n - y_n) < \frac{R}{n}$, for every $n \in \mathbb{N}$. Since B has the Mackey (α)-property associated to the Banach disk D, there exists a subsequence $(y_{n_k})_k \subset (y_n)_n$ convergent to some $y_0 \in B$, respect to ρ_D and so, respect to τ. Since $\rho_D(x_{n_k} - y_{n_k}) < \frac{R}{n_k}$ then $x_{n_k} \to y_0 \in B$, respect to ρ_D and respect to τ. Recall that A_D is ρ_D-closed and A is τ-closed which implies $y_0 \in A_D \subset A$. Hence $y_0 \in A \cap B$. It is a contradiction, since A and B are disjoint. Hence B has the quasi drop property in (E,τ) □

In [12], A. Martellotti characterizes drop property in Banach spaces in the following way:

Theorem 1. ([12], *Theorem 3.7*) *Let $(X, \|\cdot\|)$ be a Banach space and $B \subset X$ a non-empty, closed and convex subset. The following are equivalent:*

i) B has the drop property

ii) For every non-empty closed subset $C \subset X$, with $B \cap C = \emptyset$ there exists $x \in C$ such that $D(x,B) \cap C$ is compact

iii) For every non-empty closed subset $C \subset X$, with $B \cap C = \emptyset$ and for every $\varepsilon > 0$ there exists $x_\varepsilon \in C$ such that $\alpha(D(x_\varepsilon,B) \cap C) < \varepsilon$

By Martellotti's theorem and propositions 1 and 2, we obtain the following characterization of quasi drop property for disks in Frechet spaces.

Theorem 2. *Let (E,τ) be a Frechet space and $B \subset E$ a disk. The following are equivalent:*

i) B has the quasi drop property

ii) B has the Mackey (α)-property

iii) For every non-empty τ-closed subset $C \subset E$, with $B \cap C = \emptyset$ there exists $x \in C$ such that $D(x,B) \cap C$ is compact *iv) For every non-empty τ-closed subset $C \subset E$, with $B \cap C = \emptyset$ and for every $\varepsilon > 0$ there exists $x_\varepsilon \in C$ and a Banach disk $D \subset E$ such that $D(x_\varepsilon, B) \subset D$, $\tau \left|_{D(x_\varepsilon,B)} = \rho_D \right|_{D(x_\varepsilon,B)}$ and $\alpha_D(D(x_\varepsilon, B) \cap C) < \varepsilon$*

Proof. (i$\Leftrightarrow$ii) Follows from propositions 1 and 2.

(i$\Longleftrightarrow$iii$\Longleftrightarrow$iv) Follows from Martellotti's theorem, since (E, τ) satisfies the strict Mackey convergence condition and every disk $D \subset E$ is a Banach disk. □

Corollary 1. *Let (E, τ) be a Frèchet space and $B \subset E$ a disk. Then B has the quasi drop property if and only if there exists a Banach disk $D \subset E$ containing B such that $\tau\,|_B = \rho_D\,|_B$ and $\alpha_D(S(f, B, \delta)) \to 0$; as $\delta \to 0$, for every $f \in (E, \tau)' \setminus \{0\}$.*

Definition 1. *A disk C in the locally convex space (E, τ) is said to have the quasi weak (α)-property if for every $f \in (E, \tau)' \setminus \{0\}$ and every $(\alpha_n)_n \in \mathbb{R}^+$ such that $\alpha_n \to 0$, every non eventually constant sequence $(x_n)_n \in C$ such that $x_n \in S(f, C, \alpha_n)$, for every $n \in \mathbb{N}$, has a weakly cluster point in C.*

Proposition 3. *Let (E, τ) be a locally convex space and $C \subset E$ a disk with the quasi-weak drop property. Then C has the quasi-weak (α)-property.*

Proof. By [17] in the proof of Theorem 2.1(i), for every disk C with the quasi-weak drop property (and with no one additional condition) every stream in $E \setminus C$ has a weakly cluster point. Suppose that C does not have the quasi-weak (α)-property. So, there exists $f \in (E, \tau)' \setminus \{0\}$ such that for every $n \in \mathbb{N}$ there exists $x_n \in S(f, C, \frac{1}{4^n})$, $x_n \neq x_m$ if $n \neq m$ and $\{x_n : n \in \mathbb{N}\}$ has no weakly cluster points. Let $M = \sup\{f(x) : x \in C\}$. Find $x_0 \in E$ such that $f(x_0) = M + 2$. Define $y_n = \frac{1}{2^n} x_0 + \sum_{i=1}^{n} \frac{1}{2^{n-i+1}} x_i$. Then

$$\begin{aligned} f(y_n) &= \frac{1}{2^n} f(x_0) + \sum_{i=1}^{n} \frac{1}{2^{n-i+1}} f(x_i) \geq \frac{M+2}{2^n} + \sum_{i=1}^{n} \frac{1}{2^{n-i+1}} (M - \frac{1}{4^i}) \\ &= M + \frac{3}{2^{n+1}} + \frac{1}{2^{2n+1}} > M + \frac{1}{2^{n+1}} \end{aligned}$$

So, $(y_n)_n$ is a stream. If there exists a subsequence $(y_{n_k})_k \subset (y_n)_n$ with no weakly cluster points, then $A = \{y_{n_k} : k \in \mathbb{N}\}$ is weakly closed set such that does not satisfy the quasi weak drop condition for C. It would be a contradiction, and we would have finished. So, suppose that every subsequence $(y_{n_k})_k \subset (y_n)_n$ has at least a weakly cluster point. Let $(y_{n_k})_k$, $(y_{n_k+1})_k$ subsequences of $(y_n)_n$. Then there exists z_1, z_2 weakly cluster points of $(y_{n_k})_k$ and $(y_{n_k+1})_k$ respectively. Note that $y_{n_k+1} = \frac{1}{2}(y_{n_k} + x_{n_k+1})$. So, for every $\varphi \in (E, \tau)' \setminus \{0\}$, $\varphi(z_1)$ is a cluster point of the set $\{\varphi(y_{n_k}) : k \in \mathbb{N}\}$ and $\varphi(z_2)$ is a cluster point of the set $\{\varphi(y_{n_k+1}) : k \in \mathbb{N}\}$. Note that $\varphi(y_{n_k+1}) = \frac{1}{2}\varphi(y_{n_k}) + \frac{1}{2}\varphi(x_{n_k+1})$, so $2(\varphi(z_2) - \frac{1}{2}\varphi(z_1)) = \varphi(2z_2 - z_1)$ is a cluster point of $\{\varphi(2y_{n_k+1} - y_{n_k}) = \varphi(x_{n_k+1}) : k \in \mathbb{N}\}$, for every $\varphi \in (E, \tau)'$. Then $2z_2 - z_1$ is a weakly cluster point of $\{x_{n_k+1} : k \in \mathbb{N}\}$ and of $\{x_{n_k} : k \in \mathbb{N}\}$. A contradiction. Hence, C has the quasi-weak (α)-property. □

If we would want to give a converse to this result for locally convex spaces (E, τ) with sMc and local completeness conditions, we should consider $B \subset E$ a disk with the quasi-weak (α)-property. So, suppose that B does not have the quasi-weak drop property, then there exists a subset $A \subset E$, $\sigma(E, E')$-closed and disjoint from B such that $D(x, B) \cap A \neq \{x\}$ for every $x \in A$. Take $x_0 \in A$ fixed. Let $D \subset E$ be a Banach disk such that $B \cup \{x_0\} \subset D$ and $\tau \left|_{D(x_0,B)} = \rho_D \right|_{D(x_0,B)}$. Note that $A_D := A \cap E_D$ is $\sigma(E_D, E'_D)$-closed, where $E'_D = (E_D, \rho_D)'$ and A_D is disjoint from B. Moreover, $D(x_0, B) \cap A_D$ is $\sigma(E_D, E'_D)$-closed and different from $\{x_0\}$. And for every $x \in D(x_0, B) \cap A_D$ we have $\{x\} \neq D(x, B) \cap [D(x_0, B) \cap A_D] = D(x, B) \cap A_D$. By [16] lemma 2.2, $\rho_D(B, D(z, B) \cap A_D) = 0$, for every $z \in A_D$. In particular if $z = x_0$. Find $f \in (E, \tau)'$ such that $f(x_0) > M_f > 1$, where $M_f = \sup\{f(y) : y \in B\}$. Since $\rho_D(B, D(x_0, B) \cap A_D) = 0$ then there exists $x_1 \in D(x_0, B) \cap A_D$ and $y_1 \in S(f, B, 1)$ such that $\rho_D(x_1, y_1) < 1$. In an analogous way, $\rho_D(B, D(x_1, B) \cap A_D) = 0$ and there exist $x_2 \in D(x_1, B) \cap A_D$ and $y_2 \in S(f, B, \frac{1}{2})$ such that $\rho_D(x_2, y_2) < \frac{1}{2}$. So, inductively, construct sequences $(x_n)_n$, $(y_n)_n$; $x_n \neq x_m$ and $y_n \neq y_m$ if $n \neq m$, such that $x_n \in D(x_{n-1}, B) \cap A_D$ and $y_n \in S(f, B, \frac{1}{n})$ and $\rho_D(x_n, y_n) < \frac{1}{n}$. Since B has the quasi-weak (α)-property, there exists $y^* \in B$ weakly cluster point of $\{y_n : n \in \mathbb{N}\}$ and since $\rho_D(x_n, y_n) \to 0$, then $x_n - y_n \to 0$ respect to $\sigma(E_D, E'_D)$ and respect to $\sigma(E, E')$. Then $y^* \in B$ is a weakly cluster point of $\{x_n : n \in \mathbb{N}\} \subset A_D$. Since A_D is $\sigma(E_D, E'_D)$-closed, then $y^* \in A_D \subset A$, i.e. $y^* \in A \cap B$, a contradiction. And we have that B has the quasi weak drop property. But a theorem of Qiu in [16], ensures that a Mackey complete disk is weakly compact if, and only if, it has the quasi weak drop property. So, if the space (E, τ) is locally complete and satisfies the sMc, then the considered disk B is weakly compact and the equivalence should be almost obvious.

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Facultad de Ciencias, Universidad Autónoma de San Luis Potosí. Dr. Nava s/n Zona Universitaria, San Luis Potosí, SLP, México.
E-mail address: gama.slp@gmail.com

THE ORDER-PRESERVING PROPERTIES OF THE RASCH MODEL AND EXTENDED MODEL IN MARGINAL MAXIMUM LIKELIHOOD ESTIMATION

KAZUMASA MORI

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ABSTRACT. In this study, we consider the order-preserving properties of Rasch model (Rasch, 1960) and linear logistic model(Fischer, 1994) in marginal maximum likelihood estimation (MMLE). More specially, we focus on the "manifest probability," as discussed by Cressie and Holland (1983) and derive the order-preserving statistics for the item parameters. We also derive order-preserving statistics for the ability parameters in maximum likelihood estimation under the condition that the estimates of the item parameters are already given. Both sets of statistics are derived using the characteristics of arrangement increasing functions (Hollander *et al.*, 1977, Marshall *et al.*, 2011). It is notable that the order-preserving statistics of the Rasch model in MMLE coincide with those of other estimation techniques, such as joint maximum likelihood estimation and conditional maximum likelihood estimation. However, while the marginal maximum likelihood estimates and the conditional maximum likelihood estimates are consistent, the joint maximum likelihood estimates are not. Here, we discuss the reasons for such coincidences, as well as the types of bias that occur in inconsistent estimates.

1 Introduction In this study, we consider the ordering properties of Rasch model (Rasch, 1960) and linear logistic model(Fischer, 1994) in marginal maximum likelihood estimation.

First, we introduce the Rasch model. Suppose a test comprises k items administered to n examinees. Let $X_{ij} = \{0, 1\}$ be the response of the i-th examinee to the j-th item. When the i-th examinee responds with a 1 to the j-th item, the corresponding probability is

$$P_{ij}(\theta_i, \beta_j) = P(X_{ij} = 1; \boldsymbol{\theta}, \boldsymbol{\beta}) = \frac{\exp(\theta_i + \beta_j)}{1 + \exp(\theta_i + \beta_j)}, \tag{1}$$

where θ_i is the ability parameter for the i-th examinee and β_j is the item parameter for the j-th item. In addition, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ is an n-dimensional vector of ability parameters and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k)$ is a k-dimensional vector of item parameters. One of major estimation methods for the Rasch model is maximum likelihood estimation. In the Rasch model, the form of the likelihood function is

$$\begin{aligned} L(\boldsymbol{\theta}, \boldsymbol{\beta}|X) &= \prod_{i=1}^{n}\prod_{j=1}^{k} \left\{ P_{ij}(\theta_i, \beta_j)^{x_{ij}} Q_{ij}(\theta_i, \beta_j)^{1-x_{ij}} \right\} \\ &= \prod_{i=1}^{n}\prod_{j=1}^{k} \frac{\exp\{(\theta_i + \beta_j)\}^{x_{ij}}}{1 + \exp(\theta_i + \beta_j)} \\ &= \prod_{i=1}^{n}\prod_{j=1}^{k} \frac{\exp\{x_{ij}(\theta_i + \beta_j)\}}{1 + \exp(\theta_i + \beta_j)}, \end{aligned} \tag{2}$$

where X represents a matrix of all responses for the test, x_{ij} is the observed response of the i-th examinee to the j-th item, and $Q_{ij}(\theta_i, \beta_j) = 1 - P_{ij}(\theta_i, \beta_j)$.

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Three maximum likelihood estimation techniques have been proposed, all of which use (2): joint maximum likelihood estimation (JMLE), marginal maximum likelihood estimation (MMLE; Bock and Lieberman, 1970, Thissen, 1982), and conditional maximum likelihood estimation (CMLE; Andersen, 1972). The JMLE technique estimates $\boldsymbol{\theta}$ and $\boldsymbol{\beta}$ simultaneously by maximizing (2). In contrast, the CMLE and the MMLE techniques remove $\boldsymbol{\theta}$ from (2) and estimate $\boldsymbol{\beta}$ separately. Holland (1990) discussed the relationship among these estimation techniques. He compared the log-likelihood functions of the three techniques and concluded that JMLE and CMLE can both be viewed as approximations to MMLE. In other words, we can regard MMLE as being more general than JMLE and CMLE. On the other hand, Grayson (1988) and Huynh (1994) presented their basic results as the monotone likelihood ratio for the order-preserving property of the dichotomous response model. In addition, Bertoli-Barsotti (2003) derived the order-preserving property for the Rasch model using JMLE and CMLE, but not MMLE. Thus, in this study, we focus on the order-preserving property of the Rasch model based on MMLE.

In MMLE, we remove the ability parameter from the likelihood function (2) by integration. Cressie and Holland (1983) discussed the "manifest probability" for the Rasch model. The manifest probability can be obtained by integrating the ability parameter, θ, for each examinee. Thus, it corresponds to the marginal likelihood for each examinee. The form is

$$p(\boldsymbol{x}) = \int \prod_j \left[P_j(\theta,\beta_j)^{x_j}\{1-P_j(\theta,\beta_j)\}^{1-x_j}\right] dF(\theta), \tag{3}$$

where x_j is observed response for j-th item, $\boldsymbol{x} = (x_1, x_2, \cdots, x_k)$, $F(\theta)$ is the distribution function for θ and $P_j(\theta,\beta_j) = \dfrac{\exp(\theta+\beta_j)}{1+\exp(\theta+\beta_j)}$. They also derived the log-likelihood function for the Rasch model (1). Here, the form is

$$\ln L(\boldsymbol{\beta}|X) = c + n\alpha + \sum_j^k s_j\beta_j + \sum_t^k r_t\gamma(t), \tag{4}$$

where

$c = \log \dfrac{n!}{\prod_{\boldsymbol{x}} m(\boldsymbol{x})!}$,

$m(\boldsymbol{x})$ is the number of examinees whose item response vector is $\boldsymbol{x}$,

s_j is the number of examinees who answered 1 to the j-th item,

r_t is the number of examinees who answered t items as 1 on the test,

$\alpha = \ln p(\mathbf{0})$, where $\mathbf{0}$ is a k-dimensional vector all of whose elements are 0,

$\gamma(t) = \log \int_0^\infty u^t dG(u)$, with translation $u = \exp(\theta)$,

and $G(u)$ is a distribution function constructed from $dG(u)$ with

$$dG(u) = \frac{\exp\theta dF(\theta)}{p(\mathbf{0})\prod_j^k\{1+\exp(\theta+\beta_j)\}}.$$

One of extension of the Rasch model is linear logistic test model (LLTM, Fischer,1994). The LLTM is defined by adding below conditions

$$\beta_j = \sum_{l=1}^p w_{jl}\delta_l \tag{5}$$

to (1). Here, $\delta_l, l = 1, \ldots, p$ are basic parameter of the LLTM and w_{jl} are given weights for the basic parameters δ_l.

For the MMLE of the LLTM, we substitute (5) and maximum likelihood estimate of α,

$$\hat{\alpha} = \ln(r_0/n)$$

into (4). Such modification of likelihood function was also evaluated in Tjur(1982) and Andersen(1997). Then, (4) is modified as

$$\ln L(\boldsymbol{\delta}|X) = c + n\hat{\alpha} + \sum_{l}^{p} v_l \delta_l + \sum_{t}^{k} r_t \gamma(t) = \ln L(\boldsymbol{\delta}|\boldsymbol{v}), \tag{6}$$

where $v_l = \sum_i^n \sum_j^k x_{ij} w_{jl}$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_p)$.

In this study, we use the log-likelihood function (4) and (6) to derive the order-preserving properties of the MMLE technique, as well as in related maximum likelihood estimation techniques. We use the characteristics of an arrangement increasing function (Hollander *et al.*, 1977, Marshall *et al.*, 2011) for deviations among the order-preserving properties. In our results, we assume that the maximum likelihood estimates described above exist and are unique. These assumptions are related to the form of the response matrix X and the rank of weight matrix$W = [w_{jl}]$.(for details, see Fischer (1981,1994)).

The remainder of the paper is organized as follows. The preliminaries and main theorems are presented in section 2. Finally, section 3 discusses our results and concludes the paper.

2 Preliminaries and the main results As mentioned previously, we use some characteristics of arrangement increasing (AI) functions (Hollander *et al.*, 1977) to derive the order-preserving properties of the Rasch model and the LLTM. To begin with, we introduce some definitions, as per Marshall *et al.*(2011) and Boland and Proschan(1988).

Definition 1. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be n-dimensional vectors. We define equality $\stackrel{a}{=}$ as

$$(\boldsymbol{a}\Pi, \boldsymbol{b}\Pi) \stackrel{a}{=} (\boldsymbol{a}, \boldsymbol{b}),$$

where Π is an arbitrary $n \times n$ permutation matrix.

Clearly, we find $(\boldsymbol{a}, \boldsymbol{b}) \stackrel{a}{=} (\boldsymbol{a}_\uparrow, \boldsymbol{b}\Pi_1) \stackrel{a}{=} (\boldsymbol{a}_\downarrow, \boldsymbol{b}\Pi_2)$, where Π_1 is a matrix such that $\boldsymbol{a}\Pi_1 = \boldsymbol{a}_\uparrow$ and Π_2 is a matrix such that $\boldsymbol{a}\Pi_2 = \boldsymbol{a}_\downarrow$. Here, we use the ordered vectors $\boldsymbol{a}_\uparrow$ and $\boldsymbol{a}_\downarrow$, which are the vectors with components of $\boldsymbol{a}$ arranged in ascending order and descending order, respectively.

Then, we define a partial order $\stackrel{a}{\leq}$ for vector arguments. This definition corresponds to special case denoted by Boland and Proschan(1988).

Definition 2. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be n-dimensional vectors. First, we permute $\boldsymbol{a}$ and $\boldsymbol{b}$ so that

$$(\boldsymbol{a}, \boldsymbol{b}) \stackrel{a}{=} (\boldsymbol{a}_\uparrow, \boldsymbol{b}'). \tag{7}$$

Here, $\boldsymbol{b}' = \boldsymbol{b}\Pi_1$ and Π_1 is the permutation matrix such that $\boldsymbol{a}\Pi_1 = \boldsymbol{a}_\uparrow$. Then, we generate a vector $\boldsymbol{b}^*_{l,m}$ from $\boldsymbol{b}'$ in (7) by interchanging the l-th and the m-th component ($l < m$) of $\boldsymbol{b}$ such that $b_l > b_m$. Finally, we define the partial order $\stackrel{a}{\leq}$ as

$$(\boldsymbol{a}_\uparrow, \boldsymbol{b}') \stackrel{a}{\leq} (\boldsymbol{a}_\uparrow, \boldsymbol{b}^*_{l,m}).$$

Therefore, it holds that $(\boldsymbol{a}_\uparrow, \boldsymbol{b}_\downarrow) \stackrel{a}{=} (\boldsymbol{a}_\downarrow, \boldsymbol{b}_\uparrow) \stackrel{a}{\leq} (\boldsymbol{a}, \boldsymbol{b}) \stackrel{a}{\leq} (\boldsymbol{a}_\uparrow, \boldsymbol{b}_\uparrow) \stackrel{a}{=} (\boldsymbol{a}_\downarrow, \boldsymbol{b}_\downarrow)$.

Example 1. Let $\boldsymbol{a} = (7, 5, 3, 1)$ and $\boldsymbol{b} = (6, 4, 8, 2)$. Then,

$$
\begin{aligned}
&(\boldsymbol{a}, \boldsymbol{b}) \overset{a}{=} ((1, 3, 5, 7), (2, 8, 4, 6)) \overset{a}{\leq} ((1, 3, 5, 7), (2, 4, 8, 6)) \\
&\overset{a}{\leq} ((1, 3, 5, 7), (2, 4, 6, 8)) \overset{a}{=} ((7, 5, 3, 1), (8, 6, 4, 2)).
\end{aligned}
$$

Definition 3. An AI function is a function, g, with two n-dimensional vector arguments that preserves the ordering $\overset{a}{\leq}$. Thus, if g is AI, it holds that $g(\boldsymbol{a}, \boldsymbol{b}) \leq g(\boldsymbol{a}_{\uparrow}, \boldsymbol{b}^*_{l,m})$ for n-dimensional vectors $\boldsymbol{a}$,$\boldsymbol{b}$,$\boldsymbol{a}_{\uparrow}$,$\boldsymbol{b}^*_{l,m}$, such that $(\boldsymbol{a}, \boldsymbol{b}) \overset{a}{\leq} (\boldsymbol{a}_{\uparrow}, \boldsymbol{b}^*_{l,m})$.

Here, we find

$$
g(\boldsymbol{a}_{\uparrow}, \boldsymbol{b}_{\downarrow}) = g(\boldsymbol{a}_{\downarrow}, \boldsymbol{b}_{\uparrow}) \leq g(\boldsymbol{a}, \boldsymbol{b}) \leq g(\boldsymbol{a}_{\uparrow}, \boldsymbol{b}_{\uparrow}) = g(\boldsymbol{a}_{\downarrow}, \boldsymbol{b}_{\downarrow}) \tag{8}
$$

for AI function g, which describes the same case as the partial order $\overset{a}{\leq}$.

Next, we prepare a lemma (without proof) that describes the necessary and sufficient condition for AI functions containing summation forms.

Lemma 1. (Marshall *et al.*, 2011, p.233) If g has the form $g(\boldsymbol{a}, \boldsymbol{b}) = \sum_{i=1}^n \phi(a_i, b_i)$, then g is AI if and only if ϕ is L-superadditive.

In Lemma 1, L-superadditive is the function that satisfies

$$
\frac{\partial}{\partial a \partial b} \phi(a, b) \geq 0. \tag{9}
$$

On the other hand, when we consider the log likelihood function in (4), we find that c, n and $\sum_t^k r_t \gamma(t)$ do not include item parameter β_j. Also, we find that

$$
\alpha = \ln p(\mathbf{0}) = \ln \int \prod_j \frac{1}{1 + \exp(\theta_i + \beta_j)} dF(\theta)
$$

is invariant for rearrangement within $\boldsymbol{\beta}$. Thus, for considering the order-preserving properties, we focus on a part of $\ln L$:

$$
l(\boldsymbol{s}, \boldsymbol{\beta}) = \sum_j^k s_j \beta_j, \tag{10}
$$

where $\boldsymbol{s}$ is a vector consisting of $s_j (j = 1, \ldots, k)$ in (4). This means that we only need to focus on $l(\boldsymbol{s}, \boldsymbol{\beta})$ in (10) to derive $\hat{\boldsymbol{\beta}}$. Here, $\hat{\boldsymbol{\beta}}$ is a vector of maximum likelihood estimates, which maximize the log-likelihood in (4).

Now, we propose the main theorem.

Theorem 1. Let $\boldsymbol{s}^*$ be a rearranged vector such that $\boldsymbol{s}^* = \boldsymbol{s}_{\uparrow}$ and $\tilde{\boldsymbol{\beta}}$ be the marginal maximum likelihood estimates vector that maximizes $l(\boldsymbol{s}^*, \boldsymbol{\beta})$. Then, $\tilde{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}_{\uparrow}$.

Proof. First, we find that $l(\boldsymbol{s}, \boldsymbol{\beta})$ in (10) is permutation invariant in the sense that $l(\boldsymbol{s}, \boldsymbol{\beta}) = l(\boldsymbol{s}\Pi, \boldsymbol{\beta}\Pi)$ for any permutation matrix, Π. By this permutation invariance and the uniqueness of the marginal maximum likelihood estimates, we obtain

$$
l(\boldsymbol{s}, \hat{\boldsymbol{\beta}}) = l(\boldsymbol{s}^*, \hat{\boldsymbol{\beta}}\Pi^*_s) = l(\boldsymbol{s}^*, \tilde{\boldsymbol{\beta}}),
$$

where Π^*_s is a permutation matrix such that $\boldsymbol{s}\Pi^*_s = \boldsymbol{s}^*$. Thus, we find that both $\tilde{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\beta}}$ are marginal maximum likelihood estimates, and that $\tilde{\boldsymbol{\beta}}$ is a rearranged form of $\hat{\boldsymbol{\beta}}$.

On the other hand, as $s_j\beta_j$ is L-superadditive for variables s_j and β_j, from (9), it follows that $l(\boldsymbol{s},\boldsymbol{\beta})$ is AI by the Lemma 1. Then, by the property of AI functions described in (8), it holds that

$$l(\boldsymbol{s}^*,\tilde{\boldsymbol{\beta}}_{\downarrow}) \le l(\boldsymbol{s}^*,\tilde{\boldsymbol{\beta}}) \le l(\boldsymbol{s}^*,\tilde{\boldsymbol{\beta}}_{\uparrow}),$$

for given $\boldsymbol{s}^*$ and $\tilde{\boldsymbol{\beta}}$. As $\tilde{\boldsymbol{\beta}}$ is the estimate that maximizes $l(\boldsymbol{s}^*,\boldsymbol{\beta})$, it follows that $\tilde{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}}_{\uparrow}$. Consequently, it holds that $\tilde{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}_{\uparrow}$.□

Estimating the ability parameter $\boldsymbol{\theta}$ often occurs under the condition that estimates of $\boldsymbol{\beta}$ are already given. This estimation technique corresponds to maximizing the likelihood function with given item parameters $\hat{\boldsymbol{\beta}}$in terms of $\boldsymbol{\theta}$. The form of the likelihood function is

$$\begin{aligned} L(\boldsymbol{\theta}|\hat{\boldsymbol{\beta}},X) &= \prod_{i=1}^{n}\prod_{j=1}^{k}\frac{\exp\left\{x_{ij}(\theta_i+\hat{\beta}_j)\right\}}{1+\exp(\theta_i+\hat{\beta}_j)} = \frac{\exp\left\{(\sum_{i=1}^{n}\theta_i t_i \sum_{j=1}^{k}\hat{\beta}_j s_j\right\}}{\prod_{i=1}^{n}\prod_{j=1}^{k}\left\{1+\exp(\theta_i+\hat{\beta}_j)\right\}} \\ &= L(\boldsymbol{\theta}|\hat{\boldsymbol{\beta}},\boldsymbol{t}), \end{aligned} \tag{11}$$

where $t_i = \sum_j^k X_{ij}$ and $\boldsymbol{t} = (t_1,t_2,\ldots,t_n)$. In other words, this maximum likelihood estimate $\hat{\boldsymbol{\theta}}_i$ maximizes $L(\boldsymbol{\theta}|\hat{\boldsymbol{\beta}},\boldsymbol{t})$ in (11). We derive the order-preserving statistics for $\hat{\boldsymbol{\beta}}$.

Theorem 2. Let $\boldsymbol{t}^*$ be a rearranged vector such that $\boldsymbol{t}^* = \boldsymbol{t}_{\uparrow}$ and let $\tilde{\boldsymbol{\theta}}$ be a vector of the maximum likelihood estimates that maximizes $L(\boldsymbol{\theta}|\hat{\boldsymbol{\beta}},\boldsymbol{t}^*)$ in (11). Then, $\tilde{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}_{\uparrow}$.

Proof. This theorem is proved in the same way as Theorem 1. First, we evaluate the log-likelihood function of (11). We write this function as

$$\ln L(\boldsymbol{\theta}|\hat{\boldsymbol{\beta}},\boldsymbol{t}) = \sum_{i=1}^{n}\theta_i t_i + \eta - h(\boldsymbol{\theta},\hat{\boldsymbol{\beta}}), \tag{12}$$

where $\eta = \sum_{j=1}^{k}\hat{\beta}_j s_j$ is a constant under the condition that $\hat{\boldsymbol{\beta}}$ is given and $h(\boldsymbol{\theta},\hat{\boldsymbol{\beta}}) = \sum_{i=1}^{n}\sum_{j=1}^{k}\log\left\{1+\exp(\theta_i+\hat{\beta}_j)\right\}$. It is clear that $h(\boldsymbol{\theta},\hat{\boldsymbol{\beta}})$ is invariant for rearrangement within $\boldsymbol{\theta}$. Thus, we focus on

$$l(\boldsymbol{t},\boldsymbol{\theta}) = \sum_{i=1}^{n}\theta_i t_i \tag{13}$$

when estimating $\boldsymbol{\theta}$. Then, we find that $l(\boldsymbol{t},\boldsymbol{\theta})$ is permutation invariant, and that $\tilde{\boldsymbol{\theta}}$ is a rearranged vector of $\hat{\boldsymbol{\theta}}$. Here, $\hat{\boldsymbol{\theta}}$ is the conditional maximum likelihood estimates for $\log L(\boldsymbol{\theta}|\hat{\boldsymbol{\beta}},\boldsymbol{t})$ in (12). As $l(\boldsymbol{t},\boldsymbol{\theta})$ is L-superadditive, $l(\boldsymbol{t},\boldsymbol{\theta})$ is AI. Then, it holds that

$$l(\boldsymbol{t}^*,\tilde{\boldsymbol{\theta}}_{\downarrow}) \le l(\boldsymbol{t}^*,\tilde{\boldsymbol{\theta}}) \le l(\boldsymbol{t}^*,\tilde{\boldsymbol{\theta}}_{\uparrow}).$$

As $\tilde{\boldsymbol{\theta}}$ is the vector of maximum likelihood estimates that maximizes $l(\boldsymbol{t},\boldsymbol{\theta})$ in (13), and $\tilde{\boldsymbol{\theta}}$ is a rearranged vector of $\hat{\boldsymbol{\beta}}$, it follows that $\tilde{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}_{\uparrow}$.□

Analogue to the MMLE of the Rasch model, the order-preserving properties holds for the MMLE of the LLTM.

Theorem 3. Let $\boldsymbol{v}^*$ be a rearranged vector such that $\boldsymbol{v}^* = \boldsymbol{v}_{\uparrow}$ and let $\hat{\boldsymbol{\delta}}$ and $\tilde{\boldsymbol{\delta}}$ be a vector of the maximum likelihood estimates that maximizes $L(\boldsymbol{\delta}|\boldsymbol{v})$ and $L(\boldsymbol{\delta}|\boldsymbol{v}^*)$ in (6), respectively. Then, $\tilde{\boldsymbol{\delta}} = \hat{\boldsymbol{\delta}}_{\uparrow}$.

Proof. As with the proof of Theorem 1 and 2, we focus on

$$l(\boldsymbol{u},\boldsymbol{\delta}) = \sum_{l}^{p} v_l\delta_l \tag{14}$$

in (6). From such permutation invariant of $l(\boldsymbol{u}, \boldsymbol{\delta})$ and the existence and uniqueness of the MMLE of the LLTM, we find that $\tilde{\boldsymbol{\delta}}$ is a rearranged form of $\hat{\boldsymbol{\delta}}$. On the other hand, as $l(\boldsymbol{u}, \boldsymbol{\delta})$ is AI, it holds that

$$l(\boldsymbol{v}^*, \tilde{\boldsymbol{\delta}}_{\downarrow}) \leq l(\boldsymbol{v}^*, \tilde{\boldsymbol{\delta}}) \leq l(\boldsymbol{v}^*, \tilde{\boldsymbol{\delta}}_{\uparrow}).$$

Consequently, $\tilde{\boldsymbol{\delta}}$ coincides on $\hat{\boldsymbol{\delta}}_{\uparrow}$.□

Next, we consider the case when maximum likelihood estimation $\hat{\boldsymbol{\delta}}$ is already given for the the LLTM. It is clear that the same result as the Theorem 7 holds for the LLTM. We denote this result as below corollary.

Corollary 1. Define that $L(\boldsymbol{\theta}|\hat{\boldsymbol{\delta}}, \boldsymbol{t})$ is likelihood function of the LLTM provided that $\hat{\boldsymbol{\delta}}$ is already given. Let $\boldsymbol{t}^*$ be a rearranged vector such that $\boldsymbol{t}^* = \boldsymbol{t}_{\uparrow}$ and let $\tilde{\boldsymbol{\theta}}$ be a vector of the maximum likelihood estimates that maximizes $L(\boldsymbol{\theta}|\hat{\boldsymbol{\delta}}, \boldsymbol{t}^*)$. Then, $\tilde{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}_{\uparrow}$.

Lastly, we consider structurally incomplete design for the LLTM. According to Fishcer(1994), we introduce following notations:

> $B = (b_{ij})$ is an $n \times k$ design matrix. If response of j-th item by i-th examinee is presented, then $b_{ij} = 1$. Otherwise, $b_{ij} = 0$.
>
> And $x_{ij} = \{0, a, 1\}$. If $b_{ij} = 1$, then $x_{ij} = \{0, 1\}$. Otherwise ($b_{ij} = 0$) $x_{ij} = a$ with $0 < a < 1$.

Then, (6) is modified as

$$\ln L(\boldsymbol{\delta}|X) = c + n\hat{\alpha} + \sum_{l}^{p} q_l \delta_l + \sum_{t}^{k} r_t \gamma(t) = \ln L(\boldsymbol{\delta}|\boldsymbol{q}), \tag{15}$$

where $q_l = \sum_i^n \sum_j^k x_{ij} b_{ij} w_{jl}$ and $\boldsymbol{q} = (q_1, q_2, \cdots, q_p)$. From (15) we get below result as a corollary of Theorem 3.

Corollary 2. Let $\boldsymbol{q}^*$ be a rearranged vector such that $\boldsymbol{q}^* = \boldsymbol{q}_{\uparrow}$ and let $\hat{\boldsymbol{\delta}}$ and $\tilde{\boldsymbol{\delta}}$ be a vector of the maximum likelihood estimates that maximizes $L(\boldsymbol{\delta}|\boldsymbol{q})$ and $L(\boldsymbol{\delta}|\boldsymbol{q}^*)$ in (15), respectively. Then, $\tilde{\boldsymbol{\delta}} = \hat{\boldsymbol{\delta}}_{\uparrow}$.

