Online
ISSN 1346-0447

ISSN 1346-0862

Scientiae Mathematicae Japonicae

(Scientiae Mathematicae / Mathematica Japonica New Series)

Vol. 78, No. 2 Whole Number 279

October 2015

International Society for Mathematical Sciences

INTERNATIONAL SOCIETY FOR MATHEMATICAL SCIENCES
Scientiae Mathematicae Japonicae, Notices from the ISMS

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Scientiae Mathematicae Japonicae 78, No.2(2015)(113-119)

A NOTE ON INTUITIONISTIC FUZZY n-RACKS

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Received June 28,2013

Abstract. In this paper we apply the concept of intuitionistic fuzzy sets to n-racks, $n \geq 2$. Several related results are established. In particular, we discuss some properties of normality and maximality of intuitionistic fuzzy n-racks using their (α, β)-cut sets.

1 Introduction In [3], the author introduced the category of n-racks as a generalization of racks [6], and studied n-subracks in [4]. Intuitionistic fuzzy sets were introduced by Krassimiri T. Atanassov [1] as a generalization of the concept of fuzzy sets introduced by Zadeh [9] in the 60s. They have been applied to several algebraic concepts such as equivalence relations [2], congruences [7] and groups [8]. In this work, we develop this concept on n-racks. In particular we extend some results established in [5] on fuzzy n-racks to intuitionistic fuzzy n-racks.

Let us recall a few definitions. A n-rack[1][3] $(R, [-, \ldots, -]_R)$ is a set R endowed with an n-ary operation $[-, \ldots, -]_R : R \times R \times \ldots \times R \longrightarrow R$ such that

- $\big[x_1, \ldots, x_{n-1}, [y_1, \ldots, y_{n-1}]_R\big]_R = \big[[x_1, \ldots, x_{n-1}, y_1]_R, \ldots, [x_1, \ldots, x_{n-1}, y_n]_R\big]_R$ (This is the left distributive property of n-racks)
- For $a_1, \ldots, a_{n-1}, b \in R$, there is a unique $x \in R$ with $[a_1, \ldots, a_{n-1}, x]_R = b$.

If in addition there is a distinguish element $1 \in R$, such that $[1, \ldots, 1, y]_R = y$ and $[x_1 \ldots, x_{n-1}, 1]_R = 1$ for all $x_1, \ldots, x_{n-1} \in R$, then $(R, [-, \ldots, -]_R, 1)$ is said to be a pointed n-rack.

- A n-rack R is involutive if it further satisfies

$$\big[x_1, \ldots, x_{n-1}, [x_1, \ldots, x_{n-1}, y]\big] = y \quad \text{for all } x_1, \ldots, x_{n-1}, y \in R.$$

- A n-rack R is trivial if it further satisfies $[x_1, x_2, \ldots, x_{n-1}, y]_R = y$ for all $x_i, y \in R$.
- A n-rack is a n-quandle if it further satisfies $[x_1, x_2, \ldots, x_{n-1}, y]_R = y$ if $x_i = y$ for some $i \in \{1, 2, \ldots, n-1\}$.
- A non empty subset S of a n-rack (resp. pointed n-rack) R is called n-semisubrack of R if S is closed under the n-rack operation. S is called n-subrack of R if it has a n-rack structure (resp. pointed n-rack structure).

[1] In this paper, we mean by a n-rack, a left n-rack.

Keywords and phrases: Intuitionistic fuzzy n-racks, n-racks.

1991Mathematics Subject Classification: Primary 03E72, 20N15, 20N25

2 intuitionistic fuzzy n-subracks Recall from [1] that for a set R, an intuitionistic fuzzy set S in R is an object $S = \big\{\langle x, \mu_S(x), \nu_S(x)\rangle : x \in R\big\}$, where $\mu_S : R \to [0,1]$ and $\nu_S : R \to [0,1]$ are two functions satisfying $0 \leq \mu_S(x) + \nu_S(x) \leq 1$ for all $x \in R$. Also $\mu_S(x)$ and $\nu_S(x)$ define respectively the degree of membership and the degree of non-membership of $x \in R$. We say that S is constant if μ_S or ν_S is constant. Note that when $\mu_S(x) + \nu_S(x) = 1$ for all $x \in R$, S is a fuzzy set. Also for two intuitionistic fuzzy sets $S_1 = \big\{\langle x, \mu_{S_1}(x), \nu_{S_1}(x)\rangle : x \in R\big\}$ and $S_2 = \big\{\langle x, \mu_{S_2}(x), \nu_{S_2}(x)\rangle : x \in R\big\}$, one says that $S_1 \subseteq S_2$ if and only if $\mu_{S_1}(x) \leq \mu_{S_2}(x)$ and $\nu_{S_1}(x) \geq \nu_{S_2}(x)$ for all $x \in R$. Throughout the paper, we consider only intuitionistic fuzzy sets that are not fuzzy sets.

Definition 2.1. *Let R be a n-rack. An intuitionistic fuzzy set $S = \big\{\langle x, \mu_S(x), \nu_S(x)\rangle : x \in R\big\}$ in R is said to be an intuitionistic fuzzy n-semisubrack of R if for any $x_1, \dots, x_n \in R$,*

i) $\mu_S([x_1, \dots, x_n]) \geq min\{\mu_S(x_1), \dots, \mu_S(x_n)\}$

ii) $\nu_S([x_1, \dots, x_n]) \leq max\{\nu_S(x_1), \dots, \nu_S(x_n)\}$

iii) $\mu_S(1) \geq \mu_S(x)$ *and* $\nu_S(1) \leq \nu_S(x)$ *for all* $x \in R$ *if the rack is pointed by 1.*

Definition 2.2. *[8] Let S be an intuitionistic fuzzy set of a set R. The $(\alpha, \beta) - cut$ of S is a crisp subset $C_{\alpha,\beta}(S)$ of S given by*

$$C_{\alpha,\beta}(S) = \big\{x \in R \;/\; \mu_S(x) \geq \alpha,\ \nu_S(x) \leq \beta\big\}$$

where $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$.

The following is a characterization of intuitionistic fuzzy n-semisubracks by means of $(\alpha, \beta) - cut$ sets.

Proposition 2.3. *Let R be a n-rack. The intuitionistic fuzzy set $S = \big\{\langle x, \mu_S(x), \nu_S(x)\rangle : x \in R\big\}$ is an intuitionistic fuzzy n-semisubrack of R if and only if for every $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$, the $(\alpha, \beta) - cut$ of S is a n-semisubrack of R when it is non empty.*

Proof. $\Rightarrow$) Let $\alpha, \beta \in [0,1]$. Assume that $C_{\alpha,\beta}(S) \neq 0$ and let $\{a_i\}_{i=1,\dots,n-1} \subseteq C_{\alpha,\beta}(S)$. Then as S is an intuitionistic fuzzy n-semisubrack, we have

$$\mu_S([a_1, \dots, a_n]) \geq min\{\mu_S(a_1), \dots, \mu_S(a_{n-1}), \mu_S(a_n)\} \geq \alpha$$

and

$$\nu_S([a_1, \dots, a_n]) \leq max\{\nu_S(a_1), \dots, \nu_S(a_{n-1}), \nu_S(a_n)\} \leq \beta,$$

i.e. $[a_1, \dots, a_{n-1}, a_n] \in C_{\alpha,\beta}(S)$. So $C_{\alpha,\beta}(S)$ is closed under the n-rack operation and thus it is a n-semisubrack of R.

$\Leftarrow$) We proceed by contradiction. Assume S is not an intuitionistic fuzzy n-semisubrack of R. So there are $x_1, \dots, x_n \in R$ with either $\mu_S([x_1, \dots, x_n]) < min\{\mu_S(x_1), \dots, \mu_S(x_n)\}$ or $\nu_S([x_1, \dots, x_n]) > max\{\nu_S(x_1), \dots, \nu_S(x_n)\}$. Without loss of generality, consider the first case. Then setting

$$\alpha_0 = \frac{min\{\mu_S(x_1), \dots, \mu_S(x_n)\} + \mu_S([x_1, \dots, x_n])}{2}$$

yields to the compound inequality

$$0 \leq \mu_S([x_1, \dots, x_n]) < \alpha_0 \leq min\{\mu_S(x_1), \dots, \mu_S(x_n)\} \leq \mu_S(x_i)$$

for all $i = 1, \dots, n$. Choose $\beta_0 \in [0,1]$ such that $\alpha_0 + \beta_0 \leq 1$ and $\nu_S(x_i) \geq \beta_0$ for all $i = 1, \dots, n$. Hence $x_i \in C_{\alpha_0,\beta_0}(S)$ for all $i = 1, \dots, n$ and $[x_1, \dots, x_n] \notin C_{\alpha_0,\beta_0}(S)$. This contradicts the fact that $C_{\alpha_0,\beta_0}(S)$ is a n-semisubrack of R. The proof for the second case is similar.

□

Definition 2.4. *Let R be a n-rack. An intuitionistic fuzzy set $S = \big\{\langle x, \mu_S(x), \nu_S(x)\rangle : x \in R\big\}$ in R is said to be an intuitionistic fuzzy n-subrack of R if for any $x_1, \dots, x_{n-1}, y \in R$,*

i) $\mu_S(y) \geq min\{\mu_S([x_1, \dots, x_{n-1}, y]), \mu_S(x_1), \dots, \mu_S(x_{n-1})\}$

ii) $\nu_S(y) \leq max\{\nu_S([x_1, \dots, x_{n-1}, y]), \nu_S(x_1), \dots, \nu_S(x_{n-1})\}$

iii) $\mu_S(1) \geq \mu_S(x)$ *and* $\nu_S(1) \leq \nu_S(x)$ *for all* $x \in R$ *if the rack is pointed by 1.*

Example 2.5. *Consider the $(t,s)-n$-rack M of example 2.3 in [3] with $n = 4,\ s = 1\ ,t = 0$ and $M = \mathbb{N}$. Then M is a 4-rack with rack operation $[x_1, x_2, x_3, x_4] = x_1 + x_2 + x_3$. Define on M the intuitionistic fuzzy set $S = \big\{\langle x, \mu_S(x), \nu_S(x)\rangle : x \in R\big\}$ by*

$$\mu_S(x) = \begin{cases} \frac{1}{4}, & if\ \ x\ is\ odd \\ 0, & if\ \ x\ is\ even \end{cases} \qquad and \qquad \nu_S(x) = \begin{cases} 0, & if\ \ x\ is\ odd \\ \frac{1}{4}, & if\ \ x\ is\ even \end{cases}.$$

A case by case checking shows that S is an intuitionistic fuzzy 4-semisubrack. However, S is not an intuitionistic fuzzy 4-subrack because for $x_1 = 1,\ x_2 = 3,\ x_3 = 5$ and $x_4 = 2$, we have $\mu_S([x_1, x_2, x_3, x_4]) = \mu_S(9) = \frac{1}{4}$ and so
$\mu_S(x_4) = 0 < \frac{1}{4} = min\{\mu_S([x_1, x_2, x_3, x_4]), \mu_S(x_1), \mu_S(x_2), \mu_S(x_3)\}$.