3 Discussion In this study, we examined the order-preserving property of the Rasch model and the LLTM in MMLE.

Especially, for Rasch model, our results from Theorems 1 and 2 coincide with those of Bertoli-Barsotti (2003), who focused on JMLE and CMLE. It is well known that the marginal maximum likelihood (MML) estimates and conditional maximum likelihood (CML) estimates are consistent, but that the joint maximum likelihood (JML) estimates are not (Neymann and Scott, 1948, Andersen, 1970). Nevertheless, the order-preserving statistics in the three estimation techniques coincide. This is because the biases of inconsistent estimates are positive. For example, Andersen (1980, Theorem 6.1) pointed out that the JML estimates for $\beta_1, \beta_2, \cdots, \beta_k$ have an approximate asymptotic bias of $\frac{k-1}{k}$, for infinite k, corresponding to the CML estimates. Following this result, it holds that

$$\check{\beta}_j = \frac{k-1}{k} \hat{\hat{\beta}}_j, j = 1, 2, \cdots, k$$

for the JML estimate $\check{\beta}_j$ and the CML estimate $\hat{\hat{\beta}}_j$. Note that the bias $\frac{k-1}{k}$ is strictly positive. Then, if it holds that $\hat{\hat{\beta}}_u \leq \hat{\hat{\beta}}_v (u \neq v)$, it also holds that $\check{\beta}_u \leq \check{\beta}_v$, and vice versa. Thus, the ordering of the estimates of $\boldsymbol{\beta}$ is preserved between the JML and CML estimates when k is infinite. Finally, when compared to the MML estimates, the JML estimates have positive biases.

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Communicated by *Masamori Ihara*

Kazumasa Mori
Graduate school of Arts and Sciences, The university of Tokyo,
3-8-1 Komaba, Meguro-ku, Tokyo, 153-8902 JAPAN
mailto:morikzm@nifty.com

NEUROSCIENTIFIC CONSIDERATION OF THE EDUCATIONAL EFFECT ACHIEVED USING ILLUSTRATED COURSE MATERIALS

Kiyoshi Kitahara, Kunihito Usui, Masataka Kaneko, Setsuo Takato

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Abstract. It is our pedagogical challenge to introduce various mathematical concept in an educationally acceptable way and to prepare course materials that make students understand that deeply. As described in this paper, we present some of our attempts to verify the effects of using carefully prepared course materials with high-quality graphs in collegiate education of mathematics. Through our experiment, we detected the change of students' brain activity by conducting behavioral observation and neuroimaging simultaneously. In an experiment aimed at helping students understand the concept of an exponential growth comparing exponential and power functions, we prepared several graphs for that purpose. Seven students observed the graphs while we tracked their responses. Simultaneously, we monitored their brain activities using electroencephalography (EEG). Three students altered their judgments, we found, on viewing the triggering graph. Some changes in the trend of their EEG signal were recognized while they were viewing the graph. These results of our experiments show that the use of favorable graphs as course materials might promote learners' reasoning processes.

1 Introduction According to our questionnaire survey, a major opinion of teachers of collegiate mathematics in Japan denies the necessity of using high-quality graphs as course materials[4]. However our experiences indicate that the use of graphs plays a crucial role in some classroom environments. Educators usually use various graphs edited using a popular TEX tool as course materials. Additionally, we have verified the effectiveness of using graphs by comparing the responses of students to whom we showed high-quality graphs to those of students to whom we did not do that[3]. Generation of high-quality graphs is preferably done by computer algebra system (CAS) because of its computing and programming capabilities. However, it is not always easy to handle graphical images of outputs in documents edited using TEX. For instance, some elaborations are necessary to locate generated images to suitable positions in documents and to arrange the layout of other components flexibly and in a balanced manner. Although some TEX graphic systems exist such as PStricks[7] and TikZ[8], their computing capabilities remain restricted. As a handy tool for both generation of high-quality graphs with CAS and the easy arrangement of TEX document components, we have been using KETpic, a macro package designed to generate TEX-readable code for CAS-created graphical output. That package and related documentation can be freely downloaded from the website: **http://ketpic.com**.

The aim of this paper is to present some new attempts to verify the effectiveness of using carefully prepared course materials with high-quality graphs in collegiate mathematics education. Based on this experimental study, we claim that our methodology demonstrates great possibilities for providing an objective means to verify the effects of course materials of various types in collegiate education of mathematics.

2010 *Mathematics Subject Classification.* Mathematics Education .
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2 How to teach the nature of exponential growth The exponential function is an important item in science and engineering education. Exponential growth is the most fundamental and the most discriminative natures in various characteristics of the exponential function. In Section 7.2 of his book titled Calculus[6], Stewart wrote, showing Figure 4,"Figure 4 shows how the exponential function $y = 2^x$ compares with the power function $y = x^2$. The graphs intersect three times, but ultimately the exponential curve $y = 2^x$ grows far more rapidly than the parabola $y = x^2$ (see also Figure 5)".

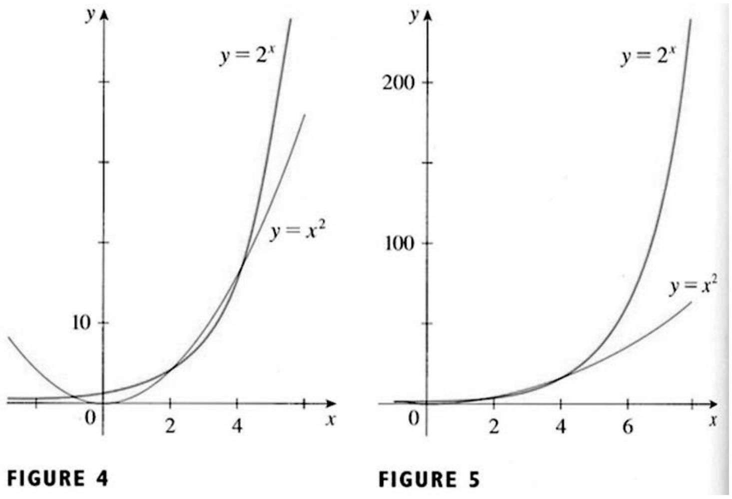

Figure 1. Graphs in Section 7.2 of Stewart's book

This nature is connected closely with the differential equation of $\frac{dy}{dx} = ky$ (k is any positive constant) that appears in phenomena treated in science and engineering. At Section 7.3* of the same book, he wrote for the case of $k = 1$. "The geometric interpretation of Formula 8 (i.e. the above Formula) is that the slope of a tangent line to the curve $y = e^x$ at any point is equal to the y-coordinate of the point. This property implies that the exponential curve $y = e^x$ grows very rapidly". Whereas the exponential growth is represented by $a^x > x^n \ : \ a > 1$, n is any natural number. Also x is large unboundedly. It is our pedagogical challenge to demonstrate this characteristic in an educationally acceptable way and to prepare course materials that make students understand that deeply.

Approaches to this matter in high school and college in our country are inappropriate. In high school mathematics textbooks (from major five textbook publishers), the description of

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty, \qquad \lim_{x\to\infty} \frac{x^n}{e^x} = 0 \qquad \text{(for every natural number } n\text{)}$$

is done to supplement the inequality of $e^x > 1 + x$, $e^x > 1 + x + \frac{1}{2}x^2$ in the section of application of differential calculus to inequality, as a tip. Furthermore, the textbook from Suken Shuppan exceptionally describes the following. "Therefore, $y = e^x$ increases more rapidly than x^n when $x \to \infty$". Our experience has just taught us that the concept of "grows more rapidly" is tough even for college students, and high school students. In standard textbooks and reference books for college mathematics, it is designated as a problem for which l'Hopital's rule is applied.

An animated course material displayed by a projector has been developed to promote deep understanding and fixing of the concept of "exponential growth". In the first frame, two graphs of power function $y = x^4$ and exponential function $y = 2^x$ are drawn with x-y axes with the same scale reduction. With scale reduction of the y-axis increase, two graphs begin to intersect with each other. Then they exchange their magnitude relation. The animation comprises 37 frames. The final scale reduction of the y-axis is $\frac{1}{10{,}000}$.

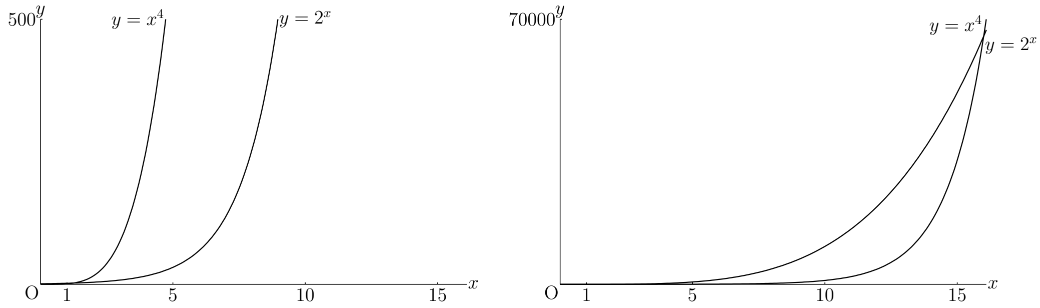

Figure 2. Examples of animation.

We used the program feature of the KETpic command "Texcom" for automatic generation of a TEX file that shows graphs for each scale reduction by the KETpic slide of TEX. The programming is extremely simple, as shown below.

```
Openfile('SisuPR.tex');
FC=0;
for K=[1,10,100,1000,10000],
  if K==10000 then JJ=[1];
  else JJ=[1,2,3,4,5,6,7,8,9]; end
  for J=JJ,
    I=J*K;FC=FC+1;
    Texcom('\newslide[0]{}%');
    Texcom('\begin{layer}{130}{0}');
    Texcom('\putnotese{35}{0}{{\Large Scale of $y=$ 1/'+string(I)+'}}');
    Texcom('\putnotese{5}{15}{\input{'+Fname+string(FC)+'Zu.tex}}');
    Texcom('\end{layer}');
    Texcom('%');
  end
end
Closefile();
```

Avoiding movement of the coordinate axis and comments in the graph is the most important tip to demonstrate a series of slides as an animation. It is the layer environment which realizes that allocating figures, mathematical symbols, and sentences to the desired place in a page[5].

Using this course material, we taught a class for about 50 students at Shibaura Institute of Technology. Then we obtained their favorable responses. The questionnaire, which surveyed their understanding of exponential growth before and after the class, revealed the following: 69.4% of students did not know that before the class but understood it correctly after the class, 14.3% students reported after the class, that their understanding of it had been inaccurate before the class. Summing the first two groups above, 83.7% of students reported gaining a correct understanding of the concept through the class. It is noteworthy that 12.2% knew the material even before the class. The remaining 4.1% students did not understand it well even after the class. Although collecting students' opinions and feelings through questionnaire surveys or interviews is important to improve course material, it might not work as an objective evaluation for a course materials. The KETpic research group has been performing an objective evaluation for the course materials using a statistical method. In academic year 2013, we taught an experimental lesson at Nagano National College of Technology and Gunma National College of Technology, making use of course materials for trigonometric functions and polar coordinates and dividing subjects into experimental and control groups for purposes of statistical analysis[2]. Although this analytical procedure is necessary to guarantee statistical objectivity, it requires too many subject students and might interfere with other ordinary lessons. Selecting a proper method to analyze statistical data from an objective standpoint is among several challenges and difficulties in this type of experiment.

Recently some researchers, mainly from Nakagawa Laboratory, Nagaoka University of Technology, have developed a method to capture a time-series of brain activities by analysis of emotional information collected through electroencephalography (EEG). This study was conducted for objective evaluation of course materials to teach the concept of "exponential growth" using evolution of time-series brain activity captured using EEG.

3 Developments until measuring experiments Recently, researchers mainly from Nakagawa Laboratory, Nagaoka University of Technology have strived to develop a method to trace activation in the brain using fractal analysis of time-series data of blood currents in the brain measured through Near Infrared Red Spectroscopy (NIRS). Employing this method, we conducted NIRS measurement for 20 first-year students in March, 2013 at Kisarazu National College of Technology. The target was to draw a graph of trigonometric functions including other fundamental items. Kurimoto Laboratory was also involved.

Experiments with and without EEG measurements are described below. The task for these experiments was exclusively to understand "exponential growth". Although the measuring task in August 2014 and the improvement to it are introduced below, the prior tasks were the same as those shown here. We passed out the first handout (sheet) describing the task. It reads as follows:

> Shall we find on the graph the difference of growth rates between power and exponential functions when x tends to infinity unboundedly?

Time measurements started when the second handout was passed out and subjects were asked to answer Sheets 1∼7 with one minute each. In sheet 1, $y = x^2$ and $y = x^4$ were compared. Two curves were drawn on the same coordinate plane in the range of $0 \leq y \leq 10$. The problem read:

1. Which one increases more rapidly when x increases unboundedly?
(1) $y = x^2$ (2) $y = x^4$

2. What is the magnitude of $y = \dfrac{x^2}{x^4}$ when x increases unboundedly?

On Sheet 2, $y = x^2$ and $y = 2^x$ were compared, two curves were drawn as in Sheet 1. The problem were similar to those in Sheet 1, as well. Sheets 3∼7 posed the main problems related to the task. There, $y = x^4$ and $y = 2^x$ were compared. Two curves were drawn on the same coordinate plain in the range of $0 \leq x \leq 15$. The scale reductions of y-axis on each sheet were $1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1{,}000}, \frac{1}{10{,}000}$, respectively. The problems were similar to those of other sheets.

In collaboration with Nakagawa Laboratory of the Nagaoka University of Technology, we monitored brain waves using two electrodes concurrently with NIRS measurements in March 2014. Subjects were three fifth-year Technical College students: two men and one woman. We later recognized the importance of recording the answering time in conjunction with the action observation. All three subjects made mistakes at the beginning, noticed the mistake, and answered correctly afterward. Comparison of records of answering time and brain waves revealed that the standard deviation SD of brain wave when they noticed their mistake was quite different from that of other instances. After they realized their mistake and answered confidently, they answered more quickly than ever. They answered slowly when wavering, we found. Table 1 presents the answering time (s) and Right-Wrong R/W ratios of each subject.

	Sheet1	Sheet2	1/1	1/10	1/100	1/1,000	1/10,000
Sub1	16, R	10, W	18, W	13, W	09, W	17, R	15, R
Sub2	28, R	32, W	33, W	36, W	45, R	15, R	16, R
Sub3	18, R	18, W	15, W	14, W	10, W	17, W	45, R

Table 1. Answering time and Right-Wrong

Motivated by this experiment, similar experiments for the same task, but without brain wave monitoring, were conducted at Pennsylvania State University and Toho University for some 20 first-year students in May, 2014. After these experiments, we improved the method of task explanation and question posing because the concept of "difference of two functions in increasing speed" was difficult and subjects did not understand that, we found. We improved the explanation of the task direction as presented below.

> Shall we find which of power function and exponential function grows larger when x is extremely large?

Problems were improved as presented below (these are questions for Sheet 4).

> Problem 1. Which of the following three ranges does include the value of x for which $y = 500$ in the graph of $y = 2^x$?
> (1-1) $0 < x < 5$ (1-2) $5 < x < 10$ (1-3) $10 < x < 15$
>
> Problem 2. Which of the following three is right when x is made very large?
> (2-1) $x^4 > 2^x$ (2-2) $x^4 < 2^x$ (2-3) None of them

Problem 1 is the reference problem for Problem 2. It is explained in the next section. In Problem 2, subjects were asked to select a correct inequality. It asked a simple magnitude relation.

At the end of this section, the major improvement on methods of selecting answers and measuring the answering time is to be reported. In the experiment in August, subjects simply pushed down switches of a Response Analyzer instead of putting a mark on a sheet by writing material. This eliminated unnecessary movement of the body and electric noise caused by it. The Response Analyzer is a test product of one of the authors (Usui). It has since been modified. The special features of this device are the following:

- Many devices (extensions) can be connected to a base unit by wireless communication, enabling simultaneous counting.
- Base unit is Raspberry Pi.
- Equipped with Linux-based OS

- High scalability
- Allow future network communication

The Response Analyzer improved the precision of the measurement of answering time and provided great merits of visualization of subjects' thinking processes. Many extensions might be used to monitor subjects with the same task. Analysis of data from such experiments will bring about a new method of evaluating course materials.

4 Method and result of measurement We conducted brain wave measurements to be discussed in this report on August 4 and 5 at Nagaoka University of Technology. To collect emotional information of five different kinds (Complete rest, Pleasant, Unpleasant, Joyful, Angry), measurements were done at 16 positions based on the International 10-20 system. Electrodes were set at positions 1∼16 as shown in Fig. 3, 18 at the top of the head were to remove noise caused by eye movement; A2 at the right ear was designed to remove that caused by heart beats (Fig. 3). Seven male subjects were tested, 6 graduate students (Sub 1 through Sub 6) and one-fourth year student of technical college (Sub 7). A TEAC Polymate V (16 channel, 8000 [Hz]) was used as the measuring instrument.

Two measurement tasks were set: one collected data used as reference data of emotion analysis and another data for mathematical task. Reference data of emotion analysis were:

Wait, Complete rest, Rest, Pleasant, Rest, Unpleasant, Rest, Joyful, Rest, Angry

All measurements were taken for one minute.

The mathematical task was done using seven handouts (sheets). Subjects were asked to be seated on their chairs along a long table. Handouts were put on the desk with the reverse side up. Therefore, the task description was hidden. Subjects flipped the handouts one by one and read the task. Each handout posed two multiple-choice questions. Sheets 1∼6 (Tasks A1 through A6) had to be answered in one minute each. Sheets 7 (Task A7) in three minutes. Although subjects were able to look at digital-clocks in front of them, the ending of answering time was declared orally by a time-keeper. Because that declaration was made to subjects, they were able to concentrate on the task with no concern about running out of time. Subjects answered by pushing down any one of four buttons of the Response Analyzer set up at the right-hand side of each subject. The Response Analyzer played an important role in this measurement, as explained in the previous section. Table 2 presents a description in this section.

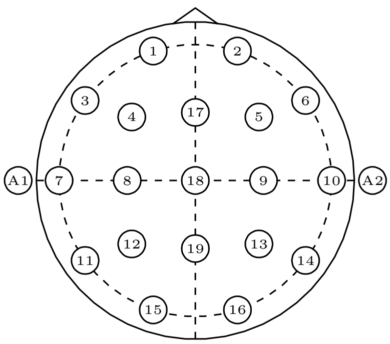

Figure 3. The positions where electrodes are attached

Term	Substance
Subject	Male
EEG Device	TEAC Polymate V(16ch, 8000[Hz])
Measuring Sites	16 positions(1∼16) in the international 10-20 system
Task	To understand the concept of "exponential growth"

Table 2. Measurement environment

The mathematical task proceeded as below. The first handout merely explained the contents of task and subjects' operations to answer questions. No measurement was done for this handout. Having written the improvement in the method of explaining task in the previous section (no description such as "difference of two functions in increasing speed") we merely introduced the tips in operation for answering questions here. These were added and modified on the introduction of Response Analyzer.

◇ You are asked to turn to the next page when you hear "All right!".
◇ Each option goes like (Number of problem−Number of answer).
◇ When you answer, you push the Number of the problem first and then the Number of the answer.
Example: For (1-3), you push button 1, 3, For (2-2), you push button 2, 2.
◇ Unless otherwise noted, you can read one page for one minute each.
When one minute has elapsed, you will hear "All right!".
◇ You can change your answer as many times as you wish during the answering time allowed.
When you change your answer, push the number of the problem first and then the number of the answer.
When you forgot how you pushed, you can push again.
Example: To change to (1-4), you push button 1, 4
Example: To select (1-3) again for confirmation, you push button 1, 3

Then timing measurement started and subjects entered into the first task (Task A1). Task A1 was a reference task for other mathematical tasks. It was similar to all other mathematical tasks. Any subject might have answered correctly. The reference task worked as the reference data for brain wave analysis later. Subjects answered, examining the graphs, two questions about magnitude relation between $y = x^2$ and $y = 2^x$. In addition, Problem 1 was the reference task for Task A1.

Problem 1. Which of the following two curves passes through origin?
(1-1) $y = x^2$ (1-2) $y = 2^x$

Problem 2. Which of the following three is right when x is very large?
(2-1) $x^2 > 2^x$ (2-2) $x^2 < 2^x$ (2-3) None of them

Tasks A1 through A6 were structured similarly to Task A1. Subjects answered, examining graphs, two questions about the magnitude relation between $y = x^4$ and $y = 2^x$. Problem 1 was the reference one; Problem 2 purposeful one. Graphs for these tasks are presented below. We monitored the change of subjects' brain activity when they examined graphs with scale reductions of y-axis $\frac{1}{1}$, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1{,}000}$, and $\frac{1}{10{,}000}$, serially. These graphs in Figure 4 respectively correspond to Tasks A2, A4, A5, and A6.

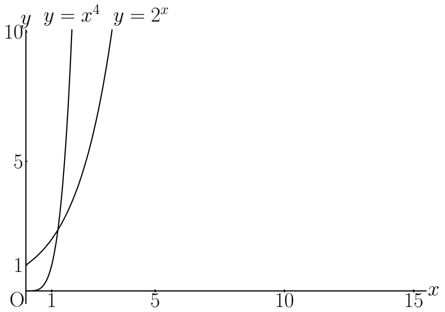

Fig.4-1. Sheet 2: 1/1

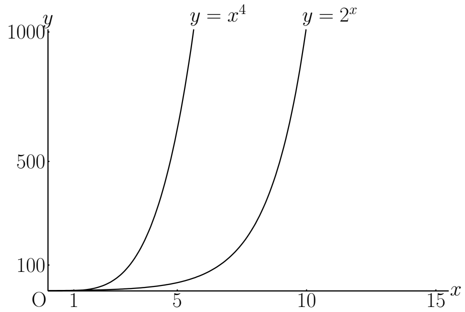

Fig.4-2. Sheet 4: 1/100

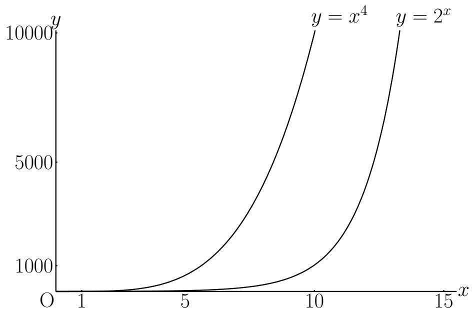

Fig.4-3. Sheet 5: 1/1,000

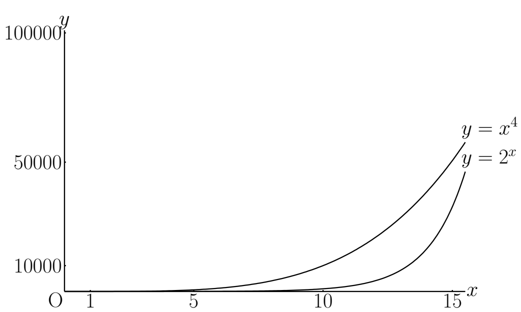

Fig.4-4. Sheet 6: 1/10,000

Figure 4. Graphs used for the EEG measurement

The last task of Task B asked subjects to understand the nature of exponential growth through Taylor's theorem. It used examples of exponential function $y = e^x$ and power function $y = x^2$ and asked subjects to read the explanation and answer questions in 3 min.

The next table presents a summary of records of answering time and right or wrong of the answer R/W measured using the Response Analyzer. Subjects were asked to push the button twice for each problem. The first push inputted the Number of the problem, which was k1 in the table. The second push inputted the Number of the answer and k2 in the table is R/W of that answer.

Task	Sub1				Sub2				Sub3				Sub4			
	Time	k1	k2	R/W	Time	k1	k2	R/W	Time	k1	k2	R/W	Time	k1	k2	R/W
A1	0:13.0	1	1	R	0:12.7	1	1	R	0:12.1	1	1	R	0:44.7	1	1	R
	0:41.7	2	2	R	0:23.4	2	2	R	0:23.3	2	2	R	1:01.9	2	2	R
A2	1:08.4	1	1	R	1:10.5	1	1	R	1:08.5	1	1	R	1:34.1	1	1	R
	1:17.3	2	1	W	1:21.2	2	2	R	1:18.4	2	2	R	1:51.0	1	1	R
													1:56.8	2	1	W
A3	2:29.6	1	2	R	2:59.5	1	2	R	2:20.9	1	2	R	2:26.3	1	2	R
	2:49.8	2	1	W					2:32.0	2	2	R	2:44.6	2	1	W
A4	3:31.6	1	3	W	3:15.3	1	2	R	3:15.3	1	2	R	3:14.3	1	2	R
	3:46.1	2	1	W	3:19.9	2	2	R	3:22.8	2	2	R	3:32.3	2	1	W
A5	4:14.5	1	3	R	4:16.7	1	3	R	4:12.5	1	3	R	4:13.1	1	3	R
	4:52.5	2	1	W	4:21.8	2	2	R	4:21.4	2	2	R	4:22.6	2	1	W
A6	5:20.5	1	3	R	5:11.3	1	3	R	5:22.6	1	3	R	5:31.4	1	3	R
	5:34.8	2	1	W	5:19.6	2	2	R	5:50.9	2	2	R	5:35.3	2	2	R
	5:52.5	2	2	R												
B	7:52.7	1	3	R	7:17.5	1	1	W	7:08.2	1	3	R	6:36.8	1	3	R
	8:22.6	2	2	R	7:39.8	2	1	W	7:19.0	2	1	W	7:06.6	2	2	R

Table 3-1. Answering time, buttons that had been pushed, and Right-Wrong

Task	Sub5				Sub6				Sub7			
	Time	k1	k2	R/W	Time	k1	k2	R/W	Time	k1	k2	R/W
A1	0:17.2	1	1	R	0:18.6	1	1	R	0:11.0	1	1	R
	0:58.9	2	2	R	0:30.5	2	2	R	0:24.5	2	2	R
A2	1:14.0	1	1	R	1:11.2	1	1	R	1:06.0	1	1	R
	1:44.2	2	1	W	1:54.2	2	2	R	1:52.0	2	2	R
A3	2:22.7	1	2	R	2:37.3	1	2	R	2:18.1	1	2	R
					2:46.0	2	2	R	2:59.2	2	1	W
A4	3:20.0	1	2	R	3:19.0	1	2	R	3:15.1	1	2	R
	3:37.5	2	2	R	3:23.8	2	2	R	3:58.0	2	1	W
A5	4:12.9	1	3	R	4:15.7	1	3	R	4:09.4	1	3	R
	4:32.3	2	2	R	4:22.7	2	2	R	4:32.0	2	2	R
A6	5:13.7	1	3	R	5:16.5	1	2	W	5:07.3	1	3	R
	5:30.5	2	2	R	5:18.0	1	3	R	5:08.2	2	2	R
					5:23.1	2	2	R				
B	6:54.9	1	3	R	7:15.0	1	3	R	6:53.1	1	3	R
	7:19.0	2	2	R	7:40.8	2	2	R	7:03.5	2	2	R

Table 3-2. Answering time, buttons that had been pushed, and Right-Wrong

Although data of brain waves at 16 positions for emotion analysis were collected, we have not conducted emotion analysis itself. We calculated the variance property $\alpha(t)$ from brain waves monitored at electrodes 4 and 5 using self-similarity analysis. Then we discussed the characteristics of brain activity. An outline of the relation between variance property and electrode voltage is shown here[1]. First we calculated the next quantity comparable to deviation of electrode voltage $x(t)$.

$$\text{a variogram at lag } \tau\text{:} \quad V(\tau) = \frac{1}{2} E\left[(x(t) - x(t+\tau))^2\right]$$

We set as $\tau = 0.25\,\mathrm{s}$. If $x(t)$ has self-similarity, then proportionality of $V(\tau) \approx |\tau|^\alpha$ holds. On taking logarithm of both sides of this equation, we obtain the following.

$$\log V(\tau) = \log A + \alpha \log|\tau| \ , \ A \text{ is a constant of proportionality}$$

Consequently, an almost linear relation is obtained when $\log V(\tau)$ is shown against $\log|\tau|$ on the logarithmic graph and α is the slope of the curve. Graphs of $\alpha(t)$ for Subject 7 are portrayed below. In these graphs red line represent the graph of $\alpha(t)$ for EEG channel 4 (left brain) and green line that for channel 5 (right brain).

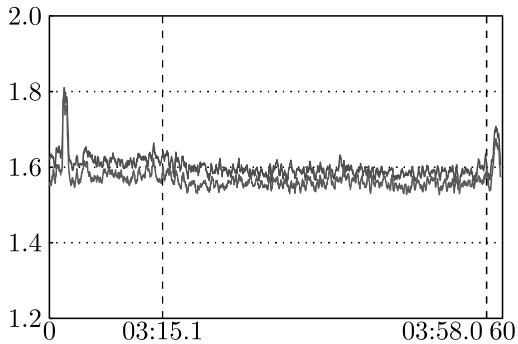

Figure 5-1. Task A4

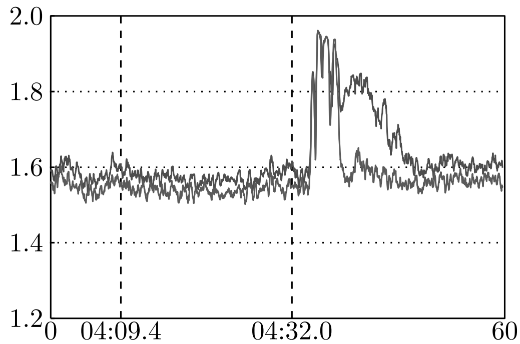

Figure 5-2. Task A5

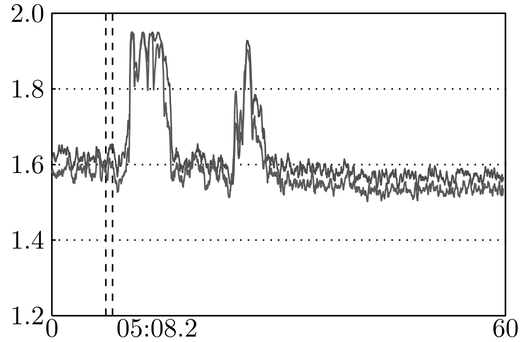

Figure 5-3. Task A6

Figure 5. Graph of $\alpha(t)$ for Subject 7

5 Summary and future challenge Making use of a few graphs of variance property $\alpha(t)$ in the previous section, we analyzed their characteristics. Herein, we provide conclusions to that analysis. The variance property is $1 \leq \alpha \leq 2$ if the input signal $x(t)$ does not include much noise and has little bias. If the brain activity is in the state of rest, then α oscillates at around 1.2. When α oscillates around a high value, the brain is estimated as activated with a number of synchronized brain signals. Herein, we examine data of Subject 7 when he engaged in Task A.

1. When he noticed the essence of the problem and changed his wrong answer to the correct one, it was reflected in the data (Fig. 5-2 Task A5).

2. Data show that he answered with certainty (Fig. 5-3 Task A6).

(1) In Fig. 5-2, we show that oscillation of α around a high value and long duration of it after answering correctly demonstrates his consent to his answer. In Figure 5-1, the rise of α value reflecting brain activity after selecting the wrong answer is apparent at left end of the figure for Task A3 and at the right end of the figure for Task A4. However, the rise of α value is not remarkable. Moreover, its duration is extremely short .
(2) Figure 5-3 shows oscillation of α around a high value and long duration of it when answered correctly to Problem 2 immediately after answering Problem 1 correctly.

Subject 7 answered our questionnaire after the tasks describing that "Although I did not know the magnitude relation between power function and exponential function, I noticed it at Task A5 and was convinced it at Task A6".

He answered all tasks of Problem 1 correctly. His brain wave was stable after pushing the button. No special activation was observed. Furthermore, because he answered Problem 2 of Task A6 just 0.9 s after answering Problem 1 of the same task, no change of α value reflecting activity related to Problem 1 was extracted.

The following three points remain as important challenges for future study.

1. Performing emotion analysis using brain wave data collected in this study to find new information.

2. Continuing similar experimental measurements using proper course materials to facilitate the development of course materials.

3. Continual improvement of the Response Analyzer to establish a course material evaluation method.

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Communicated by *Tadashi Takahashi*

Kiyoshi Kitahara
kitahara@cc.kogakuin.ac.jp
Division of Liberal ArtsKogakuin University
1-24-2 Nishi-Shinjuku, Shinjuku-kuTokyo 163-8677 Japan

A NOTE ON A TWO-PERSON ZERO-SUM STOPPING GAME

Cloud Makasu

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Abstract. In this short note, a two-person zero-sum stopping game when the observation processes are exponential Brownian motions is formulated and solved explicitly under certain conditions. The present problem extends the planar Brownian motion case treated by Vinnichenko and Mazalov [6].

1 Introduction For a planar Brownian motion on a closed interval $[0,1]$ and absorbed at the end points, a two-person zero-sum stopping game is considered by Vinnichenko and Mazalov [6]. It is proved therein that the optimal stopping times are the so-called Azéma-Yor stopping times [2] and the value of the game is the smallest upper convex function of the payoff (see Dynkin and Yushkevich [3]), both given explicitly. The proof is essentially a result of decomposing the zero-sum stopping game into two pure optimal stopping problems [5]. In the one-dimensional case, a variant of the zero-sum stopping game in [6] is treated by Yasuda [7]. The main purpose of this note is to extend the result by Vinnichenko and Mazalov [6] to the case of exponential Brownian motions. We note that our method of proof is based on similar arguments used in [6]. However, it must be stressed that our main result is not contained in [6].

For the reader's convenience, we shall use similar notation as in [6]. We consider the following two-person zero sum stopping game. Let $i = 1, 2$ and $x_t^{(i)}$ be a geometric Brownian motion starting at a_i in the closed interval $[0,1]$ and absorbed at the end points, with drift-diffusion coefficients $(\mu_i x^i, \sigma_i x^i)$ where $\mu_i < 0$ and $\sigma_i > 0$ are fixed and given constants. The strategy of player i is the stopping time τ_i with respect to the process $x_t^{(i)}$. Players stop their observation processes at the states $x_{\tau_1}^{(1)}$, $x_{\tau_2}^{(2)}$, respectively. Then if $x_{\tau_1}^{(1)} > x_{\tau_2}^{(2)}$ the payoff of player 1 is $+1$; if $x_{\tau_1}^{(1)} < x_{\tau_2}^{(2)}$ then the payoff is -1. Otherwise, the payoff is assumed to be zero. Player 1 seeks to maximize the expected payoff

$$H(\tau_1, \tau_2) = \mathbf{E}\left\{ I_{\{x_{\tau_1}^{(1)} > x_{\tau_2}^{(2)}\}} - I_{\{x_{\tau_1}^{(1)} < x_{\tau_2}^{(2)}\}} \right\}$$

with player 2 seeking to minimize it, where I_A denotes the indicator function of the set A.

The motivation of the present problem arises from an application in mathematical finance. Consider two investors observing the evolution of prices of two types of stocks. The problem is that of deciding when to invest depending on the values of the two stocks.

2 Main result Fix $i = 1, 2$. Let $\Delta_i = 1 - \frac{2\mu_i}{\sigma_i^2}$, where $\mu_i < 0$ and $\sigma_i > 0$ are fixed and given constants. Now following [6], we let $a_1 \le a_2$, $\Delta_1 \le \Delta_2$ and put

$$a = \min\left\{ \left(\frac{2}{\Delta_2}(\Delta_2 + 1)a_2^2 \right)^{1/(\Delta_2+1)}, (2 - 2a_2)^{1/\Delta_2} \right\}.$$

Define

$$\psi_1(x) = 1 - \frac{a_1}{a_2} + \frac{a_1}{2a_2^2} x^{\Delta_1}, \quad \psi_2(x) = \frac{1}{2a_2} x^{\Delta_2}$$

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and

$$
(1) \qquad s_i^*(x) = \begin{cases} 0 & \text{if } x < 0 \\ \psi_i(x) & \text{if } 0 \le x < a \\ \psi_i(a) & \text{if } a \le x < 1 \\ 1 & \text{if } x \ge 1. \end{cases}
$$

To this end, we shall also introduce the following barycenter function

$$
g_i(x) = a_i + \frac{1}{1 - s_i^*(x-0)} \int_x^1 (u - a_i) ds_i^*(u) = \begin{cases} a_i & \text{if } x = 0 \\ \min\left\{1, a_2 + \frac{\Delta_2 x^{\Delta_2}}{2a_2 - x^{\Delta_2}} \left(\frac{a_2}{\Delta_2} - \frac{x}{\Delta_2+1}\right)\right\} & \text{if } 0 < x \le 1 \end{cases}
$$

and the Azéma-Yor stopping times

$$
(2) \qquad \tau_i^* = \inf\left\{t : g_i(x_t^{(i)}) \le \sup_{0\le s\le t} x_s^{(i)}\right\}.
$$

The main result of this note is stated in the next theorem.

THEOREM 2.1. Let $i = 1, 2$ and $x_t^{(i)}$ be an exponential Brownian motion starting at a_i in the closed interval $[0, 1]$ and absorbed at the end points, with drift-diffusion coefficients $(\mu_i x^i, \sigma_i x^i)$ where $\mu_i < 0$ and $\sigma_i > 0$ are fixed and given constants. For $a_1 \le a_2$ and $\Delta_1 \le \Delta_2$, the value of the two-person zero-sum stopping game

$$
\sup_{\tau_1}\inf_{\tau_2} \mathbf{E}\left\{I_{\{x_{\tau_1}^{(1)} > x_{\tau_2}^{(2)}\}} - I_{\{x_{\tau_1}^{(1)} < x_{\tau_2}^{(2)}\}}\right\} = \inf_{\tau_2}\sup_{\tau_1} \mathbf{E}\left\{I_{\{x_{\tau_1}^{(1)} > x_{\tau_2}^{(2)}\}} - I_{\{x_{\tau_1}^{(1)} < x_{\tau_2}^{(2)}\}}\right\}
$$

is given by $H^* = \frac{a_1^{\Delta_2} - a_2}{a_2}$ and the pair of stopping times (τ_1^*, τ_2^*) given by (2) is the equilibrium point provided that

$$
2a_1a_2 - a_1a_2^{\Delta_1} - a_2a_1^{\Delta_2} = 0.
$$

Proof. This follows with minor modifications of the proof in [6]. Hence, we shall omit the details.

REMARK 2.1. In the special case when $\Delta_1 = \Delta_2 = 1$, then the above result coincides with the result in [6] for the planar Brownian motion case.

3 Conclusion A two-person zero-sum stopping game when the observation processes are assumed to be exponential Brownian motions is solved explicitly, under certain conditions. The present result extends the one obtained by Vinnichenko and Mazalov [6] for the planar Brownian motion case, and has several applications in mathematical finance.

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Communicated by *Vladimir V. Mazalov*

Cloud Makasu
Dept. of Maths and Applied Maths
University of the Western Cape
Private Bag X17
Bellville 7535
Cape Town, South Africa
E-mail: cmakasu@uwc.ac.za

SOFT BCI-POSITIVE IMPLICATIVE IDEALS OF SOFT BCI-ALGEBRAS

M. ASLAM MALIK *AND M. TOUQEER †

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Abstract

The notion of soft BCI-positive implicative ideals and BCI-positive implicative idealistic soft BCI-algebras is introduced and their basic properties are discussed. Relations between soft ideals and soft BCI-positive implicative ideals of soft BCI-algebras are provided. Also idealistic soft BCI-algebras and BCI-positive implicative idealistic soft BCI-algebras are being related. The intersection, union, "AND" operation and "OR" operation of soft BCI-positive implicative ideals and BCI-positive implicative idealistic soft BCI-algebras are established. The characterizations of (fuzzy) BCI-positive implicative ideals in BCI-algebras are given by using the concept of soft sets. Relations between fuzzy BCI-positive implicative ideals and BCI-positive implicative idealistic soft BCI-algebras are discussed.

Keywords: Soft set; (BCI-positive implicative idealistic) soft BCI-algebra; Soft ideal; Soft BCI-positive implicative ideal.

1 Introduction

The real world is inherently uncertain, imprecise and vague. Because of various uncertainties, classical methods are not successful for solving complicated problems in economics, engineering and environment. The theories such as the probability theory, the (intuitionistic) fuzzy sets theory, the vague set theory, the theory of interval mathematics and the rough set theory, which are used for handling uncertainties have their own difficulties. One of the reasons for these difficulties is due to the inadequacy of the parametrization tool of the theory, which was pointed out by Molodtsov [16]. To overcome these difficulties, Molodtsov introduced the concept of soft sets as a new mathematical tool for dealing with uncertainties. Soft set is a parameterized general mathematical tool which deals with a collection of approximate description of objects. In the soft set theory, the initial description of the object has an approximate nature and there is no need to introduce the notion of exact solution. The absence of any restrictions on the approximate description in soft set theory makes this theory very convenient and easily applicable in practice. Applications of soft set theory in different disciplines and real life problems are now catching momentum some of which are being discussed here.

Min [15] studied the concept of similarity between soft sets, which is an extension of the equality for soft set theory. He introduced the concepts of conjunction parameter and disjunction parameter of ordered pair parameter for soft set theory and investigated modified operations of soft set theory in terms of ordered parameters. Yang and Guo [19] introduced the notions of anti-reflexive kernel, symmetric kernel, reflexive closure and symmetric closure of a soft set relation. Soft set relation mappings and inverse soft set relation mappings were also discussed. Kalayathankal and Singh [9] introduced a fuzzy soft flood alarm model which was applied to five selected sites of Karal, India to predict potential flood. Shabir and Naz [18] introduced soft topological spaces defined over an initial universe with a fixed set of parameters. They introduced the notions of soft open sets, soft closed sets, soft closure, soft interior points, soft neighborhood of a point and soft separation axioms. Zhan and Jun [20] discussed soft BL-algebras based on fuzzy sets. They proved that a soft set is an implicative filteristic soft BL-algebra if and only if it is both a positive implicative filteristic soft BL-algebra and a fantastic filteristic soft BL-algebra. Z. Zhang [21] presented a rough set approach to intuitionistic fuzzy soft set based decision making. Jiang et al. [4] discussed interval-valued intuitionistic fuzzy soft sets and their properties. Agarwal et al. [1] generalized the concept of intuitionistic fuzzy soft set by including a parameter reflecting a moderator's opinion about the validity of the information provided. We refer the readers to [2, 17] for further informa-

tion regarding development of soft set theory.
Jun [5] applied the concept of soft sets by Molodtsov to the theory of BCK/BCI-algebras. He introduced the notion of soft BCK/BCI-algebras and soft subalgebras. Jun and song [8] defined soft subalgebras and soft ideals in BCK/BCI-algebras related to fuzzy set theory. Jun et al. [6] introduced the notion of soft p-ideals and p-idealistic soft BCI-algebras and provided the relations between fuzzy p-ideals and p-idealistic soft BCI-algebras. In this paper, we introduce the notion of soft BCI-positive implicative ideals and BCI-positive implicative idealistic soft BCI-algebras. Using soft sets, we give characterizations of (fuzzy) BCI-positive implicative ideals in BCI-algebras. We provide relations between fuzzy BCI-positive implicative ideals and BCI-positive implicative idealistic soft BCI-algebras.

2. Basic results on BCI-algebras

BCK/BCI-algebras are important classes of logical algebras introduced by Y. Imai and K. Iséki [3] and were extensively investigated by several researchers.

An algebra $(X, *, 0)$ of type $(2, 0)$ is called a BCI-algebra if it satisfies the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0$
(II) $(x * (x * y)) * y = 0$
(III) $x * x = 0$
(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y, z \in X$. In a BCI-algebra X, we can define a partial ordering "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$.

If a BCI-algebra X satisfies the identity:
(V) $0 * x = 0$,
for all $x \in X$, then X is called a BCK-algebra.

In any BCI-algebra the following hold:

(VI) $(x * y) * z = (x * z) * y$
(VII) $x * 0 = x$
(VIII) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$

(IX) $0 * (x * y) = (0 * x) * (0 * y)$
(X) $x * (x * (x * y)) = (x * y)$
(XI) $(x * z) * (y * z) \leq x * y$

for all $x, y, z \in X$.

A non-empty subset S of a BCI-algebras X is called a subalgebra of X if $x * y \in S$ for all $x, y \in S$. A non-empty subset $\mathcal{I}$ of a BCI-algebra X is called an ideal of X if for any $x \in X$

($\mathcal{I}1$) $0 \in \mathcal{I}$
($\mathcal{I}2$) $x * y \in \mathcal{I}$ and $y \in \mathcal{I}$ implies $x \in \mathcal{I}$

Any ideal $\mathcal{I}$ of a BCI-algebra X satisfies the following implication:

$$x \leq y \text{ and } y \in \mathcal{I} \Rightarrow x \in \mathcal{I}, \forall\ x \in X$$

A non-empty subset $\mathcal{I}$ of a BCI-algebra X is called an BCI-positive implicative ideal (see Liu and Zhang [12]) of X if it satisfies ($\mathcal{I}1$) and

($\mathcal{I}3$) $((x * z) * z) * (y * z) \in \mathcal{I}$ and $y \in \mathcal{I} \Rightarrow x * z \in \mathcal{I}$ for all $x, z \in X$.

We know that every BCI-positive implicative ideal of a BCI-algebra X is also an ideal of X.

We refer the readers to [11, 14] for further study about ideals in BCK/BCI-algebras.

3. Basic results on soft sets

In [16] the soft set is defined in the following way: Let U be an initial universe set and E be a set of parameters. Let $\mathfrak{P}(U)$ denotes the power set of U and $A \subset E$.

Definition 3.1 (Molodtsov [16]). A pair $(\mathcal{F},\ A)$ is called a soft set over U, where $\mathcal{F}$ is a mapping given by

$\mathcal{F} : A \rightarrow \mathfrak{P}(U)$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $a \in A$, $\mathcal{F}(a)$ may be considered as the set of a-approximate elements of the soft set $(\mathcal{F},\ A)$.

Definition 3.2 (Maji et al. [13]). Let $(\mathcal{F},\ A)$ and $(\mathcal{G},\ B)$ be two soft sets over a common universe U. The intersection of $(\mathcal{F},\ A)$ and $(\mathcal{G},\ B)$ is defined to be the soft set $(\mathcal{H},\ C)$ satisfying the following conditions:

(i) $C = A \cap B$
(ii) $\mathcal{H}(x) = \mathcal{F}(x)$ or $\mathcal{G}(x)$ for all $x \in C$, (as both are same sets)

In this case, we write $(\mathcal{F},\ A)\ \tilde{\cap}\ (\mathcal{G},\ B) = (\mathcal{H},\ C)$.

Definition 3.3 (Maji et al. [13]). Let $(\mathcal{F},\ A)$ and $(\mathcal{G},\ B)$ be two soft sets over a common universe U. The union of $(\mathcal{F},\ A)$ and $(\mathcal{G},\ B)$ is defined to be the soft set $(\mathcal{H},\ C)$ satisfying the following conditions:

(i) $C = A \cup B$
(ii) for all $x \in C$,

$$\mathcal{H}(x) = \begin{cases} \mathcal{F}(x) & if\ x \in A \setminus B \\ \mathcal{G}(x) & if\ x \in B \setminus A \\ \mathcal{F}(x) \cup \mathcal{G}(x) & if\ x \in A \cap B \end{cases}$$

In this case, we write $(\mathcal{F},\ A)\ \tilde{\cup}\ (\mathcal{G},\ B) = (\mathcal{H},\ C)$.

Definition 3.4 (Maji et al. [13]). Let $(\mathcal{F},\ A)$ and $(\mathcal{G},\ B)$ be two soft sets over a common universe U. Then "$(\mathcal{F},\ A)\ AND\ (\mathcal{G},\ B)$" denoted by $(\mathcal{F},\ A)\ \tilde{\wedge}\ (\mathcal{G},\ B)$ is defined as $(\mathcal{F},\ A)\ \tilde{\wedge}\ (\mathcal{G},\ B) = (\mathcal{H},\ A \times B)$, where $\mathcal{H}(x, y) = \mathcal{F}(x)\ \cap\ \mathcal{G}(y)$ for all $(x, y) \in A \times B$.

Definition 3.5 (Maji et al. [13]). Let $(\mathcal{F},\ A)$ and $(\mathcal{G},\ B)$ be two soft sets over a common universe U. Then "$(\mathcal{F},\ A)\ OR\ (\mathcal{G},\ B)$" denoted by $(\mathcal{F},\ A)\ \tilde{\vee}\ (\mathcal{G},\ B)$ is defined as $(\mathcal{F},\ A)\ \tilde{\vee}\ (\mathcal{G},\ B) = (\mathcal{H},\ A \times B)$, where $\mathcal{H}(x, y) = \mathcal{F}(x)\ \cup\ \mathcal{G}(y)$ for all $(x, y) \in A \times B$.

Definition 3.6 (Maji et al. [13]). For two soft sets $(\mathcal{F},\ A)$ and $(\mathcal{G},\ B)$ over a common universe U, we say that $(\mathcal{F},\ A)$ is a soft subset of $(\mathcal{G},\ B)$, denoted by $(\mathcal{F},\ A)\ \tilde{\subset}\ (\mathcal{G},\ B)$, if it satisfies:

(i) $A \subset B$
(ii) For every $a \in A$, $\mathcal{F}(a)$ and $\mathcal{G}(a)$ are identical approximations.

4. Soft BCI-positive implicative ideals

In what follows let X and A be a BCI-algebra and a nonempty set, respectively and R will refer to an arbitrary binary relation between an element of A and an element of X, that is, R is a subset of $A \times X$ without otherwise specified. A set valued function $\mathcal{F} : A \to \mathfrak{P}(X)$ can be defined as $\mathcal{F}(x) = \{y \in X \mid xRy\}$ for all $x \in A$. The pair $(\mathcal{F},\ A)$ is then a soft set over X.

Definition 4.1 (Jun and Park [7]). Let S be a subalgebra of X. A subset $\mathcal{I}$ of X is called an ideal of X related to S (briefly, S-ideal of X), denoted by $\mathcal{I} \lhd S$, if it satisfies:

(i) $0 \in \mathcal{I}$
(ii) $x * y \in \mathcal{I}$ and $y \in \mathcal{I} \Rightarrow x \in \mathcal{I}$ for all $x \in S$

Definition 4.2. Let S be a subalgebra of X. A subset $\mathcal{I}$ of X is called a BCI-positive implicative ideal of X related to S (briefly, $S-(BCI-PI)-$ideal of X), denoted by $\mathcal{I} \lhd_{bci-pi} S$, if it satisfies:

(i) $0 \in \mathcal{I}$

(ii) $((x * z) * z) * (y * z) \in \mathcal{I}$ and $y \in \mathcal{I} \Rightarrow x * z \in \mathcal{I}$ for all $x, z \in S$

Example 4.3. Let $X = \{0, a, b, c\}$ be a BCI-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	c
a	a	0	0	c
b	b	b	0	c
c	c	c	c	0

Then $S = \{0, b\}$ is a subalgebra of X and $\mathcal{I} = \{0, a, b\}$ is an $S-(BCI-PI)-$ideal of X.

Note that every $S-(BCI-PI)-$ideal of X is an S-ideal of X.

Definition 4.4 (Jun [5]). Let $(\mathcal{F},\ A)$ be a soft set over X. Then $(\mathcal{F},\ A)$ is called a soft BCI-algebra over X if $\mathcal{F}(x)$ is a subalgebra of X for all $x \in A$.

Definition 4.5 (Jun and Park [7]). Let $(\mathcal{F},\ A)$ be a soft BCI-algebra over X. A soft set $(\mathcal{G},\ \mathcal{I})$ over X is called a soft ideal of $(\mathcal{F},\ A)$, denoted $(\mathcal{G},\ \mathcal{I})\ \tilde{\lhd}\ (\mathcal{F},\ A)$, if it satisfies:

(i) $\mathcal{I} \subset A$
(ii) $\mathcal{G}(x) \lhd \mathcal{F}(x)$ for all $x \in \mathcal{I}$

Definition 4.6. Let $(\mathcal{F},\ A)$ be a soft BCI-algebra over X. A soft set $(\mathcal{G},\ \mathcal{I})$ over X is called a soft BCI-positive implicative ideal of $(\mathcal{F},\ A)$, denoted $(\mathcal{G},\ \mathcal{I})\ \tilde{\lhd}_{bci-pi}\ (\mathcal{F},\ A)$, if it satisfies:

(i) $\mathcal{I} \subset A$
(ii) $\mathcal{G}(x) \lhd_{bci-pi} \mathcal{F}(x)$ for all $x \in \mathcal{I}$

Let us illustrate this definition using the following example.

Example 4.7. Consider a BCI-algebra $X = \{0, a, b, c\}$ which is given in Example 4.3. Let $(\mathcal{F},\ A)$ be a soft set over X, where $A = X$ and $\mathcal{F} : A \to \mathfrak{P}(X)$ is a set-valued function defined by:

$$\mathcal{F}(x) = \{0\} \cup \{y \in X \mid y * (y * x) \in \{0, a\}\}$$

for all $x \in A$. Then $\mathcal{F}(0) = \mathcal{F}(a) = X,\ \mathcal{F}(b) = \{0, a, c\},\ \mathcal{F}(c) = \{0\},$

which are subalgebras of X. Hence $(\mathcal{F},\ A)$ is a soft BCI-algebra over X. Let $\mathcal{I} = \{0, a, b\} \subset A$ and $\mathcal{G} : \mathcal{I} \to \mathfrak{P}(X)$ be a set-valued function defined by:

$$\mathcal{G}(x) = \begin{cases} Z(\{0, a\}) & if\ x = b \\ \{0\} & if\ x \in \{0, a\} \end{cases}$$

where $Z(\{0, a\}) = \{x \in X \mid 0 * (0 * x) \in \{0, a\}\}$. Then $\mathcal{G}(0) = \{0\} \lhd_{bci-pi} X = \mathcal{F}(0)$, $\mathcal{G}(a) = \{0\} \lhd_{bci-pi} X = \mathcal{F}(a)$, $\mathcal{G}(b) = \{0, a, b\} \lhd_{bci-pi} \{0, a, c\} = \mathcal{F}(b)$. Hence $(\mathcal{G},\ \mathcal{I})$ is a soft BCI-positive implicative ideal of $(\mathcal{F},\ A)$.

Note that every soft BCI-positive implicative ideal is a soft ideal but the converse is not true as seen in the following example.

Example 4.8. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra and hence a BCI-algebra, with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	c	0

Let $(\mathcal{F},\ A)$ be a soft set over X, where $A = X$ and $\mathcal{F} : A \to \mathfrak{P}(X)$ is a set-valued function defined by:

$$\mathcal{F}(x) = \{y \in X \mid y * (y * x) \in \{0, a\}\}$$

for all $x \in A$. Then $\mathcal{F}(0) = \mathcal{F}(a) = X$, $\mathcal{F}(b) = \{0, a, c, d\}$ and $\mathcal{F}(c) = \mathcal{F}(d) = \{0, a\}$, which are subalgebras of X. Hence $(\mathcal{F},\ A)$ is a soft BCI-algebra over X.
Let $(\mathcal{G},\ \mathcal{I})$ be a soft set over X, where $\mathcal{I} = \{a, b\} \subset A$ and $\mathcal{G} : \mathcal{I} \to \mathfrak{P}(X)$ be a set-valued function defined by:

$$\mathcal{G}(x) = \{y \in X \mid y * x = 0\}$$

for all $x \in \mathcal{I}$. Then $\mathcal{G}(a) = \{0, a\} \lhd X = \mathcal{F}(a)$, $\mathcal{G}(b) = \{0, a, b\} \lhd$

$\{0, a, c, d\} = \mathcal{F}(b)$. Hence $(\mathcal{G},\ \mathcal{I})$ is a soft ideal of $(\mathcal{F},\ A)$ but it is not a soft BCI-positive implicative ideal of $(\mathcal{F},\ A)$ because $\mathcal{G}(a)$ is not a BCI-positive implicative ideal of X related to $\mathcal{F}(a)$ since $((d * c) * c) * (0 * c) = 0 \in \mathcal{G}(a)$ and $0 \in \mathcal{G}(a)$ but $d * c = c \notin \mathcal{G}(a)$.

Theorem 4.9. Let $(\mathcal{F},\ A)$ be a soft BCI-algebra over X. For any soft sets $(\mathcal{G}_1,\ \mathcal{I}_1)$ and $(\mathcal{G}_2,\ \mathcal{I}_2)$ over X where $\mathcal{I}_1 \cap \mathcal{I}_2 \neq \emptyset$, we have

$$(\mathcal{G}_1,\ \mathcal{I}_1)\ \tilde{\lhd}_{bci-pi}\ (\mathcal{F},\ A),\ (\mathcal{G}_2,\ \mathcal{I}_2)\ \tilde{\lhd}_{bci-pi}\ (\mathcal{F},\ A) \Rightarrow (\mathcal{G}_1,\ \mathcal{I}_1)\ \tilde{\cap}\ (\mathcal{G}_2,\ \mathcal{I}_2)\ \tilde{\lhd}_{bci-pi}\ (\mathcal{F},\ A)$$

Proof. Using Definition 3.2, we can write

$$(\mathcal{G}_1,\ \mathcal{I}_1)\ \tilde{\cap}\ (\mathcal{G}_2,\ \mathcal{I}_2) = (\mathcal{G},\ \mathcal{I})$$

where $\mathcal{I} = \mathcal{I}_1 \cap \mathcal{I}_2$ and $\mathcal{G}(e) = \mathcal{G}_1(e)$ or $\mathcal{G}_2(e)$ for all $e \in \mathcal{I}$. Obviously, $\mathcal{I} \subset A$ and $\mathcal{G} : \mathcal{I} \to \mathfrak{P}(X)$ is a mapping. Hence $(\mathcal{G},\ \mathcal{I})$ is a soft set over X. Since $(\mathcal{G}_1,\ \mathcal{I}_1)\ \tilde{\lhd}_{bci-pi}\ (\mathcal{F},\ A)$ and $(\mathcal{G}_2,\ \mathcal{I}_2)\ \tilde{\lhd}_{bci-pi}\ (\mathcal{F},\ A)$, it follows that $\mathcal{G}(e) = \mathcal{G}_1(e) \lhd_{bci-pi} \mathcal{F}(e)$ or $\mathcal{G}(e) = \mathcal{G}_2(e) \lhd_{bci-pi} \mathcal{F}(e)$ for all $e \in \mathcal{I}$. Hence

$$(\mathcal{G}_1,\ \mathcal{I}_1)\ \tilde{\cap}\ (\mathcal{G}_2,\ \mathcal{I}_2) = (\mathcal{G},\ \mathcal{I})\ \tilde{\lhd}_{bci-pi}\ (\mathcal{F},\ A)$$

This completes the proof. $\square$

Corollary 4.10. Let $(\mathcal{F},\ A)$ be a soft BCI-algebra over X. For any soft sets $(\mathcal{G},\ \mathcal{I})$ and $(\mathcal{H},\ \mathcal{I})$ over X, we have

$$(\mathcal{G},\ \mathcal{I})\ \tilde{\lhd}_{bci-pi}\ (\mathcal{F},\ A),\ (\mathcal{H},\ \mathcal{I})\ \tilde{\lhd}_{bci-pi}\ (\mathcal{F},\ A) \Rightarrow (\mathcal{G},\ \mathcal{I})\ \tilde{\cap}\ (\mathcal{H},\ \mathcal{I})\ \tilde{\lhd}_{bci-pi}\ (\mathcal{F},\ A)$$

Proof. Straightforward. $\square$

Theorem 4.11. Let $(\mathcal{F},\ A)$ be a soft BCI-algebra over X. For any soft sets $(\mathcal{G},\ \mathcal{I})$ and $(\mathcal{H},\ \mathcal{J})$ over X in which $\mathcal{I}$ and $\mathcal{J}$ are disjoint, we have

$$(\mathcal{G},\ \mathcal{I})\ \tilde{\lhd}_{bci-pi}\ (\mathcal{F},\ A),\ (\mathcal{H},\ \mathcal{J})\ \tilde{\lhd}_{bci-pi}\ (\mathcal{F},\ A) \Rightarrow (\mathcal{G},\ \mathcal{I})\ \tilde{\cup}\ (\mathcal{H},\ \mathcal{J})\ \tilde{\lhd}_{bci-pi}\ (\mathcal{F},\ A)$$

Proof. Assume that $(\mathcal{G},\ \mathcal{I})\ \tilde{\lhd}_{bci-pi}\ (\mathcal{F},\ A)$ and $(\mathcal{H},\ \mathcal{J})\ \tilde{\lhd}_{bci-pi}\ (\mathcal{F},\ A)$. By means of Definition 3.3, we can write $(\mathcal{G},\ \mathcal{I})\ \tilde{\cup}\ (\mathcal{H},\ \mathcal{J}) = (\mathcal{R},\ \mathcal{U})$, where $\mathcal{U} = \mathcal{I} \cup \mathcal{J}$ and for every $e \in \mathcal{U}$,

$$\mathcal{R}(x) = \begin{cases} \mathcal{G}(e) & if\ e \in \mathcal{I} \setminus \mathcal{J} \\ \mathcal{H}(e) & if\ e \in \mathcal{J} \setminus \mathcal{I} \\ \mathcal{G}(e) \cup \mathcal{H}(e) & if\ e \in \mathcal{I} \cap \mathcal{J} \end{cases}$$

Since $\mathcal{I} \cap \mathcal{J} = \emptyset$, either $e \in \mathcal{I} \setminus \mathcal{J}$ or $e \in \mathcal{J} \setminus \mathcal{I}$ for all $e \in \mathcal{U}$. If $e \in \mathcal{I} \setminus \mathcal{J}$, then $\mathcal{R}(e) = \mathcal{G}(e) \lhd_{bci-pi} \mathcal{F}(e)$ since $(\mathcal{G},\ \mathcal{I})\ \tilde{\lhd}_{bci-pi}\ (\mathcal{F},\ A)$. If $e \in \mathcal{J} \setminus \mathcal{I}$, then $\mathcal{R}(e) = \mathcal{H}(e) \lhd_{bci-pi} \mathcal{F}(e)$ since $(\mathcal{H},\ \mathcal{J})\ \tilde{\lhd}_{bci-pi}\ (\mathcal{F},\ A)$. Thus $\mathcal{R}(e) \lhd_{bci-pi} \mathcal{F}(e)$ for all $e \in \mathcal{U}$ and so

$$(\mathcal{G},\ \mathcal{I})\ \tilde{\cup}\ (\mathcal{H},\ \mathcal{J}) = (\mathcal{R},\ \mathcal{U})\ \tilde{\lhd}_{bci-pi}\ (\mathcal{F},\ A) \qquad \square$$

It $\mathcal{I}$ and $\mathcal{J}$ are not disjoint in Theorem 4.11, then Theorem 4.11 is not true in general as seen in the following example.

Example 4.12. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra and hence a BCI-algebra, with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	b	0
c	c	c	c	0	0
d	d	d	c	b	0

Let $(\mathcal{F},\ A)$ be a soft set over X, where $A = X$ and $\mathcal{F} : A \to \mathfrak{P}(X)$ is a set-valued function defined by:

$$\mathcal{F}(x) = \{y \in X \mid y * (y * x) \in \{0, b\}\}$$

for all $x \in A$. Then $\mathcal{F}(0) = X$, $\mathcal{F}(a) = \mathcal{F}(b) = \{0, b, c, d\}$ and $\mathcal{F}(c) = \mathcal{F}(d) = \{0, b\}$, which are subalgebras of X. Hence $(\mathcal{F},\ A)$ is a soft BCI-algebra over X.
Let $(\mathcal{G},\ \mathcal{I})$ be a soft set over X, where $\mathcal{I} = \{b, c, d\} \subset A$ and $\mathcal{G} : \mathcal{I} \to \mathfrak{P}(X)$ be a set-valued function defined by:

$$\mathcal{G}(x) = \{y \in X \mid y * x = 0\}$$

for all $x \in \mathcal{I}$. Then $\mathcal{G}(b) = \{0, a, b\} \lhd_{bci-pi} \{0, b, c, d\} = \mathcal{F}(b)$, $\mathcal{G}(c) = \{0, a, c\} \lhd_{bci-pi} \{0, b\} = \mathcal{F}(c)$, $\mathcal{G}(d) = X \lhd_{bci-pi} \{0, b\} = \mathcal{F}(d)$. Hence $(\mathcal{G}, \mathcal{I})$ is a soft BCI-positive implicative ideal of $(\mathcal{F}, A)$.
Now consider $\mathcal{J} = \{b\}$ which is not disjoint with $\mathcal{I}$ and let $\mathcal{H} : \mathcal{J} \to \mathfrak{P}(X)$ be a set valued function by:

$$\mathcal{H}(x) = \{y \in X \mid y * (y * x) = 0\}\}$$

for all $x \in \mathcal{J}$. Then $\mathcal{H}(b) = \{0, c\} \lhd_{bci-pi} \{0, b, c, d\} = \mathcal{F}(b)$. Hence $(\mathcal{H}, \mathcal{J})$ is a soft BCI-positive implicative ideal of $(\mathcal{F}, A)$. But if $(\mathcal{R}, \mathcal{U}) = (\mathcal{G}, \mathcal{I}) \tilde{\cup} (\mathcal{H}, \mathcal{J})$, then $\mathcal{R}(b) = \mathcal{G}(b) \cup \mathcal{H}(b) = \{0, a, b, c\}$, which is not a BCI-positive implicative ideal of X related to $\mathcal{F}(b)$ since
$((d * 0) * 0) * (b * 0) = d * b = c \in \mathcal{R}(b)$ and $b \in \mathcal{R}(b)$ but $d * 0 = d \notin \mathcal{R}(b)$. Hence $(\mathcal{R}, \mathcal{U}) = (\mathcal{G}, \mathcal{I}) \tilde{\cup} (\mathcal{H}, \mathcal{J})$ is not a soft BCI-positive implicative ideal of $(\mathcal{F}, A)$.

5. BCI-positive implicative idealistic soft BCI-algebras

Definition 5.1 (Jun and Park [7]). Let $(\mathcal{F}, A)$ be soft set over X. Then $(\mathcal{F}, A)$ is called an idealistic soft BCI-algebra over X if $\mathcal{F}(x)$ is an ideal of X for all $x \in A$.

Definition 5.2. Let $(\mathcal{F}, A)$ be soft set over X. Then $(\mathcal{F}, A)$ is called a BCI-positive implicative idealistic soft BCI-algebra over X if $\mathcal{F}(x)$ is a BCI-positive implicative ideal of X for all $x \in A$.

Example 5.3. Consider a BCI-algebra $X = \{0, a, b, c\}$ which is given in Example 4.3. Let $(\mathcal{F}, A)$ be a soft set over X, where $A = X$ and $\mathcal{F} : A \to \mathfrak{P}(X)$ is a set-valued function defined by:

$$\mathcal{F}(x) = \begin{cases} Z(\{0, a\}) & if\ x \in \{b, c\} \\ X & if\ x \in \{0, a\} \end{cases}$$

where $Z(\{0, a\}) = \{x \in X \mid 0 * (0 * x) \in \{0, a\}\}$. Then $(\mathcal{F}, A)$ is a BCI-positive implicative idealistic soft BCI-algebra over X.

For any element x of a BCI-algebra X, we define the order of x, denoted by $o(x)$, as

$$o(x) = min\{n \in N \mid 0 * x^n = 0\}$$

where $0 * x^n = (...((0 * x) * x)...) * x$, in which x appears n-times.

Example 5.4. Let $X = \{0, a, b, c, d, e, f, g\}$ be a BCI-algebra defined by the following Cayley table:

$*$	0	a	b	c	d	e	f	g
0	0	0	0	0	d	d	d	d
a	a	0	0	0	e	d	d	d
b	b	b	0	0	f	f	d	d
c	c	b	a	0	g	f	e	d
d	d	d	d	d	0	0	0	0
e	e	d	d	d	a	0	0	0
f	f	f	d	d	b	b	0	0
g	g	f	e	d	c	b	a	0

Let $(\mathcal{F},\ A)$ be a soft set over X, where $A = \{a, b, c\} \subset X$ and $\mathcal{F} : A \to \mathfrak{P}(X)$ is a set-valued function defined by:

$$\mathcal{F}(x) = \{y \in X \mid o(x) = o(y)\}$$

for all $x \in A$. Then $\mathcal{F}(a) = \mathcal{F}(b) = \mathcal{F}(c) = \{0, a, b, c\}$ is a BCI-positive implicative ideal of X. Hence $(\mathcal{F},\ A)$ is a BCI-positive implicative idealistic soft BCI-algebra over X. But if we take $B = \{a, b, f, g\} \subset X$ and defined a set-valued function $\mathcal{G} : B \to \mathfrak{P}(X)$ by:

$$\mathcal{G}(x) = \{0\} \cup \{y \in X \mid o(x) = o(y)\}$$

for all $x \in B$, then $(\mathcal{G},\ B)$ is not a BCI-positive implicative idealistic soft BCI-algebra over X, since $\mathcal{G}(f) = \{0, d, e, f, g\}$ is not a BCI-positive implicative ideal of X because $((g * d) * d) * (f * d) = g * b = e \in \mathcal{G}(f)$ and $f \in \mathcal{G}(f)$ but $g * d = c \notin \mathcal{G}(f)$.

Example 5.5 Consider a BCI-algebra X = $\{0, a, b, c\}$ with the following cayley table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $(\mathcal{F},\ A)$ be a soft set over X, where $A = X$ and $\mathcal{F} : A \to \mathfrak{P}(X)$ is a set-valued function defined by:

$$\mathcal{F}(x) = \{y \in X \mid y = x^n,\ n \in N\}$$

for all $x \in A$. Then $\mathcal{F}(0) = \{0\}$, $\mathcal{F}(a) = \{0, a\}$, $\mathcal{F}(b) = \{0, b\}$, $\mathcal{F}(c) = \{0, c\}$, which are BCI-positive implicative ideals of X. Hence $(\mathcal{F},\ A)$ is a BCI-positive implicative idealistic soft BCI-algebra over X.

Obviously, every BCI-positive implicative idealistic soft BCI-algebra over X is an idealistic soft BCI-algebra over X, but the converse is not true in general as seen in the following example.

Example 5.6. Consider a BCI-algebra $X := Y \times Z$, where $(Y, *, 0)$ is a BCI-algebra and $(Z, -, 0)$ is the adjoint BCI-algebra of the additive group $(Z, +, 0)$ of integers. Let $\mathcal{F} : X \to \mathfrak{P}(X)$ be a set-valued function defined as follows:

$$\mathcal{F}(y, n) = \begin{cases} Y \times N_\circ & if\ n \in N_\circ \\ \{(0, 0)\} & otherwise \end{cases}$$

for all $(y, n) \in X$, where $N_\circ$ is the set of all non-negative integers. Then $(\mathcal{F},\ X)$ is an idealistic soft BCI-algebra over X but it is not a BCI-positive implicative idealistic soft BCI-algebra over X since $\{(0, 0)\}$ may not be a BCI-positive implicative ideal of X.

Proposition 5.7. Let $(\mathcal{F},\ A)$ and $(\mathcal{F},\ B)$ be soft sets over X where $B \subseteq A \subseteq X$. If $(\mathcal{F},\ A)$ is a BCI-positive implicative idealistic soft BCI-algebra over X, then so is $(\mathcal{F},\ B)$.

Proof. Straightforward. $\square$

The converse of Proposition 5.7 is not true in general as seen in the following example.

Example 5.8. Consider a BCI-positive implicative idealistic soft BCI-algebra over X which is described in Example 5.4. If we take $B = \{a, b, c, d\} \supseteq A$, then $(\mathcal{F},\ B)$ is not a BCI-positive implicative idealistic soft BCI-algebra over X since $\mathcal{F}(d) = \{d, e, f, g\}$ is not a BCI-positive implicative ideal of X.

Theorem 5.9. Let $(\mathcal{F},\ A)$ and $(\mathcal{G},\ B)$ be two BCI-positive implicative idealistic soft BCI-algebras over X. If $A \cap B \neq \emptyset$, then the intersection $(\mathcal{F},\ A)\ \tilde{\cap}\ (\mathcal{G},\ B)$ is a BCI-positive implicative idealistic soft BCI-algebra over X.

Proof. Using Definition 3.2, we can write

$$(\mathcal{F},\ A)\ \tilde{\cap}\ (\mathcal{G},\ B) = (\mathcal{H},\ C)$$

where $C = A \cap B$ and $\mathcal{H}(e) = \mathcal{F}(e)$ or $\mathcal{G}(e)$ for all $e \in C$. Note that $\mathcal{H} : C \to \mathfrak{P}(X)$ is a mapping, therefore $(\mathcal{H},\ C)$ is a soft set over X. Since $(\mathcal{F},\ A)$ and $(\mathcal{G},\ B)$ are BCI-positive implicative idealistic soft BCI-algebras over X, it follows that $\mathcal{H}(e) = \mathcal{F}(e)$ is a BCI-positive implicative ideal of X or $\mathcal{H}(e) = \mathcal{G}(e)$ is a BCI-positive implicative ideal of X for all $e \in C$. Hence $(\mathcal{H},\ C) = (\mathcal{F},\ A)\ \tilde{\cap}\ (\mathcal{G},\ B)$ is a BCI-positive implicative idealistic soft BCI-algebra over X. $\square$

Corollary 5.10. Let $(\mathcal{F},\ A)$ and $(\mathcal{G},\ A)$ be two BCI-positive implicative idealistic soft BCI-algebras over X. Then their intersection $(\mathcal{F},\ A)\ \tilde{\cap}\ (\mathcal{G},\ A)$ is a BCI-positive implicative idealistic soft BCI-algebra over X.