Example 2.6. *Consider the quandle (containing the dihedral rack $D = \{a, b, c\}$ as a subquandle) $(R = \{1, a, b, c\}, \circ)$ whose Cayley table is given by:*

$\circ$	1	a	b	c
1	1	a	b	c
a	1	a	c	b
b	1	c	b	a
c	1	b	a	c

It is easy to show that the intuitionistic fuzzy set $S = \big\{\langle x, \mu_S(x), \nu_S(x)\rangle : x \in R\big\}$ on R defined by

$$\mu_S(x) = \begin{cases} \frac{1}{2}, & if\ \ x = 1, a \\ \frac{1}{8}, & if\ \ x = b, c \end{cases} \quad and \quad \nu_S(x) = \begin{cases} \frac{1}{2}, & if\ \ x = 1, a \\ \frac{3}{4}, & if\ \ x = b, c \end{cases}$$

is an intuitionistic fuzzy subrack of R.

Theorem 2.7. *[4] A n-semisubrack S of a pointed n-rack $(R, [-, \dots, -], 1)$ is a n-subrack if and only if for all $b \in R$, $[a_1, a_2, \dots, a_{n-1}, b] \in S$ and $\{a_i\}_{i=1,\dots,n-1} \subseteq S$ implies $b \in S$.*

The following is a characterization of intuitionistic fuzzy n-subracks by means of $(\alpha, \beta)-$ *cut* sets.

Proposition 2.8. *Let R be a n-rack. The intuitionistic fuzzy set $S = \big\{\langle x, \mu_S(x), \nu_S(x)\rangle : x \in R\big\}$ is an intuitionistic fuzzy n-subrack of R if and only if for every $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$, the $(\alpha, \beta) - cut$ of S is a n-subrack of R when it is non empty.*

Proof. $\Rightarrow$) Let $\alpha, \beta \in [0,1]$. Assume that $C_{\alpha,\beta}(S) \neq 0$ and let $\{a_i\}_{i=1,\ldots,n-1} \subseteq C_{\alpha,\beta}(S)$ with $[a_1, \ldots, a_{n-1}, b] \in C_{\alpha,\beta}(S)$. Then $\mu_S([a_1, \ldots, a_{n-1}, b]) \geq \alpha$, $\mu_S(a_i) \geq \alpha$ and $\nu_S([a_1, \ldots, a_{n-1}, b]) \leq \beta$, $\nu_S(a_i) \leq \beta$ for $i = 1, \ldots, n-1$. Now as S is an intuitionistic fuzzy n-subrack of R, we have

$$\mu_S(b) \geq min\{\mu_S([a_1, \ldots, a_{n-1}, b]), \mu_S(a_1), \ldots, \mu_S(a_{n-1})\} \geq \alpha$$

and

$$\nu_S(b) \leq max\{\nu_S([a_1, \ldots, a_{n-1}, b]), \nu_S(a_1), \ldots, \nu_S(a_{n-1})\} \leq \beta,$$

i.e. $b \in C_{\alpha,\beta}(S)$. So $C_{\alpha,\beta}(S)$ is a n-subrack of R.
$\Leftarrow$) We proceed by contradiction. Assume S is not an intuitionistic fuzzy n-subrack of R. So there are $x_1^0, \ldots, x_{n-1}^0, y_0 \in R$ with either

$$\mu_S(y_0) < min\{\mu([x_1^0, \ldots, x_{n-1}^0, y_0]), \mu(x_1^0), \ldots, \mu(x_{n-1}^0)\}$$

or

$$\nu_S(y_0) > max\{\nu([x_1^0, \ldots, x_{n-1}^0, y_0]), \nu(x_1^0), \ldots, \nu(x_{n-1}^0)\}.$$

Without loss of generality, consider the first case. Setting

$$\alpha_0 = \frac{min\{\mu_S([x_1^0, \ldots, x_{n-1}^0, y_0]), \mu_S(x_1^0), \ldots, \mu_S(x_{n-1}^0)\} + \mu_S(y_0)}{2}$$

yields to the compound inequality

$$0 \leq \mu_S(y_0) < \alpha_0 \leq min\{\mu_S([x_1^0, \ldots, x_{n-1}^0, y_0]), \mu_S(x_1^0), \ldots, \mu_S(x_{n-1}^0)\} \leq \mu_S(x_i^0).$$

Choose $\beta_0 \in [0,1]$ such that $\alpha_0 + \beta_0 \leq 1$ and $\nu_S(x_i^0) \geq \beta_0$ for all $i = 1, \ldots, n-1$. So $[x_1^0, \ldots, x_{n-1}^0, y_0] \in C_{\alpha_0,\beta_0}(S)$, $x_i^0 \in C_{\alpha_0,\beta_0}(S)$ for all $i = 1, \ldots, n-1$ and $y_0 \notin C_{\alpha_0,\beta_0}(S)$. This contradicts by theorem 2.7 the fact that $C_{\alpha_0,\beta_0}(S)$ is a n-subrack of R. The proof for the second case is similar.

□

Remark 2.9. *If R is an involutive n-subrack, one shows by theorem 2.7 that n-semisubracks and n-subracks coincide. It follows by proposition 2.8 and proposition 2.3 that intuitionistic fuzzy n-subracks and intuitionistic fuzzy n-semisubracks coincide in involutive n-racks (thus in trivial n-racks).*

Proposition 2.10. *Let S be a n-subrack of R. Then S can be realized as a $(\alpha, \beta) - cut$ of some intuitionistic fuzzy n-subrack of R.*

Proof. Choose $r, s \in [0,1]$ with $s < r$. Consider the fuzzy set on R defined by

$$\mu_S(x) = \begin{cases} r, & \text{if } x \in S \\ s, & \text{else.} \end{cases} \quad and \quad \nu_S(x) = \begin{cases} s, & \text{if } x \in S \\ r, & \text{else.} \end{cases}$$

We claim that the set $\tilde{S} = \{\langle x, \mu_S, \nu_S\rangle : x \in R\}$ is an intuitionistic fuzzy n-subrack of R. In fact, a case by case checking shows that the inequalities
$\mu_S(x_n) \geq min\{\mu_S([x_1, \ldots, x_{n-1}, x_n]), \mu_S(x_1), \ldots, \mu_S(x_{n-1})\}$ and
$\nu_S(x_n) \leq max\{\nu_S([x_1, \ldots, x_{n-1}, x_n]), \nu_S(x_1), \ldots, \nu_S(x_{n-1})\}$ fail only if $x_n \notin S$, $[x_1, \ldots, x_n] \in S$ and $x_i \in S$ for all $i = 1, \ldots, n-1$. But this can't occur by theorem 2.7 as S is a n-subrack of R. Moreover, it is clear that for any choice of $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1, \alpha \leq r$ and $\beta \geq s$, we have $C_{\alpha,\beta}(\tilde{S}) = S$. □

Corollary 2.11. *Let S be a n-subrack of R. For each $\alpha, \beta \in (0,1]$ with $\alpha+\beta \leq 1$, there is an intuitionistic fuzzy n-subrack $\tilde{S} = \{\langle x, \mu_S(x), \nu_S(x)\rangle : x \in R\}$ of R with $C_{\alpha,\beta}(\tilde{S}) = S$.*

Proof. The result follows by the proof of Proposition 2.10. □

3 Normal and Maximal Intuitionistic Fuzzy n-Subracks Throughout this section, R denotes a pointed n-rack.

Definition 3.1. *A normal intuitionistic fuzzy n-subrack of R is an intuitionistic fuzzy n-subrack $S = \{\langle x, \mu_S(x), \nu_S(x)\rangle : x \in R\}$ of R such that $1 \in (\mu_S^{-1} + \nu_S^{-1})(1)$.*

Proposition 3.2. *Every intuitionistic fuzzy n-subrack of R can be embedded into a normal intuitionistic fuzzy n-subrack of R.*

Proof. Let S be an intuitionistic fuzzy n-subrack of R. If S is normal, there is nothing to prove. Otherwise, let $p, q \in [0,1]$ such that $\mu_S(1) \leq p,\ \nu_S(1) \geq q$ and $p+q=1$. Consider on R the functions ζ_S and ζ'_S defined by $\zeta_S(x) = \mu_S(x) - \mu_S(1) + p$ and $\zeta'_S(x) = \nu_S(x) - \nu_S(1) + q$. Clearly, ζ_S and ζ'_S are well-defined, $(\zeta_S + \zeta'_S)(1) = 1$, $\zeta_S(x) \geq \mu_S(x)$ and $\zeta'_S(x) \leq \nu_S(x)$ for all $x \in R$. Also, for $x_1, x_2, \ldots, x_n \in R$ we have

$$\begin{aligned}
\zeta_S(x_n) &= \mu_S(x_n) - \mu_S(1) + p \\
&\geq min\{\mu_S([x_1,\ldots,x_n]), \mu_S(x_1), \ldots, \mu_S(x_{n-1})\} - \mu_S(1) + p \\
&\geq min\{\mu_S([x_1,\ldots,x_n]) - \mu_S(1) + p, \mu_S(x_1) - \mu_S(1) + p, \ldots, \mu_S(x_{n-1}) - \mu_S(1) + p\} \\
&\geq min\{\zeta_S([x_1,\ldots,x_n]), \zeta_S(x_1), \ldots, \zeta_S(x_{n-1})\},
\end{aligned}$$

$$\begin{aligned}
\zeta'_S(x_n) &= \nu_S(x_n) - \nu_S(1) + q \\
&\leq max\{\nu_S([x_1,\ldots,x_n]), \nu_S(x_1), \ldots, \nu_S(x_{n-1})\} - \nu_S(1) + q \\
&\leq max\{\nu_S([x_1,\ldots,x_n]) - \nu_S(1) + q, \nu_S(x_1) - \nu_S(1) + q, \ldots, \nu_S(x_{n-1}) - \nu_S(1) + q\} \\
&\leq max\{\zeta'_S([x_1,\ldots,x_n]), \zeta'_S(x_1), \ldots, \zeta'_S(x_{n-1})\},
\end{aligned}$$

and $\zeta_S(1) \geq \zeta_S(x)$ and $\zeta'_S(1) \leq \zeta'_S(x)$ for all $x \in R$ since $\mu_S(1) \geq \mu_S(x)$ and $\nu_S(1) \leq \nu_S(x)$ for all $x \in R$.