Proof. Straightforward. $\square$

Theorem 5.11. Let $(\mathcal{F},\ A)$ and $(\mathcal{G},\ B)$ be two BCI-positive implicative idealistic soft BCI-algebras over X. If A and B are disjoint, then the union $(\mathcal{F},\ A)\ \tilde{\cup}\ (\mathcal{G},\ B)$ is a BCI-positive implicative idealistic soft BCI-algebra over X.

Proof. By means of Definition 3.3, we can write $(\mathcal{F},\ A)\ \tilde{\cup}\ (\mathcal{G},\ B) = (\mathcal{H},\ C)$, where $C = A \cup B$ and for every $e \in C$,

$$\mathcal{H}(x) = \begin{cases} \mathcal{F}(e) & if\ e \in A \setminus B \\ \mathcal{G}(e) & if\ e \in A \setminus B \\ \mathcal{F}(e) \cup \mathcal{G}(e) & if\ e \in A \cap B \end{cases}$$

Since $A \cap B = \emptyset$, either $e \in A \setminus B$ or $e \in B \setminus A$ for all $e \in C$. If $e \in A \setminus B$, then $\mathcal{H}(e) = \mathcal{F}(e)$ is a BCI-positive implicative ideal of X since $(\mathcal{F},\ A)$ is a BCI-positive implicative idealistic soft BCI-algebra over X. If $e \in B \setminus A$, then $\mathcal{H}(e) = \mathcal{G}(e)$ is a BCI-positive implicative ideal of X since $(\mathcal{G},\ B)$ is a BCI-positive implicative idealistic soft BCI-algebra over X. Hence $(\mathcal{H},\ C) = (\mathcal{F},\ A)\ \tilde{\cup}\ (\mathcal{G},\ B)$ is a BCI-positive implicative idealistic soft BCI-algebra over X. □

Theorem 5.12. Let $(\mathcal{F},\ A)$ and $(\mathcal{G},\ B)$ be two BCI-positive implicative idealistic soft BCI-algebras over X, then $(\mathcal{F},\ A)\ \tilde{\wedge}\ (\mathcal{G},\ B)$ is a BCI-positive implicative idealistic soft BCI-algebra over X.

Proof. By means of Definition 3.4, we know that

$$(\mathcal{F},\ A)\ \tilde{\wedge}\ (\mathcal{G},\ B) = (\mathcal{H},\ A \times B),$$

where $H(x,y) = \mathcal{F}(x)\ \cap\ \mathcal{G}(y)$ for all $(x,y) \in A \times B$. Since $\mathcal{F}(x)$ and $\mathcal{G}(y)$ are BCI-positive implicative ideals of X, the intersection $\mathcal{F}(x)\ \cap\ \mathcal{G}(y)$ is also a BCI-positive implicative ideal of X. Hence $H(x,y)$ is a BCI-positive implicative ideal of X for all $(x,y) \in A \times B$.
Hence $(\mathcal{F},\ A)\ \tilde{\wedge}\ (\mathcal{G},\ B) = (\mathcal{H},\ A \times B)$ is a BCI-positive implicative idealistic soft BCI-algebra over X. □

Definition 5.13. A BCI-positive implicative idealistic soft BCI-algebra $(\mathcal{F},\ A)$ over X is said to be trivial (resp., whole) if $\mathcal{F}(x) = 0$ (resp., $\mathcal{F}(x) = X$) for all $x \in A$.

Example 5.14. Let X be a BCI-algebra which is given in Example 5.5 and let $\mathcal{F} : X \to \mathfrak{P}(X)$ be a set-valued function defined by

$$\mathcal{F}(x) = \{0\} \cup \{y \in X \mid o(x) = o(y)\}$$

for all $x \in X$. Then $\mathcal{F}(0) = \{0\}$ and $\mathcal{F}(a) = \mathcal{F}(b) = \mathcal{F}(c) = X$, which are BCI-positive implicative ideals of X. Hence $(\mathcal{F},\ \{0\})$ is a trivial BCI-positive implicative idealistic soft BCI-algebra over X and $(\mathcal{F},\ X \setminus \{0\})$ is a whole BCI-positive implicative idealistic soft BCI-algebra over X.

The proofs of the following three lemmas are straightforward, so they are omitted.

Lemma 5.15. Let $f : X \to Y$ be an onto homomorphism of BCI-algebras. If I is an ideal of X, then $f(I)$ is an ideal of Y.

Lemma 5.16. Let $f : X \to Y$ be an isomorphism of BCI-algebras. If I is a BCI-positive implicative ideal of X, then $f(I)$ is a BCI-positive implicative ideal of Y.

Let $f : X \to Y$ be a mapping of BCI-algebras. For a soft set $(\mathcal{F},\ A)$ over X, $(f(\mathcal{F}),\ A)$ is soft set over Y, where $f(\mathcal{F}) : A \to \mathfrak{P}(Y)$ is defined by $f(\mathcal{F})(x) = f(\mathcal{F}(x))$ for all $x \in A$.

Lemma 5.17 Let $f : X \to Y$ be an isomorphism of BCI-algebras. If $(\mathcal{F},\ A)$ is a BCI-positive implicative idealistic soft BCI-algebra over X, then $(f(\mathcal{F}),\ A)$ is a BCI-positive implicative idealistic soft BCI-algebra over Y.

Theorem 5.18. Let $f : X \to Y$ be an isomorphism of BCI-algebras and let $(\mathcal{F},\ A)$ be a BCI-positive implicative idealistic soft BCI-algebra over X.

(1) If $\mathcal{F}(x) = ker(f)$ for all $x \in A$, then $(f(\mathcal{F}),\ A)$ is a trivial BCI-positive implicative idealistic soft BCI-algebra over Y.

(2) If $(\mathcal{F},\ A)$ is whole, then $(f(\mathcal{F}),\ A)$ is a whole BCI-positive implicative idealistic soft BCI-algebra over Y.

Proof. (1) Assume that $\mathcal{F}(x) = ker(f)$ for all $x \in A$. Then $f(\mathcal{F})(x) = f(\mathcal{F}(x)) = \{0_Y\}$ for all $x \in A$. Hence $(\mathcal{F},\ A)$ is a trivial BCI-positive implicative idealistic soft BCI-algebra over Y by Lemma 5.17 and Definition 5.13.

(2) Suppose that $(\mathcal{F},\ A)$ is whole. Then $\mathcal{F}(x) = X$ for all $x \in A$ and so $f(\mathcal{F})(x) = f(\mathcal{F}(x)) = f(X) = Y$ for all $x \in A$. It follows from Lemma 5.17 and Definition 5.13 that $(f(\mathcal{F}),\ A)$ is a whole BCI-positive implicative idealistic soft BCI-algebra over Y. □

Definition 5.19 (Liu and Meng [11]). A fuzzy set μ in X is called a fuzzy BCI-positive implicative ideal of X, if for all $x, y, z \in X$,

(i) $\mu(0) \geq \mu(x)$
(ii) $\mu(x * z) \geq min\{\mu(((x * z) * z) * (y * z)),\ \mu(y)\}$

The transfer principle for fuzzy sets described in [10] suggest the following theorem.

Lemma 5.20 (Liu and Meng [11]). A fuzzy set μ in X is a fuzzy BCI-positive implicative ideal of X if and only if for any $t \in [0, 1]$, the level subset $U(\mu; t) := \{x \in X \mid \mu(x) \geq t\}$ is either empty or a BCI-positive implicative ideal of X.

Theorem 5.21. For every fuzzy BCI-positive implicative ideal μ of X, there exists a BCI-positive implicative idealistic soft BCI-algebra $(\mathcal{F},\ A)$ over X.

Proof. Let μ be a fuzzy BCI-positive implicative ideal of X. Then $U(\mu; t) := \{x \in X \mid \mu(x) \geq t\}$ is an BCI-positive implicative ideal of X for all $t \in Im(\mu)$. If we take $A = Im(\mu)$ and consider a set valued function $\mathcal{F} : A \to \mathfrak{P}(X)$ given by $\mathcal{F}(t) = U(\mu; t)$ for all $t \in A$, then $(\mathcal{F},\ A)$ is a BCI-positive implicative idealistic soft BCI-algebra over X. □

Conversely, the following theorem is straightforward.

Theorem 5.22. For any fuzzy set μ in X, if a BCI-positive implicative idealistic soft BCI-algebra $(\mathcal{F},\ A)$ over X is given by $A = Im(\mu)$ and $\mathcal{F}(t) = U(\mu; t)$ for all $t \in A$, then μ is a fuzzy BCI-positive implicative ideal of X.

Let μ be a fuzzy set in X and let $(\mathcal{F},\ A)$ be a soft set over X in which $A = Im(\mu)$ and $\mathcal{F} : A \to \mathfrak{P}(X)$ is a set-valued function defined by

$$\mathcal{F}(t) = \{x \in X \mid \mu(x) + t > 1\} \tag{5.2}$$

for all $t \in A$. Then there exists $t \in A$ such that $\mathcal{F}(t)$ is not a BCI-positive implicative ideal of X as seen in the following example.

Example 5.23. For any BCI-algebra X, define a fuzzy set μ in X by $\mu(0) = t_\circ < 0.5$ and $\mu(x) = 1 - t_\circ$ for all $x \neq 0$. Let $A = Im(\mu)$ and $\mathcal{F} : A \to \mathfrak{P}(X)$ be a set-valued function defined by (5.2). Then $\mathcal{F}(1 - t_\circ) = X \setminus \{0\}$, which is not a BCI-positive implicative ideal of X.

Theorem 5.24. Let μ be a fuzzy set in X and let $(\mathcal{F},\ A)$ be a soft set over X in which $A = [0, 1]$ and $\mathcal{F} : A \to \mathfrak{P}(X)$ is given by (5.2). Then the following assertions are equivalent:

(1) μ is a fuzzy BCI-positive implicative ideal of X.

(2) for every $t \in A$ with $\mathcal{F}(t) \neq \emptyset$, $\mathcal{F}(t)$ is an BCI-positive implicative ideal of X.

Proof. Assume that μ is a fuzzy BCI-positive implicative ideal of X. Let $t \in A$ be such that $\mathcal{F}(t) \neq \emptyset$. Then for any $x \in \mathcal{F}(t)$, we have $\mu(0)+t \geq \mu(x)+t > 1$, that is, $0 \in \mathcal{F}(t)$. Let $((x*z)*z)*(y*z) \in \mathcal{F}(t)$ and $y \in \mathcal{F}(t)$ for any $t \in A$ and $x, y, z \in X$. Then $\mu(((x*z)*z)*(y*z))+t > 1$ and $\mu(y) + t > 1$. Since μ is a fuzzy BCI-positive implicative ideal of X, it follows that

$$\mu(x*z) + t \ \geq min\{\mu(((x*z)*z)*(y*z)),\ \mu(y)\} + t$$

$$= min\{\mu(((x*z)*z)*(y*z)) + t,\ \mu(y) + t\} > 1$$

so that $x * z \in \mathcal{F}(t)$. Hence $\mathcal{F}(t)$ is a BCI-positive implicative ideal of X for all $t \in A$ such that $\mathcal{F}(t) \neq \emptyset$.

Conversely, suppose that (2) is valid. If there exists $x_\circ \in X$ such that $\mu(0) < \mu(x_\circ)$, then there exists $t_\circ \in A$ such that $\mu(0) + t_\circ \leq 1 < \mu(x_\circ) + t_\circ$. It follows that $x_\circ \in \mathcal{F}(t_\circ)$ and $0 \notin \mathcal{F}(t_\circ)$, which is a contradiction. Hence $\mu(0) \geq \mu(x)$ for all $x \in X$. Now assume that

$$\mu(x_\circ * z_\circ) < min\{\mu(((x_\circ * z_\circ) * z_\circ) * (y_\circ * z_\circ)),\ \mu(y_\circ)\}$$

for some $x_\circ, y_\circ, z_\circ \in X$. Then there exists some $s_\circ \in A$ such that

$$\mu(x_\circ * z_\circ) + s_\circ \leq 1 < min\{\mu(((x_\circ * z_\circ) * z_\circ) * (y_\circ * z_\circ)),\ \mu(y_\circ)\} + s_\circ$$

$$\Rightarrow \mu(x_\circ * z_\circ) + s_\circ \leq 1 < min\{\mu(((x_\circ * z_\circ) * z_\circ) * (y_\circ * z_\circ)) + s_\circ,\ \mu(y_\circ) + s_\circ\}$$

which implies that $((x_\circ * z_\circ) * z_\circ) * (y_\circ * z_\circ) \in \mathcal{F}(s_\circ)$ and $y_\circ \in \mathcal{F}(s_\circ)$ but $x_\circ * z_\circ \notin \mathcal{F}(s_\circ)$. This is a contradiction. Therefore

$\mu(x * z) \geq min\{\mu(((x * z) * z) * (y * z)),\ \mu(y)\}$ for all $x, y, z \in X$

Thus μ is fuzzy BCI-positive implicative ideal of X. □

Corollary 5.25. Let μ be a fuzzy set in X such that $\mu(x) > 0.5$ for all $x \in X$ and let $(\mathcal{F},\ A)$ be a soft set over X in which

$$A := \{t \in Im(\mu) \mid t > 0.5\}$$

and $\mathcal{F} : A \to \mathfrak{P}(X)$ is given by (5.2). If μ is a fuzzy BCI-positive implicative ideal of X, then $(\mathcal{F},\ A)$ is a BCI-positive implicative idealistic soft BCI-algebra over X.

Proof. Straightforward. □

Theorem 5.26. Let μ be a fuzzy set in X and let $(\mathcal{F},\ A)$ be a soft set over X in which $A = (0.5, 1]$ and $\mathcal{F} : A \to \mathfrak{P}(X)$ is defined by

$$\mathcal{F}(t) = U(\mu; t) \text{ for all } t \in A$$

Then $\mathcal{F}(t)$ is a BCI-positive implicative ideal of X for all $t \in A$ with $\mathcal{F}(t) \neq \emptyset$ if and only if the following assertions are valid:

(1) $max\{\mu(0),\ 0.5\} \geq \mu(x)$ for all $x \in X$.

(2) $max\{\mu(x*z),\ 0.5\} \geq min\{\mu(((x*z)*z)*(y*z)),\ \mu(y)\}$ for all $x, y, z \in X$.

Proof. Assume that $\mathcal{F}(t)$ is a BCI-positive implicative ideal of X for all $t \in A$

with $\mathcal{F}(t) \neq \emptyset$. If there exists $x_\circ \in X$ such that $max\{\mu(0),\ 0.5\} < \mu(x_\circ)$, then there exists $t_\circ \in A$ such that $max\{\mu(0),\ 0.5\} < t_\circ \leq \mu(x_\circ)$. It follows that $\mu(0) < t_\circ$, so that $x_\circ \in \mathcal{F}(t_\circ)$ and $0 \notin \mathcal{F}(t_\circ)$. This is a contradiction. Therefore (1) is valid. Suppose that there exist $a, b, c \in X$ such that

$$max\{\mu(a * c),\ 0.5\} < min\{\mu(((a * c) * c) * (b * c)),\ \mu(b)\}$$

Then there exists $s_\circ \in A$ such that

$$max\{\mu(a * c),\ 0.5\} < s_\circ \leq min\{\mu(((a * c) * c) * (b * c)),\ \mu(b)\}$$

which implies that $((a*c)*c)*(b*c) \in \mathcal{F}(s_\circ)$ and $b \in \mathcal{F}(s_\circ)$, but $a*c \notin \mathcal{F}(s_\circ)$. This is a contradiction. Hence (2) is valid.
Conversely, suppose that (1) and (2) are valid. Let $t \in A$ with $\mathcal{F}(t) \neq \emptyset$. Then for any $x \in \mathcal{F}(t)$, we have

$$max\{\mu(0),\ 0.5\} \geq \mu(x) \geq t > 0.5$$

which implies $\mu(0) \geq t$ and thus $0 \in \mathcal{F}(t)$. Let $((x * z) * z) * (y * z) \in \mathcal{F}(t)$ and $y \in \mathcal{F}(t)$, for any $x, y, z \in X$. Then $\mu(((x * z) * z) * (y * z)) \geq t$ and $\mu(y) \geq t$. It follows from the second condition that

$$max\{\mu(x * z),\ 0.5\} \geq min\{\mu(((x * z) * z) * (y * z)),\ \mu(y)\} \geq t > 0.5$$

$$\Rightarrow \mu(x * z) \geq t$$

so that $x * z \in \mathcal{F}(t)$. Therefore $\mathcal{F}(t)$ is a BCI-positive implicative ideal of X for all $t \in A$ with $\mathcal{F}(t) \neq \emptyset$. $\square$

CONCLUSION

The concept of soft set, which is introduced by Molodtsov [16], is a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Soft sets are deeply related to fuzzy sets and rough sets. We introduced the notion of soft BCI-positive implicative ideals and BCI-positive implicative idealistic soft BCI-algebras and discussed related properties. We established the intersection, union, "AND" operation and "OR" operation of soft BCI-positive implicative ideals and BCI-positive implicative idealistic soft BCI-algebras. From the above discussion it can be observed that fuzzy BCI-positive implicative ideals can be characterized using the concept of soft sets. For a soft set $(\mathcal{F},\ A)$ over X, a fuzzy set μ in X is a fuzzy BCI-positive implicative ideal of X if and only if for every $t \in A$ with $\mathcal{F}(t) = \{x \in X \mid \mu(x) + t > 1\} \neq \emptyset$, $\mathcal{F}(t)$ is a BCI-positive implicative ideal of X. Finally we have discussed the relations between fuzzy BCI-positive implicative ideals and BCI-positive implicative idealistic soft BCI-algebras.

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Communicated by *Arto Salomaa*

M.AslamMalik* and M.Touqeer†

*malikpu@math.pu.edu.pk †touqeer-fareed@yahoo.com

Department of Mathematics, University of the Punjab, Quaid-e-AzamCampus, Lahore-54590,Pakistan.

CLASSIFICATION OF TOPOLOGICAL MANIFOLDS BY THE EULER CHARACTERISTIC AND THE K-THEORY RANKS OF C^*-ALGEBRAS

Takahiro Sudo

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Abstract. We consider classification of homeomorphism classes of connected sums of closed surfaces by the Euler characteristic and the K-theory ranks of C^*-algebras. We next consider classification of those of connected sums of higher dimensional, closed topological manifolds by the Euler characteristic and the K-theory ranks of C^*-algebras.

1 Introduction In this paper, first of all, we consider classification (of homeomorphism classes) of connected sums of closed surfaces such as the real two-dimensional, sphere, torus, and the real projective plane, by the Euler characteristic in the K-theory of C^*-algebras. We obtain the Euler characteristic formula for the C^*-algebras corresponding to connected sums of closed surfaces and show that the classification list obtained by Euler characteristic in K-theory for the corresponding C^*-algebras is the same as the classification list for connected sums of closed surfaces by the Euler characteristic in homology, well known (see for instance, [5] inspired or [4]). As well, we consider another classification of these two-dimensional topological manifolds by the K-theory group (free) ranks (i.e. the Betti numbers) of the corresponding C^*-algebras.

We next consider classification (of homeomorphism classes) of connected sums of higher dimensional, closed topological manifolds such as higher dimensional, spheres, tori, and real projective spaces, by the Euler characteristic in the K-theory of C^*-algebras. We obtain the Euler characteristic formulae for the C^*-algebras corresponding to connected sums of the closed topological manifolds and do the classification list for connected sums of the closed topological manifolds by the K-theory Euler characteristic for the corresponding C^*-algebras. As well, we consider another classification of these higher dimensional, closed topological manifolds by the K-theory group (free) ranks (i.e. the Betti numbers) (as well as the K-theory torsion ranks in some cases) of the corresponding C^*-algebras.

In the process, and in the end we obtain the list of K-theory groups for the C^*-algebras considered so far in this paper, and as well we show that those closed topological manifolds are classifiable (up to homeomorphism) by K-theory data for C^*-algebras (together with dimension for spaces).

As a generalization from connected sums of topological manifolds, in the final section we define and consider connected sums of C^*-algebras viewed as noncommutative connected sums and obtain their Euler characteristic formula and K-theory rank formulae.

Now recall that the Euler characteristic (in K-theory) of a C^*-algebra $\mathfrak{A}$ is (first introduced by Hiroshi Takai and) defined to be the alternative sum of the Betti numbers of the K-theory groups of $\mathfrak{A}$:

$$\chi(\mathfrak{A}) = b_0(\mathfrak{A}) - b_1(\mathfrak{A}) \equiv \operatorname{rank}_{\mathbb{Z}} K_0(\mathfrak{A}) - \operatorname{rank}_{\mathbb{Z}} K_1(\mathfrak{A}),$$

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where each $\mathrm{rank}_{\mathbb{Z}} K_j(\mathfrak{A})$ means the (free) $\mathbb{Z}$-rank of the free summmand of the abelian group $K_j(\mathfrak{A})$ of $\mathfrak{A}$ ($j = 0, 1$). We say that a C^*-algebra $\mathfrak{A}$ has Euler number n if $\chi(\mathfrak{A}) = n$, where n is an integer, or may as well be $+\infty$ or $-\infty$ (or formally $\pm\infty - \pm\infty$).

We denote by $\mathbb{K}$ the C^*-algebra of all compact operators on a separable, infinite dimensional, Hilbert space. Denote by $M_n(\mathbb{C})$ the $n \times n$ matrix C^*-algebra over the complex field $\mathbb{C}$. We denote by $C(X)$ the C^*-algebra of all complex-valued, continuous functions on a compact Hausdorff space X. Denote by $C_0(X)$ the C^*-algebra of all $\mathbb{C}$-valued, continuous functions on a locally compact Hausdorff space X vanishing at infinity.

Recall some basic facts on the Euler characteristic for C^*-algebras, which can be found in [8] or [9] and are used in the following sections without mentioning.

• Group isomorphisms $K_0(\mathbb{C}) \cong K_0(M_n(\mathbb{C})) \cong K_0(\mathbb{K}) \cong \mathbb{Z}$, and $K_1(\mathbb{C}) \cong K_1(M_n(\mathbb{C})) \cong K_1(\mathbb{K}) \cong 0$, so that $\chi(\mathbb{C}) = \chi(M_n(\mathbb{C})) = \chi(\mathbb{K}) = 1$.

• $K_0(C_0(\mathbb{R}^{2n})) \cong \mathbb{Z}$ and $K_1(C_0(\mathbb{R}^{2n})) \cong 0$ by the Bott periodicity, so that $\chi(C_0(\mathbb{R}^{2n})) = 1$. Also, $K_0(C_0(\mathbb{R}^{2n+1})) \cong 0$ and $K_1(C_0(\mathbb{R}^{2n+1})) \cong \mathbb{Z}$, so that $\chi(C_0(\mathbb{R}^{2n+1})) = -1$. Indeed, $\chi(C_0(X \times \mathbb{R})) = \chi(SC_0(\mathbb{R})) = -\chi(C_0(X))$ for X a locally compact Hausdorff space X. Moreover, $\chi(S\mathfrak{A}) = -\chi(\mathfrak{A})$ for a C^*-algebra $\mathfrak{A}$, with the suspension $S\mathfrak{A} = C_0(\mathbb{R}) \otimes \mathfrak{A}$, because $K_j(S\mathfrak{A}) = K_{j+1}(\mathfrak{A})$ with $j + 1 \bmod 2$.

• We have $\chi(C(X)) = \chi(X)$ the Euler characteristic of X in homology (or cohomology in the several definitions) via the Euler-Poincaré formula.

• For a short exact sequence $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$ of C^*-algebras, we have $\chi(\mathfrak{A}) = \chi(\mathfrak{I}) + \chi(\mathfrak{A}/\mathfrak{I})$ if each term is finite.

• For a tensor product $\mathfrak{A} \otimes \mathfrak{B}$ of C^*-algebras, we have $\chi(\mathfrak{A} \otimes \mathfrak{B}) = \chi(\mathfrak{A}) \cdot \chi(\mathfrak{B})$ if each term is finite and if one of the tensor product factors belongs to the bootstrap category or the UCT class, which is deduced from the Künneth formula in K-theory of C^*-algebras.

Refer to [1] or [10] for some facts on K-theory of C^*-algebras, used below.

2 Connected sums of closed surfaces A closed surface is a compact (real) 2-dimensional topological manifold without boundary. Let M, N be closed surfaces. The connected sum $M\#N$ of M and N is defined to be the closed surface obtained by removing the 2-dimensional closed unit disks D viewed on M and N from themselves and gluing the open differences $M \setminus D$ and $N \setminus D$ by attaching the unit circle S^1 (or the 1-dimensional torus $\mathbb{T}$) to them as their boundaries.

Theorem 2.1. *Let M, N be closed surfaces and $M\#N$ be their connected sum. Then*

$$\chi(C(M\#N)) = \chi(C(M)) + \chi(C(N)) - 2.$$

Proof. By the definition of $M\#N$, we have the following short exact sequence of C^*-algebras:

$$0 \to C_0((M \setminus D) \sqcup (N \setminus D)) \xrightarrow{\ i\ } C(M\#N) \xrightarrow{\ q\ } C(\mathbb{T}) \to 0,$$

where i is the inclusion map and q is the quotient map since $\mathbb{T}$ attached in gluing is closed in $M\#N$, and $\sqcup$ means the disjoint union, and the closed ideal is isomorphic to the direct sum $C_0(M \setminus D) \oplus C_0(N \setminus D)$. Therefore, it follows that

$$\chi(C(M\#N)) = \chi(C_0(M \setminus D)) + \chi(C_0(N \setminus D)) + \chi(C(\mathbb{T})),$$

with $\chi(C(\mathbb{T})) = 0$, which follows from that there is the short exact sequence $0 \to C_0(\mathbb{R}) \to C(\mathbb{T}) \to \mathbb{C} \to 0$ of C^*-algebras.

Moreover, we have

$$0 \to C_0(M \setminus D) \xrightarrow{\ i\ } C(M) \xrightarrow{\ q\ } C(D) \to 0,$$

so that

$$\chi(C(M)) = \chi(C_0(M \setminus D)) + \chi(C(D)),$$

with $\chi(C(D)) = 1$, which follows from that D is contractible.

The same holds for N. $\square$

Note that there is a homeomorphism between (2 times)-successive connected sums of three closed surfaces M_1, M_2, M_3, denoted as

$$(M_1 \# M_2) \# M_3 \approx M_1 \# (M_2 \# M_3).$$

We denote both sides by $M_1 \# M_2 \# M_3$ or $\#_{i=1}^3 M_i$ (called a 3-connected sum) and apply this convention for more successive connected sums of finitely many closed surfaces.

Corollary 2.2. *Let $M_1 \# M_2 \cdots \# M_n$ be an $(n-1)$-successive connected sum of closed surfaces $M_1, M_2, \cdots, M_n$. Then*

$$\chi(C(M_1 \# M_2 \cdots \# M_n)) = \sum_{k=1}^{n} \chi(C(M_k)) - 2(n-1).$$

Example 2.3. Let $\mathbb{T}^2$ be the 2-dimensional torus. Denote by $\#^n \mathbb{T}^2$ the $(n-1)$-successive connected sum of n-copies of $\mathbb{T}^2$, and we call it the n-connected sum of $\mathbb{T}^2$. A closed surface is said to be an orientable closed surface with genus n if it is homeomorphic to the n-connected sum $\#^n \mathbb{T}^2$, and we denote it by $T(n)$. Set $T(1) \approx \mathbb{T}^2$. Then, by Theorem 2.1,

$$\chi(C(T(n))) = \chi(C(\#^n \mathbb{T}^2)) = -2(n-1) = 2 - 2n,$$

since $C(T(n))$ is isomorphic to $C(\#^n \mathbb{T}^2)$ and $\chi(C(\mathbb{T}^2)) = \chi(C(\mathbb{T} \times \mathbb{T})) = \chi(C(\mathbb{T}) \otimes C(\mathbb{T})) = \chi(C(\mathbb{T})) \cdot \chi(C(\mathbb{T})) = 0$.

A closed surface is said to be a closed surface with genus zero if it is homeomorphic to the 2-dimensional sphere S^2, and we denote it by $S(0)$. There is the following short exact sequence of C^*-algebras:

$$0 \to C_0(\mathbb{R}^2) \to C(S^2) \to \mathbb{C} \to 0,$$

so that

$$\chi(C(S(0))) = \chi(C(S^2)) = \chi(C_0(\mathbb{R}^2)) + \chi(\mathbb{C}) = 1 + 1 = 2.$$

Note that $M \# S^2$ is homeomorphic to M for any closed surface M and that

$$\chi(C(M \# S(0))) = \chi(C(M)) + \chi(C(S(0))) - 2 = \chi(C(M)).$$

Let P^2 be the real 2-dimensional projective plane, obtained (in $\mathbb{R}^4$) by gluing the boundary of the Möbius band M_b with that of the closed unit disk D of $\mathbb{R}^2$, where the Möbius band M_b is obtained from a 2-dimensional closed interval I by identifying one edge E of four edges of I with the opposite edge with one twist. Denote by $\#^n P^2$ the $(n-1)$-successive connected sum of n-copies of P^2, and we call it the n-connected sum of P^2. A closed surface is said to be a non-orientable closed surface with genus n if it is homeomorphic to the n-connected sum $\#^n P^2$, and we denote it by $P(n)$. There is the following short exact sequence of C^*-algebras:

$$0 \to C_0(M_b^\circ) \to C(P^2) \to C(D) \to 0,$$

where M_b° is the interior of M_b. Also,

$$0 \to C_0(I^\circ) \to C_0(M_b^\circ) \to C_0(E^\circ) \to 0$$

with $I^\circ \approx \mathbb{R}^2$ and $E^\circ \approx \mathbb{R}$. Therefore,

$$\begin{aligned}\chi(C(P^2)) &= \chi(C_0(M_b^\circ)) + \chi(C(D)) \\ &= \chi(C_0(I^\circ)) + \chi(C_0(E^\circ)) + 1 \\ &= \chi(C_0(\mathbb{R}^2)) + \chi(C_0(\mathbb{R})) + 1 = 1 - 1 + 1 = 1.\end{aligned}$$

Moreover, by Theorem 2.1,

$$\chi(C(P(n))) = \chi(C(\#^n P^2)) = n \cdot 1 - 2(n-1) = 2 - n.$$

Note that the 2-connected sum $P^2 \# P^2$ is homeomorphic to the Klein bottle K^2, obtained by gluing two Möbius bands M_b along their boundaries homeomorphic to S^1. Note also that $\mathbb{T}^2 \# P^2 \approx \#^3 P^2$. An intuitive explanation for this fact is that $\mathbb{T}^2$ viewed as a square 2-dimensional closed interval with four edges identified with opposites is transformed by cutting the interval on the diagonal to the Klein bottle K^2 viewed as a square 2-dimensional closed interval with two edges identified with opposites (with no twist) and with the other two edges identified with opposites with one twist in the connected sum $\mathbb{T}^2 \# P^2$. Moreover, $(\#^m \mathbb{T}^2) \# (\#^n P^2) \approx \#^{n+2m} P^2$.

Refer to [5] or [4] about connected sums of closed surfaces.

It is well known as a remarkable fact in (low dimensional) algebraic topology that closed surfaces (or compact 2-dimensional topological manifolds without boundary) X are classified as in the list of the Table 1, which becomes the same list as obtained by our Euler numbers $\chi(C(X))$ for the C^*-algebras $C(X)$:

Table 1: Classification for closed surfaces

Euler number	Orientable	Non-orientable
2	$S^2 \approx S(0)$	No
1	No	$P^2 \approx P(1)$
0	$\mathbb{T}^2 \approx T(1)$	$P^2 \# P^2 \approx P(2) \approx K^2$
-1	No	$\#^3 P^2 \approx P(3) \approx \mathbb{T}^2 \# P^2$
-2	$\mathbb{T}^2 \# \mathbb{T}^2 \approx T(2)$	$\#^4 P^2 \approx P(4)$
$3 - 2n$	No	$\#^{2n-1} P^2 \approx P(2n-1)$
$2 - 2n$	$\#^n \mathbb{T}^2 \approx T(n)$	$\#^{2n} P^2 \approx P(2n)$

We now compute K-theory groups.

Theorem 2.4. *Let M, N be closed surfaces and $M \# N$ be their connected sum. Then*

$$\begin{aligned}&K_0(C(M\#N)) \cong \mathbb{Z} \oplus \{[(K_0(C(M))/\mathbb{Z}[1]) \oplus (K_0(C(N))/\mathbb{Z}[1])]/\partial\mathbb{Z}[z]\}, \\ &K_1(C(M\#N)) \cong K_1(C(M)) \oplus K_1(C(N)) \oplus \ker(\partial),\end{aligned}$$

where each $[1]$ means the K_0-class of the unit 1 of $C(M)$ or $C(N)$, with $\mathbb{Z}[1] \cong \mathbb{Z}$, and $[z]$ means the K_1-class of $K_1(C(\mathbb{T}))$ corresponding to the coordinate function on $\mathbb{T}$, with $\mathbb{Z}[z] = K_1(C(\mathbb{T})) \cong \mathbb{Z}$, and $\partial\mathbb{Z}[z]$ is the image under the associated boundary map ∂, whose kernel is denoted by $\ker(\partial)$ and is isomorphic to (a subgroup of) $\mathbb{Z}$ or zero.

Remark. The image $\partial\mathbb{Z}[z]$ may not be isomorphic to $\mathbb{Z}$ in general, which depends on $\ker(\partial)$, and the image may not split into a direct summand of $K_0(C(M\#N))$ in general, while each $\mathbb{Z}[1] \cong \mathbb{Z}$ always splits in its K_0-group. Also, the K-theory groups may have torsion in general. Examples are given after the proof below or later.

Proof. The six-term exact sequence of K-theory groups follows from the short exact sequence of $C(M\#N)$ in the proof of Theorem 2.1:

$$\begin{array}{ccccc}
K_0(\mathfrak{I}) & \xrightarrow{i_*} & K_0(C(M\#N)) & \xrightarrow{q_*} & K_0(C(\mathbb{T})) \\
\partial \uparrow & & & & \downarrow \partial \\
K_1(C(\mathbb{T})) & \xleftarrow{q_*} & K_1(C(M\#N)) & \xleftarrow{i_*} & K_1(\mathfrak{I})
\end{array}$$

with $\mathfrak{I} = C_0((M \setminus D) \sqcup (N \setminus D))$ and $K_j(C(\mathbb{T})) = \mathbb{Z}[z] \cong \mathbb{Z}$ $(j = 0, 1)$ and

$$K_j(C_0((M \setminus D) \sqcup (N \setminus D))) \cong K_j(C_0(M \setminus D)) \oplus K_j(C_0(N \setminus D))$$

$(j = 0, 1)$, where the maps i_* and q_* are induced from the maps i and q, and ∂ are boundary maps (or the index map on the left and the exponential map on the right). The map q_* on K_0 sends the K_0-class of the unit of $C(M\#N)$ to that of $C(\mathbb{T})$, and hence is onto. Thus, ∂ on the right is zero by exactness of the diagram.

Moreover, we also have the following diagram:

$$\begin{array}{ccccc}
K_0(C_0(M \setminus D)) & \xrightarrow{i_*} & K_0(C(M)) & \xrightarrow{q_*} & K_0(C(D)) \\
\partial \uparrow & & & & \downarrow \partial \\
K_1(C(D)) & \xleftarrow{q_*} & K_1(C(M)) & \xleftarrow{i_*} & K_1(C_0(M \setminus D))
\end{array}$$

with $K_0(C(D)) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$ and $K_1(C(D)) \cong K_1(\mathbb{C}) \cong 0$ since D is contractible. The map q_* on K_0 sends the K_0-class of the unit of $C(M)$ to that of $C(D)$, and hence is onto. Thus, both of the boundary maps ∂ are zero. Therefore, the diagram implies that

$$\begin{aligned}
K_0(C(M)) &\cong K_0(C_0(M \setminus D)) \oplus \mathbb{Z}, \\
K_1(C(M)) &\cong K_1(C_0(M \setminus D)),
\end{aligned}$$

where the direct summand $\mathbb{Z}$ corresponds to the K_0-class $[1]$ of the unit 1 of $C(M)$. Note also that $K_1(C_0(M \setminus D)) \cong K_1(C_0(M \setminus D)^+)$, where the C^*-algebra unitization $C_0(M \setminus D)^+$ by $\mathbb{C}$ is isomorphic to $C((M \setminus D)^+)$, with the one-point compactification $(M \setminus D)^+$, which is in fact homeomorphic to M.

The same holds for N.

It then follows from the first six-term diagram above that

$$\begin{aligned}
K_0(C(M\#N)) &\cong \mathbb{Z} \oplus \{[(K_0(C(M))/\mathbb{Z}[1]) \oplus (K_0(C(N))/\mathbb{Z}[1])]/\partial\mathbb{Z}[z]\}, \\
K_1(C(M\#N)) &\cong K_1(C(M)) \oplus K_1(C(N)) \oplus \ker(\partial).
\end{aligned}$$

□

Example 2.5. There is the following short exact sequence of C^*-algebras:

$$0 \to C_0((S^2 \setminus D) \sqcup (S^2 \setminus D)) \xrightarrow{i} C(S^2\#S^2) \xrightarrow{q} C(\mathbb{T}) \to 0$$

with $S^2 \setminus D \approx \mathbb{R}^2$ and $S^2\#S^2 \approx S^2$. The six-term exact sequence of K-theory groups, associated, becomes:

$$\begin{array}{ccccc}
\mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{i_*} & \mathbb{Z}^2 & \xrightarrow{q_*} & \mathbb{Z} \\
\partial \uparrow & & & & \downarrow \partial \\
\mathbb{Z}[z] & \xleftarrow{q_*} & 0 & \xleftarrow{i_*} & 0 \oplus 0.
\end{array}$$

Therefore, the boundary map ∂ on the left is nonzero, and as in Theorem 2.4,

$$\begin{aligned} K_0(C(S^2\#S^2)) &\cong \mathbb{Z}\oplus\{[(K_0(C(S^2))/\mathbb{Z})\oplus(K_0(C(S^2))/\mathbb{Z})]\}/\partial\mathbb{Z}[z] \\ &\cong \mathbb{Z}\oplus\{[\mathbb{Z}\oplus\mathbb{Z}]/\partial\mathbb{Z}[z]\}\cong\mathbb{Z}^2, \\ K_1(C(S^2\#S^2)) &\cong K_1(C(S^2))\oplus K_1(C(S^2))\oplus\ker(\partial)\cong 0\oplus 0\oplus 0\cong 0. \end{aligned}$$

Note that $\mathbb{Z}\oplus\mathbb{Z}$ is torsion free, so that $\partial\mathbb{Z}[z]\cong\mathbb{Z}$ and $\ker(\partial)\cong 0$, and that the diagram above does not involve torsion, so that $[\mathbb{Z}\oplus\mathbb{Z}]/\partial\mathbb{Z}[z]\cong\mathbb{Z}$.

Next, there is the following short exact sequence of C^*-algebras:

$$0\to C_0((\mathbb{T}^2\setminus D)\sqcup(S^2\setminus D))\xrightarrow{i} C(\mathbb{T}^2\#S^2)\xrightarrow{q} C(\mathbb{T})\to 0$$

with $\mathbb{T}^2\#S^2\approx\mathbb{T}^2$. The six-term exact sequence of K-theory groups, associated, becomes:

$$\begin{array}{ccccc} K_0(C_0(\mathbb{T}^2\setminus D))\oplus\mathbb{Z} & \xrightarrow{i_*} & \mathbb{Z}^2 & \xrightarrow{q_*} & \mathbb{Z} \\ \partial\uparrow & & & & \downarrow\partial=0 \\ \mathbb{Z}[z] & \xleftarrow{q_*} & \mathbb{Z}^2 & \xleftarrow{i_*} & K_1(C_0(\mathbb{T}^2\setminus D))\oplus 0. \end{array}$$

Moreover, the exact sequence $0\to C_0(\mathbb{T}^2\setminus D)\to C(\mathbb{T}^2)\to C(D)\to 0$ implies

$$\begin{array}{ccccc} K_0(C_0(\mathbb{T}^2\setminus D)) & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} \\ \partial=0\uparrow & & & & \downarrow\partial=0 \\ 0 & \longleftarrow & \mathbb{Z}^2 & \longleftarrow & K_1(C_0(\mathbb{T}^2\setminus D)) \end{array}$$

so that $K_0(C_0(\mathbb{T}^2\setminus D))\cong\mathbb{Z}$ and $K_1(C_0(\mathbb{T}^2\setminus D))\cong\mathbb{Z}^2$. Therefore, the boundary map ∂ on the left in the second six-term diagram in this example is nonzero, and as in Theorem 2.4, by the same reasoning as above,

$$\begin{aligned} K_0(C(\mathbb{T}^2\#S^2)) &\cong \mathbb{Z}\oplus\{[(K_0(C(\mathbb{T}^2))/\mathbb{Z})\oplus(K_0(C(S^2))/\mathbb{Z})]\}/\partial\mathbb{Z}[z] \\ &\cong \mathbb{Z}\oplus\{[\mathbb{Z}\oplus\mathbb{Z}]/\partial\mathbb{Z}[z]\}\cong\mathbb{Z}^2, \\ K_1(C(\mathbb{T}^2\#S^2)) &\cong K_1(C(\mathbb{T}^2))\oplus K_1(C(S^2))\oplus\ker(\partial)\cong\mathbb{Z}^2\oplus 0\oplus 0\cong\mathbb{Z}^2. \end{aligned}$$

Now, let M be a closed surface. There is the following short exact sequence of C^*-algebras:

$$0\to C_0((M\setminus D)\sqcup(S^2\setminus D))\xrightarrow{i} C(M\#S^2)\xrightarrow{q} C(\mathbb{T})\to 0$$

with $S^2\setminus D\approx\mathbb{R}^2$ and $M\#S^2\approx M$. The six-term exact sequence of K-theory groups, associated, becomes:

$$\begin{array}{ccccc} K_0(C_0(M\setminus D))\oplus\mathbb{Z} & \xrightarrow{i_*} & K_0(C(M\#S^2)) & \xrightarrow{q_*} & \mathbb{Z} \\ \partial\uparrow & & & & \downarrow\partial=0 \\ \mathbb{Z}[z] & \xleftarrow{q_*} & K_1(C(M\#S^2)) & \xleftarrow{i_*} & K_1(C_0(M\setminus D))\oplus 0. \end{array}$$

Moreover, $K_1(C_0(M\setminus D))\cong K_1(C_0(M\setminus D)^+)\cong K_1(C(M))$. Therefore, the map q_* on K_1 is zero, so that $\ker(\partial)=0$. Also, $K_0(C(M))\cong K_0(C_0(M\setminus D)^+)\cong K_0(C_0(M\setminus D))\oplus\mathbb{Z}$. Thus, as in Theorem 2.4,

$$\begin{aligned} K_0(C(M\#S^2)) &\cong \mathbb{Z}\oplus\{[K_0(C_0(M\setminus D))\oplus\mathbb{Z}]/\partial\mathbb{Z}[z]\} \\ &\cong \mathbb{Z}\oplus K_0(C_0(M\setminus D))\cong K_0(C(M)), \\ K_1(C(M\#S^2)) &\cong K_1(C(M))\oplus K_1(C(S^2))\oplus\ker(\partial)\cong K_1(C(M)). \end{aligned}$$

Corollary 2.6. *The formula obtained in Theorem* 2.1 *follows from Theorem* 2.4.

Proof. Note that each quotient by $\mathbb{Z}[1]$, or by $\partial\mathbb{Z}[z]$ together with $\ker(\partial)$ as a set, in Theorem 2.4 corresponds to either one rank lowering the free ranks of those K_0-groups, or either the same or one rank raising the free ranks of those K_1-groups, respectively, where either $\partial\mathbb{Z}[z]$ or $\ker(\partial)$ has rank one, Hence,

$$\chi(C(M\#N)) = 1 + \chi(C(M)) + \chi(C(N)) - 3 = \chi(C(M)) + \chi(C(N)) - 2.$$

□

Corollary 2.7. *Let M_i $(1 \leq i \leq n)$ be closed surfaces and $\#_{i=1}^n M_i$ be their successive connected sum. Then, inductively,*

$$\begin{aligned}
&K_0(C(\#_{i=1}^n M_i))\\
&\cong \mathbb{Z} \oplus \{[(K_0(C(\#_{i=1}^{n-1} M_i))/\mathbb{Z}[1]) \oplus (K_0(C(M_n))/\mathbb{Z}[1])]/\partial_{n-1}\mathbb{Z}[z]\},\\
&\cong \mathbb{Z} \oplus \{[((\cdots(\mathbb{Z} \oplus \{[(K_0(C(M_1))/\mathbb{Z}[1]) \oplus (K_0(C(M_2))/\mathbb{Z}[1])]/\partial_1\mathbb{Z}[z]\})\\
&\quad \cdots/\mathbb{Z}[1]) \oplus (K_0(C(M_n))/\mathbb{Z}[1])]/\partial_{n-1}\mathbb{Z}[z]\},\\
&K_1(C(\#_{i=1}^n M_i)) \cong [\oplus_{i=1}^n K_1(C(M_i))] \oplus [\oplus_{i=1}^{n-1} \ker(\partial_i)],
\end{aligned}$$

where each $[1]$ *means the* K_0*-class of the unit* 1 *of* $C(M_i)$*, and* $[z] \in K_1(C(\mathbb{T}))$ *the generating class, and each* $\partial_i = \partial$ *is the boundary map in each step in induction.*

Corollary 2.8. *That Corollary* 2.2 *follows from this Corollary* 2.7.

Proof. Note that each quotient by $\mathbb{Z}[1]$, or by $\partial_i\mathbb{Z}[z]$ together with $\ker(\partial_i)$ as a set, in Corollary 2.7 corresponds to either one rank lowering the free ranks of those K_0-groups, or either the same or one rank raising the free ranks of those K_1-groups, respectively, where either $\partial_i\mathbb{Z}[z]$ or $\ker(\partial_i)$ has rank one. Hence, inductively,

$$\begin{aligned}
\chi(C(\#_{i=1}^n M_i)) &= \chi(C(\#_{i=1}^{n-1} M_i)) + \chi(C(M_i)) - 2\\
&= \cdots = \sum_{i=1}^n \chi(C(M_i)) - 2(n-1).
\end{aligned}$$

□

Example 2.9. Since $K_j(C(\mathbb{T}^2)) \cong \mathbb{Z}^2$ $(j = 0, 1)$, it is obtained by Theorem 2.4 that

$$\begin{aligned}
K_0(C(\mathbb{T}^2\#\mathbb{T}^2)) &\cong \mathbb{Z} \oplus \{[(\mathbb{Z}^2/\mathbb{Z}) \oplus (\mathbb{Z}^2/\mathbb{Z})]\}/\partial\mathbb{Z}\\
&\cong \mathbb{Z} \oplus \{[\mathbb{Z} \oplus \mathbb{Z}]/\partial\mathbb{Z}\} \cong \mathbb{Z}^2,\\
K_1(C(\mathbb{T}^2\#\mathbb{T}^2)) &\cong \mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \ker(\partial) \cong \mathbb{Z}^4.
\end{aligned}$$

Hence, $\chi(C(\mathbb{T}^2\#\mathbb{T}^2)) = (3-1) - 4 = -2$.

Moreover, it is obtained by Corollary 2.7 that

$$\begin{aligned}
&K_0(C(\#^n\mathbb{T}^2)) \cong \mathbb{Z} \oplus \{[(\cdots([\mathbb{Z} \oplus \mathbb{Z}]/\partial_1\mathbb{Z})\cdots) \oplus \mathbb{Z}]/\partial_{n-1}\mathbb{Z}\},\\
&K_1(C(\#^n\mathbb{T}^2)) \cong [\oplus_{i=1}^n \mathbb{Z}^2] \oplus [\oplus_{i=1}^{n-1} \ker(\partial_i)] \cong \mathbb{Z}^{2n}.
\end{aligned}$$

Hence, $\chi(C(\#^n\mathbb{T}^2)) = (1 + n - (n-1)) - 2n = 2 - 2n$. Indeed, it is deduced that $K_0(C(\#^n\mathbb{T}^2)) \cong \mathbb{Z}^2$ by repeating the reasoning as before.

As for the real 2-dimensional, projective plane P^2, the six-term exact sequence of K-theory groups, associated to the short exact sequence of $C(P^2)$ in Example 2.3, becomes

$$\begin{tikzcd}
K_0(C_0(M_b^\circ)) \arrow[r] & K_0(C(P^2)) \arrow[r] & \mathbb{Z} \arrow[d, "\partial=0"] \\
0 \arrow[u, "\partial=0"] & K_1(C(P^2)) \arrow[l] & K_1(C_0(M_b^\circ)) \arrow[l]
\end{tikzcd}$$

so that

$$K_0(C(P^2)) \cong \mathbb{Z} \oplus K_0(C_0(M_b^\circ)),$$
$$K_1(C(P^2)) \cong K_1(C_0(M_b^\circ)).$$

Indeed, P^2 is viewed as the one-point compactification of the interior M_b° of the Möbius band M_b. Moreover, the six-term exact sequence of K-theory groups, associated to the short exact sequence of $C_0(M_b^\circ)$ in Example 2.3, becomes

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] & K_0(C_0(M_b^\circ)) \arrow[r] & 0 \arrow[d, "\partial"] \\
\mathbb{Z} \arrow[u, "\partial"] & K_1(C_0(M_b^\circ)) \arrow[l] & 0. \arrow[l]
\end{tikzcd}$$

Furthermore, it follows from the diagram that $K_1(C_0(M_b^\circ))$ viewed as a subgroup of $\mathbb{Z}$ without torsion is isomorphic to $\mathbb{Z}$ or zero. On the other hand, there is the quotient map from $C_0(M_b^\circ)^+ \cong C((M_b^\circ)^+)$ to $C_0(\mathbb{R})^+ \cong C(\mathbb{T})$, which induces the induced map q_* on K_1-groups to be zero. In fact, $(M_b^\circ)^+$ is homeomorphic to the Moore space of order two, so that $K_0(C((M_b^\circ)^+)) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ and $K_1(C((M_b^\circ)^+)) \cong 0$ (see [7] and [6, 12.3]). Hence it follows that $K_0(C_0(M_b^\circ)) \cong \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and $K_1(C_0(M_b^\circ)) \cong 0$. Therefore, we do have

$$K_0(C(P^2)) \cong \mathbb{Z} \oplus (\mathbb{Z}/\partial\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2, \quad K_1(C(P^2)) \cong 0.$$

Hence $\chi(C(P^2)) = 1$. (Note that these results are compatible with those in homology theory for P^2 in the sense that the Euler characteristic obtained in homology theory for P^2 coincides with our Euler number for $C(P^2)$. See for instance, [3].)

It is obtained by Theorem 2.4 that

$$\begin{aligned}
K_0(C(P^2 \# P^2)) &\cong \mathbb{Z} \oplus \{[(([\mathbb{Z} \oplus (\mathbb{Z}/\partial\mathbb{Z})]/\mathbb{Z}) \oplus ([\mathbb{Z} \oplus (\mathbb{Z}/\partial\mathbb{Z})]/\mathbb{Z})]/\partial_1\mathbb{Z}\} \\
&\cong \mathbb{Z} \oplus \{[(\mathbb{Z}/\partial\mathbb{Z}) \oplus (\mathbb{Z}/\partial\mathbb{Z})]/\partial_1\mathbb{Z}\} \\
&\cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \mathbb{Z} \oplus \mathbb{Z}_2^2, \\
K_1(C(P^2 \# P^2)) &\cong 0 \oplus 0 \oplus \ker(\partial_1) \cong \mathbb{Z}
\end{aligned}$$

with in fact $\mathbb{Z}/\partial\mathbb{Z} \cong \mathbb{Z}_2$, so that $\partial_1\mathbb{Z} \cong 0$ and $\ker(\partial_1) \cong \mathbb{Z}$. Hence $\chi(C(P^2 \# P^2)) = 1-1 = 0$. Indeed, it follows from [6, 12.3] that the image $\partial_1\mathbb{Z}$ is zero.

Moreover, it is obtained by Corollary 2.7 that

$$\begin{aligned}
K_0(C(\#^n P^2)) &\cong \mathbb{Z} \oplus \{[(\cdots([(\mathbb{Z}/\partial\mathbb{Z}) \oplus (\mathbb{Z}/\partial\mathbb{Z})]/\partial_1\mathbb{Z}) \cdots) \oplus (\mathbb{Z}/\partial\mathbb{Z})]/\partial_{n-1}\mathbb{Z}\} \\
&\cong \mathbb{Z} \oplus \mathbb{Z}_2^n, \\
K_1(C(\#^n P^2)) &\cong [\oplus_{i=1}^n 0] \oplus [\oplus_{i=1}^{n-1} \ker(\partial_i)] \cong \mathbb{Z}^{n-1}.
\end{aligned}$$

Hence $\chi(C(\#^n P^2)) = 1-(n-1) = 2-n$. In fact, since $\mathbb{Z}/\partial\mathbb{Z} \cong \mathbb{Z}_2$, we have the image $\partial_1\mathbb{Z}$ isomorphic to 0, and inductively, the image $\partial_{n-1}\mathbb{Z}$ isomorphic to 0, so that each $\ker(\partial_i)$ is isomorphic to $\mathbb{Z}$.

Table 2: Classification for closed surfaces

K_0 rank	Orientable	Non-orientable
2	$S^2 \approx S(0)$ $\mathbb{T}^2 \approx T(1)$ $\#^n\mathbb{T}^2 \approx T(n)$	No
1	No	$P^2 \approx P(1)$ $P^2\#P^2 \approx P(2) \approx K^2$ $\#^n P^2 \approx P(n)$

It follows from the Table 2 that:

Corollary 2.10. *The rank of K_0-groups for C^*-algebras can not classify homeomorphism classes of orientable closed surfaces, and as well, the rank of K_0-groups for C^*-algebras does not classify homeomorphism classes of non-orientable closed surfaces.*

But, the rank of K_0-groups for C^*-algebras does distinguish the class of homeomorphism classes of orientable, closed surfaces from the class of those of non-orientable, closed surfaces.

Table 3: Classification for closed surfaces

K_1 rank	Orientable	Non-orientable
$2n$	$\#^n\mathbb{T}^2 \approx T(n)$	$\#^{2n+1}P^2 \approx P(2n+1)$
$2n-1$	No	$\#^{2n}P^2 \approx P(2n)$
4	$\mathbb{T}^2\#\mathbb{T}^2 \approx T(2)$	$\#^5 P^2 \approx P(5)$
3	No	$\#^4 P^2 \approx P(4)$
2	$\mathbb{T}^2 \approx T(1)$	$\#^3 P^2 \approx P(3) \approx \mathbb{T}^2\#P^2$
1	No	$P^2\#P^2 \approx P(2) \approx K^2$
0	$S^2 \approx S(0)$	$P^2 \approx P(1)$

It follows from the Table 3 that:

Corollary 2.11. *The rank of K_1-groups for C^*-algebras does classsify homeomorphism classes of orientable, closed surfaces and as well, those of non-orientable, closed surfaces.*

3 Connected sums of closed topological manifolds A closed topological manifold is a compact real n-dimensional topological manifold without boundary ($n \geq 1$). Let M, N be n-dimensional closed topological manifolds. The connected sum $M\#N$ of M and N is defined to be the closed topological manifold obtained by removing the n-dimensional closed unit balls D^n (of $\mathbb{R}^n$) viewed on M and N from themselves and gluing the open differences $M \setminus D^n$ and $N \setminus D^n$ by attaching the $(n-1)$-dimensional sphere S^{n-1} (or the boundary ∂D^n of D^n) to them as their boundaries ($n \geq 2$), where when $n = 1$, $D^1 = [-1, 1]$ the closed interval and $S^0 = \partial D^1 = \{-1, 1\}$ the set of two points.

Theorem 3.1. *Let M, N be n-dimensional closed topological manifolds and $M\#N$ be their connected sum ($n \geq 1$). If n is even, then*

$$\chi(C(M\#N)) = \chi(C(M)) + \chi(C(N)) - 2,$$

and if n is odd, then

$$\chi(C(M\#N)) = \chi(C(M)) + \chi(C(N)).$$

Proof. By the definition of $M\#N$, we have the following short exact sequence of C^*-algebras:

$$0 \to C_0((M \setminus D^n) \sqcup (N \setminus D^n)) \to C(M\#N) \to C(S^{n-1}) \to 0,$$

where $\sqcup$ means the disjoint union, and the closed ideal is isomorphic to the direct sum $C_0(M \setminus D^n) \oplus C_0(N \setminus D^n)$. Therefore, it follows that

$$\chi(C(M\#D)) = \chi(C_0(M \setminus D^n)) + \chi(C_0(N \setminus D^n)) + \chi(C(S^{n-1})).$$

Moreover, we have

$$0 \to C_0(M \setminus D^n) \to C(M) \to C(D^n) \to 0,$$

so that

$$\chi(C(M)) = \chi(C_0(M \setminus D^n)) + \chi(C(D^n)),$$

with $\chi(C(D^n)) = 1$ since D^n is contractible to a point, so that $C(D^n)$ is contractible to $\mathbb{C}$, and the Euler characteristic is stable under homotopy equivalence in C^*-algebras.

The same holds for N. Also, we have

$$0 \to C_0(\mathbb{R}^{n-1}) \to C(S^{n-1}) \to \mathbb{C} \to 0$$

since S^{n-1} is viewed as the one-point compactification of $\mathbb{R}^{n-1}$, so that

$$\chi(C(S^{n-1})) = \begin{cases} -1+1=0 & \text{if } n \text{ is even,} \\ 1+1=2 & \text{if } n \text{ is odd} \end{cases}$$

($n \geq 2$), where note that $C(S^0) \cong \mathbb{C}^2$, and hence the equation above for $n = 1$ also holds. $\square$

Corollary 3.2. *Let $M_1\#M_2 \cdots \#M_l$ be an $(l-1)$-successive connected sum of closed n-dimensional topological manifolds $M_1, M_2, \cdots, M_l$. If n is even, then*

$$\chi(C(M_1\#M_2 \cdots \#M_l)) = \sum_{k=1}^{l} \chi(C(M_k)) - 2(l-1),$$

and if n is odd, then

$$\chi(C(M_1\#M_2 \cdots \#M_l)) = \sum_{k=1}^{l} \chi(C(M_k)).$$

As an interest, we consider higher dimensional analogues of closed surfaces.

Example 3.3. Let $\mathbb{T}^n$ be the n-dimensional torus. Then $\chi(C(\mathbb{T}^n)) = 0$ since $\chi(C(\mathbb{T}^n)) = \chi(C(\Pi^n\mathbb{T})) = \chi(\otimes^n C(\mathbb{T})) = \chi(C(\mathbb{T}))^n = 0^n$, with $\Pi^n\mathbb{T} = \mathbb{T} \times \cdots \times \mathbb{T}$. Therefore, we have

$$\chi(C(\#^l\mathbb{T}^n)) = \begin{cases} -2(l-1) = 2-2l & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Let S^n be the n-dimensional sphere. Then

$$\chi(C(\#^l S^n)) = \begin{cases} 2l - 2(l-1) = 2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Note that for M a closed n-dimensional topological manifold,

$$\chi(C(M\#S^n)) = \begin{cases} \chi(C(M)) + \chi(C(S^n)) - 2 = \chi(C(M)) & \text{if } n \text{ is even,} \\ \chi(C(M)) + \chi(C(S^n)) = \chi(C(M)) & \text{if } n \text{ is odd.} \end{cases}$$

Indeed, $M\#S^n \approx M$. Thus, in particular, $\#^l S^n \approx S^n$.

Let P^n be the real n-dimensional projective space, which is a quotient space of $\mathbb{R}^{n+1}\setminus\{0\}$ (or of S^n), where two points x, y in $\mathbb{R}^{n+1} \setminus \{0\}$ are equivalent if there is $t \in \mathbb{R}$ such that $x = ty$. Now we view P^n as obtained by gluing the boundary of the n-dimensional Möbius band M_b^n with that of the closed unit ball D^n of $\mathbb{R}^n$, where the n-dimensional Möbius band M_b^n defined by us is obtained from the product space $I = [0,1] \times [\{-\infty\} \cup (P^{n-1})^- \cup \{\infty\}]$ by identifying one edge $E \approx [0,1]$ at $-\infty$ with the opposite edge at ∞ with one twist, so that $(P^{n-1})^- \cup \{\pm\infty\} \approx P^{n-1}$ with $+\infty = -\infty$ identified, where $(P^{n-1})^-$ means our uncompactification of P^{n-1} by removing one point. One can check that this should be true, as follows. We have the decomposition $S^n = S^n_+ \cup S^{n-1} \cup S^n_-$ as a disjoint union, where $S^n_+ \cup S^n_- = S^n \setminus S^{n-1}$ with the north and south poles contained in S^n_+ and S^n_- respectively. Then S^n_+ is homeomorphic to the interior of D^n and is identified with S^n_- in P^n, and P^n is obtained by gluing the boundary P^{n-1} of the n-dimensional Möbius band M_b^n with that of D^n mapped in P_n by the quotient map from S^n to P^n. Refer also [3, Section 2.8] for the cell decomposition for P^n as $P^n = P^{n-1} \sqcup \mathbb{R}^n$ a disjoint union.

There is the following short exact sequence of C^*-algebras:

$$0 \to C_0((M_b^n)^\circ) \to C(P^n) \to C(D^n) \to 0,$$

where $(M_b^n)^\circ$ is the interior of M_b^n. Also, we have

$$0 \to C_0(I^\circ) \to C_0((M_b^n)^\circ) \to C_0(E^\circ) \to 0.$$

Therefore, we get

$$\begin{aligned} \chi(C(P^n)) &= \chi(C_0((M_b^n)^\circ)) + \chi(C(D^n)) \\ &= \chi(C_0(I^\circ)) + \chi(C_0(E^\circ)) + 1 \\ &= \chi(C_0((P^{n-1})^- \times \mathbb{R})) + \chi(C_0(\mathbb{R})) + 1 \\ &= -\chi(C_0((P^{n-1})^-)) - 1 + 1 \\ &= -[\chi(C(P^{n-1})) - 1] \\ &= -\chi(C(P^{n-1})) + 1, \end{aligned}$$

($n \geq 1$), where we have $0 \to C_0((P^{n-1})^-) \to C(P^{n-1}) \to \mathbb{C} \to 0$ split, and $P^1 \approx S^1 = \mathbb{T}$, and P^0 is identified with the quotient $\{-1 = +1\}$ of S^0. The equation obtained above is converted to

$$a_n - \frac{1}{2} = -\left(a_{n-1} - \frac{1}{2}\right)$$

($n \geq 1$) with $a_n = \chi(C(P^n))$ for $n \geq 1$, so that

$$\begin{aligned} \chi(C(P^n)) &= \frac{1}{2}(1 + (-1)^n) \quad (n \geq 1) \\ &= \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

(Note that this result is compatible with that in homology theory for P^n in the sense that the Euler characteristic in homology theory for P^n coincides with our Euler number for $C(P^n)$. Indeed,

$$H_0(P^{2n}) \cong \mathbb{Z}, \quad H_{2k}(P^{2n}) \cong 0, \quad \text{and} \quad H_{2k-1}(P^{2n}) \cong \mathbb{Z}/2\mathbb{Z} \quad (k = 1, \cdots, n),$$

so that $\chi(P^{2n}) = 1$. Also,

$$\begin{aligned} &H_0(P^{2n+1}) \cong \mathbb{Z}, \quad H_{2k}(P^{2n+1}) \cong 0, \quad H_{2n+1}(P^{2n+1}) \cong \mathbb{Z}, \\ &\text{and} \quad H_{2k-1}(P^{2n+1}) \cong \mathbb{Z}/2\mathbb{Z} \quad (k = 1, \cdots, n), \end{aligned}$$

so that $\chi(P^{2n+1}) = 0$ (see [3, Section 3.7]).)

Moreover, if n is even, then

$$\chi(C(\#^l P^n)) = l - 2(l-1) = 2 - l,$$

and if n is odd, then

$$\chi(C(\#^l P^n)) = 0.$$

Note that $P^n \# P^n$ is homeomorhic to the closed topological manifold obtained by glueing two n-dimensional Möbius bands M_b^n along with their boundaries homeomorphic to P^{n-1}, which we may call it the n-dimensional Klein bottle, and denote it by K^n. Note also that $\mathbb{T}^2 \# P^2 \approx \#^3 P^2$, and as well, we may have $\mathbb{T}^n \# P^n \approx \#^3 P^n$ $(n \geq 3)$ (as our consequence). (Our intuitive explanation for this is that $\mathbb{T}^n$ viewed as a square n-dimensional closed interval with edges identified with opposites (with no twist) is transformed by cutting the interval on the diagonal to the Klein bottle K^n viewed as a square n-dimensional closed interval with edges identified with opposites with half twisted and with half no twisted alternatively in the connected sum $\mathbb{T}^n \# P^n$.) Indeed, if n is even,

$$\begin{aligned} \chi(C(\mathbb{T}^n \# P^n)) &= \chi(C(\mathbb{T}^n)) + \chi(C(P^n)) - 2 = 0 + 1 - 2 = -1, \\ \chi(C(\#^3 P^n)) &= 2 - 3 = -1, \end{aligned}$$

and if n is odd, then

$$\begin{aligned} \chi(C(\mathbb{T}^n \# P^n)) &= \chi(C(\mathbb{T}^n)) + \chi(C(P^n)) \\ &= 0 + 0 = 0, \\ \chi(C(\#^3 P^n)) &= 0. \end{aligned}$$

Table 4: Classification for closed topological manifolds with dimension even

Euler number	Orientable	Non-orientable
2	$S^{2n} \approx \#^l S^{2n}$, S^0	No
1	No	P^{2n}, P^0
0	$\mathbb{T}^{2n}$	$P^{2n} \# P^{2n} = K^{2n}$
-1	No	$\#^3 P^{2n}$
-2	$\mathbb{T}^{2n} \# \mathbb{T}^{2n}$	$\#^4 P^{2n}$
$3 - 2l$	No	$\#^{2l-1} P^{2n}$
$2 - 2l$	$\#^l \mathbb{T}^{2n}$	$\#^{2l} P^{2n}$

Note that $\#^3 P^{2n} \approx \mathbb{T}^{2n} \# P^{2n}$ by the reason given above. It follows from the Table 4 that

Corollary 3.4. *Let n be a natural number. The Euler characteristic in K-theory of C^*-algebras classifies even $2n$-dimensional orientable topological manifolds in the homeomorphism classes of connected sums of the even $2n$-dimensional torus $\mathbb{T}^{2n}$ and the even $2n$-dimensional sphere S^{2n}, and also does even $2n$-dimensional non-orientable topological manifolds in the homeomorphism classes of connected sums of the even $2n$-dimensional projective space P^{2n}.*

Table 5: Classification for closed topological manifolds with dimension odd

Euler number	Orientable	Orientable
0	$\mathbb{T}^{2n+1}$, $\#^l\mathbb{T}^{2n+1}$ $S^{2n+1} \approx \#^l S^{2n+1}$	P^{2n+1} $\#^l P^{2n+1}$

Note that P^n is orientable if n is odd but not if n is even (see [3]). We also have $P^{2n+1}\#P^{2n+1} = K^{2n+1}$ and $\mathbb{T}^{2n+1}\#P^{2n+1} \approx \#^3 P^{2n+1}$. It follows from the Table 5 that

Corollary 3.5. *Let n be a natural number. The Euler characteristic in K-theory of C^*-algebras can not classify odd $(2n+1)$-dimensional orientable topological manifolds in the homeomorphism classes of connected sums of the odd $(2n+1)$-dimensional torus $\mathbb{T}^{2n+1}$ and the odd $(2n+1)$-dimensional sphere S^{2n+1}, and does not classify odd $(2n+1)$-dimensional orientable topological manifolds in the homeomorphism classes of connected sums of the odd $(2n+1)$-dimensional projective space P^{2n+1}.*

We now consider another decomposition for a sort of substitute of P^n in our sense. As well, one may use the K-theory of its corresponding (different) C^*-algebras to classify homeomorphism classes of connected sums of P^n.

Example 3.6. As a contrast to P^n, and as a sort of substitute of P^n, we may define P_n to be a closed topological manifold obtained by gluing the boundary of the n-dimensional Möbius band $M_{b,n}$ (the same name as before, but with different fibers) with that of the closed unit ball D^n of $\mathbb{R}^n$, where the n-dimensional Möbius band $M_{b,n}$ defined by us is obtained from the product space $I = [0,1] \times [\{-\infty\} \cup \mathbb{R}^{n-1} \cup \{\infty\}]$ by identifying one edge $E \approx [0,1]$ at $-\infty$ with the opposite edge at ∞ with one twist, so that $\mathbb{R}^{n-1} \cup \{\pm\infty\} \approx S^{n-1}$ with $+\infty = -\infty$ identified. We have the decomposition $S^n = S^n_+ \cup S^{n-1} \cup S^n_-$ as a disjoint union, where $S^n_+ \cup S^n_- = S^n \setminus S^{n-1}$ with the north and south poles contained in S^n_+ and S^n_- respectively. Then S^n_+ is homeomorphic to the interior of D^n and is identified with S^n_- in P_n, and P_n is obtained by gluing the boundary S^{n-1} of the n-dimensional Möbius band $M_{b,n}$ with that of D^n.

There is the following short exact sequence of C^*-algebras:

$$0 \to C_0((M_{b,n})^\circ) \to C(P_n) \to C(D^n) \to 0,$$

where $(M_{b,n})^\circ$ is the interior of $M_{b,n}$. Also,

$$0 \to C_0(I^\circ) \to C_0((M_{b,n})^\circ) \to C_0(E^\circ) \to 0.$$

Therefore, we do obtain

$$\begin{aligned}\chi(C(P_n)) &= \chi(C_0((M_{b,n})^\circ)) + \chi(C(D^n)) \\ &= \chi(C_0(I^\circ)) + \chi(C_0(E^\circ)) + 1 \\ &= \chi(C_0(\mathbb{R}^n)) + \chi(C_0(\mathbb{R})) + 1 \\ &= \begin{cases} 1 - 1 + 1 = 1 & \text{if } n \text{ is even,} \\ -1 - 1 + 1 = -1 & \text{if } n \text{ is odd.} \end{cases}\end{aligned}$$

(Note that these results are not compatible with those in homology theory for P^n, so that the decomposition for P_n should be not applied to P^n, but one can use this result for a more better classification for connected sums of P_n as compared with that given above, when n is odd.) Indeed, moreover,

$$\chi(C(\#^l P_n)) = \begin{cases} l - 2(l-1) = 2 - l & \text{if } n \text{ is even,} \\ -l \quad (\neq 0) & \text{if } n \text{ is odd.} \end{cases}$$

(As a question, there must be a suitable topological reason for the last inequality. The reason may be that P_n is less twisted than P^n.)

Note that $P_n \# P_n$ is homeomorhic to the closed topological manifold obtained by glueing two n-dimensional Möbius bands $M_{b,n}$ along with their boundaries homeomorphic to S^{n-1}, which we may call it the n-dimensional Klein bottle (the same name as before), and denote it by K_n (slightly different). Note also that we may have that $\mathbb{T}^n \# P_n \approx \#^3 P_n$ if n is even, but not if n is odd. Indeed, if n is even, then

$$\begin{aligned}\chi(C(\mathbb{T}^n \# P_n)) &= 0 + 1 - 2 = -1, \\ \chi(C(\#^3 P_n)) &= 2 - 3 = -1,\end{aligned}$$

but if n is odd, then

$$\begin{aligned}\chi(C(\mathbb{T}^n \# P_n)) &= 0 + (-1) = -1, \\ \chi(C(\#^3 P_n)) &= -3 \neq -1.\end{aligned}$$

In the following we compute K-theory groups.