Hence the set $\{\langle x, \zeta_S(x), \zeta'_S(x)\rangle : x \in R\}$ is a normal intuitionistic fuzzy n-subrack containing S. □

Definition 3.3. *Let S_1 and S_2 be two intuitionistic fuzzy n-subracks of R. We say*[2] *that $S_1 \subseteq_{ae} S_2$ if the set $\{x \in R \ /\ \mu_{S_1}(x) \geq \mu_{S_2}(x),\ \nu_{S_1}(x) \leq \nu_{S_2}(x)\} = \{1\}$.*

Remark 3.4. *It is not hard to check that this relation is an order. Under this relation, the intuitionistic fuzzy set $\{\langle x, \zeta_S(x), \zeta'_S(x)\rangle : x \in R\}$ above in the proof of proposition 3.2 is the smallest normal intuitionistic fuzzy n-subrack of R containing S. Denote it $\bar{S} = \{\langle x, \bar{\mu}_S(x), \bar{\nu}_S(x)\rangle : x \in R\}$.*

Definition 3.5. *When $p = \frac{1}{2}$ and $q = \frac{1}{2}$, $\bar{S}$ is called the normal closure of S.*

Definition 3.6. *A non constant intuitionistic fuzzy n-subrack of R is said to be maximal if its normal closure is maximal among normal intuitionistic fuzzy n-subracks of R.*

Theorem 3.7. *Every maximal intuitionistic fuzzy n-subrack of R is normal.*

[2]Read $S_1 \subseteq_{ae} S_2$ as "$S_1 \subseteq S_2$" almost everywhere

Proof. Let S be a maximal intuitionistic fuzzy n-subrack of R. If $\mu_S(1)+\nu_S(1)=1$, then S is normal and $S=\bar{S}$. Assume $\mu_S(1)+\nu_S(1)\neq 1$ and define an intuitionistic fuzzy set S_0 on R by $S_0=\big\{\langle x,\zeta_{S_0}(x),\zeta'_{S_0}(x)\rangle : x\in R\big\}$ with $\zeta_{S_0}(x)=\frac{\mu_S(x)+\mu_S(1)}{2}$ and $\zeta'_{S_0}(x)=\frac{\nu_S(x)+\nu_S(1)}{2}$. Clearly, S_0 is an intuitionistic fuzzy n-subrack of R since for $x_1,x_2,\dots,x_n\in R$ we have

$$\begin{aligned}
\zeta_{S_0}(x_n) &= \frac{\mu_S(x_n)+\mu_S(1)}{2}\\
&\geq \frac{min\{\mu_S([x_1,\dots,x_n]),\mu_S(x_1),\dots,\mu_S(x_{n-1})\}+\mu_S(1)}{2}\\
&\geq min\{\frac{\mu_S([x_1,\dots,x_n])+\mu_S(1)}{2},\frac{\mu_S(x_1)+\mu_S(1)}{2},\dots,\frac{\mu_S(x_{n-1})+\mu_S(1)}{2}\}\\
&\geq min\{\zeta_{S_0}([x_1,\dots,x_n]),\zeta_{S_0}(x_1),\dots,\zeta_{S_0}(x_{n-1})\},
\end{aligned}$$

$$\begin{aligned}
\zeta'_{S_0}(x_n) &= \frac{\nu_S(x_n)+\nu_S(1)}{2}\\
&\leq \frac{max\{\nu_S([x_1,\dots,x_n]),\nu_S(x_1),\dots,\nu_S(x_{n-1})\}+\nu_S(1)}{2}\\
&\leq max\{\frac{\nu_S([x_1,\dots,x_n])+\nu_S(1)}{2},\frac{\nu_S(x_1)+\nu_S(1)}{2},\dots,\frac{\nu_S(x_{n-1})+\nu_S(1)}{2}\}\\
&\leq max\{\zeta'_{S_0}([x_1,\dots,x_n]),\zeta'_{S_0}(x_1),\dots,\zeta'_{S_0}(x_{n-1})\}.
\end{aligned}$$

and $\zeta_{S_0}(1)\geq\zeta_{S_0}(x)$ and $\zeta'_{S_0}(1)\leq\zeta'_{S_0}(x)$ for all $x\in R$ since $\mu_S(1)\geq\mu_S(x)$ and $\nu_S(1)\leq\nu_S(x)$ for all $x\in R$. Moreover, $\zeta_{S_0}(1)=\mu_S(1)$, $\zeta'_{S_0}(1)=\nu_S(1)$ and $\mu_S(x_0)<\mu_S(1)$ and $\nu_S(x_0)>\nu_S(1)$ for some $x_0\in R$ as S is non constant. Let $\bar{S}_0=\big\{\langle x,\bar{\zeta}_{S_0}(x),\bar{\zeta}'_{S_0}(x)\rangle : x\in R\big\}$ be the normal closure of S_0. Then

$$\bar{\zeta}_{S_0}(x_0)=\zeta_{S_0}(x_0)-\zeta_{S_0}(1)+\frac{1}{2}=\zeta_{S_0}(x_0)-\mu_S(1)+\frac{1}{2}>\mu_S(x_0)-\mu_S(1)+\frac{1}{2}=\bar{\mu}_S(x_0)$$

and

$$\bar{\zeta}'_{S_0}(x_0)=\zeta'_{S_0}(x_0)-\zeta'_{S_0}(1)+\frac{1}{2}=\zeta'_{S_0}(x_0)-\nu_S(1)+\frac{1}{2}<\nu_S(x_0)-\nu_S(1)+\frac{1}{2}=\bar{\nu}_S(x_0).$$

This contradicts the maximality of $\bar{S}$ among the normal intuitionistic fuzzy n-subracks of R. Hence $\mu_S(1)+\nu_S(1)=1$ and S is normal. □

Theorem 3.8. *If S is a maximal intuitionistic fuzzy n-subrack of R, then*

$$Im(\mu_s+\nu_S)=\{0,1\}.$$

Proof. Assume S is a maximal intuitionistic fuzzy n-subrack of R. Then $\mu_S(1)+\nu_S(1)=1$ and $S=\bar{S}$ by theorem 3.7. Now let $x\in R$ with $0<\mu_S(x)+\nu_S(x)<1$. Define an intuitionistic fuzzy set S_0 on R by $S_0=\big\{\langle x,\zeta_{S_0}(x),\zeta'_{S_0}(x)\rangle : x\in R\big\}$ with $\zeta_{S_0}(x)=\frac{\mu_S(x)+\frac{1}{2}}{2}$ and $\zeta'_{S_0}(x)=\frac{\nu_S(x)+\frac{1}{2}}{2}$. Clearly, S_0 is an intuitionistic fuzzy n-subrack of R by the proof of theorem 3.7. Moreover S_0 is normal as S is normal. In addition, $\bar{\zeta}_{S_0}(x)=\zeta_{S_0}(x)>\mu_S(x)=\bar{\mu}_S(x)$ since $0<\mu_S(x)<\frac{1}{2}$ for all $x\in R$, and $\bar{\zeta}'_{S_0}(x)=\zeta'_{S_0}(x)<\nu_S(x)=\bar{\nu}_S(x)$ since $\nu_S(x)>\frac{1}{2}$ for all $x\in R$. Thus $\bar{S}\subseteq_{ae}\bar{S}_0$ because the set $\big\{x\in R\ /\ \bar{\zeta}_{S_0}(x)\geq\bar{\mu}_{S_2}(x),\ \bar{\zeta}'_{S_0}(x)\leq\bar{\nu}_{S_2}(x)\big\}\neq\{1\}$. This contradicts the maximality of $\bar{S}$ among the normal intuitionistic fuzzy n-subracks of R. Hence $\mu_S(1)+\nu_S(1)=0$ or $\mu_S(1)+\nu_S(1)=1$. □

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Communicated by *Junzo Watada*

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LOCAL COMPLETENESS AND PARETO EFFICIENCY IN PRODUCT SPACES

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Received August 27,2013 ; Revised June 7,2014

Abstract. In this paper we study the Pareto efficiency with respect to a locally nuclear cone in the product of two locally convex spaces with the restricted assumption that only some related sets are locally complete.

1. Introduction

Kuhn and Tucker in their famous paper [14] considered "proper solutions" for vector maximum problems and introduced the concept of proper efficient points. Hurwicz [7] introduced the notion of a proper maximal point with respect to ordering cones to characterize maximal points as solutions for optimization problems. Isac [8, 9, 11] used a method based on a general existence theorem for critical points of dynamical systems to obtain several general results on the existence of solutions of the general optimization problem in sequentially complete locally convex spaces. He introduced the concept of nuclear cone [9] in a locally convex space, intimately related to Pareto efficiency [8, 9, 10, 11, 12]. Also he defined a nuclear cone in a product of two locally convex spaces [12] to obtain maximal point theorems and a vectorial Ekeland type theorem.

After it was discovered, the Ekeland's principle [5] has had many different applications and extensions [6, 10, 11, 12, 20]. Qiu [21, 22, 23] and Bosch, García et al.[2, 3, 4] found some extensions of Ekeland's variational principle and Pareto efficiency assuming only local completeness conditions. In this paper by adapting ideas of Isac [12] we extend Pareto efficiency respect to locally nuclear cones in the product of two locally convex spaces only assuming that some related sets are locally complete. Also we stablish a vectorial Ekeland type theorem for locally complete spaces.

2. Preliminaries

Througout this paper (E, τ) will denote a locally convex space E, with topology τ generated by a family of seminorms $\{\rho_\alpha : \alpha \in \Lambda\}$ with Λ a set of indexes. A disk B in E is a closed, bounded and absolutely convex set. We denote by (E_B, ρ_B) the linear span of B endowed with the topology defined by the Minkowski functional associated with B. If (E_B, ρ_B) is complete then B is called a Banach disk. E' will denote the topological dual of (E, τ) and $(E_B, \rho_B)'$ will denote the topological dual of E_B with respect to the norm ρ_B.

Date: August 20, 2013.

1991 *Mathematics Subject Classification.* Primary 49J53; Secondary 46N10.

Key words and phrases. Banach disk, Local completeness, Generalized dynamical system, Nuclear cone, Pareto efficiency.

A sequence $(x_n)_n$ in E is said to be locally convergent or Mackey convergent to an element x in E if there exists a disk B in E such that the sequence converges to x in E_B with respect to ρ_B. A sequence is called locally Cauchy or Mackey Cauchy if it is a ρ_B-Cauchy sequence in E_B for a certain disk B in E.

Let C be a non-void subset of E. A point x is a local limit point of C if there is a sequence in C that is locally convergent to x. A set C is locally closed if every local limit point of C belongs to C.

A subset A of a space E is said to be locally complete if every local Cauchy sequence in A converges locally to a point of A. It is clear that every locally complete subset of a space is locally closed. For the whole space (E, τ), it is locally complete if and only if every disk B in E is in a Banach disk. And a locally closed subset A of a locally complete space E is locally complete. For more details on local completeness see [13, 17].

A closed pointed convex cone in a locally convex space is a nonempty subset $K \subset E$ such that:

(1) K is a closed convex subset,
(2) $K + K \subset K$,
(3) $\lambda K \subset K$ for all $\lambda \in \mathbb{R}^+$,
(4) $K \cap (-K) = \{0\}$.

If a closed, pointed convex cone $K \subset E$ is given we can define an ordering in E by $x \preceq_K y$ if and only if $y - x \in K$. For more details on order and cones see [16].

If $A \subset E$ is a nonempty subset we say that $a \in A$ is an efficient (maximal) point of A if $A \cap (a + K) = \{a\}$. We denote by $E(A; K)$ the set of efficient points of A with respect to K.

We say that $\Gamma : A \to 2^A$ is a dynamical system (in the generalized sense) if for every $x \in A$, $\Gamma(x)$ is a nonempty subset of A, and $x^* \in A$ is a critical point for Γ if $\Gamma(x^*) = \{x^*\}$. We can see easily that $\Gamma_A(x) = A \cap (x + K)$ for every $x \in A$ is a dynamical system. Note that for $y \in \Gamma_A(x) = A \cap (x + K)$ and $z \in \Gamma_A(y) = A \cap (y + K)$, we have $z \in A \cap ((x + K) + K) = A \cap (x + K)$. So $\Gamma_A(y) \subset \Gamma_A(x)$. The reader can verify that an element $x^* \in A$ is an efficient point of A if and only if x^* is a critical point of Γ_A. Following Aubin-Siegel, Muntean, Petrusel, Rus and Yao a critical point for a dynamical system is also known in the literature as an end or stationary point (see [1]) or a strict fixed point for a set valued operator (see [15, 18, 19]).

In [9] G. Isac introduced the concept of nuclear cone. The cone $K \subset (E, \tau)$ is said to be nuclear if for every ρ_α in the family of seminorms wich defines the topology τ there exists $f_\alpha \in E'$ such that $\rho_\alpha(x) \leq f_\alpha(x)$, for every $x \in K$. In [2], is proved the following

Corollary 1. *Let (E, τ) be a locally convex space and $K \subset E$ a closed, pointed convex cone. Suppose that there exists a non-zero Banach disk D in E and $f \in (E_D, \rho_D)'$, such that $K \cap E_D \neq \{0\}$ and $\rho_D(x) \leq f(x)$, for every $x \in K \cap E_D$. Suppose that for a nonempty locally closed subset $B \subset E$ we have $B \cap E_D \neq \emptyset$ and that f is bounded above in $B \cap E_D$; then for every $x_0 \in B \cap E_D$, there exists an element $x^* \in E(B; K)$ such that $x^* \in x_0 + K$.*

Note that in this corollary the property of nuclearity is applied *locally* to the cone in the space (E_D, ρ_D). Motivated by this condition, we say that a closed, pointed convex cone $K \subset E$ is locally nuclear with respect to the disk $D \subset E$ if there exists $f \in (E_D, \rho_D)'$ such that $\rho_D(x) \leq f(x)$, for every $x \in K \cap E_D$.

3. Main Results

In [2], the author and C. Bosch proved the following theorem

Theorem 1. *Let (E,τ) be a locally convex space, $A \subset E$ a nonempty subset and $K \subset E$ a closed, pointed convex cone. Suppose there exists A_0, a nonempty subset of A, such that:*
a) A_0 is locally complete,
b) $\Gamma_A(A_0) \subset A_0$,
c) There exists a Banach disk $D \subset E$ and $f \in (E_D, \rho_D)'$ such that $A_0 \subset E_D$ and
i) $\rho_D(v) \leq f(v)$, for $v \in K(A_0) = \{v \in K : v = v_1 - v_2;\, v_1, v_2 \in A_0\} \subset E_D$
ii) $\sup\{f(x) : x \in A_0\} < \infty$
Then $E(A,K)$ is nonempty.

We note, from the local completeness of A in the proof of this theorem, that is sufficient to ask B to be a disk, that is, the completeness of (E_B, ρ_B) is unnecessary. Now, from this theorem and the locally nuclear property for a cone, we obtain

Corollary 2. *Let (E,τ) be a locally convex space, $A \subset E$ a nonempty subset and $B \subset E$ a disk such that $A \subset E_B$. Let $K \subset E$ be a closed, pointed convex cone locally nuclear respect to B. Suppose there exists $x_0 \in A$ such that $A \cap (x_0 + K)$ is locally complete and bounded. Then $E\,(A,K)$ is nonempty.*

Theorem 2. *Let (E,τ) be a locally convex space, $K \subset E$ a closed, pointed convex cone and $A \subset E$ a locally complete subset. Suppose that given $x_0 \in A$ there exists a sequence $(x_n)_n \in A$ such that $x_{n+1} \in \Gamma_A(x_n)\setminus\{x_n\}$, for every $n \in \mathbb{N}$. Suppose there exists a disk $D \subset E$ such that $\lim_n R_D\left(\Gamma_A(x_n)\right) = 0$, where $R_D\left(\Gamma_A(x_n)\right) = \sup\{\rho_D(x-y) : x, y \in \Gamma(x_n)\}$. Then there exists $x^* \in E(A;K)$ such that $x_0 \preceq_K x^*$.*

Proof. According to the hypothesis, $x_{n+k} \in \Gamma(x_n)$ for every $n \in \mathbb{N}$, $k \in \mathbb{N}$. Since $\lim_n R_D\left(\Gamma_A(x_n)\right) = 0$, then there exists $n_0 \in \mathbb{N}$ such that $\Gamma_A(x_{n_0+k}) \subset E_D$, for every $k \in \mathbb{N}$. Then $\rho_D(x_{n+k} - x_n) \leq R_D\left(\Gamma_A(x_n)\right)$ for $n \geq n_0$ and $k \in \mathbb{N}$. So, the sequence $(x_n)_n \in A$ is locally Cauchy. Since A is locally complete, there exists $x^* \in A$ such that $\rho_D(x_n - x^*)$ converges to zero. Clearly, $x^* \in \bigcap_{n\in\mathbb{N}\cup\{0\}} \Gamma_A(x_n)$. Since $\lim_n R_D\left(\Gamma_A(x_n)\right) = 0$ then $R_D\left(\Gamma(x^*)\right) = 0$ and $\{x^*\} = \Gamma(x^*) = \bigcap_{n\in\mathbb{N}\cup\{0\}} \Gamma(x_n)$. And $x^* \in \Gamma_A(x_0)$ implies $x^* - x_0 \in K$, so $x_0 \preceq_K x^*$. □

Let (E,τ), (F,τ') be locally convex spaces and suppose that F is ordered by a closed, pointed convex cone $K \subset F$. Let $B \subset E$ and $D \subset F$ be disks. So, the space $E_B \times F_D$ is a normed space endowed with the topology generated by $\rho_B + q_D$. Let $K_D = K \cap F_D$ and suppose $K_D \neq \{0\}$. Find $k_0 \in K_D$ such that $q_D(k_0) = 1$. Consider the set $K_D^* = \{f \in (F_D, q_D)' \colon f(y) \geq 0 \text{ for every } y \in K_D\}$ and $\psi \in K_D^*$ such that $\psi(k_0) = 1$. Let $1 > \varepsilon > 0$. In $E_B \times F_D$ consider the set

$$K(\varepsilon, B, D) = \left\{(x,y) \in E_B \times F_D \colon\, y + \sqrt{\varepsilon}\,(\rho_B(x) + q_D(y))\, k_0 \in -K_D\right\}.$$

Proposition 1. *The set $K(\varepsilon, B, D)$ is a non-trivial, closed, pointed and nuclear cone in $(E_B \times F_D, \rho_B + q_D)$.*

Proof. Let (x,y); $(u,v) \in K(\varepsilon,B,D)$.

Since $\rho_B(x+u)+q_D(y+v) \le \rho_B(x)+\rho_B(u)+q_D(y)+q_D(v)$,
then $(\ y+v)+\sqrt{\varepsilon}\,(\rho_B(x+u)+q_D(y+v))\,k_0$
$= (y+v)+\sqrt{\varepsilon}\,(\rho_B(x)+\rho_B(u)+q_D(y)+q_D(v)-\gamma)\,k_0$, for some $\gamma \ge 0$,
$= (y+\sqrt{\varepsilon}\,(\rho_B(x)+q_D(y))\,k_0)+\ (v+\sqrt{\varepsilon}\,(\rho_B(u)+q_D(v))\,k_0)-\gamma k_0$
$\in -K_D-K_D-K_D=-K_D$.

Then $(x,y)+(u,v) \in K(\varepsilon,B,D)$.

Let $\lambda \in \mathbb{R}^+$ and $(x,y) \in K(\varepsilon,B,D)$, so $\lambda(x,y) \in K(\varepsilon,B,D)$, since
$\lambda y+\sqrt{\varepsilon}\,(\rho_B(\lambda x)+q_D(\lambda y))\,k_0 = \lambda\,(\ y+\sqrt{\varepsilon}\,(\rho_B(x)+q_D(y))\,k_0)$
$\in \lambda(-K_D)=-K_D$.

Note that, $K_D \cap (-K_D) = (K \cap (-K)) \cap F_D = \{0\}$.

Let $(x_n,y_n) \in K(\varepsilon,B,D)$ such that $(x_n,y_n) \to (x_0,y_0)$ with respect to the norm ρ_B+q_D. Then $x_n \to x_0$ respect to ρ_B and respect to τ and $y_n \to y_0$ respect to q_D and respect to τ'. Then $y_n+\sqrt{\varepsilon}\,(\rho_B(x_n)+q_D(y_n))\,k_0 \in -K_D$ converges to $y_0+\sqrt{\varepsilon}\,(\rho_B(x_0)+q_D(y_0))\,k_0$ respect to τ'. And $y_0+\sqrt{\varepsilon}\,(\rho_B(x_0)+q_D(y_0))\,k_0$ belongs to $-K_D = -K\cap F_D$ since K is τ'-closed in F and then $-K\cap F_D$ is q_D-closed in F_D.