Theorem 3.7. *Let M, N be n-dimensional closed topological manifolds and $M \# N$ be their connected sum. If n is even, then*

$$\begin{aligned}K_0(C(M\#N)) &\cong \mathbb{Z} \oplus \{[(K_0(C(M))/\mathbb{Z}[1]) \oplus (K_0(C(N))/\mathbb{Z}[1])]/\partial\mathbb{Z}[z]\}, \\ K_1(C(M\#N)) &\cong K_1(C(M)) \oplus K_1(C(N)) \oplus \ker(\partial),\end{aligned}$$

and if n is odd, then

$$\begin{aligned}K_0(C(M\#N)) &\cong \mathbb{Z} \oplus [(K_0(C(M))/\mathbb{Z}[1]) \oplus (K_0(C(N))/\mathbb{Z}[1])], \\ K_1(C(M\#N)) &\cong [K_1(C(M)) \oplus K_1(C(N))]/\partial\mathbb{Z}[p],\end{aligned}$$

where each $[1]$ means the K_0-class of the unit 1 (of $C(M)$, $C(N)$, and $C(S^{n-1})$), and $[z]$ means the generating K_1-class of $K_1(C(S^{n-1})) \cong \mathbb{Z}$ when n is even, and $[p]$ means the non-trivial K_0-class of $K_0(C(S^{n-1})) \cong \mathbb{Z}[1] \oplus \mathbb{Z}[p] \cong \mathbb{Z}^2$ when n is odd, with the image $\partial\mathbb{Z}[p] \cong \mathbb{Z}$.

Remark. See the Remark for Theorem 2.4 for more details on notes, also applied to the image $\partial\mathbb{Z}[p]$.

Proof. The six-term exact sequence of K-theory groups follows from the short exact sequence of $C(M\#N)$ in the proof of Theorem 3.1:

$$\begin{array}{ccccc} K_0(\mathfrak{I}) & \xrightarrow{i_*} & K_0(C(M\#N)) & \xrightarrow{q_*} & K_0(C(S^{n-1})) \\ \partial\uparrow & & & & \downarrow\partial \\ K_1(C(S^{n-1})) & \xleftarrow{q_*} & K_1(C(M\#N)) & \xleftarrow{i_*} & K_1(\mathfrak{I}) \end{array}$$

where $\mathfrak{I} = C_0((M \setminus D^n) \sqcup (N \setminus D^n))$, and if $n = 2k$ even and if $n = 2k+1$ odd, then respectively,

$$K_j(C(S^{2k-1})) \cong \begin{cases} \mathbb{Z}[1] & j = 0, \\ \mathbb{Z}[z] & j = 1, \end{cases} \quad K_j(C(S^{2k})) \cong \begin{cases} \mathbb{Z}[1] \oplus \mathbb{Z}[p] & j = 0, \\ 0 & j = 1, \end{cases}$$

since $0 \to C_0(\mathbb{R}^{n-1}) \to C(S^{n-1}) \to \mathbb{C} \to 0$ is a split, short exact sequence of C^*-algebras, and also

$$K_j(C_0((M \setminus D^n) \sqcup (N \setminus D^n))) \cong K_j(C_0(M \setminus D^n)) \oplus K_j(C_0(N \setminus D^n))$$

$(j = 0, 1)$, and where the maps i_* and q_* are induced from the maps i and q, and ∂ are boundary maps (or the index map on the left and the exponential map on the right). The map q_* on K_0 sends the K_0-class of the unit of $C(M\#N)$ to that of $C(S^{n-1})$, and hence is onto if n is even. Thus, ∂ on the right is zero if n is even. If n is odd, one can see in general that the kernel $\ker(\partial)$ contains $\mathbb{Z}[1]$.

Moreover, we also have the following diagram:

$$\begin{array}{ccccc} K_0(C_0(M \setminus D^n)) & \xrightarrow{i_*} & K_0(C(M)) & \xrightarrow{q_*} & K_0(C(D^n)) \\ \partial\uparrow & & & & \downarrow\partial \\ K_1(C(D^n)) & \xleftarrow{q_*} & K_1(C(M)) & \xleftarrow{i_*} & K_1(C_0(M \setminus D^n)) \end{array}$$

with $K_0(C(D^n)) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$ and $K_1(C(D^n)) \cong K_1(\mathbb{C}) \cong 0$ since D^n is contractible. The map q_* on K_0 sends the K_0-class of the unit of $C(M)$ to that of $C(D^n)$, and hence is onto. Thus, both of the boundary maps ∂ are zero. Therefore, the diagram implies that

$$\begin{aligned} K_0(C(M)) &\cong K_0(C_0(M \setminus D^n)) \oplus \mathbb{Z}, \\ K_1(C(M)) &\cong K_1(C_0(M \setminus D^n)), \end{aligned}$$

where the direct summand $\mathbb{Z}$ corresponds to the K_0-class $[1]$ of the unit 1 of $C(M)$. Note also that $K_1(C_0(M\setminus D^n)) \cong K_1(C_0(M\setminus D^n)^+)$, where the unitization $C_0(M\setminus D^n)^+$ is isomorphic to $C((M \setminus D^n)^+)$, with the one-point compactification $(M \setminus D^n)^+$ homeomorphic to M.

The same holds for N.

Furthermore, the map q_* on K_1 in the first diagram in this proof is zero when n is odd, so that the kernel $\ker(\partial) = 0$.

It then follows consequently that if n is even, then

$$\begin{aligned} K_0(C(M\#N)) &\cong \mathbb{Z} \oplus \{[(K_0(C(M))/\mathbb{Z}[1]) \oplus (K_0(C(N))/\mathbb{Z}[1])]/\partial\mathbb{Z}[z]\}, \\ K_1(C(M\#N)) &\cong K_1(C(M)) \oplus K_1(C(N)) \oplus \ker(\partial), \end{aligned}$$

and also, if n is odd, then

$$\begin{aligned} K_0(C(M\#N)) &\cong \mathbb{Z} \oplus [(K_0(C(M))/\mathbb{Z}[1]) \oplus (K_0(C(N))/\mathbb{Z}[1])], \\ K_1(C(M\#N)) &\cong [K_1(C(M)) \oplus K_1(C(N))]/\partial\mathbb{Z}[p], \end{aligned}$$

which follows from exactness of the six-term diagram of K-theory groups above in the first of the proof, where $K_0(C(S^{2k})) \cong \mathbb{Z}[1] \oplus \mathbb{Z}[p] \cong \mathbb{Z}^2$ with $n = 2k+1$, for which $\partial[1] = [0]$ the zero class, but $\partial[p] \neq [0]$. This follows by considering $M\#N \cong (M\#N)\#S^n$ and by the several cases and the general case in Example 3.8 below. We indeed have the following commutative diagram:

$$\begin{tikzcd} C(M\#N) \arrow[r, "q"] \arrow[d, "f"'] & C(S^{n-1}) \arrow[r] \arrow[d, "g"] & 0 \arrow[d, equal] \\ C((M\#N)\#S^n) \arrow[r, "q"] & C(S^{n-1}) \arrow[r] & 0 \end{tikzcd}$$

with f the isomorphism induced from the homeomorphism and g the isomorphism to make the diagram commutative, so that the following diagram commutes

$$\begin{tikzcd} K_0(C(M\#N)) \arrow[r, "q_*"] \arrow[d, "f_*"'] & K_0(C(S^{n-1})) \arrow[d, "g_*"] \\ K_0(C((M\#N)\#S^n)) \arrow[r, "q_*"] & K_0(C(S^{n-1})) \end{tikzcd}$$

with f_* and g_* isomorphisms induced from f and g respectively. □

Example 3.8. There is the following short exact sequence of C^*-algebras:

$$0 \to C_0((S^n \setminus D^n) \sqcup (S^n \setminus D^n)) \xrightarrow{i} C(S^n\#S^n) \xrightarrow{q} C(S^{n-1}) \to 0$$

with $S^n \setminus D^n \approx \mathbb{R}^n$ and $S^n\#S^n \approx S^n$. The six-term exact sequence of K-theory groups, associated, becomes, if n is even,

$$\begin{tikzcd} \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "i_*"] & \mathbb{Z}^2 \arrow[r, "q_*"] & \mathbb{Z} \arrow[d, "\partial=0"] \\ \mathbb{Z}[z] \arrow[u, "\partial"] & 0 \arrow[l, "q_*"'] & 0 \oplus 0 \arrow[l, "i_*"'] \end{tikzcd}$$

and if n is odd,

$$\begin{tikzcd} 0 \oplus 0 \arrow[r, "i_*"] & \mathbb{Z} \arrow[r, "q_*"] & \mathbb{Z}[1] \oplus \mathbb{Z}[p] \arrow[d, "\partial"] \\ 0 \arrow[u, "\partial=0"] & \mathbb{Z} \arrow[l, "q_*"'] & \mathbb{Z} \oplus \mathbb{Z}. \arrow[l, "i_*"'] \end{tikzcd}$$

Therefore, if n is even, then

$$\begin{aligned} K_0(C(S^n\#S^n)) &\cong \mathbb{Z} \oplus \{[(K_0(C(S^n))/\mathbb{Z}) \oplus (K_0(C(S^n))/\mathbb{Z})]/\partial\mathbb{Z}[z]\} \\ &\cong \mathbb{Z} \oplus \{[\mathbb{Z} \oplus \mathbb{Z}]/\partial\mathbb{Z}[z]\} \cong \mathbb{Z}^2, \\ K_1(C(S^n\#S^n)) &\cong K_1(C(S^n)) \oplus K_1(C(S^n)) \oplus \ker(\partial) \cong 0. \end{aligned}$$

and if n is odd, then

$$\begin{aligned} K_0(C(S^n\#S^n)) &\cong \mathbb{Z}\oplus[(K_0(C(S^n))/\mathbb{Z})\oplus(K_0(C(S^n))/\mathbb{Z})]\oplus(\ker(\partial)/\mathbb{Z}[1]) \\ &\cong \mathbb{Z}\oplus[0\oplus 0]\oplus 0\cong\mathbb{Z}, \\ K_1(C(S^n\#S^n)) &\cong [K_1(C(S^n)\oplus K_1(C(S^n))]/\partial\mathbb{Z}[p] \\ &\cong [\mathbb{Z}\oplus\mathbb{Z}]/\partial\mathbb{Z}[p]\cong\mathbb{Z}. \end{aligned}$$

Note that the diagram above when n is odd involves no torsion.

Next, there is the following short exact sequence of C^*-algebras:

$$0\to C_0((\mathbb{T}^n\setminus D^n)\sqcup(S^n\setminus D^n)) \xrightarrow{\ i\ } C(\mathbb{T}^n\#S^n) \xrightarrow{\ q\ } C(S^{n-1})\to 0$$

with $\mathbb{T}^n\#S^n\approx\mathbb{T}^n$. The six-term exact sequence of K-theory groups, associated, becomes, if n is even,

$$\begin{array}{ccccc} K_0(C_0(\mathbb{T}^n\setminus D^n))\oplus\mathbb{Z} & \xrightarrow{i_*} & \mathbb{Z}^{2^{n-1}} & \xrightarrow{q_*} & \mathbb{Z} \\ \partial\uparrow & & & & \downarrow\partial=0 \\ \mathbb{Z}[z] & \xleftarrow{q_*} & \mathbb{Z}^{2^{n-1}} & \xleftarrow{i_*} & K_1(C_0(\mathbb{T}^n\setminus D^n))\oplus 0 \end{array}$$

and if n is odd,

$$\begin{array}{ccccc} K_0(C_0(\mathbb{T}^n\setminus D^n))\oplus 0 & \xrightarrow{i_*} & \mathbb{Z}^{2^{n-1}} & \xrightarrow{q_*} & \mathbb{Z}[1]\oplus\mathbb{Z}[p] \\ \partial=0\uparrow & & & & \downarrow\partial \\ 0 & \xleftarrow{q_*} & \mathbb{Z}^{2^{n-1}} & \xleftarrow{i_*} & K_1(C_0(\mathbb{T}^n\setminus D^n))\oplus\mathbb{Z}. \end{array}$$

Moreover, the exact sequence $0\to C_0(\mathbb{T}^n\setminus D^n)\to C(\mathbb{T}^n)\to C(D^n)\to 0$ implies

$$\begin{array}{ccccc} K_0(C_0(\mathbb{T}^n\setminus D^n)) & \longrightarrow & \mathbb{Z}^{2^{n-1}} & \longrightarrow & \mathbb{Z} \\ \partial=0\uparrow & & & & \downarrow\partial=0 \\ 0 & \longleftarrow & \mathbb{Z}^{2^{n-1}} & \longleftarrow & K_1(C_0(\mathbb{T}^n\setminus D^n)) \end{array}$$

so that $K_0(C_0(\mathbb{T}^n\setminus D^n))\cong\mathbb{Z}^{2^{n-1}-1}$ and $K_1(C_0(\mathbb{T}^n\setminus D^n))\cong\mathbb{Z}^{2^{n-1}}$. Therefore, the boundary map ∂ on the left in the first diagram in this case is nonzero when n is even, and if n is even, then

$$\begin{aligned} K_0(C(\mathbb{T}^n\#S^n)) &\cong \mathbb{Z}\oplus\{[(K_0(C(\mathbb{T}^n))/\mathbb{Z})\oplus(K_0(C(S^n))/\mathbb{Z})]\}/\partial\mathbb{Z}[z] \\ &\cong \mathbb{Z}\oplus\{[\mathbb{Z}^{2^{n-1}-1}\oplus\mathbb{Z}]/\partial\mathbb{Z}[z]\}\cong\mathbb{Z}^{2^{n-1}}, \\ K_1(C(\mathbb{T}^n\#S^n)) &\cong K_1(C(\mathbb{T}^n))\oplus K_1(C(S^n))\oplus\ker(\partial)\cong\mathbb{Z}^{2^{n-1}}\oplus 0\oplus 0, \end{aligned}$$

and if n is odd, then

$$\begin{aligned} K_0(C(\mathbb{T}^n\#S^n)) &\cong \mathbb{Z}\oplus[(K_0(C(\mathbb{T}^n))/\mathbb{Z})\oplus(K_0(C(S^n))/\mathbb{Z})] \\ &\cong \mathbb{Z}\oplus[\mathbb{Z}^{2^{n-1}-1}\oplus 0]\cong\mathbb{Z}^{2^{n-1}}, \\ K_1(C(\mathbb{T}^n\#S^n)) &\cong [K_1(C(\mathbb{T}^n))\oplus K_1(C(S^n))]/\partial\mathbb{Z}[p] \\ &\cong [\mathbb{Z}^{2^{n-1}}\oplus\mathbb{Z}]/\partial\mathbb{Z}[p]\cong\mathbb{Z}^{2^{n-1}}. \end{aligned}$$

Note that $\ker(\partial) = \mathbb{Z}[1]$ follows from that $\partial\mathbb{Z}[p] \cong \mathbb{Z}$ by the diagram.

Let M be an n-dimensional closed topological manifold. There is the following short exact sequence of C^*-algebras:

$$0 \to C_0((M \setminus D^n) \sqcup (S^n \setminus D^n)) \xrightarrow{\ i\ } C(M\#S^n) \xrightarrow{\ q\ } C(S^{n-1}) \to 0$$

with $S^n \setminus D^n \approx \mathbb{R}^n$ and $M\#S^n \approx M$. The six-term exact sequence of K-theory groups, associated, becomes, if n is even,

$$\begin{tikzcd}
K_0(C_0(M \setminus D^n)) \oplus \mathbb{Z} \arrow[r, "i_*"] & K_0(C(M\#S^n)) \arrow[r, "q_*"] & \mathbb{Z} \arrow[d, "\partial=0"] \\
\mathbb{Z}[z] \arrow[u, "\partial"] & K_1(C(M\#S^n)) \arrow[l, "q_*"'] & K_1(C_0(M \setminus D^n)) \oplus 0 \arrow[l, "i_*"']
\end{tikzcd}$$

and if n is odd,

$$\begin{tikzcd}
K_0(C_0(M \setminus D^n)) \oplus 0 \arrow[r, "i_*"] & K_0(C(M\#S^n)) \arrow[r, "q_*"] & \mathbb{Z}[1] \oplus \mathbb{Z}[p] \arrow[d, "\partial"] \\
0 \arrow[u, "\partial=0"] & K_1(C(M\#S^n)) \arrow[l, "q_*"'] & K_1(C_0(M \setminus D^n)) \oplus \mathbb{Z}. \arrow[l, "i_*"']
\end{tikzcd}$$

Moreover, $K_1(C_0(M \setminus D^n)) \cong K_1(C_0(M \setminus D^n)^+) \cong K_1(C(M))$. Therefore, the map q_* on K_1 is zero, when n is even, so that $\ker(\partial) = 0$. Also, $K_0(C(M)) \cong K_0(C_0(M \setminus D)^+) \cong K_0(C_0(M \setminus D)) \oplus \mathbb{Z}$. Thus, if n is even, then

$$\begin{aligned}
K_0(C(M\#S^n)) &\cong \mathbb{Z} \oplus \{[K_0(C_0(M \setminus D^n)) \oplus \mathbb{Z}]/\partial\mathbb{Z}[z]\} \\
&\cong \mathbb{Z} \oplus K_0(C_0(M \setminus D^n)) \oplus 0 \cong K_0(C(M)), \\
K_1(C(M\#S^n)) &\cong K_1(C(M)) \oplus K_1(C(S^n)) \oplus \ker(\partial) \cong K_1(C(M)),
\end{aligned}$$

and if n is odd, then

$$\begin{aligned}
K_0(C(M\#S^n)) &\cong \mathbb{Z} \oplus [K_0(C_0(M \setminus D^n)) \oplus 0] \\
&\cong \mathbb{Z} \oplus K_0(C_0(M \setminus D^n)) \cong K_0(C(M)), \\
K_1(C(M\#S^n)) &\cong [K_1(C_0(M \setminus D^n)) \oplus \mathbb{Z}]/\partial\mathbb{Z}[p] \cong K_1(C(M)).
\end{aligned}$$

Note that $\ker(\partial) = \mathbb{Z}[1]$ follows from that $\partial\mathbb{Z}[p] \cong \mathbb{Z}$ by the diagram.

Corollary 3.9. *The formula obtained in Theorem* 3.1 *follows from Theorem* 3.7.

Proof. Note that each quotient by $\mathbb{Z}[1]$, $\partial\mathbb{Z}[z]$ together with $\ker(\partial)$, or $\partial\mathbb{Z}[p]$ in Theorem 3.7, respectively, corresponds to one rank lowering or one rank raising of the free ranks of those K_0-groups or K_1-groups, respectively. Hence, if n is even,

$$\chi(C(M\#N)) = 1 + \chi(C(M)) + \chi(C(N)) - 3 = \chi(C(M)) + \chi(C(N)) - 2,$$

and if n is odd,

$$\chi(C(M\#N)) = 1 + \chi(C(M)) + \chi(C(N)) - 2 - (-1) = \chi(C(M)) + \chi(C(N)).$$

□

Corollary 3.10. *Let M_i $(1 \leq i \leq l)$ be n-dimensional closed topological manifolds and $\#_{i=1}^{l} M_i$ be their connected sum. If n is even, then inductively,*

$$
\begin{aligned}
&K_0(C(\#_{i=1}^{l} M_i)) \\
&\cong \mathbb{Z} \oplus \{[(K_0(C(\#_{i=1}^{l-1} M_i))/\mathbb{Z}[1]) \oplus (K_0(C(M_l))/\mathbb{Z}[1])]/\partial_{l-1}\mathbb{Z}[z]\} \\
&\cong \mathbb{Z} \oplus \{[((\cdots(\mathbb{Z} \oplus \{[(K_0(C(M_1))/\mathbb{Z}[1]) \oplus (K_0(C(M_2))/\mathbb{Z}[1])]/\partial_1\mathbb{Z}[z]\}) \\
&\quad \cdots)/\mathbb{Z}[1]) \oplus (K_0(C(M_l))/\mathbb{Z}[1])]/\partial_{l-1}\mathbb{Z}[z]\}, \\
&K_1(C(\#_{i=1}^{l} M_i)) \cong [\oplus_{i=1}^{l} K_1(C(M_i))] \oplus [\oplus_{i=1}^{l-1} \ker(\partial_i)],
\end{aligned}
$$

and if n is odd, then inductively,

$$
\begin{aligned}
&K_0(C(\#_{i=1}^{l} M_i)) \\
&\cong \mathbb{Z} \oplus [(K_0(C(\#_{i=1}^{l-1} M_i))/\mathbb{Z}[1]) \oplus (K_0(C(M_l))/\mathbb{Z}[1])] \\
&\cong \mathbb{Z} \oplus [((\cdots(\mathbb{Z} \oplus [(K_0(C(M_1))/\mathbb{Z}[1]) \oplus (K_0(C(M_2))/\mathbb{Z}[1])]) \\
&\quad \cdots)/\mathbb{Z}[1]) \oplus (K_0(C(M_l))/\mathbb{Z}[1])], \\
&K_1(C(\#_{i=1}^{l} M_i)) \cong [K_1(C(\#_{i=1}^{l-1} M_i)) \oplus K_1(C(M_l))]/\partial_{l-1}\mathbb{Z}[p] \\
&\cong [((\cdots([K_1(C(M_1)) \oplus K_1(C(M_2))]/\partial_1\mathbb{Z}[p]) \\
&\quad \cdots)/\partial_{l-2}\mathbb{Z}[p]) \oplus K_1(C(M_l))]/\partial_{l-1}\mathbb{Z}[p],
\end{aligned}
$$

where each $[1]$ means the K_0-class of the unit 1 of $C(M_i)$, and $[z] \in K_1(C(S^{n-1})) \cong \mathbb{Z}$ when n is even, and $[p] \in K_1(C(S^{n-1})) = \mathbb{Z}[1] \oplus \mathbb{Z}[p] \cong \mathbb{Z}^2$ when n is odd, and each $\partial_i = \partial$ is the boundary map in each step in induction.

Corollary 3.11. *That Corollary 3.2 follows from this Corollary 3.10.*

Proof. Note that each quotient by $\mathbb{Z}[1]$, $\partial_i\mathbb{Z}[z]$ together with $\ker(\partial_i)$, or $\partial_i\mathbb{Z}[p]$ in Corollary 3.10, repsectively, corresponds to one rank lowering or one rank raising of the free ranks of those either K_0-groups or K_1-groups, respectively. Hence, if n is even, then inductively,

$$
\begin{aligned}
\chi(C(\#_{i=1}^{l} M_i)) &= 1 + \chi(C(\#_{i=1}^{l-1} M_i)) + \chi(C(M_l)) - 3 \\
&= \cdots = \sum_{i=1}^{l} \chi(C(M_i)) - 2(l-1),
\end{aligned}
$$

and if n is odd, then inductively,

$$
\begin{aligned}
\chi(C(\#_{i=1}^{l} M_i)) &= 1 + \chi(C(\#_{i=1}^{l-1} M_i)) + \chi(C(M_l)) - 2 + 1 \\
&= \cdots = \sum_{i=1}^{l} \chi(C(M_i)).
\end{aligned}
$$

$\square$

Example 3.12. Since $K_j(C(\mathbb{T}^n)) \cong \mathbb{Z}^{2^{n-1}}$ $(j = 0, 1)$, it is obtained by Theorem 3.7 that if n is even, then

$$
\begin{aligned}
K_0(C(\mathbb{T}^n \# \mathbb{T}^n)) &\cong \mathbb{Z} \oplus \{[(\mathbb{Z}^{2^{n-1}}/\mathbb{Z}) \oplus (\mathbb{Z}^{2^{n-1}}/\mathbb{Z})]/\partial\mathbb{Z}[z]\} \\
&\cong \mathbb{Z} \oplus \{[\mathbb{Z}^{2^{n-1}-1} \oplus \mathbb{Z}^{2^{n-1}-1}]/\partial\mathbb{Z}[z]\} \cong \mathbb{Z}^{2^n-2}, \\
K_1(C(\mathbb{T}^n \# \mathbb{T}^n)) &\cong \mathbb{Z}^{2^{n-1}} \oplus \mathbb{Z}^{2^{n-1}} \oplus \ker(\partial) \cong \mathbb{Z}^{2^n},
\end{aligned}
$$

with $\partial\mathbb{Z}[z] \cong \mathbb{Z}$ since a finitely many times direct sum of $\mathbb{Z}$ is torsion free, so that $\ker(\partial) = 0$, and as well, the K_0-group above can not involve torsion because its quotient by q_* is $\mathbb{Z}$, and if n is odd, then

$$\begin{aligned} K_0(C(\mathbb{T}^n \# \mathbb{T}^n)) &\cong \mathbb{Z} \oplus [(\mathbb{Z}^{2^{n-1}}/\mathbb{Z}) \oplus (\mathbb{Z}^{2^{n-1}}/\mathbb{Z})] \\ &\cong \mathbb{Z} \oplus [\mathbb{Z}^{2^{n-1}-1} \oplus \mathbb{Z}^{2^{n-1}-1}] \cong \mathbb{Z}^{2^n-1}, \\ K_1(C(\mathbb{T}^n \# \mathbb{T}^n)) &\cong [\mathbb{Z}^{2^{n-1}} \oplus \mathbb{Z}^{2^{n-1}}]/\partial\mathbb{Z}[p] \cong \mathbb{Z}^{2^n-1}. \end{aligned}$$

Hence,

$$\chi(C(\mathbb{T}^n \# \mathbb{T}^n)) = \begin{cases} (2^n - 2) - 2^n = -2 & \text{if } n \text{ is even}, \\ (2^n - 1) - (2^n - 1) = 0 & \text{if } n \text{ is odd}. \end{cases}$$

Moreover, it is obtained by Corollary 3.10 that if n is even, then

$$\begin{aligned} K_0(C(\#^l \mathbb{T}^n)) &\cong \mathbb{Z} \oplus \{[(\cdots([\mathbb{Z}^{2^{n-1}-1} \oplus \mathbb{Z}^{2^{n-1}-1}]/\partial_1\mathbb{Z}[z])\cdots) \oplus \mathbb{Z}^{2^{n-1}-1}]/\partial_{l-1}\mathbb{Z}[z]\} \\ &\cong \mathbb{Z}^{1+l(2^{n-1}-1)-(l-1)} \cong \mathbb{Z}^{2+l(2^{n-1}-2)}, \\ K_1(C(\#^l \mathbb{T}^n)) &\cong [\oplus_{i=1}^{l} \mathbb{Z}^{2^{n-1}}] \oplus [\oplus_{i=1}^{l-1} \ker(\partial_i)] \cong \mathbb{Z}^{l2^{n-1}}, \end{aligned}$$

and if n is odd, then

$$\begin{aligned} K_0(C(\#^l \mathbb{T}^n)) &\cong \mathbb{Z} \oplus [(\cdots[\mathbb{Z}^{2^{n-1}-1} \oplus \mathbb{Z}^{2^{n-1}-1}]\cdots) \oplus \mathbb{Z}^{2^{n-1}-1}] \\ &\cong \mathbb{Z}^{l2^{n-1}-l+1}, \\ K_1(C(\#^l \mathbb{T}^n)) &\cong [(\cdots([\mathbb{Z}^{2^{n-1}} \oplus \mathbb{Z}^{2^{n-1}}]/\partial_1\mathbb{Z}[p])\cdots) \oplus \mathbb{Z}^{2^{n-1}}]/\partial_{l-1}\mathbb{Z}[p] \cong \mathbb{Z}^{l2^{n-1}-(l-1)}. \end{aligned}$$

Hence,

$$\chi(C(\#^l \mathbb{T}^n)) = \begin{cases} [2 + l(2^{n-1} - 2)] - l2^{n-1} = 2 - 2l & \text{if } n \text{ is even}, \\ [l2^{n-1} - l + 1] - (l2^{n-1} - (l-1)) = 0 & \text{if } n \text{ is odd}. \end{cases}$$

As for the real n-dimensional projective space P^n, the six-term exact sequence of K-theory groups, associated to the short exact sequence of $C(P^n)$ in Example 3.3, becomes

$$\begin{tikzcd} K_0(C_0((M_b^n)^\circ)) \arrow[r] & K_0(C(P^n)) \arrow[r] & \mathbb{Z} \arrow[d, "\partial=0"] \\ 0 \arrow[u, "\partial=0"] & K_1(C(P^n)) \arrow[l] & K_1(C_0((M_b^n)^\circ)) \arrow[l] \end{tikzcd}$$

so that

$$\begin{aligned} K_0(C(P^n)) &\cong \mathbb{Z} \oplus K_0(C_0((M_b^n)^\circ)), \\ K_1(C(P^n)) &\cong K_1(C_0((M_b^n)^\circ)). \end{aligned}$$

Indeed, P^n is viewed as the one-point compactification of the interior $(M_b^n)^\circ$ of the n-dimensional Möbius band M_b^n. Moreover, the six-term exact sequence of K-theory groups, associated to the short exact sequence of $C_0((M_b^n)^\circ)$ in Example 3.3, becomes

$$\begin{tikzcd} K_1(C_0((P^{n-1})^-)) \arrow[r] & K_0(C_0((M_b^n)^\circ)) \arrow[r] & 0 \arrow[d, "\partial"] \\ \mathbb{Z} \arrow[u, "\partial"] & K_1(C_0((M_b^n)^\circ)) \arrow[l] & K_0(C_0((P^{n-1})^-)) \arrow[l] \end{tikzcd}$$

and

$$\begin{aligned}&K_1(C_0((P^{n-1})^-)) \cong K_1(C_0((P^{n-1})^-)^+) \cong K_1(C(P^{n-1})) \quad \text{and}\\&K_0(C_0((P^{n-1})^-)) \cong K_0(C(P^{n-1}))/\mathbb{Z}[1].\end{aligned}$$

We now determine the K-theory groups of $C(P^n)$ inductively by the diagram above, as follows. Since $K_0(C(P^2)) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ and $K_1(C(P^2)) \cong 0$,

$$\begin{aligned}K_0(C(P^3)) &\cong \mathbb{Z} \oplus 0 \cong \mathbb{Z},\\K_1(C(P^3)) &\cong \mathbb{Z} \oplus \mathbb{Z}_2,\end{aligned}$$

so that it follows that

$$\begin{aligned}K_0(C(P^4)) &\cong \mathbb{Z} \oplus \mathbb{Z}_2,\\K_1(C(P^4)) &\cong 0\end{aligned}$$

and hence we obtain inductively that for $k \geq 1$,

$$\begin{cases}K_0(C(P^{2k})) \cong \mathbb{Z} \oplus \mathbb{Z}_2,\\K_1(C(P^{2k})) \cong 0,\end{cases}$$

and

$$\begin{cases}K_0(C(P^{2k+1})) \cong \mathbb{Z},\\K_1(C(P^{2k+1})) \cong \mathbb{Z} \oplus \mathbb{Z}_2.\end{cases}$$

(Note that this result is compatible with that in homology mentioned in Example 3.3. In fact, the compatiblity does imply it.) It is deduced that

$$\chi(C(P^n)) = 1 - \chi(C(P^{n-1})) \quad (n \geq 1)$$

with $\chi(C(P^2)) = 1$ and $\chi(C(P^1)) = \chi(C(S^1)) = 0$ and $C(P^0) \cong \mathbb{C}$. Moreover, the equation obtained above is converted to

$$a_n - \frac{1}{2} = -\left(a_{n-1} - \frac{1}{2}\right)$$

$(n \geq 1)$ with $a_n = \chi(C(P^n))$, so that we obtain the same formula for $\chi(C(P^n))$ as in Example 3.3. Let

$$\alpha_n = \mathrm{rank}_{\mathbb{Z}} K_0(C(P^n)) \text{ and } \beta_n = \mathrm{rank}_{\mathbb{Z}} K_1(C(P^n)).$$

If n is even ($n \geq 2$), then

$$\alpha_n = 1 \quad \text{and} \quad \beta_n = 0,$$

and if n is odd ($n \geq 3$), then

$$\alpha_n = 1 \quad \text{and} \quad \beta_n = 1.$$

It is obtained by Theorem 3.7 that if n is even, then

$$\begin{aligned}K_0(C(P^n \# P^n)) &\cong \mathbb{Z} \oplus \{[(K_0(C(P^n))/\mathbb{Z}[1]) \oplus (K_0(C(P^n))/\mathbb{Z}[1])]/\partial\mathbb{Z}[z]\}\\&\cong \mathbb{Z} \oplus \{[\mathbb{Z}_2 \oplus \mathbb{Z}_2]/\partial\mathbb{Z}[z]\}\\&\cong \mathbb{Z} \oplus [\oplus^2 \mathbb{Z}_2],\\K_1(C(P^n \# P^n)) &\cong K_1(C(P^n)) \oplus K_1(C(P^n)) \oplus \ker(\partial)\\&\cong 0 \oplus 0 \oplus \mathbb{Z} \cong \mathbb{Z},\end{aligned}$$

and if n is odd, then

$$\begin{aligned}
K_0(C(P^n\#P^n)) &\cong \mathbb{Z}\oplus[(K_0(C(P^n))/\mathbb{Z}[1])\oplus(K_0(C(P^n))/\mathbb{Z}[1])] \\
&\cong \mathbb{Z}\oplus[0\oplus 0]\oplus(\ker(\partial)/\mathbb{Z}[1])\cong\mathbb{Z}, \\
K_1(C(P^n\#P^n)) &\cong [K_1(C(P^n))\oplus K_1(C(P^n))]/\partial\mathbb{Z}[p] \\
&\cong [(\mathbb{Z}\oplus\mathbb{Z}_2)\oplus(\mathbb{Z}\oplus\mathbb{Z}_2)]/\partial\mathbb{Z}[p] \\
&\cong [\mathbb{Z}^2\oplus\mathbb{Z}_2^2]/\partial\mathbb{Z}[p]\cong\mathbb{Z}\oplus\mathbb{Z}_2^2.
\end{aligned}$$

Hence, if n is even, then $\chi(C(P^n\#P^n)) = 1-1 = 0$, and if n is odd, then $\chi(C(P^n\#P^n)) = 1-(2-1) = 0$.

Moreover, it is obtained by Corollary 3.10 that if n is even, then

$$\begin{aligned}
K_0(C(\#^l P^n)) &\cong \mathbb{Z}\oplus\{[((\cdots\mathbb{Z}\oplus([(K_0(C(P^n))/\mathbb{Z}[1])\oplus(K_0(C(P^n))/\mathbb{Z}[1])]/\partial_1\mathbb{Z}[z]) \\
&\qquad\cdots)/\mathbb{Z}[1])\oplus(K_0(C(P^n))/\mathbb{Z}[1])]/\partial_{l-1}\mathbb{Z}[z]\} \\
&\cong \mathbb{Z}\oplus\{[((\cdots(\mathbb{Z}\oplus([\mathbb{Z}_2\oplus\mathbb{Z}_2]/\partial_1\mathbb{Z}[z]) \\
&\qquad\cdots)/\mathbb{Z}[1])\oplus\mathbb{Z}_2]/\partial_{l-1}\mathbb{Z}[z]\}\cong\mathbb{Z}\oplus(\mathbb{Z}_2)^l, \\
K_1(C(\#^l P^n)) &\cong [\oplus_{i=1}^l K_1(C(P^n))]\oplus[\oplus_{i=1}^{l-1}\ker(\partial_i)] \\
&\cong 0\oplus[\oplus_{i=1}^{l-1}\ker(\partial_i)]\cong\mathbb{Z}^{l-1},
\end{aligned}$$

and if n is odd, then

$$\begin{aligned}
K_0(C(\#^l P^n)) &\cong \mathbb{Z}\oplus[((\cdots(\mathbb{Z}\oplus[(K_0(C(P^n))/\mathbb{Z}[1])\oplus(K_0(C(P^n))/\mathbb{Z}[1])] \\
&\qquad\cdots)/\mathbb{Z}[1])\oplus(K_0(C(P^n))/\mathbb{Z}[1])] \\
&\cong \mathbb{Z}\oplus[(\cdots(0\oplus[0\oplus 0])\cdots)\oplus 0]\cong\mathbb{Z}, \\
K_1(C(\#^l P^n)) &\cong [(\cdots[K_1(C(P^n))\oplus K_1(C(P^n))]/\partial_1\mathbb{Z}[p] \\
&\qquad\cdots)\oplus K_1(C(P^n))]/\partial_{l-1}\mathbb{Z}[p] \\
&\cong [(\cdots[(\mathbb{Z}\oplus\mathbb{Z}_2)\oplus(\mathbb{Z}\oplus\mathbb{Z}_2)]/\partial_1\mathbb{Z}[p] \\
&\qquad\cdots)\oplus(\mathbb{Z}\oplus\mathbb{Z}_2)]/\partial_{l-1}\mathbb{Z}[p]\cong\mathbb{Z}\oplus\mathbb{Z}_2^l.
\end{aligned}$$

Hence, if n is even, then $\chi(C(\#^l P^n)) = 1-(l-1) = 2-l$, and if n is odd, then $\chi(C(\#^l P^n)) = 1-(l-(l-1)) = 0$.