Let $x \in E_B\backslash\{0\}$ and $y \in K_D\backslash\{0\}$. Since $\rho_B(x) > 0$ and $q_D(y) > 0$ then $y+\sqrt{\varepsilon}\,(\rho_B(x)+q_D(y))\,k_0 \in K_D\backslash\{0\}$, that is $(x,y) \in (E_B,F_D)\backslash K(\varepsilon,B,D)$. Recall that $q_D(k_0) = 1$, then $-k_0+\ \sqrt{\varepsilon}q_D(-k_0)k_0 = -(1-\sqrt{\varepsilon})k_0 \in -K_D$ and $k_0\ +\sqrt{\varepsilon}q_D(k_0)k_0 = (1+\sqrt{\varepsilon})k_0 \in K_D$. That means, $(0,-k_0) \in K(\varepsilon,B,D)$ and $(0,k_0) \notin K(\varepsilon,B,D)$. So, $K(\varepsilon,B,D)$ is a non-trivial, pointed and closed cone in $(E_B\times F_D,\rho_B+q_D)$.

Let us see it is nuclear in this space. For $\pi_2 : E_B\times F_D \to F_D$, where $\pi_2(x,y)=y$ and $\psi \in K_D^*$ such that $\psi(k_0)=1$. Let $\Psi : E_B\times F_D \to \mathbb{R}$, given by $\Psi(x,y)=\psi\circ\pi_2(x,y)=\psi(y)$, So, $\Psi \in (E_B\times F_D,\rho_B+q_D)'$. Then for every $(u,v)\in K(\varepsilon,B,D)$ there exists $k \in K_D$ such that $v+\sqrt{\varepsilon}\,(\rho_B(u)+q_D(v))\,k_0 = -k \in -K_D$. Then $\sqrt{\varepsilon}\,(\rho_B(u)+q_D(v))\,k_0 = -v-k$ and applying ψ we obtain $\sqrt{\varepsilon}\,(\rho_B(u)+q_D(v)) = \psi\,(\sqrt{\varepsilon}\,(\rho_B(u)+q_D(v))\,k_0) = -\psi(v)-\psi(k) \le -\psi(v) = -\Psi(u,v)$.

Let $T : E_B\times F_D \to \mathbb{R}$, such that $T=-\Psi$. Then $\sqrt{\varepsilon}\,(\rho_B(u)+q_D(v)) \le T(u,v)$, for every $(u,v)\in K(\varepsilon,B,D)$. So, $T \in (K(\varepsilon,B,D))^*$ and $K(\varepsilon,B,D)$ is nuclear in $(E_B\times F_D,\rho_B+q_D)$. □

Theorem 3. *Let (E,τ), (F,τ') be locally convex spaces and suppose that F is ordered by a closed, pointed convex cone $K \subset F$. Let $A \subset E\times F$ be a nonempty locally complete subset and $B \subset E$, $D \subset F$ disks such that $A \subset E_B\times F_D$. Let $k_0 \in K_D = K\cap F_D$ be an element such that $q_D(k_0)=1$. For $1 > \varepsilon > 0$ consider $K(\varepsilon,B,D)$. Suppose there exists $z_0 \in F_D$ such that $\{y \in F_D\colon (x,y) \in A$ for some $x \in E_B\} \subset z_0+K_D$. Then for every $(x_0,y_0) \in A$ there exists $(x^*,y^*) \in A$ satisfying*

i) $(x^*,y^*) \in A\cap[(x_0,y_0)+K(\varepsilon,B,D)]$

ii) $A\cap[(x^*,y^*)+K(\varepsilon,B,D)] = \{(x^*,y^*)\}$

Proof. Let $T \in [K(\varepsilon,B,D)]^*$ be as the constructed in Proposition 5. Then
$K(\varepsilon,B,D) \subset \{(x,y)\in E_B\times F_D\colon \sqrt{\varepsilon}\,(\rho_B(x)+q_D(y)) \le T(x,y)\}$
$= \{(x,y)\in E_B\times F_D\colon \Psi(x,y)+\sqrt{\varepsilon}\,(\rho_B(x)+q_D(y)) \le 0\}$;
for $\Psi(x,y) = -T(x,y) = \psi(y)$.

Let $(u,v) \in A$, then $v \in z_0+K_D$, and $v-z_0 \in K_D$. Since $\psi \in K_D^*$, then $\psi(v-z_0) \ge 0$ and $\Psi(u,v) = \psi(v) \ge \psi(z_0)$. Hence Ψ is bounded from below on A. Consider the generalized dynamical system $\Gamma : A \to 2^A$, such that $\Gamma(x,y) =$

$A \cap [(x,y) + K(\varepsilon, B, D)]$. We will define an inductive sequence in A. Starting from (x_0, y_0), suppose $(x_n, y_n) \in A$ is defined and $\Gamma(x_{k+1}, y_{k+1}) \subset \Gamma(x_k, y_k)$, for $k = 0, 1, ..., n-1$. If we have $\Gamma(x_n, y_n) = \{(x_n, y_n)\}$, then we have finished. So, if we have $\Gamma(x_n, y_n) \neq \{(x_n, y_n)\}$ for every $n \in \mathbb{N}$, we have to find (x^*, y^*).

For every $(x,y) \in \Gamma(x_n, y_n) \setminus \{(x_n, y_n)\}$, $\rho_B(x_n - x) + q_D(y_n - y) > 0$.

Since $K(\varepsilon, B, D) \subset \{(x,y) \in E_B \times F_D\colon\ \Psi(x,y) + \sqrt{\varepsilon}\,(\rho_B(x) + q_D(y)) \leq 0\}$ then $\Psi(x,y) - \Psi(x_n, y_n) + \sqrt{\varepsilon}\,(\rho_B(x - x_n) + q_D(y - y_n)) \leq 0$, for $(x,y) \in \Gamma(x_n, y_n) \setminus \{(x_n, y_n)\}$. So,

$$\Psi(x,y) \leq \Psi(x_n, y_n) - \sqrt{\varepsilon}\,(\rho_B(x - x_n) + q_D(y - y_n)) < \Psi(x_n, y_n), \tag{3.1}$$

for $(x,y) \in \Gamma(x_n, y_n) \setminus \{(x_n, y_n)\}$. Then

$$0 < \Psi(x_n, y_n) - \Psi(x,y) \leq \Psi(x_n, y_n) - \inf_{(x,y)\in\Gamma(x_n,y_n)} \Psi(x,y).$$

So, for every $n \in \mathbb{N} \cup \{0\}$ there exists $(x_{n+1}, y_{n+1}) \in \Gamma(x_n, y_n)$ such that

$$\Psi(x_{n+1}, y_{n+1}) < \inf_{(x,y)\in\Gamma(x_n,y_n)} \Psi(x,y) + \frac{1}{2}\left[\Psi(x_n, y_n) - \inf_{(x,y)\in\Gamma(x_n,y_n)} \Psi(x,y)\right],$$

And $\Gamma(x_{k+1}, y_{k+1}) \subset \Gamma(x_k, y_k)$, for every $k \in \mathbb{N} \cup \{0\}$. And from the previous inequality, for $(s,t) \in \Gamma(x_{k+1}, y_{k+1})$ we have

$$\Psi(x_{k+1}, y_{k+1}) - \Psi(s,t) \leq \Psi(x_{k+1}, y_{k+1}) - \inf_{(v,w)\in\Gamma(x_{k+1},y_{k+1})} \Psi(v,w)$$

$$\leq \Psi(x_{k+1}, y_{k+1}) - \inf_{(v,w)\in\Gamma(x_k,y_k)} \Psi(v,w) \leq \frac{1}{2}\left[\Psi(x_k, y_k) - \inf_{(v,w)\in\Gamma(x_k,y_k)} \Psi(v,w)\right]$$

$$\leq \frac{1}{2}\left[\Psi(x_0, y_0) - \inf_{(v,w)\in\Gamma(x_k,y_k)} \Psi(v,w)\right]$$

by (3.1), since $(x_k, y_k) \in \Gamma(x_0, y_0)$. So, for $n \in \mathbb{N}$ and $(s,t) \in \Gamma(x_n, y_n)$ we have

$$\begin{aligned}
\Psi(x_n, y_n) - \Psi(s,t) &\leq \frac{1}{2}\left[\Psi(x_0, y_0) - \inf_{(v,w)\in\Gamma(x_{n-1},y_{n-1})} \Psi(v,w)\right] \\
&\leq \frac{1}{2^2}\left[\Psi(x_0, y_0) - \inf_{(v,w)\in\Gamma(x_{n-2},y_{n-2})} \Psi(v,w)\right] \leq \cdots \\
\cdots &\leq \frac{1}{2^n}\left[\Psi(x_0, y_0) - \inf_{(v,w)\in\Gamma(x_0,y_0)} \Psi(v,w)\right]
\end{aligned}$$

Recall $K(\varepsilon, B, D) \subset \{(x,y) \in E_B \times F_D\colon\ \Psi(x,y) + \sqrt{\varepsilon}\,(\rho_B(x) + q_D(y)) \leq 0\}$.

If $(s,t) \in \Gamma(x_n, y_n) = A \cap [(x_n, y_n) + K(\varepsilon, B, D)]$ then $(s - x_n, t - y_n) \in K(\varepsilon, B, D)$, which implies

$$\Psi(s - x_n, t - y_n) + \sqrt{\varepsilon}\,(\rho_B(s - x_n) + q_D(t - y_n)) \leq 0.$$

Then for $(s,t) \in \Gamma(x_n, y_n)$ and for every $n \in \mathbb{N}$ we have

$$\begin{aligned}
\rho_B(s - x_n) + q_D(t - y_n) &\leq \frac{1}{\sqrt{\varepsilon}}\,[\Psi(x_n, y_n) - \Psi(s,t)] \\
&\leq \frac{1}{\sqrt{\varepsilon}}\frac{1}{2^n}\left[\Psi(x_0, y_0) - \inf_{(v,w)\in\Gamma(x_0,y_0)} \Psi(v,w)\right].
\end{aligned}$$

Since $(x_{k+1}, y_{k+1}) \in \Gamma(x_n, y_n) = A \cap [(x_n, y_n) + K(\varepsilon, B, D)]$, for every $n \in \mathbb{N} \cup \{0\}$, then $\rho_B(x_{n+1} - x_n) + q_D(y_{n+1} - y_n) \leq R_{B\times D}\,(\Gamma(x_n, y_n))$ which is small if n is large enough. So, the sequence $(x_n, y_n) \in A$ is a locally Cauchy sequence with respect to $\rho_B(\cdot)+q_D(\cdot)$ and convergent to some $(x^*,y^*)\in A$, since A is locally complete. Then by Theorem 4, $(x^*,y^*)\in E(A,K(\varepsilon,B,D))$. □

Let (E,τ) and (F,τ') be locally convex spaces. F ordered by a closed pointed convex cone K. Recall $f:E\to F$ is bounded from below if there exists $z_*\in K$ such that $f(x)\succeq_K z_*$, for every $x\in E$, that is, $f(E)\subset z_*+K$. Also, an element $f(x_\varepsilon)$ is an approximately efficient point of $f(E)$ with respect to K, $k_0\in K$ and $\varepsilon\in[0,1)$ if $f(E)\cap[f(x_\varepsilon)-\varepsilon k_0-(K\setminus\{0\})]=\emptyset$. The set of approximately efficient points of $f(E)$ with respect to K, $k_0\in K$ and $\varepsilon\in[0,1)$ is denoted by $Eff\,(f(E),K_{\varepsilon k_0})$ where $K_{\varepsilon k_0}=\varepsilon k_0+K$. Note that $Eff\,(f(E),K_{\varepsilon k_0})=E\,(f(E),-K)$ (minimal), for $\varepsilon=0$.

As an application of the previous Theorem, under these conditions, we prove the following vectorial Ekeland type Theorem.

Theorem 4. *Let (E,τ), (F,τ') be locally complete locally convex spaces and F ordered by a closed, pointed convex cone K. Let $k_0,z_*\in K$ and $f:E\to F$ be such that $f(x)\succeq_K z_*$, for every $x\in E$, and assume $Graph(f)=\{(x,f(x)):x\in E\}$ is locally closed in $E\times F$. Let $\varepsilon\in(0,1)$ and $f(x_0)\in Eff\,(f(E),K_{\varepsilon k_0})$. Let $D\subset F$ be a non-zero disk such that $k_0,z_*,f(x_0)\in F_D$. Then for every disk non-zero $B\subset E$ such that $x_0\in E_B$ and $f(E_B)\subset z_*+K_D=z_*+(K\cap F_D)$, there exists $x_\varepsilon\in E_B$ satisfying:*

1. $f(x_\varepsilon)\in f(x_0)-\sqrt{\varepsilon}\rho_B(x_\varepsilon-x_0)k_0-K_D$
2. $f(x_\varepsilon)\in E\left(f^{B,D}_{\varepsilon k_0}(E_B),-K_D\right)$*; where*

$$f^{B,D}_{\varepsilon k_0}(x)=f(x)+\sqrt{\varepsilon}\left[\rho_B(x-x_\varepsilon)+q_D(f(x)-f(x_\varepsilon))\right]k_0$$

for every $x\in E_B$.

Proof. We may assume D is a disk such that $q_D(k_0)=1$. In order to apply the previous theorem, we verify those hypotheses. For $\varepsilon\in(0,1)$ consider the corresponding $K(\varepsilon,B,D)$. As the locally complete set A, now consider $A=\{(x,f(x)):x\in E_B\}\subset E_B\times F_D$ which we will denote by $Graph(f^{B,D})$, and $(x_0,f(x_0))\in Graph(f^{B,D})$. Since $Graph(f)$ is locally closed in the locally complete space $(E\times F)$ then $Graph(f^{B,D})$ is locally complete. Recall B and D are Banach disks, since E and F are locally complete. Then according to the previous theorem, there exists $(x_\varepsilon,f(x_\varepsilon))\in Graph(f^{B,D})$ such that

i) $(x_\varepsilon,f(x_\varepsilon))\in Graph(f^{B,D})\cap[(x_0,f(x_0))+K(\varepsilon,B,D)]$

ii) $Graph(f^{B,D})\cap[(x_\varepsilon,f(x_\varepsilon))+K(\varepsilon,B,D)]=\{(x_\varepsilon,f(x_\varepsilon))\}$.

From (ii), for $x\in E_B\setminus\{x_\varepsilon\}$ we have $(x,f(x))-(x_\varepsilon,f(x_\varepsilon))\notin K(\varepsilon,B,D)$, that is $f(x)-f(x_\varepsilon)+\sqrt{\varepsilon}\left[\rho_B(x-x_\varepsilon)+q_D(f(x)-f(x_\varepsilon))\right]k_0\notin -K_D$.

Then $f^{B,D}_{\varepsilon k_0}(x)\notin f^{B,D}_{\varepsilon k_0}(x_\varepsilon)-K_D$, for every $x\in E_B\setminus\{x_\varepsilon\}$.

Hence $f^{B,D}_{\varepsilon k_0}(E_B)\cap\left[f^{B,D}_{\varepsilon k_0}(x_\varepsilon)-(K_D\setminus\{0\})\right]=\emptyset$, and $f^{B,D}_{\varepsilon k_0}(x_\varepsilon)=f(x_\varepsilon)$ is a minimal efficient point, according to (2).

To see (1), from (i) we have $(x_\varepsilon,f(x_\varepsilon))\in(x_0,f(x_0))+K(\varepsilon,B,D)$. Then $f(x_\varepsilon)-f(x_0)+\sqrt{\varepsilon}\left[\rho_B(x_0-x_\varepsilon)+q_D(f(x_0)-f(x_\varepsilon))\right]k_0\in -K_D$. Then $f(x_\varepsilon)+\sqrt{\varepsilon}\left[\rho_B(x_0-x_\varepsilon)\right]k_0\in f(x_0)-\sqrt{\varepsilon}\left[q_D(f(x_0)-f(x_\varepsilon))\right]k_0-K_D\subset f(x_0)-K_D-K_D$. Hence $f(x_\varepsilon)\in f(x_0)-\sqrt{\varepsilon}\rho_B(x_0-x_\varepsilon)k_0-K_D$. □

Acknowledgement 1. *The author is very grateful to the Departament de Matemàtiques of the Universitat Autònoma de Barcelona for the kind hospitality during the preparation of this work.*

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Communicated by *Adrian Petrusel*

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Scientiae Mathematicae Japonicae 78, No.2(2015)(129-134)

Fixed points of multifunctions on COTS with end points *

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October 4, 2013

Abstract. We prove that if F and G are multifunctions from X to Y, with connected values, where X is connected, Y a space admitting a continuous bijection to a connected space Z with endpoints, and Z is T_0 whenever $|Z| = 2$ such that both F, G are either upper semicontinuous with compact values, or, are lower semicontinuous with one of F and G onto, then $F(w) \cap G(w) \neq \emptyset$ for some $w \in X$. We proved that if a multifunction F on a connected space X with endpoints such that X is T_0 whenever $|X| = 2$, has a connected multigraph, then there exists some $w \in X$ such that $w \in F(w)$.

1 Introduction COTS (=connected ordered topological space), defined by Khalimsky, Kopperman and Meyer [6], is an integral part of any study of cut points. Topological spaces are assumed to be connected for any consideration of cut points. By Theorem 2.7 of [6], there are two total orders (or linear orders) on every COTS and each of these orders is the reverse of the other. A COTS can have at most two endpoints [6, Proposition 2.5]. A set with a total order has a topology called interval topology. A topological space is a LOTS (=linearly ordered topological space) if its topology equals some interval topology. Multifunctions are considered on LOTS by Park in [8]. The main result (Theorem 1) of Park [8] about fixed point requires the space to be a connected LOTS having two end points. It can be seen that every LOTS is Hausdorff (without assuming it to be connected). As noted in Proposition 2.9 of [6], the topology of a T_1 COTS is finer than the interval topology given by any of its two orders, so a COTS need not be a LOTS. The concept of COTS does not require any separation axiom. In view of the applications of cut points (see e.g. [6]) and the fact that the many connected topological spaces used for cut points like the Khalimsky line, are not T_1, the assumption of separation axioms is avoided as far as possible. There is the concept of strong cut points for connected topological spaces. Without assuming cut points to be strong cut points, a topological space with endpoints is defined in [2]. Since by Theorem 3.4 of [2], $H(i)$ connected topological spaces have at least two non-cut points, it follows from Remark 4.5 of [2] that such topological spaces with at most two non-cut points turn out to be COTS with endpoints. It is shown in [3] that a connected topological space with endpoints is a COTS with endpoints. It is proved in [4] that a connected topological space is a COTS with endpoints iff it admits a continuous bijection onto a topological space with endpoints. In [4] and [5] there are obtained several classes of connected topological spaces where the members are COTS with endpoints. In this paper, we study multifunction on COTS with endpoints.

Notation, definitions and preliminaries are given in Section 2. The main results of the paper appear in Section 3. In Section 3, we prove that if F and G are multifunctions from X to Y, with connected values, where X is connected, Y a space admitting a continuous bijection to a connected space Z with endpoints and Z is T_0 whenever $|Z| = 2$ has only two points such that both F, G are either upper semicontinuous with compact values, or, are

*2010 Math. Subject Classification — Primary: 54C60, 54F05, 54H25.
Key words and phrases —Connected space, COTS with end points, Multifunctions

lower semicontinuous with one of F and G onto, then $F(w) \cap G(w) \neq \emptyset$ for some $w \in X$. It is proved that if, for a connected space X with endpoints such that X is T_0 whenever $|X| = 2$, F is a multifunction from X to X with connected multigraph, then there exists some $w \in X$ such that $w \in F(w)$. This gives a sort of fixed point theorem. Some results are obtained in the presence of a connected space with endpoints and/or multifunctions.