Table 6: Classification for closed topological manifolds with dimension even

K_0 rank	Orientable	Non-orientable
$2+l(2^{2n-1}-2)$	$\#^l\mathbb{T}^{2n}$	No
$2^{2n}-2$	$\mathbb{T}^{2n}\#\mathbb{T}^{2n}$	No
2^{2n-1}	$\mathbb{T}^{2n}$	No
8	$\mathbb{T}^4\#\mathbb{T}^4$	No
2	$S^{2n}\approx\#^l S^{2n}$, S^0 $\#^n\mathbb{T}^2\approx T(n)$	No
1	No	P^0, P^{2n}, $\#^n P^2\approx P(n)$ $P^{2n}\#P^{2n}$, $\#^l P^{2n}$

Note that $S^0 = \{-1,1\}$ and P^0 is the one point set as the quotient of S^0.

It follows from the Table 6 that:

Corollary 3.13. *Let n be a natural number with $n \geq 1$. The rank of K_0-groups for C^*-algebras can not classify homeomorphism classes of connected sums of the 2-dimensional, orientable closed topological manifolds $\mathbb{T}^2$ and S^2.*

But, if $n \geq 2$, it does classify homeomorphism classes of connected sums of the $2n$-dimensional, orientable closed topological manifolds $\mathbb{T}^{2n}$ and S^{2n}.

And the rank of K_0-groups can not classify homeomorphism classes of connected sums of the $2n$-dimensional, non-orientable closed topological manifold P^{2n} $(n \geq 1)$.

Table 7: Classification for closed topological manifolds with dimension even

K_1 rank	Orientable	Non-orientable
$l2^{2n-1}$	$\#^l\mathbb{T}^{2n}$	$\#^{l2^{2n-1}+1}P^2$, $\#^{l2^{2n-1}+1}P^{2k}$
$l2n$	$\#^{ln}\mathbb{T}^2$	$\#^{l2n+1}P^2$, $\#^{l2n+1}P^{2k}$
2^{2n}	$\mathbb{T}^{2n}\#\mathbb{T}^{2n}$	$\#^{2^{2n}+1}P^2$, $\#^{2^{2n}+1}P^{2k}$
2^{2n-1}	$\mathbb{T}^{2n}$	$\#^{2^{2n-1}+1}P^2$, $\#^{2^{2n-1}+1}P^{2k}$
$2l$	$\#^l\mathbb{T}^2 \approx T(n)$	$\#^{2l+1}P^2$, $\#^{2l+1}P^{2n}$
$2l-1$	No	$\#^{2l}P^2$, $\#^{2l}P^{2n}$
4	$\mathbb{T}^2\#\mathbb{T}^2 \approx T(2)$	$\#^5P^2$, $\#^5P^{2n}$
3	No	$\#^4P^2$, $\#^4P^{2n}$
2	$\mathbb{T}^2 \approx T(1)$	$\#^3P^2$, $\#^3P^{2n}$
1	No	$P^2\#P^2 \approx P(2) \approx K^2$ $P^{2n}\#P^{2n}$
0	$S^{2n} \approx \#^lS^{2n}$, S^0	P^0, $P^2 \approx P(1)$, P^{2n}

It follows from the Table 7 that:

Corollary 3.14. *Let n be a natural number with $n \geq 1$. The rank of K_1-groups for C^*-algebras does classsify homeomorphism classes of connected sums of the $2n$-dimensional, orientable closed topological manifolds $\mathbb{T}^{2n}$ and S^{2n}, and does classify homeomorphism classes of connected sums of the $2n$-dimensional, non-orientable closed topological manifold P^{2n}.*

Table 8: Classification for closed topological manifolds with dimension odd

K_0 rank	Orientable	Orientable
$1+l(2^{2n}-1)$	$\#^l\mathbb{T}^{2n+1}$	No
$2^{2n+1}-1$	$\mathbb{T}^{2n+1}\#\mathbb{T}^{2n+1}$	No
2^{2n}	$\mathbb{T}^{2n+1}$	No
4	$\mathbb{T}^3$	No
1	$S^1 = \mathbb{T}$, $S^{2n+1} \approx \#^lS^{2n+1}$	$S^1 \approx P^1$, P^{2n+1}, $\#^lP^{2n+1}$

It follows from the Table 8 that:

Corollary 3.15. *Let n be a natural number with $n \geq 1$. The rank of K_0-groups for C^*-algebras does classify homeomorphism classes of connected sums of the $(2n+1)$-dimensional, orientable closed topological manifolds $\mathbb{T}^{2n+1}$ and S^{2n+1}, but the rank of K_0-groups does not classify homeomorphism classes of connected sums of $(2n+1)$-dimensional, orientable closed topological manifold P^{2n+1}.*

Table 9: Classification for closed topological manifolds with dimension odd

K_1 rank	Orientable	Orientable
$1+l(2^{2n}-1)$	$\#^l\mathbb{T}^{2n+1}$	No
$2^{2n+1}-1$	$\mathbb{T}^{2n+1}\#\mathbb{T}^{2n+1}$	No
2^{2n}	$\mathbb{T}^{2n+1}$	No
4	$\mathbb{T}^3$	No
1	$S^1=\mathbb{T}$, $S^{2n+1}\approx\#^lS^{2n+1}$	$S^1\approx P^1$, P^{2n+1}, $\#^lP^{2n+1}$

Note that the list of ranks and items in the Table 8 is exactly the same as that in the Table 9. It follows from the Table 9 that:

Corollary 3.16. *Let n be a natural number with $n\geq 1$. The rank of K_1-groups for C^*-algebras does classsify homeomorphism classes of connected sums of the $(2n+1)$-dimensional, orientable closed topological manifolds $\mathbb{T}^{2n+1}$ and S^{2n+1}, but does not classify homeomorphism classes of connected sums of the $(2n+1)$-dimensional, orientable closed topological manifold P^{2n+1}.*

As in the examples considered so far, consequently, one can say that

Corollary 3.17. *The free ranks of K-theory groups K_0 or K_1 of C^*-algebras are more classifiable or the same level invariants for closed topological manifolds than or as the Euler characteristic of C^*-algebras, respectively.*

But the Euler characteristic of C^*-algebras are more easily computable and more beautiful numerically than the K-theory group ranks of C^*-algebras.

Remark. It follows from our K-theory group formulae obtained so far that the K-theory groups written as quotients in some examples and cases may have torsion in general (but may not in some corresponding cases). But without knowing its information, we could determine the K-theory group ranks and the Euler characteristic for C^*-algebras. As a question, it should be of interest to understand more about the K-theory torsion (or torsion freeness), which in fact could be known from more about the boundary maps.

Table 10: The K-theory groups for the C^*-algebras of topological manifolds

C^*-algebra	K_0-group	K_1-group
$C(S^{2n})$	$\mathbb{Z}^2$	0
$C(S^{2n+1})$	$\mathbb{Z}$	$\mathbb{Z}$
$C(\mathbb{T}^n)$	$\mathbb{Z}^{2^{n-1}}$	$\mathbb{Z}^{2^{n-1}}$
$C(\#^l\mathbb{T}^{2n})$	$\mathbb{Z}^{2+l(2^{2n-1}-2)}$	$\mathbb{Z}^{l2^{2n-1}}$
$C(\#^l\mathbb{T}^{2n+1})$	$\mathbb{Z}^{1+l(2^{2n}-1)}$	$\mathbb{Z}^{1+l(2^{2n}-1)}$
$C(P^{2n})$	$\mathbb{Z}\oplus\mathbb{Z}_2$	0
$C(P^{2n+1})$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}_2$
$C(\#^lP^{2n})$	$\mathbb{Z}\oplus\mathbb{Z}_2^l$	$\mathbb{Z}^{l-1}$
$C(\#^lP^{2n+1})$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}_2^l$

It follows from the Table 10 that

Corollary 3.18. *The torsion rank $t_0(\mathfrak{A})$ of K_0-groups of C^*-algebras $\mathfrak{A}$ with respect to $\mathbb{Z}_2$ (or any other torsion groups in general) classifies homeomorphism classes of connected sums of the $2n$-dimensional, non-orientable closed topological manifold P^{2n}.*

The torsion rank $t_1(\mathfrak{A})$ of K_1-groups of C^*-algebras $\mathfrak{A}$ with respect to $\mathbb{Z}_2$ (or any other torsion groups in general) classifies homeomorphism classes of connected sums of the $(2n+1)$-dimensional, orientable closed topological manifold P^{2n+1}.

As well, the torsion freeness for both K_0 and K_1-groups of C^*-algebras distinguish homeomorphism classes of connected sums of spheres S^n and tori $\mathbb{T}^n$ from those of connected sums of projective spaces P^n, and becomes a more better invariant than orientaion for manifolds in this case.

Added before the last minute, as a summary we obtain, with $\emptyset$ to mean empty,

Table 11: Do or not classify the closed topological manifolds

Manifolds	χ	b_0	b_1	t_0	t_1
Orientable closed surfaces S^2, $\#^l\mathbb{T}^2$	Yes	No	Yes	$\emptyset$	$\emptyset$
Non-orientable closed surfaces $\#^l P^2$	Yes	No	Yes	Yes	$\emptyset$
Even $2n(\geq 4)$ dimensional, orientable closed manifolds S^{2n}, $\#^l\mathbb{T}^{2n}$	Yes	Yes	Yes	$\emptyset$	$\emptyset$
Even $2n(\geq 4)$ dimensional, non-orientable closed manifolds $\#^l P^{2n}$	Yes	No	Yes	Yes	$\emptyset$
Odd $2n+1(\geq 3)$ dimensional, orientable closed manifolds S^{2n+1}, $\#^l\mathbb{T}^{2n+1}$	No	Yes	Yes	$\emptyset$	$\emptyset$
Odd $2n+1(\geq 3)$ dimensional, orientable closed manifolds $\#^l P^{2n+1}$	No	No	No	$\emptyset$	Yes

The last table shows that

Corollary 3.19. *All the closed topological manifolds X as in the list are classifiable (up to homeomorphism) by using K-theory data such as either the Euler charactersistic $\chi(C(X))$, the Betti numbers b_j of $K_j(C(X))$ $(j = 0, 1)$, or the torsion ranks t_j of $K_j(C(X))$ $(j = 0, 1)$, together with dimension of X (not K-theoretic) and torsion freeness for both $K_0(C(X))$ and $K_1(C(X))$ (or orientaion of X in part).*

Remark. Now comes out a natural question (to be considered), whether one can know that the converse of that corollary holds or not. Namely, determine the (suitable) class of closed topological manifolds, which are classifiable by those data as complete invariants.

Furthermore, a moment of thought implies that, as a class to answer the question,

Theorem 3.20. *For X and Y two closed topological manifolds as in the list above, X is homeomophic to Y if and only if*

$$K_0(C(X)) \oplus K_1(C(X)) \cong K_0(C(Y)) \oplus K_1(C(Y))$$

and $\dim X = \dim Y$.

Remark. Moreover, the (covering) dimension for spaces can be replaced with the real rank for C^*-algebras. Indeed, for X a compact Hausdorff space, $\dim X = \mathrm{RR}(C(X))$ (see [2]).

Therefore,

Corollary 3.21. *K-theory groups and real rank for C^*-algebras are complete invariants for those closed topological manifolds X (up to homeomorphism).*

4 Noncommutative connected sums As a generalization of connected sums of closed topological manifolds to C^*-algebras, we define a connected sum $\mathfrak{A}\#\mathfrak{B}$ of two unital C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ with a unital C^*-algebra $\mathfrak{D}$ as a common quotient, having a quotient $\mathfrak{E}$, to be the following extension of $\mathfrak{E}$ by the direct sum $\mathfrak{J} \oplus \mathfrak{K}$:

$$0 \to \mathfrak{J} \oplus \mathfrak{K} \xrightarrow{\quad i \quad} \mathfrak{A}\#\mathfrak{B} \xrightarrow{\quad q \quad} \mathfrak{E} \to 0,$$

where we have

$$0 \to \mathfrak{J} \xrightarrow{\quad i \quad} \mathfrak{A} \xrightarrow{\quad q \quad} \mathfrak{D} \to 0,$$

$$0 \to \mathfrak{K} \xrightarrow{\quad i \quad} \mathfrak{B} \xrightarrow{\quad q \quad} \mathfrak{D} \to 0,$$

and $\mathfrak{E}$ is a quotient of $\mathfrak{D}$, where each i is the inclusion map and each q is the quotient map.

As a note, in the definition we may replace the closed ideals $\mathfrak{J}$, $\mathfrak{K}$, and $\mathfrak{J} \oplus \mathfrak{K}$ with $\mathfrak{J} \otimes \mathbb{K}$, $\mathfrak{K} \otimes \mathbb{K}$, and $[\mathfrak{J} \oplus \mathfrak{K}] \otimes \mathbb{K}$, respectively, if necessary as in the extension theory of C^*-algebras. Also, the connected sum $\mathfrak{A}\#\mathfrak{B}$ defined may not be unique, which depends on the extension theory of C^*-algebras and can be unique as an equivalence class in the theory, so that $\mathfrak{A}\#\mathfrak{B}$ is one representative of the connected sums defined above. Also, C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{D}$ in the definition may not be unital. Note that the unital case of C^*-algebras corresponds to the compact case of spaces, as in this paper, and the non-unital case does to the non-compact case, not dealt with here.

Theorem 4.1. *Let $\mathfrak{A}\#\mathfrak{B}$ be the connected sum of two unital C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ with a common quotient $\mathfrak{D}$ having a quotient $\mathfrak{E}$. Then*

$$\chi(\mathfrak{A}\#\mathfrak{B}) = \chi(\mathfrak{A}) + \chi(\mathfrak{B}) - 2 \cdot \chi(\mathfrak{D}) + \chi(\mathfrak{E}).$$

Proof. It follows from the definition of $\mathfrak{A}\#\mathfrak{B}$ above that

$$\chi(\mathfrak{A}\#\mathfrak{B}) = \chi(\mathfrak{J}) + \chi(\mathfrak{K}) + \chi(\mathfrak{E}),$$

and also that

$$\chi(\mathfrak{A}) = \chi(\mathfrak{J}) + \chi(\mathfrak{D}), \quad \text{and} \quad \chi(\mathfrak{B}) = \chi(\mathfrak{K}) + \chi(\mathfrak{D}).$$

Therefore, we obtain

$$\chi(\mathfrak{A}\#\mathfrak{B}) = \chi(\mathfrak{A}) + \chi(\mathfrak{B}) - 2 \cdot \chi(\mathfrak{D}) + \chi(\mathfrak{E}).$$

□

Example 4.2. Let X, Y be compact Hausdorff spaces and $C(X), C(Y)$ be the C^*-algebras of all continuous, complex-valued functions on X, Y respectively. Assume that there is a closed subset D of X which is identified with a closed subset of Y. Let $E = \partial D$ be the boundary of D, which is closed in D. Then one can define the connected sum $C(X)\#C(Y)$ in our sense to be the following extension:

$$0 \to C_0(X \setminus D) \oplus C_0(Y \setminus D) \to C(X)\#C(Y) \to C(E) \to 0.$$

It follows that

$$\chi(C(X)\#C(Y)) = \chi(C(X)) + \chi(C(Y)) - 2 \cdot \chi(C(D)) + \chi(C(\partial D)).$$

Compare with those formulae in Theorems 2.1 and 3.1, contained in this formula and in that of Theorem 4.1.

To define the 2-successive connected sum of three unital C^*-algebras $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$ wiht a common quotient $\mathfrak{D}$ having a quotient $\mathfrak{E}$, we assume that there are short exact sequences of C^*-algebras:

$$0 \to \mathfrak{I}_j \to \mathfrak{A}_j \to \mathfrak{D} \to 0$$

($j = 1, 2, 3$) and

$$0 \to \mathfrak{I}_{j,k} \to \mathfrak{A}_j \# \mathfrak{A}_k \to \mathfrak{D} \to 0$$

($1 \leq j, k \leq 3$ and $j \neq k$). We then define the 2-successive connected sum $(\mathfrak{A}_j \# \mathfrak{A}_k) \# \mathfrak{A}_l$ to be the following extension:

$$0 \to \mathfrak{I}_{j,k} \oplus \mathfrak{I}_l \to (\mathfrak{A}_j \# \mathfrak{A}_k) \# \mathfrak{A}_l \to \mathfrak{E} \to 0,$$

which may not be unique. Also, the associativity for the connected sum may not holds, i.e., $(\mathfrak{A}_j \# \mathfrak{A}_k) \# \mathfrak{A}_l \not\cong \mathfrak{A}_j \# (\mathfrak{A}_k \# \mathfrak{A}_l)$ in general. (Checking this should be another task to be continued elsewhere.) Anyhow, we can define the $(n-1)$-successive connected sum of unital C^*-algebras $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ with a common quotient $\mathfrak{D}$ having a quotient $\mathfrak{E}$ to be inductively as

$$\mathfrak{A}_1 \# \mathfrak{A}_2 \cdots \# \mathfrak{A}_n = (\cdots (\mathfrak{A}_1 \# \mathfrak{A}_2) \# \mathfrak{A}_3 \cdots) \# \mathfrak{A}_n$$

in this order, where we need to assume that there are short exact sequences of C^*-algebras:

$$0 \to \mathfrak{I}_j \to \mathfrak{A}_j \to \mathfrak{D} \to 0$$

for $1 \leq j \leq n$ and

$$0 \to \mathfrak{I}_{1,2,\cdots,k} \to (\cdots (\mathfrak{A}_1 \# \mathfrak{A}_2) \cdots) \# \mathfrak{A}_k \to \mathfrak{D} \to 0$$

($k = 2, \cdots, n-1$), so that one can define the following extensions:

$$0 \to \mathfrak{I}_{1,2,\cdots,k} \oplus \mathfrak{I}_{k+1} \to ((\cdots (\mathfrak{A}_1 \# \mathfrak{A}_2) \cdots) \# \mathfrak{A}_k) \# \mathfrak{A}_{k+1} \to \mathfrak{E} \to 0$$

for $1 \leq k \leq n-1$. We omit to write this assumption in what follows.

Corollary 4.3. *Let $\mathfrak{A}_1 \# \mathfrak{A}_2 \cdots \# \mathfrak{A}_n$ be the $(n-1)$-successive connected sum of unital C^*-algebras $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ by a common quotient $\mathfrak{D}$ with a quotient $\mathfrak{E}$. Then*

$$\chi(\mathfrak{A}_1 \# \mathfrak{A}_2 \cdots \# \mathfrak{A}_n) = \sum_{i=1}^n \chi(\mathfrak{A}_i) + (n-1)[\chi(\mathfrak{E}) - 2 \cdot \chi(\mathfrak{D})].$$

Proposition 4.4. *Let $\mathfrak{A}_1 \# \mathfrak{A}_2$ be the connected sum of two unital C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ with a common quotient $\mathfrak{D} = \mathfrak{A}_1/\mathfrak{I}_1 = \mathfrak{A}_2/\mathfrak{I}_2$ with a quotient $\mathfrak{E}$. Then*

$$\begin{tikzcd} K_0(\mathfrak{I}_1) \oplus K_0(\mathfrak{I}_2) \arrow[r, "i_*"] & K_0(\mathfrak{A}_1 \# \mathfrak{A}_2) \arrow[r, "q_*"] & K_0(\mathfrak{E}) \arrow[d, "\partial"] \\ K_1(\mathfrak{E}) \arrow[u, "\partial"] & K_1(\mathfrak{A}_1 \# \mathfrak{A}_2) \arrow[l, "q_*"'] & K_1(\mathfrak{I}_1) \oplus K_1(\mathfrak{I}_2) \arrow[l, "i_*"'] \end{tikzcd}$$

and

$$\begin{tikzcd} K_0(\mathfrak{I}_j) \arrow[r, "i_*"] & K_0(\mathfrak{A}_j) \arrow[r, "q_*"] & K_0(\mathfrak{D}) \arrow[d, "\partial"] \\ K_1(\mathfrak{D}) \arrow[u, "\partial"] & K_1(\mathfrak{A}_j) \arrow[l, "q_*"'] & K_1(\mathfrak{I}_j) \arrow[l, "i_*"'] \end{tikzcd}$$

$(j = 1, 2)$, *from which* $K_l(\mathfrak{I}_j)$ $(j, l = 0, 1)$ *are computable in terms of the K-theory groups of given* C^**-algebras, so that*

$$\begin{array}{ccccc} 0 \to [K_l(\mathfrak{I}_1 \oplus \mathfrak{I}_2)]/\partial K_{l+1}(\mathfrak{E}) & \longrightarrow & K_l(\mathfrak{A}_1 \# \mathfrak{A}_2) & \longrightarrow & q_*(K_l(\mathfrak{A}_1 \# \mathfrak{A}_2)) \to 0 \\ \| & & & & \| \\ \mathrm{coker}(\partial) & & & & \ker(\partial) \end{array}$$

$(l = 0, 1)$ *and*

$$\begin{array}{ccccc} 0 \to K_l(\mathfrak{I}_j)/\partial K_{l+1}(\mathfrak{D}) & \longrightarrow & K_l(\mathfrak{A}_j) & \longrightarrow & q_*(K_l(\mathfrak{A}_j)) \to 0 \\ \| & & & & \| \\ \mathrm{coker}(\partial) & & & & \ker(\partial) \end{array}$$

$(l = 0, 1)$, *where* $l + 1$ *(mod 2). It follows that the K-theory groups* $K_l(\mathfrak{A}_1 \# \mathfrak{A}_2)$ *as well as* $K_l(\mathfrak{A}_j)$ *are determined by the cokernels* $\mathrm{coker}(\partial)$ *and the kernels* $\ker(\partial)$ *of the boundary maps* ∂ (*up and down arrows*) *in the left and right sides* (*that are index and exponential maps, respectively*).

Proposition 4.5. *Let* $\#_{j=1}^n \mathfrak{A}_j$ *be the* $(n-1)$*-successive connected sum of unital* C^**-algebras* $\mathfrak{A}_j$ $(1 \leq j \leq n)$ *by a common quotient* $\mathfrak{D}$ *with a quotient* $\mathfrak{E}$*. Then inductively,*

$$\begin{array}{ccccc} K_0(\mathfrak{I}_{1,\cdots,n-1} \oplus \mathfrak{I}_n) & \xrightarrow{i_*} & K_0((\#_{j=1}^{n-1}\mathfrak{A}_j)\#\mathfrak{A}_n) & \xrightarrow{q_*} & K_0(\mathfrak{E}) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(\mathfrak{E}) & \xleftarrow{q_*} & K_1((\#_{j=1}^{n-1}\mathfrak{A}_j)\#\mathfrak{A}_n) & \xleftarrow{i_*} & K_1(\mathfrak{I}_{1,\cdots,n-1} \oplus \mathfrak{I}_n) \end{array}$$

and

$$\begin{array}{ccccc} K_0(\mathfrak{I}_j) & \xrightarrow{i_*} & K_0(\mathfrak{A}_j) & \xrightarrow{q_*} & K_0(\mathfrak{D}) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(\mathfrak{D}) & \xleftarrow{q_*} & K_1(\mathfrak{A}_j) & \xleftarrow{i_*} & K_1(\mathfrak{I}_j) \end{array}$$

$(1 \leq j \leq n)$, *in particular, when* $j = n$, *and*

$$\begin{array}{ccccc} K_0(\mathfrak{I}_{1,\cdots,n-1}) & \xrightarrow{i_*} & K_0(\#_{j=1}^{n-1}\mathfrak{A}_j) & \xrightarrow{q_*} & K_0(\mathfrak{D}) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(\mathfrak{D}) & \xleftarrow{q_*} & K_1(\#_{j=1}^{n-1}\mathfrak{A}_j) & \xleftarrow{i_*} & K_1(\mathfrak{I}_{1,\cdots,n-1}) \end{array}$$

with $\#_{j=1}^{n-1}\mathfrak{A}_j = (\#_{j=1}^{n-2}\mathfrak{A}_j)\#\mathfrak{A}_{n-1}$ *as the next step, from which* $K_l(\mathfrak{I}_{1,\cdots,n-1})$ $(l = 0, 1)$ *are computed inductively in terms of the K-theory groups of given* C^**-algebras, so that*

$$\begin{array}{c} 0 \to K_l(\mathfrak{I}_{1,\cdots,n-1} \oplus \mathfrak{I}_n)/\partial K_{l+1}(\mathfrak{E}) = \mathrm{coker}(\partial) \\ \downarrow \\ K_l((\#_{j=1}^{n-1}\mathfrak{A}_j)\#\mathfrak{A}_n) \\ \downarrow \\ q_*(K_l((\#_{j=1}^{n-1}\mathfrak{A}_j)\#\mathfrak{A}_n)) = \ker(\partial) \to 0 \end{array}$$

$(l = 0, 1)$ *and*

$$\begin{tikzcd} 0 \to K_l(\mathfrak{I}_j)/\partial K_{l+1}(\mathfrak{D}) \arrow[r] \arrow[d, equal] & K_l(\mathfrak{A}_j) \arrow[r] & q_*(K_l(\mathfrak{A}_j)) \to 0 \arrow[d, equal] \\ \mathrm{coker}(\partial) & & \ker(\partial) \end{tikzcd}$$

$(l = 0, 1)$, *where* $l + 1$ $(mod\ 2)$, *in particular, when* $j = n$, *and*

$$\begin{tikzcd} 0 \to K_l(\mathfrak{I}_{1,\cdots,n-1})/\partial K_{l+1}(\mathfrak{D}) = \mathrm{coker}(\partial) \arrow[d] \\ K_l(\#_{j=1}^{n-1}\mathfrak{A}_j) \arrow[d] \\ q_*(K_l(\#_{j=1}^{n-1}\mathfrak{A}_j)) = \ker(\partial) \to 0 \end{tikzcd}$$

$(l = 0, 1)$ *with* $\#_{j=1}^{n-1}\mathfrak{A}_j = (\#_{j=1}^{n-2}\mathfrak{A}_j)\#\mathfrak{A}_{n-1}$, *for which its K-theory groups are computed similarly as the case of* $\#_{j=1}^{n}\mathfrak{A}_j$ *above. It follows that the K-theory groups* $K_l(\#_{j=1}^{n}\mathfrak{A}_j)$, $K_l(\#_{j=1}^{n-1}\mathfrak{A}_j)$, $\cdots$, *as well as* $K_l(\mathfrak{A}_j)$ *are determined inductively by the cokernels* coker(∂) *and the kernels* ker(∂) *of the boundary maps* ∂ (*up and down arrows*) *in the left and right sides* (*that are index and exponential maps, respectively*).

Corollary 4.6. *The K-theory groups of successive connected sums of* C^**-algebras in our sense is computable inductively if the six-term diagrams in the proposition are computable inductively in the sense that the cokernels and the kernels of the boundary maps associated with the diagrams can be determined.*

Example 4.7. Principal examples in the commutative case should be those in Sections 2 and 3 and that of Example 4.2. Principal examples in the even noncommutative case should be from the tenor product C^*-algebras of the commutative C^*-algebras in the commutative case, respectively tensored with noncommutative C^*-algebras such as $M_n(\mathbb{C})$, $\mathbb{K}$ and any other C^*-algebras with their K-theory groups computable.

In the case of $M_n(\mathbb{C})$ and $\mathbb{K}$, the noncommutative connected sums have the same Euler characteristic and the same K-theory as the commutative connected sums without tensoring with $M_n(\mathbb{C})$ and $\mathbb{K}$ by the stability of K-theory groups.

Some more complicated examples can be given by replacing the tensor product C^*-algebras viewed as the trivial bundle C^*-algebras, with more general bundle C^*-algebras (or continuous fields of C^*-algebras) (or with crossed product C^*-algebras with suitable actions, viewed as skewed tensor product C^*-algebras) but their base spaces should have the topological strucure of connected sums of spaces involved.

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Department of Mathematical Sciences, Faculty of Science, University of the Ryukyus, Senbaru 1, Nishihara, Okinawa 903-0213, Japan.
Email: sudo@math.u-ryukyu.ac.jp

ON THE DECOMPOSITION OF CONTRACTIONS AND ISOMETRIES

G. A. BAGHERI-BARDI

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ABSTRACT. It is proved (with given different proofs) that the von Neumann-Wold and the Nagy-Foias-Langer decompositions are valid in more general classes than the classical W*-algebras.

INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space and let $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. The aim of the structure theory analysis is the structure of operators in $B(\mathcal{H})$. The structure of some operators are well-understood. As for unitaries a spectral theory and effective functional calculus are available. Another part is unilateral shifts. An operator a on $\mathcal{H}$ is called a unilateral shift if there is a sequence of pairwise orthogonal subspaces $\mathcal{H}_0, \mathcal{H}_1, \cdots$ such that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots$ and a maps $\mathcal{H}_n$ isometrically onto $\mathcal{H}_{n+1}$. For a comprehensive discussion about unilateral shifts, we refer to section 23 of [1].

Two fundamental theorems make the cornerstone of the structure theory. The first one provides the largest reducing subspace for a given contraction $a \in B(\mathcal{H})$ on which a will be unitary [5][8] and the second one gives much more details when a is an isometry[10].

Theorem 0.1. ***The Nagy-Foias-Langer Decomposition*** *To every contraction a on the Hilbert space $\mathcal{H}$ there corresponds a decomposition of $\mathcal{H}$ into an orthogonal sum of two subspaces reducing a, say $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, such that the restriction of a to $\mathcal{H}_0$ is unitary, and the restriction of a to $\mathcal{H}_1$ is completely non-unitary. This decomposition is uniquely determined.*

Theorem 0.2. ***The von Neumann-Wold Decomposition.*** *If x is an isometry on the Hilbert space $\mathcal{H}$ and $\mathcal{H}_0 = \bigcap_n x^n \mathcal{H}$, then $\mathcal{H}_0$ reduces x, $x_{|_{\mathcal{H}_0}}$ is unitary and $x_{|_{\mathcal{H}_0^\perp}}$ is a unilateral shift.*

The strategy of the original proofs of these decompositions are completely based on the geometry of the underlying Hilbert space. In this discussion, we give different proofs of these results which are more algebraic in nature than the the well-known proofs. These proof therefore offer valuable insight as to how one can extend the results to non-normed topological algebras. This is demonstrated in Section 2, where it is shown that the results are valid for locally W*-algebras[2], a class of (generally non-normed) topological *-algebras more general than W*-algebras.

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1. Wold decomposition

Throughout this section $\mathcal{M}$ stands for a W*-algebra with the unit 1. At first, we deal with the decomposition of contractions. To begin we need a convention. Let x be in $\mathcal{M}$. We denote by $[x]$ the relative unit of the w^*-closed algebra generated by xx^* in $\mathcal{M}$ and call it the range projection of x.

Remark 1.1. To make an illustration what the projection $[x]$ is in the concrete case, assume that x is a bounded linear operator on the Hilbert space $\mathcal{H}$. In this case $[x]$ will be the relative unit of the von Neumann algebra generated by xx^* in $B(\mathcal{H})$. One may check that $[x]$ is just the orthogonal projection onto $\overline{x\mathcal{H}}$.

Let a be a contraction in $\mathcal{M}$. A sequence of projections $\{e_n\}_{n\in\mathbb{Z}}$ in $\mathcal{M}$ is called a $U(a)$-solution if

$$\begin{cases} {a^*}^n a^n e_n = e_n & n \geq 0 \\ a^{-n} {a^*}^{-n} e_n = e_n & n \leq 0 \end{cases}$$

For two given solutions $\{e_n^j\}_{n\in\mathbb{Z}}$ $(j=1,2)$, we write $\{e_n^1\}_{n\in\mathbb{Z}} \preceq \{e_n^2\}_{n\in\mathbb{Z}}$ if $e_n^1 \leq e_n^2$ for all $n \in \mathbb{Z}$. Clearly $\preceq$ defines a partial order relation on $U(a)$-solutions.

Lemma 1.2. *Let a be a contraction. The set of $U(a)$-solutions has a maximal element.*

Proof. Let us consider

$$\begin{cases} p_n = 1 - [1 - {a^*}^n a^n] & n \geq 0 \\ p_n = 1 - [1 - a^{-n} {a^*}^{-n}] & n \leq 0. \end{cases}$$

Since $[1 - {a^*}^n a^n]$ is the relative unit of the w^*-closed algebra generated by $1 - {a^*}^n a^n$, then

$$\underbrace{(1 - (1 - {a^*}^n a^n))}_{{a^*}^n a^n} \underbrace{(1 - [1 - {a^*}^n a^n])}_{p_n} = p_n.$$

Similarly one may see that $a^{-n} {a^*}^{-n} p_n = p_n$ when $n \leq 0$. It means that $\{p_n\}_{n\in\mathbb{Z}}$ is a $U(a)$-solution. Assume $\{q_n\}_{n\in\mathbb{Z}}$ is another $U(a)$-solution. We have then

$$\begin{aligned} (1 - {a^*}^n a^n) q_n = 0 &\Longrightarrow [1 - {a^*}^n a^n] q_n = 0 \\ &\Longrightarrow q_n \leq 1 - [1 - {a^*}^n a^n] = p_n \end{aligned}$$

for all $n \geq 0$. Similarly $q_n \leq p_n$ for negative integers n. $\square$

We put $v := \inf_{n\in\mathbb{N}} p_n$ where $\{p_n\}$ is the maximum of $U(a)$-solutions and call v the unitary projection of a. The unitary projection of a is zero if and only if either $\{{a^*}^n a^n\}$ or $\{a^{-n} {a^*}^{-n}\}$ converges to zero in the w^*-topology. Such a contraction is called completely non-unitary.

Lemma 1.3. *The unitary projection v of a commutes with a. Moreover*

$$va^*av = vaa^*v = v$$

Proof. We combine some points to obtain the assertion.

- Since v is majorized by p_1 then $a^*av = v$. Therefore ava^* is a projection, and hence the unit element of the von Neumann algebra generated by $(ava^*)(ava^*)^*$, being $[av]$, is ava^*.

- We now show that $[av]$ (as a constant sequence) is a $U(a)$-solution too: when n is negative:

$$\begin{aligned} a^{-n}{a^*}^{-n}\underbrace{(ava^*)}_{[av]} &= a^{-n}a^{*-n-1}\underbrace{a^*(ava^*)}_{va^*} \\ &= a^{-n}a^{*-n-1}(va^*) && (v \leq p_1 \to a^*av = v) \\ &= ava^* && (v \leq p_{-n-1} \to a^{-n-1}{a^*}^{-n-1}v = v) \end{aligned}$$

  Let n be a positive integer. Since ${a^*}^{n+1}a^{n+1}v = v$ we have then

$$(ava^*)(1 - {a^*}^n a^n)(ava^*) = 0$$

  On the other hand

$$\begin{aligned} (ava^*)(1 - {a^*}^n a^n)(ava^*) = 0 &\iff \sqrt{(1 - a^{*n}a^n)}(ava^*) = 0 \\ &\implies (1 - {a^*}^n a^n)(ava^*) = 0 \end{aligned}$$

  Similarly one may see that $[a^*v]$ is a $U(a)$-solution too. Therefore v majorizes both $[av]$ and $[av^*]$.
- Finally we have

$$\begin{aligned} [av] \leq v &\iff (1 - v)[av] = 0 \\ &\iff (1 - v)av = 0, \end{aligned}$$

  which implies that $av = vav$. We apply $[a^*v] \leq v$ to conclude that $a^*v = va^*v$. These two earlier identities finish the proof.

□

Combination of these two lemmas implies the following result:

Theorem 1.4. *Let a be a contraction in $\mathcal{M}$. Then v, the unitary projection of a is uniquely determined with the following properties:*

(1) *v commutes with a.*
(2) *$(1 - v)a(1 - v)$ is completely non unitary in W^*-algebra $(1 - v)\mathcal{M}(1 - v)$.*
(3) *vav is unitary in W^*-algebra $v\mathcal{M}v$ provided that v is non zero.*

Proof. Assume that w is another projection which satisfies the above axioms. Based on Lemma 1.2, v majorizes w. Therefore $v - w$ is a projection which commutes with a, majorized by $1 - w$ and $(v - w)a(v - w)$ is unitary in $(v - w)\mathcal{M}(v - w)$. It is contradiction with the axiom (2). □

Remark 1.5. Let $\mathcal{A}$ be a von Neumann subalgebra in $B(\mathcal{H})$. Let a be a contraction in $\mathcal{A}$ and consider v, the unitary projection of a, obtained in Lemma 1.2. The identity $va = av$ is equivalent to the point that $\mathcal{H}_0 = v\mathcal{H}$ is reduced by a. A glance at the proof of the Nagy-Foias-Langer decomposition theorem (see [9] page 9) shows that $\mathcal{H}_0$ is just the largest reducing subspace such that $a_{|\mathcal{H}_0}$ is unitary.

We now commence with the process of decomposition of an isometry x. To begin, the analogue of unilateral shifts in any arbitrary W^*-algebra is introduced.

Lemma 1.6. *Let x be an isometry in $\mathcal{M}$ and p be a projection in $\mathcal{M}$. We have*

$$[xp] = xpx^*$$

Proof. Since xpx^* is itself a projection, then similar to the argument in Lemma 1.3, $x = xpx^*$. □

Let x be an isometry. We shift forward the projection p to the projection $[xp]$ by x and continue this process to obtain the following sequence of projections

$$p_0 = p, p_1 = [xp] = xpx^*, \cdots, p_n = [xp_{n-1}] = x^n p {x^*}^n, \cdots$$

Such a sequence is called the p-shift spectrum of x with the initial projection p. This sequence is called an orthogonal p-shift spectrum if the projections p_n are pairwise mutually orthogonal.

Definition 1.7. Assume x is an isometry in $\mathcal{M}$. A projection p in $\mathcal{M}$ is called wandering for x if the corresponding shift spectrum (with the initial projection p) is orthogonal. If the total summation $\sum_0^\infty p_n$ (in the sense of w^*-topology) is just the unit of $\mathcal{M}$ then x is called an abstract unilateral shift.

Proposition 1.8. *Let x be an isometry in $\mathcal{M}$.*

(1) $1 - [x]$ *is a wandering projection of* x.
(2) *If x is an abstract unilateral shift, then there is unique orthogonal shift spectrum of x with total summation* 1. *Moreover the initial projection is* $1 - [x]$.

Proof. We apply Lemma 1.6 to obtain the following points:

i) $[x^{n+1}] = x[x^n]x^*$
ii) $[x([x^{n-1}] - [x^n])] = [x^n] - [x^{n+1}]$

Since $[x^{n+1}]$ is majorized by $[x^n]$, then ii) shows that $1-[x]$ is a wandering projection of x. As for the second item (2), assume x is an abstract unilateral shift. Let p be a wandering projection of x whose orthogonal shift spectrum has total summation 1. It means in the sense of w^*-topology that

$$\begin{aligned} 1 &= p + xpx^* + x^2 p {x^*}^2 + \cdots \\ &= p + x(p + xpx^* + x^2 p {x^*}^2 + \cdots)x^* \\ &= p + xx^* = p + [x] \end{aligned}$$

Therefore the initial projection p should be $1 - [x]$. □

Remark 1.9. Assume that x is a unilateral shift. We have then for every positive integer n that

i′) $[x^n] = x^*[x^{n+1}]x$
ii′) $[x^*([x^{n+1}] - [x^{n+2}])] = [x^n] - [x^{n+1}]$: To prove this, note that item i′) shows that the absolute value $|x^*([x^{n+1}]-[x^{n+2}])|$ is the just the projection $[x^n] - [x^{n+1}]$.
iii′) $[x^*(1 - [x])] = 0$

Based on these relations one may say that x^* acts as a backward shift.

Let x be an isometry and consider the following projections

$$\begin{aligned} s &= \sum [x^n] - [x^{n+1}] = 1 - \lim\ [x^n] \\ u &= \lim\ [x^n] = \inf [x^n] \end{aligned}$$

where the limits are taken in the w^*-topology. The pair (s, u) is called the Wold pair of x. We have the following main result when both s, u are non-trivial.

Theorem 1.10. *Let x be an isometry in a W^*-algebra $\mathcal{M}$. The Wold pair (s,u) of x is uniquely determined with the following properties*

(1) *s and u are mutually orthogonal and $s+u=1$.*
(2) *Both projections s and u commute with x.*
(3) *sxs is an abstract unilateral shift in the W^*-algebra $s\mathcal{M}s$ and uxu is a unitary in the W^*-algebra $u\mathcal{M}u$.*

Proof. The first item is clear and the second one is directly obtained by the definition of the Wold pair (s,u). As for (3), since s commutes with x, then sxs is an isometry in the W*-algebra $s\mathcal{M}s$. The projection $1-[x]$ is majorized by s and so is a projection in $s\mathcal{M}s$. We apply the second item of the Proposition 1.8 to obtain $1-[x]$ is a wandering projection for sxs. Moreover

$$\sum(sx^n s)([x^n]-[x^{n+1}])(sx^{*^n}s)=s(\sum x^n([x^n]-[x^{n+1}])x^{*^n})s=s$$

which shows that sxs is an abstract unilateral shift in $s\mathcal{M}s$. The definition of u shows that $u\le[x]$. Since u commutes with x

$$(uxu)(ux^*u)=u[x]u=u$$

Therefore uxu is unitary in $u\mathcal{M}u$.
Assume that (s_1,u_1) is a pair satisfying in conditions $(1),(2)$ and (3). We have then

$$\begin{aligned}1-[x]&=(s_1\oplus u_1)-[x(s_1\oplus u_1)]\\&=(s_1\oplus u_1)-(s_1[x]s_1\oplus u_1[x]u_1)\\&=s_1-s_1[x]s_1.\end{aligned}$$

We conclude x and s_1xs_1 have the same orthogonal shift spectrum with intial projection $1-[x]$ which implies that $s_1=s$ and so $u_1=u$. □

This theorem says that any isometry x is decomposed into an abstract unilateral shift and a unitary, which is exactly the von-Neumann-Wold decomposition:

$$x=sxs\oplus uxu.$$

Remark 1.11. Assume x is an isometry in $B(\mathcal{H})$ and p is a wandering projection of x. Let us denote $\mathcal{H}_n$ to be the range of the projection $p_n=x^npx^{*^n}$. Then $\mathcal{H}_0,\mathcal{H}_1,\cdots$ form pairwise orthogonal closed subspaces and x maps $\mathcal{H}_n$ isometrically onto $\mathcal{H}_{n+1}$. Therefore abstract unilateral shifts coincide with the unilateral shifts in $B(\mathcal{H})$. Moreover the corresponding Wold pair of x induces reducing subspaces $\mathcal{H}_0$ and $\mathcal{H}_1$ on which x is decomposed into a unitary and a unilateral shift.

The Wold decomposition is concerned with the structure of an isometry. It is extended for a particular finite sequence of isometries. We examine the current method for two such items.

The structure of an isometric tuple of operators in $B(\mathcal{H})$ is given in [6] as an extension of the Wold decomposition of an isometry. An n-tuple of operators $(x_1,\cdots,x_n)$ acting on $\mathcal{H}$ is said to be isometric if the row operator $[x_1,\cdots,x_n]:\mathcal{H}^n\to\mathcal{H}$ is an isometry. In fact, an isometric n-tuple is a sequence of isometries $x_1,\cdots,x_n$ such that the x_i's have pairwise orthogonal ranges. It is equivalent to say that the sequence of isometries $x_1,\cdots,x_n$ satisfies the Cuntz relations:

$$x_i^*x_j=\begin{cases}1 & i=j\\0 & i\ne j.\end{cases}$$

We now follow the structure of an isometric n-tuple in any arbitrary W^*-algebra.

Let $x_1, \cdots, x_n$ be a sequence of isometries in $\mathcal{M}$. The following are equivalent:

(1) The Cuntz relations hold for the sequence $x_1, \cdots, x_n$.
(2) If i and j are distinct then $[x_i][x_j] = 0$.
(3) $\sum_{i=1}^{n}[x_i] \leq 1$.

Assume that $x_1, \cdots, x_n$ is a sequence of isometries satisfying the Cuntz relations. We take $F_{m,n}$ to be the set of all functions from $\{1, \cdots, m\}$ to $\{1, \cdots, n\}$. For given $f \in F$, we set

$$x_f := x_{f(1)} \cdots x_{f(m)} \ , \ x_f^* := x_{f(m)}^* \cdots x_{f(1)}^*.$$

We also put $x_0 = x_{f(0)} = 1$ and $F = \bigcup_{m\geq 0} F_m$. Let us consider $p = 1 - \sum[x_i]$. A direct calculation shows that $[x_f p][x_g p] = 0$ (see Lemma 1.6) for all distinct functions $f, g \in F$. It allows us to say that $p = 1 - \sum[x_i]$ plays the role of a wandering projection for the isometric n-tuple $(x_1, \cdots, x_n)$. Let us consider the total summation $s := \sum_{f\in F}[x_f p]$. If $s = 1$ then $x_1, \cdots, x_n$ is called an n-orthogonal shift.

Just like the case $n = 1$, an n-orthogonal shift has a unique wandering projection with the total summation 1. To see this, assume q is a wandering projection for an isometric n-tuple $x_1, \cdots, x_n$ with $\sum_{f\in F}[x_f q] = 1$. Then in the sense of w^*-topology

$$\begin{aligned} 1 &= \sum_{f\in F}[x_f q] \\ &= q + (\sum_{f(1)=1}[x_f q]) + \cdots + (\sum_{f(1)=n}[x_f q]) \\ &= q + x_1(\sum_{f\in F}[x_f q])x_1^* + \cdots + x_n(\sum_{f\in F}[x_f q])x_n^* \\ &= q + \sum_{i=1}^{n}[x_i]. \end{aligned}$$

We put $u := 1 - s$ and call (s, u) the Wold pair of $x_1, \cdots, x_n$. One may apply Lemma 1.6 to conclude that both projections u, s commute with all x_i's. To sum up:

Theorem 1.12. *Let $x_1, \cdots, x_n$ be an isometric n-tuple in $\mathcal{M}$. Then the Wold pair (s, u) is uniquely determined with the following properties*

(1) *s and u are mutually orthogonal and $s + u = 1$.*
(2) *Both projections s and u commute with all x_i's.*
(3) *$sx_1s, \cdots, sx_ns$ is an n-orthogonal unilateral shift in the W^*-algebra $s\mathcal{M}s$ and*

$$\sum ux_ix_i^*u = u$$

Finally we examine the method for doubly commuting isometries in W*-algebras. A pair of commuting isometries (x_1, x_2) is called double commuting if $x_ix_j^* = x_j^*x_i$. In [7] Slocinski obtained an analogous result of the Wold decomposition for a pair of doubly commuting isometries.

Theorem 1.13. *Let $x = (x_1, x_2)$ be a pair of doubly commuting isometries on the Hilbert space $\mathcal{H}$. Then there exists a unique decomposition*

$$\mathcal{H} = \mathcal{H}_{ss} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{uu}$$

where $\mathcal{H}_{ij}$ are joint x-reducing subspaces of $\mathcal{H}$. Moreover x_1 on $\mathcal{H}_{ij}$ is a shift if $i = 1$ and unitary if $i = u$ and x_2 is a shift if $j = s$ and unitary if $j = u$.

Let $x = (x_1, x_2)$ be a pair of doubly commuting isometries in W*-algebra $\mathcal{M}$. Let (s_1, u_1) be the Wold pair of x_1 (see Theorem 1.10). Both projections s_1 and u_1 commute with x_2, since $x_i x_j^* = x_j^* x_i$. We again apply Theorem 1.10 for isometries $s_1 x_2 s_1$ and $u_1 x_2 u_1$ in W*-algebras $s_1 \mathcal{M} s_1$ and $u_1 \mathcal{M} u_1$ respectively. We then obtain two Wold pairs as follow

$$\begin{cases} u_1 = w_{uu} \oplus w_{us} \\ s_1 = w_{su} \oplus w_{ss} \end{cases}$$

One may check all these projections $w_{\alpha\beta}$'s commute with both x_1 and x_2. Moreover

$$\begin{cases} w_{\alpha\beta} x_1 w_{\alpha\beta} \text{ is a unitary in } w_{\alpha\beta} \mathcal{M} w_{\alpha\beta} \text{ if } \alpha = u \\ w_{\alpha\beta} x_1 w_{\alpha\beta} \text{ is a unilateral shift in } w_{\alpha\beta} \mathcal{M} w_{\alpha\beta} \text{ if } \alpha = s \\ w_{\alpha\beta} x_2 w_{\alpha\beta} \text{ is a unitary in } w_{\alpha\beta} \mathcal{M} w_{\alpha\beta} \text{ if } \beta = u \\ w_{\alpha\beta} x_2 w_{\alpha\beta} \text{ is a unilateral shift in } w_{\alpha\beta} \mathcal{M} w_{\alpha\beta} \text{ if } \beta = s \end{cases}$$

2. APPLICATION

Let us have a look at the proof of lemmas 1.2 and 1.3 again. We observe that the following points are used.

(1) Every W*-algebra is unital.
(2) The lattice of projections in any W*-algebra is complete.
(3) There is a partial ordered relation on the hermitian part of $\mathcal{A}$ and any positive element has unique square root.

In the current decomposition of an isometry in any W*-algebra, in addition to the above points, the following are also applied

(4) Any monotone sequence of projections is w^*-convergent to a projection.
(5) Assume $a \leq b$. For given $x \in \mathcal{M}$ we have then $x^* a x \leq x^* b x$.

Hence one may conclude fundamental decompositions theorems 0.1 and 0.2 in any dual topological *-algebra satisfying these properties. A well behaved of these structures are locally W*-algebras. We recall these structures.

In [3] Inoue introduced the notion of locally Hilbert space and the analogue of $B(\mathcal{H})$ as well. Let Λ be a directed index set and $\{\mathcal{H}_\alpha\}_{\alpha \in \Lambda}$ a family of Hilbert spaces such that $\mathcal{H}_\alpha$ is embedded in $\mathcal{H}_\beta$ where $\alpha \leq \beta$.

Let $\mathbf{H}$ be the direct limit of $\{\mathcal{H}_\alpha\}_{\alpha \in \Lambda}$

$$\mathbf{H} := \lim_{\rightarrow} \mathcal{H}_\alpha = \bigcup_{\alpha \in \Lambda} \mathcal{H}_\alpha.$$

Endow $\mathbf{H}$ with the inductive limit topology, that is the finest locally convex topology making the injections $\mathcal{H}_\alpha \hookrightarrow \mathbf{H}$ continuous. Then $\mathbf{H}$ is called a locally Hilbert space which is not a Hilbert space in general. Let $\iota_{\alpha\beta} : \mathcal{H}_\alpha \hookrightarrow \mathcal{H}_\beta$ be the canonical injection, and define $\mathbf{L}(\mathbf{H})$ to be the set of all continuous linear maps $T : \mathbf{H} \to \mathbf{H}$ for which $T_\beta \circ \iota_{\alpha\beta} = \iota_{\alpha\beta} \circ T_\alpha$, where $T_\alpha \in B(\mathcal{H}_\alpha)$ is the restriction of T to $\mathcal{H}_\alpha$. We have then that $\mathbf{L}(\mathbf{H})$ is the inverse limit of $\{\mathbf{B}(\mathcal{H}_\alpha)\}_{\alpha \in \Lambda}$ that is,

$$\mathbf{L}(\mathbf{H}) = \lim_{\leftarrow} B(\mathcal{H}_\alpha),$$

where $\mathbf{L}(\mathbf{H})$ is endowed with the inverse limit topology

$$\sigma := \lim_{\leftarrow} \sigma_\alpha \text{ with } \sigma_\alpha = \sigma(B(\mathcal{H}_\alpha), B(\mathcal{H}_\alpha)_*).$$

The topology σ on $\mathbf{L}(\mathbf{H})$ is called σ-weak topology. The σ-weakly closed *-subalgebras of $\mathbf{L}(\mathbf{H})$ are concrete locally W*-algebras [2]:

Theorem 2.1. *Every locally W*-algebra $\mathcal{M}$ endowed with the inverse limit topology σ coincides, within an isomorphism of topological *-algebras, with a σ-weakly closed *-subalgebra of $\boldsymbol{L}(\boldsymbol{H})$ for some locally Hilbert space $\boldsymbol{H}$.*

A continuous linear map $x : \mathbf{H} \to \mathbf{H}$ in $\mathbf{L}(\mathbf{H})$ is called an isometry (contraction) if x_α (the restriction of x on $\mathcal{H}_\alpha$) is an isometry (contraction) in $B(\mathcal{H}_\alpha)$. Equivalently, x is an isometry (contraction) if $x^*x = 1$ ($x^*x \leq 1$).

Inoue proved any locally C*-algebra satisfies in (3) and (5). To conclude, for Theorems 0.1 and 0.2, it is enough to show that items (1), (2) and (4) are also valid in locally W*-algebras. They are routine based on Theorem 2.1.

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Communicated by *Maria Fragoulopoulou*

Department of Mathematics, Persian Gulf University, Boushehr 75168, Iran
E-mail address: bagheri@mail.pgu.ac.ir , bagheri@gmail.com

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SOME RESULTS ON $\mathrm{BN_1}$-ALGEBRAS

ANDRZEJ WALENDZIAK

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ABSTRACT. $\mathrm{BN_1}$-algebras have been introduced by C. B. Kim and H. S. Kim. Here we give an equivalent definition of $\mathrm{BN_1}$-algebras and show that every $\mathrm{BN_1}$-algebra is a loop. Moreover we prove that an algebra is $\mathrm{BN_1}$-algebra if and only if it is a commutative BG-algebra. We also prove that the class of associative $\mathrm{BN_1}$-algebras coincides with the class of Coxeter algebras. Finally we indicate the interrelationships between $\mathrm{BN_1}$-algebras and several algebras.

1 Introduction In 1966, K. Iséki introduced in [3] the concept of BCI-algebras as algebras connected with some logics. Next, in 1983, Q. P. Hu and X. Li ([1]) defined BCH-algebras which are a generalization of BCI-algebras. Several years later, Y. B. Jun, E. H. Roh and H. S. Kim ([4]) introduced a wide class of abstract algebras called BH-algebras. Recently, C. B. Kim and H. S. Kim introduced in [7] the notion of a $\mathrm{BN_1}$-algebra. They defined a *BN_1-algebra* as an algebra $(A;*,0)$ of type $(2,0)$ (i.e., a nonempty set A with a binary operation $*$ and a constant 0) satisfying the following axioms:

(B1) $x * x = 0$,
(B2) $x * 0 = x$,
(BN) $(x * y) * z = (0 * z) * (y * x)$,
($\mathrm{BN_1}$) $x = (x * y) * y$.

Every Boolean group (that is, Abelian group all of whose elements have order 2) is a $\mathrm{BN_1}$-algebra. The class of all $\mathrm{BN_1}$-algebras is a proper subclass of the class of BN-algebras defined in [7]. A. Walendziak introduced in [12] BF-algebras which are a generalization of BN-algebras and B-algebras ([10]). C. B. Kim and H. S. Kim defined in [6] BM-algebras and proved that every BM-algebra is a B-algebra. They also introduced BG-algebras ([5]) as a generalization of B-algebras.

We will denote by **BCI** (resp., **BCH/BH/B/BM/BG/BF/BN/**$\mathbf{BN_1}$) the class of all BCI-algebras (resp., BCH/BH/B/BM/BG/BF/BN/$\mathrm{BN_1}$-algebras). The interrelationships between some classes of algebras mentioned before are visualized in Figure 1. (An arrow indicates proper inclusion, that is, if $\mathbf{X}$ and $\mathbf{Y}$ are classes of algebras, then $\mathbf{X} \to \mathbf{Y}$ means $\mathbf{X} \subset \mathbf{Y}$.)

In this paper we study $\mathrm{BN_1}$-algebras. We give another axiomatization of $\mathrm{BN_1}$-algebras and prove that every $\mathrm{BN_1}$-algebra is a loop. Moreover we show that the concept of a $\mathrm{BN_1}$-algebra is equivalent to the concept of a commutative BG-algebra. We also show that the class of associative $\mathrm{BN_1}$-algebras coincides with the class of Coxeter algebras. Finally we consider the relationships between $\mathrm{BN_1}$-algebras and several algebras.

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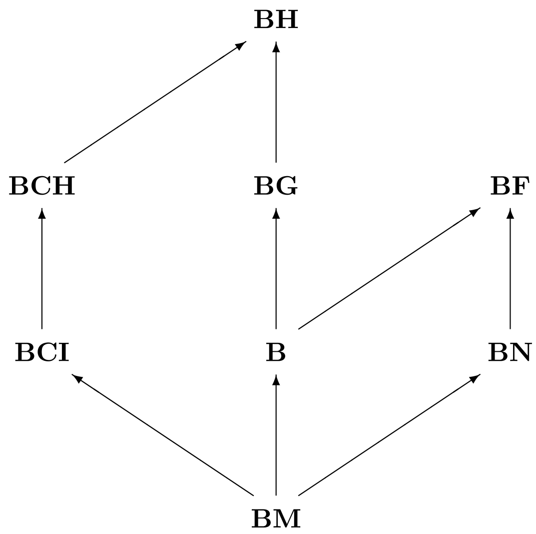

Figure 1

2 Preliminaries Throughout this paper $\mathcal{A}$ will denote an algebra $(A; *, 0)$ of type $(2, 0)$.

An algebra $\mathcal{A}$ is said to be a *BH-algebra* ([4]) if it satisfies (B1), (B2) and the following axiom:

(BH) $\quad x * y = y * x = 0 \Longrightarrow x = y.$

A BH-algebra $\mathcal{A}$ with the condition

(BCH) $\quad (x * y) * z = (x * z) * y$

(for all $x, y, z \in A$) is called a *BCH-algebra.* In [1], it is proved that $\mathcal{A}$ is a BCH-algebra if and only if it satisfies (B1), (BH), and (BCH).

A BH-algebra $\mathcal{A}$ satisfying the identity

(BCI) $\quad ((x * y) * (x * z)) * (z * y) = 0$

is called a *BCI-algebra.* Recall that according to the H. S. Li's axiom system ([9]), an algebra $\mathcal{A}$ is a BCI-algebra if and only if it satisfies (B2), (BH), and (BCI).

Remark 2.1. We know that every BCI-algebra is a BCH-algebra and every BCH-algebra is a BH-algebra.

Let an algebra $\mathcal{A}$ satisfy identities (B1) and (B2). We say that $\mathcal{A}$ is a *B-algebra* (resp., *BF*/*BG*/*BN-algebra*) if $\mathcal{A}$ satisfies axiom (B) (resp., (BF)/(BG)/(BN)), where:

(B) $\quad (x * y) * z = x * [z * (0 * y)],$
(BF) $\quad 0 * (x * y) = y * x,$
(BG) $\quad x = (x * y) * (0 * y),$
(BN) $\quad (x * y) * z = (0 * z) * (y * x),$

An algebra $\mathcal{A}$ is called a *BM-algebra* ([6]) if it satisfies (B2) and the following axiom:

(BM) $\quad (x * y) * (x * z) = z * y.$

Remark 2.2. From Theorem 2.6 of [6] it follows that every BM-algebra is a B-algebra. By Theorem 2.2 and Proposition 2.8 of [5], every B-algebra is a BG-algebra and every BG-algebra is a BH-algebra. It is easy to see that (BM) implies (BCI). Therefore the class of BM-algebras is a subclass of the class of BCI-algebras.

An algebra $\mathcal{A}$ is said to be 0-*commutative* (resp., *commutative*) if $x * (0 * y) = y * (0 * x)$ (resp., $x * y = y * x$) for any $x, y \in A$.

Remark 2.3. In [6], it is proved that $\mathcal{A}$ is a BM-algebra if and only if it is a 0-commutative B-algebra. C. B. Kim and H. S. Kim ([7]) showed that an algebra is a BN-algebra if and only if it is a 0-commutative BF-algebra (therefore, every BN-algebra is a BF-algebra). By Corollary 2.12 of [7], every BM-algebra is a BN-algebra.

Proposition 2.4. ([7]) *If* $(A; *, 0)$ *is a BN-algebra, then*

(a) $0 * (0 * x) = x$,
(b) $y * x = (0 * x) * (0 * y)$

for all $x, y \in A$.

H. S. Kim, Y. H. Kim and J. Neggers introduced the concepts of Coxeter algebras and pre-Coxeter algebras. A *Coxeter algebra* ([8]) is an algebra $\mathcal{A}$ satisfying identities (B1), (B2) and

(As) $x * (y * z) = (x * y) * z$.

It is known that a Coxeter algebra is a special type of abelian groups (see [8]). In [7], it is proved that $\mathcal{A}$ is a Coxeter algebra if and only if it is a BN-algebra satisfying the following axiom:

(D) $(x * y) * z = x * (z * y)$.

Proposition 2.5. ([6]) *Every Coxeter algebra is a BM-algebra.*

Proposition 2.6. ([6]) *If* $\mathcal{A}$ *is a BM-algebra satisfying the condition*

(B2') $0 * x = x$,

then it is a Coxeter algebra.

A commutative BH-algebra is called a *pre-Coxeter algebra* (shortly, *PC-algebra*). The class of all Coxeter algebras (resp., pre-Coxeter algebras) we denote by **CA** (resp., **PC**). Every Coxeter algebra is a PC-algebra and there is a PC-algebra which is not a Coxeter algebra (see [8]). Consequently, **CA** is a proper subclass of **PC**. Every BM-algebra satisfying the condition (B2') is a PC-algebra (see Theorem 3.7 of [6]). In general, a PC-algebra need not be a BM-algebra (see Example 3.8 of [6]).

From Proposition 2.5 and Remark 2.3 we obtain

$$\mathbf{CA} \subset \mathbf{BM} \subset \mathbf{BN} \subset \mathbf{BF}. \tag{1}$$

Let $\mathcal{A}$ be a PC-algebra. Observe that $\mathcal{A}$ is a BN-algebra. Indeed, $(x * y) * z = z * (y * x) = (0 * z) * (y * x)$ for all $x, y, z \in A$. Therefore, $\mathcal{A}$ satisfies (BN) and consequently, $\mathcal{A}$ is a BN-algebra. Thus

$$\mathbf{PC} \subset \mathbf{BN}. \tag{2}$$

3. On BN$_1$-algebras

By definition, $\mathcal{A} = (A; *, 0)$ is a BN$_1$-algebra if and only if it is a BN-algebra satisfying (BN$_1$).

Example 3.1. Let $A = \{0, 1\}$ and $*$ be defined by the following table:

$*$	0	1
0	0	1
1	1	0

Then $(A; *, 0)$ is a BN_1-algebra.

Example 3.2. Let $A = \{0, 1, 2, 3\}$ and define the binary operation "$*$" on A by the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(A; *, 0)$ is a BN_1-algebra (In fact, A is the Klein 4-group.)

Example 3.3. Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $*$ be defined by the following table:

$*$	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	0	3	2	6	7	4	5	9	8
2	2	3	0	1	8	6	5	9	4	7
3	3	2	1	0	7	8	9	4	5	6
4	4	6	8	7	0	9	1	3	2	5
5	5	7	6	8	9	0	2	1	3	4
6	6	4	5	9	1	2	0	8	7	3
7	7	5	9	4	3	1	8	0	6	2
8	8	9	4	5	2	3	7	6	0	1
9	9	8	7	6	5	4	3	2	1	0

It is easy to check that $(A; *, 0)$ is a BN_1-algebra.

Proposition 3.4. *If* $(A; *, 0)$ *is a* BN_1*-algebra, then*

(P1) $0 * x = x$,
(P2) $x = (x * y) * (0 * y)$,
(P3) $x * y = y * x$,
(P4) $x = y * (y * x)$.
(P5) $x * y = 0 \Longrightarrow x = y$,
(P6) $x * y = y \Longrightarrow x = 0$,
(P7) $x * y = x \Longrightarrow y = 0$,
(P8) $x * y = x * z \Longrightarrow y = z$,

for all $x, y, z \in A$.

Proof. Let $x, y, z \in A$.

(P1) Applying (BN_1) and (B1) we have $x = (x * x) * x = 0 * x$, that is, (P1) holds.
(P2) By (BN_1) and (P1).
(P3) From (P1) and Theorem 2.4 (b) we obtain

$$x * y = (0 * y) * (0 * x) = y * x.$$

(P4) Clear.

(P5) By Corollary 3.10 of [7].

(P6) Let $x * y = y$. Using (BN$_1$) and (B1) we get $x = (x * y) * y = y * y = 0$. Therefore (P6) is satisfied.

(P7) The proof is similar to the proof of (P6).

(P8) Let $x * y = x * z$. Hence $x * (x * y) = x * (x * z)$. By (P4), $y = z$. Thus (P8) holds. □

Proposition 3.5. *Every BN$_1$-algebra has the unique solution property.*

Proof. Let $\mathcal{A}$ be a BN$_1$-algebra and $a, b \in A$. It is easy to see that the equations $x * b = a$ and $b * x = a$ have solutions given by $x = a * b$ and $x = b * a$, respectively. (P8) implies that in each case, such x is unique. □

Theorem 3.6. *Every BN$_1$-algebra is a loop.*

Proof. Let $\mathcal{A}$ be a BN$_1$-algebra. Since $x * 0 = 0 * x = x$ for each $x \in A$ and $\mathcal{A}$ has the unique solution property, we conclude that $\mathcal{A}$ is a loop. □

Remark 3.7. There is a loop which is not a BN$_1$-algebra. Let $A = \{0, 1, 2, 3, 4\}$ and define the binary operation "$*$" on A by the following table:

$*$	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	4	2
2	2	4	0	1	3
3	3	2	4	0	1
4	4	3	1	2	0

Then $(A; *, 0)$ is a loop but it is not a BN$_1$-algebra, since $(1 * 2) * 2 = 3 * 2 = 4 \neq 1$.

Theorem 3.8. *An algebra $\mathcal{A}$ is a BN$_1$-algebra if and only if it satisfies the following axioms:*

(B1) $x * x = 0$,

(C) $x * y = y * x$,

(BN$_1$) $(x * y) * y = x$.

Proof. Let $\mathcal{A}$ be a BN$_1$-algebra. By definition and property (P3), $\mathcal{A}$ satisfies (B1), (BN$_1$) and (C).

Conversely, suppose that the above identities hold in $\mathcal{A}$. From (BN$_1$) and (B1) we have $x = (x * x) * x = 0 * x$ for all $x \in A$, that is, (B2') is satisfied. Using commutativity of $*$ we get (B2). Observe that (BN) also holds in $\mathcal{A}$. Let $x, y, z \in A$. Applying (C) and (B2') we obtain

$$(x * y) * z = z * (y * x) = (0 * z) * (y * x).$$

Thus $\mathcal{A}$ is a BN-algebra and finally, $\mathcal{A}$ is a BN$_1$-algebra. □

Theorem 3.9. *An algebra $\mathcal{A}$ is a BN$_1$-algebra if and only if it is a commutative BG-algebra.*

Proof. Let $\mathcal{A}$ be a BN$_1$-algebra. By (P2), $\mathcal{A}$ satisfies (BG). From property (P3) we see that the operation $*$ is commutative.

Conversely, if $\mathcal{A}$ is a commutative BG-algebra, then $\mathcal{A}$ satisfies (B1), (C) and (BN_1). From Theorem 3.8 it follows that $\mathcal{A}$ is a BN_1-algebra. □

It is easy to see that every Coxeter algebra is a BN_1-algebra, that is,

$$\mathbf{CA} \subset \mathbf{BN}_1. \tag{3}$$

Proposition 3.10. *If $\mathcal{A}$ is a BN_1-algebra, then it is a PC-algebra.*

Proof. From (P5) it follows that $\mathcal{A}$ satisfies the condition (BH). Since the operation $*$ is commutative, we see that $\mathcal{A}$ is a commutative BH-algebra, that is, $\mathcal{A}$ is a PC-algebra. □

The converse of Proposition 3.10 does not hold in general. The PC-algebra $(A; *, 0)$ given in Example 4.7 of [8] is not a BN_1-algebra, since $(2 * 1) * 1 = 3 \neq 2$.

Remark 3.11. Let $A = \{0, 1, 2\}$ and $*$ be defined by the following table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(A; *, 0)$ is a BM-algebra (see [6]) but it is not a PC-algebra. Consequently, $\mathbf{BM} \not\subseteq \mathbf{PC}$. Hence $\mathbf{BM} \not\subseteq \mathbf{BN}_1$.

Remark 3.12. The BN_1-algebra given in Example 3.3 is not a BM-algebra, since $(1 * 3) * (1 * 4) = 2 * 6 = 5 \neq 7 = 4 * 3$. Therefore, $\mathbf{BN}_1 \not\subseteq \mathbf{BM}$ and hence $\mathbf{PC} \not\subseteq \mathbf{BM}$.

From (1)–(3), Proposition 3.10, and Remarks 3.11 and 3.12 we obtain the following interrelationships between some of the class of algebras mentioned above.

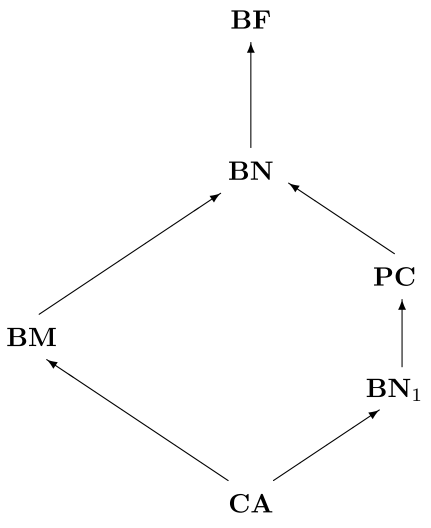

Figure 2

Lemma 3.13. *If an algebra $\mathcal{A}$ satisfies the commutative law* (C) *and* (B2'), *then condition* (As) *implies condition* (B).

Proof. Using associativity, commutativity and (B2') we obtain

$$(x * y) * z = x * (y * z) = x * (z * y) = x * (z * (0 * y))$$

for all $x, y, z \in A$, that is, (B) holds.

Lemma 3.14. *If $\mathcal{A}$ satisfies* (B2'), *then condition* (B) *implies condition* (D).

Proof. Let $x, y, z \in A$. We have

$$(x * y) * z = x * (z * (0 * y)) = x * (z * y),$$

i.e., (D) is true in $\mathcal{A}$. □

Lemma 3.15. *Let $\mathcal{A}$ satisfy* (C). *Then* (D) *implies* (BCH).

Proof. Let $x, y, z \in A$. Applying commutativity of $*$ and (D) we get

$$(x * y) * z = (y * x) * z = y * (z * x) = (x * z) * y.$$

Thus (BCH) is valid in $\mathcal{A}$. □

Lemma 3.16. *Let $\mathcal{A}$ satisfy* (C) *and* (BN$_1$). *Then* (BCH) *implies* (BM).

Proof. Let $x, y, z \in A$. Using (BCH), (C) and (BN$_1$) we have

$$(x * y) * (x * z) = (x * (x * z)) * y = ((z * x) * x)) * y = z * y,$$

i.e., (BM) holds in $\mathcal{A}$. □

Lemma 3.17. *Let* (B1) *hold in $\mathcal{A}$. Then condition* (BM) *implies condition* (BCI).

Proof. By (BM) and (B1),

$$((x * y) * (x * z)) * (z * y) = (z * y) * (z * y) = 0$$

for all $x, y, z \in A$. This means that $\mathcal{A}$ satisfies (BCI). □

Lemma 3.18. *In BN_1-algebras,* (BCI) *implies* (As).

Proof. Let $\mathcal{A}$ be a BN$_1$-algebra satisfying (BCI) and $x, y, z \in A$. Then $((x * y) * (x * z)) * (z * y) = 0$. By (P5), $(x * y) * (x * z) = z * y$. Therefore $\mathcal{A}$ is a BM-algebra. From (P1) we see that (B2') holds in $\mathcal{A}$. Applying Proposition 2.6 we get (As). □

From Lemmas 3.13 – 3.18 we have the following result.

Theorem 3.19. *In a BN_1-algebra, the conditions* (As), (B), (D), (BCH), (BM), *and* (BCI) *are all equivalent.*

Corollary 3.20. *An algebra $\mathcal{A} = (A; *, 0)$ is a Coxeter algebra if and only if it is a BN_1-algebra with the associative law for $*$.*

Remark 3.21. From Theorem 3.19 it follows that

$$\mathbf{B} \cap \mathbf{BN}_1 = \mathbf{BCH} \cap \mathbf{BN}_1 = \mathbf{BM} \cap \mathbf{BN}_1 = \mathbf{BCI} \cap \mathbf{BN}_1 = \mathbf{CA}.$$

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ANDRZEJ WALENDZIAK
INSTITUTE OF MATHEMATICS AND PHYSICS
SIEDLCE UNIVERSITY OF NATURAL SCIENCES AND HUMANITIES
3 MAJA 54, PL-08110 SIEDLCE, POLAND
E-mail address: walent@interia.pl

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