2 Notation, definitions and preliminaries Some of the standard notation and definitions have been included here for completeness sake. Let X be a space. X is called $T_{1/2}$([6]) if every singleton set is either open or closed. Let $\Delta = \{(x,x) : x \in X\}$ and $\Delta(O) = \{(x,x) : x \in X, \{x\}$ is open in $X\}$. Let $A \subset X$. For $K \subset X$, if need be, A^{+K} is used for the set $A \cup K$, and A^{-K} for the set $A - K$. If X is disconnected, a *separation* of X is denoted by $A|B$, and each one of A and B is a called a *separating set* of X. If A is a separating set of X and $K \subset X$ is connected, if need be, we write $A(K)$ for A if $K \subset A$, and $A(-K)$ for A if $K \subset X - A$. If $K = \{x\}$ for some $x \in X$, then $A^{+x}, A^{-x}, A(x)$ and $A(-x)$ are respectively used for $A^{+K}, A^{-K}, A(K)$ and $A(-K)$. For $x \in X$, if the dependence of a separation $A|B$ of X^{-x} on x is to be specified, then $A|B$ is denoted by $A_x|B_x$. Let $x \in X$. x is called a *cut point* of X if X^{-x} is disconnected. x is called *strong cut point* of X, if X^{-x} has a separation with connected separating sets. ctX is used to denote the set of all cut points of X. A space X is called COTS (=*connected ordered topological space*) ([6]) if it is connected and has the property: if Y is a three-point subset of X, then there is a point x in Y such that Y meets two connected components of X^{-x}. Let X be a space. Let $a, b \in X$. A point $x \in X - \{a, b\}$, is said to be a *separating point* between a and b or x *separates* a and b if there exists a separation $A|B$ of X^{-x} with $a \in A$ and $b \in B$. $S(a,b)$ is used to denote the set of all separating points between a and b. Clearly $S(a,b) \subset ctX$. If we adjoin the points a and b to $S(a,b)$, then the new set is denoted by $S[a,b]$. A space X is called a *space with endpoints* if there exist a and $b \in X$ such that $X = S[a,b]$. For $x \in S(a,b)$, we shall write $X^{-x} = A(a) \cup B(b)$ for a separation $A|B$ of X^{-x}.

For spaces X and Y, a *multifunction* ([7]) from X to Y is a function F from X to $P(Y)$ (= the set of all subsets of Y) with $F(x) \neq \emptyset$ for every $x \in X$, (written as $F : X\text{-}\circ Y$). Let $F : X\text{-}\circ Y$ be a multifunction. F has *compact* (*connected*) *values* if $F(x)$ is compact (connected) for every $x \in X$. For $V \subset Y, F^{\subset}(V)$ (resp. $F^{\cap}(V)$) denotes the set $\{x \in X : F(x) \subset V\}$ (resp. $\{x \in X : F(x) \cap V \neq \emptyset\}$). For $A \subset X, F(A)$ denotes the subset $\cup\{F(x) : x \in A\}$ of Y. For a subset A of X, *multigraph of* F *over* A is the subset $\{(x,y) \in X \times Y : x \in A, y \in F(x)\} = \bigcup\{\{x\} \times F(x) : x \in A\}$, it is denoted by $mgrA$, or $F\text{-}mgrA(F\text{-}mgrA(Y))$ if the dependence on $F(F$ and $Y)$ is to be specified; multigraph of F over X is called the *multigraph* of F. F is said to be *lower* (resp *upper*) *semicontinuous* ([7]) if for each open (resp. closed) set V of Y, the set $F^{\cap}(V)$ is open (resp. closed) in X. F is called a *connectivity multifunction* ([8]) if its multigraph over each connected subset of X is a connected set. F is called *closed* ([8]) if multigraph of F is closed in $X \times Y$; F is called compact ([8]) if $cl_Y(F(X))$ is a compact subset of Y. For sets X and Y, let $p_1 : X \times Y \to X$, and $p_2 : X \times Y \to Y$ be the projection maps. Let $T \subset X \times Y$. For a multifunction F from X to Z (resp. G from Y to Z), F^1 (resp. G^2) denotes the multifunction $F \circ p_1$ from T to Z (resp. $G \circ p_2$ from T to Z).

For a set X, a multifunction F from X to X is called a multifunction on X. A multifunction F on X is said to have a *fixed point* if there exists some $w \in X$ such that $w \in F(w)$. The multifunction on X taking $x \in X$ to $\{x\}$ is denoted by i_X.

Remark 2.1 Let F be multifunction from X to Y. (i) For $A \subset Y, F^{\cap}(\{A\}) = \bigcup\{F^{\cap}(\{y\}) : y \in A\}$. (ii) For $A \subset X, p_2(F\text{-}mgrA) = F(A)$.

Let $h : Y \to Z$. Define $h^p : P(Y) \to P(Z)$ as $h^p(A) = h(A)$ for $A \in P(Y)$. Let F be a multifunction from X to Y. $h^p \circ F$ is a multifunction from X to Z.
Let X, Y and Z be spaces. Let F be a multifunction from X to Y and G a multifunction from X to Z. For $x \in X$, if we define $(F \times G)(x) = F(x) \times G(x) (\in P(Y) \times P(Z) \subset P(Y \times Z))$, then $F \times G$ is a multifunction from X to $Y \times Z$.

Let F and G be multifunctions from X to Y. $(h^p \circ F) \times (h^p \circ G) : X \to P(Z \times Z)$.
The following lemma is a modified version of some results (i.e., Theorems 7.3.12, 7.3.14 and 7.4.4) of [7] in our notation.

Lemma 2.2 *Let X, Y and Z be spaces. For a function $h : Y \to Z$ and multifunctions F, G from X into Y, let $H = (h^p \circ F) \times (h^p \circ G)$. Let h be continuous.*
(a) *If F, G are lower semicontinuous, then $F \times G$ and H are lower semicontinuous.*
(b) *If F and G are upper semicontinuous with compact values, then $F \times G$ and H are upper semicontinuous with compact values.*
(c) *Let F and G be with connected values. Then H has connected values.*

Let X and Y be spaces and T a subset of $X \times Y$. For $x \in X$, let $T^m(x) = \{y \in Y : (x, y) \in T\}$. $T^m(x)$ may not be non-empty for every $x \in X$. For T^m to be a multifunction, $T^m(x)$ should be non-empty for every $x \in X$. For this we may consider only those $x \in X$ such that $(x, y) \in T$ for some $y \in Y$. Let $X_T = p_1(T) = \{x \in X : (x, y) \in T$ for some $y \in Y\}$. Then T^m is a multifunction from X_T to Y and $T \subset X_T \times Y$. In order that concepts concerning a multifunction make sense for T^m, we need to consider X_T in place of X. For $y \in Y$, let $T_y = \{x \in X : (x, y) \in T\}$. Let $Y_T = p_2(T) = \{y \in Y : (x, y) \in T$ for some $x \in X\}$. Note that $T \subset X_T \times Y_T$.

Lemma 2.3 *Let X, Y be two spaces, and let T be a subset of $X \times Y$.*
(a) *If T is closed in $X_T \times Y$, then for every compact subset A of $X_T, T^m(A)$ is a closed subset of Y.*
(b) *If T is closed in $X_T \times Y$, then $T^{m\cap}(B)$ is closed in X_T for every compact subset B of Y.*

Now we note that every multifunction is of the form T^m. Let F be a multifunction from X to Y. Let $T_F = F\text{-}mgrX = \{(x, y) : x \in X, y \in F(x)\}$. Let $x \in X$. Since $F(x) \neq \emptyset$, $(T_F)^m$ is a multifunction from X to Y.

Remark 2.4 (a) $F = (T_F)^m$.
(b) $p_2(T_F) = F(X)$.
Proof. (a) Let $x \in X$. For $y \in Y, y \in (T_F)^m(x)$ iff $(x, y) \in T_F$, i.e iff $y \in F(x)$.
(b) Since $T_F = F\text{-}mgrX$, by Remark 2.1(ii), $p_2(T_F) = F(X)$.

We note the following before the next observation.
Let F be a multifunction from X to Y. For $F(X) \subset Z \subset Y$, F is a multifunction from X to Z, and $F\text{-}mgrX(Y) = F\text{-}mgrX(Z)$.

Lemma 2.5 *For spaces X and Y, with X connected, let F be a multifunction from X to Y with connected values. Then the multigraph of F is connected if one of the following conditions hold:*
(i) *F is a connectivity multifunction.*
(ii) *F is lower semicontinuous.*
(iii) *F is upper semicontinuous with compact values.*
(iv) *$F^\cap(\{y\})$ is open in X for $y \in Y$.*
(v) *F is a closed compact multifunction.*

Proof. (i) Since F is a connectivity multifunction and X is connected, F has connected multigraph.
(ii) and (iii). By Theorem 3.2 of [1], multigraph of F is connected.
(iv) By Remark 2.1(i), (iv)$\Rightarrow$(ii).
(v) Let $Z = cl_Y(F(X))$, F be a compact multifunction form X to Z. Since $T_F = F$-$mgrX, T_F$ is closed. Now by Lemma 2.3(b) and Remark 2.4(a), F is upper semicontinuous. By (a) of Lemma 2.3, F has compact values. Now by (iii), multigraph of F is connected.

3 Connected spaces with endpoints and Multifunctions Let X be a set with a total order $<$ on it. For $x \in X$, let $L(x) = \{y \in X : y < x\}, U(x) = \{y \in X : x < y\}$ [6]. Let $L = \{(s,t) \in X \times X : t < s\}$ and $U = \{(s,t) \in X \times X : s < t\}$. Then it can be seen that $L = \bigcup\{\{s\} \times L(s) : s \in X\} = \bigcup\{U(s) \times \{s\} : s \in X\}$ and $U = \bigcup\{\{s\} \times U(s) : s \in X\} = \bigcup\{L(s) \times \{s\} : s \in X\}$.

We denote the cardinality of a set X by $|X|$.

Lemma 3.1 *Let X be a COTS such that X is T_0 whenever $|X| = 2$. Then $U \cup \Delta(O)$ and $L \cup \Delta(O)$ are open in $X \times X$.*

Proof. Case (i): $|X| = 2$, i.e., X has only two points. Since X is a connected non-indiscrete space, it follows that $X = \{s,t\}$, with a Sierpinski topology, say $\{\emptyset, \{t\}, X\}$ and $s < t$. Then $U \cup \Delta(O) = X \times \{t\}$, which is open in $X \times X$. That $L \cup \Delta(O)$ is open is proved similarly.

Case (ii): $|X| > 2$, i.e., X has at least three points. Let $(s,t) \in U \cup \Delta(O)$. Then X is $T_{1/2}$ by Proposition 2.9 of [6]. Now if $\{s\}$ and $\{t\}$ are open in X, then $\{(s,t)\} = \{s\} \times \{t\}$ is open in $X \times X$. If $\{s\}$ is open and $\{t\}$ is closed, using Theorem 2.7 and Lemma 2.8 of [6], $\{s\} \times (U(s))^{+s}$ is open in $X \times X$ and $(s,t) \in \{s\} \times (U(s))^{+s} \subset U \cup \Delta(O)$. If $\{s\}$ is closed and $\{t\}$ is open, using Theorem 2.7 and Lemma 2.8 of [6], $(L(t))^{+t} \times \{t\}$ is open in $X \times X$ and $(s,t) \in (L(t))^{+t} \times \{t\} \in U \cup \Delta(O)$. In the case when $\{s\}$ and $\{t\}$ are closed, there is some point y of X such that $s < y < t$ by Lemma 2.8(b) and (c) of [6]. Since $\{y\}$ is either open or closed in X, by Theorem 2.7 and Lemma 2.8 of [6], either $(U(y))^{+y}$ and $(L(y))^{+y}$ or $U(y)$ and $L(y)$ are open in X. So either $(L(y))^{+y} \times (U(y))^{+y}$ or $L(y) \times U(y)$ is open in $X \times X$ and $(s,t) \in L(y) \times U(y) \subset (L(y))^{+y} \times (U(y))^{+y} \subset U \cup \Delta(O)$. Thus $U \cup \Delta(O)$ is open in $X \times X$. Since, in a COTS there are two total orders and each of these orders is the reverse of the other, $L \cup \Delta(O)$ is open in $X \times X$.

Theorem 3.2 *For two multifunctions F, G from a space X to a connected space Y with endpoints such that Y is T_0 whenever $|Y| = 2$, one of which is onto, if either $(F \times G)(X)$ is connected or $F \times G$ has a connected multigraph, then there exists some $w \in X$ such that $F(w) \cap G(w) \neq \emptyset$.*

Proof. In view of Remark 2.4(b), we prove the result by contradiction under the assumption that $(F \times G)(X)$ is connected. Suppose not; then $F(w) \cap G(w) = \emptyset$ for every $w \in X$. By the given condition Y is a space with endpoints, so $Y = S[a,b]$. Let $H = F \times G$. Since, by Theorem 3.2 of [3], Y is a COTS with end points a and b (with $a < b$), $H(X) \subset L \cup U$ in $Y \times Y$. So $(L \cup \Delta(O)) \cap H(X) = L \cap H(X)$ and $(U \cup \Delta(O)) \cap H(X) = U \cap H(X)$. Using Lemma 3.1, $L \cup \Delta(O)$ and $U \cup \Delta(O)$ are open in $Y \times Y$. By given condition, either $F(X) = Y$ or $G(X) = Y$. First assume that $F(X) = Y$. Then we pick $x_a, x_b \in X$ such that $a \in F(x_a)$ and $b \in F(x_b)$. Let $y_a \in G(x_a)$ and $y_b \in G(x_b)$. Since $F(x_a) \cap G(x_a) = \emptyset$, so $a < y_a$. Similarly $y_b < b$. This implies that $(a, y_a) \in U \cap H(X)$ and $(b, y_b) \in L \cap H(X)$. Thus we get a separation of $H(X)$ as $L \cap H(X)$ and $U \cap H(X)$ are disjoint non-empty open subsets of $H(X)$. This gives a contradiction as $H(X)$ is connected by Remark 2.4(b). Thus $F(X) \neq Y$. Similarly we have $G(X) \neq Y$. This leads to again a contradiction to the given condition. The proof is complete.

Theorem 1 of [8] gives a sort of fixed point theorem for a multifunction on a connected LOTS with two end points. Every connected LOTS with end points is a connected space with endpoints, but the converse need not be true. The following theorem and corollary are about a connected space with endpoints; so they strengthen Theorems 1 and 2 of [8] respectively.

Theorem 3.3 *Let X be a connected space with endpoints such that X is T_0 whenever $|X| = 2$. Let F be a multifunction on X with connected multigraph. Then there exists some $w \in X$ such that $w \in F(w)$.*

Proof. The theorem follows by taking $X = Y$ and $G(x) = \{x\}$ for $x \in X$ in Theorem 3.2.

Corollary 3.4 *Let X be a connected space with endpoints such that X is T_0 whenever $|X| = 2$. Let F be a multifunction on X with connected values. Then there exists some $w \in X$ such that $w \in F(w)$, if one of the following conditions hold:*
(i) *F is a connectivity multifunction.*
(ii) *F is lower semicontinuous.*
(iii) *F is upper semicontinuous with compact values.*
(iv) *$F^{\cap}(y)$ is open in X for $y \in X$.*
(v) *F is a closed compact multifunction.*

Proof. The result follows by Lemma 2.5 and Theorem 3.3.

The following two theorems respectively strengthen Theorems 2.1 and 2.2 of [9] because here $[0, 1]$ is replaced by a connected space with endpoints (with no separation axioms assumed).

Theorem 3.5 *Let X be a connected space and Y be a space admitting a continuous bijection to a connected space Z with endpoints such that Z is T_0 whenever $|Z| = 2$. Let F, G be two multifunctions from X to Y, with connected values and one of which is onto. Assume that both F and G are either upper semicontinuous with compact values, or lower semicontinuous. Then there exists some $w \in X$ such that $F(w) \cap G(w) \neq \emptyset$.*

Proof. By the given condition we have a connected space Z with endpoints, say a and b and a one-one, onto and continuous function $h : Y \to Z$. Let $H = (h^p \circ F) \times (h^p \circ G)$. By Lemmas 2.2 and 2.5, multigraph of H is connected. Now by Theorem 3.2, there exists some $w \in X$ such that $h(F(w)) \cap h(G(w)) \neq \emptyset$. This implies that $F(w) \cap G(w) \neq \emptyset$ as h is one-one.

Below we have some results in which we assume a subset of a product space of two spaces to be connected. It may be added that Theorem 2.5 of [9] is handy to know the connectedness of a given set in a product space.

Theorem 3.6 *Let X, Y be two spaces, with Y admitting a continuous bijection to a connected space Z with endpoints such that Z is T_0 whenever $|Z| = 2$, and let T be a connected subset of $X \times Y$. Let Φ be a multifunction from X to Y, with connected values. Assume that Φ is either upper semicontinuous with compact values, or lower semicontinuous.*
(i) *If $Y_T = Y$ or $\Phi(X_T) = Y$, then $T \cap (\Phi\text{-}mgrX) \neq \emptyset$.*
(ii) *If $X_T = X$ and Φ is onto, then $T \cap (\Phi\text{-}mgrX) \neq \emptyset$.*

Proof. (i) $F = (i_Y)^2 (= i_Y \circ p_2)$ and $G = \Phi^1 (= \Phi \circ p_1)$ are multifunctions from T to Y. So using the given condition, F and G are either upper semicontinuous with compact values, or lower semicontinuous. Also F and G have connected values and so by the given condition, one of F and G is onto. Now by applying Theorem 3.5 to F and G, the result follows.

(ii) It follows from the assumption of (ii) that the hypothesis $\Phi(X_T) = Y$ of (i) is satisfied.

The following particular case of theorem 3.6 is about fixed point of a multifunction.

Corollary 3.7 *Let X be a space admitting a continuous bijection to a connected space Z with endpoints such that Z is T_0 whenever $|Z| = 2$. Let Φ be a multifunction from X to X, with connected values. Assume that Φ is either upper semicontinuous with compact values, or lower semicontinuous. If Δ is a connected set of $X \times X$, then there exists some $x_0 \in X$ such that $x_0 \in \Phi(x_0)$.*

Proof. Since $X_\Delta = X$, the result follows by taking $Y = X$ and $T = \Delta$ in Theorem 3.6.

Acknowledgement. The authors are grateful to the referee for his very valuable suggestions which resulted in the improvement of the paper.

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Communicated by *Haruo Maki*

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PERIPHERAL SPECTRUM FOR $A \times B$

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Received January 18, 2014

Abstract.

We discuss the idea of peripheral spectrum and related concepts such as Maximum modulus set, peak sets etc. for a function algebra. We study the interrelation of them. We further study these concepts for the Cartesian product $A \times B$ of two function algebras.

1 Introduction

The spectrum of an element of a Banach algebra unveils the algebraic structure of the Banach algebras. However, sometimes a subset, the peripheral spectrum of the spectrum suffices for the purpose. This concept was introduced in [1].

We shall assume throughout that A is a function algebra on a compact Hausdorff space X.

Definition 1.1 *Let A be a function algebra on X. For $f \in A$, the peripheral spectrum is the set, $\sigma_\pi(f) = \sigma(f) \cap \{z \in \mathbb{C} : |z| = \|f\|\}$, where $\sigma(f)$ is the spectrum of f, and the set $\{z \in \mathbb{C} : |z| = \|f\|\}$ is the circle centered at origin and having radius $\|f\|$, denoted by $\Gamma_{\|f\|}$.*

To emphasize on the algebra we denote the peripheral spectrum with respect to algebra A by $\sigma_{\pi_A}(f)$.

Remarks 1.2 *(1) $\sigma_\pi(f)$ is a nonempty compact subset of $\sigma(f)$.*
(2) The concept of peripheral spectrum can be defined for any Banach algebra. However, it is non-empty only if the spectral radius $r(f)$ equals the norm $\|f\|$.
e.g. Take $A = C^1[0,1]$ with norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ and $f(t) = t, t \in [0,1]$.

2 Peripheral spectrum and peaking functions

We have studied certain properties for the Cartesian product of two function algebras [2]. Let A and B be function algebras on X and Y respectively. Then $A \times B$ with coordinatewise operations and $\|(f,g)\| = \max\{\|f\|_\infty, \|g\|_\infty\}$ is a function algebra on $X + Y$. It was proved in general setting [3], that $\sigma((f,g)) = \sigma(f) \cup \sigma(g)$, $\forall f \in A, g \in B$. Here we discuss peripheral spectrum and related concepts for $A \times B$.

Theorem 2.1 *For $h = (f,g) \in A \times B$,*
(a) $\sigma_{\pi_{A\times B}}(h) \subset \sigma_{\pi_A}(f) \cup \sigma_{\pi_B}(g)$
(b) $\sigma_{\pi_{A\times B}}(h) = \sigma_{\pi_A}(f) \cup \sigma_{\pi_B}(g)$ iff $\|f\| = \|g\|$
(c) $\sigma_{\pi_{A\times B}}(h) = \begin{cases} \sigma_{\pi_A}(f), & \text{if } \|f\| > \|g\|; \\ \sigma_{\pi_B}(g), & \text{if } \|f\| < \|g\|. \end{cases}$

2010 *Mathematics Subject Classification.* Primary 46J10.
Key words and phrases. Peripheral spectrum, Maximum modulus set, Peripheral range.

Proof. (a) Let $\lambda \in \sigma_{\pi_{A\times B}}(h)$. Now

$$\begin{aligned}\sigma_{\pi_{A\times B}}(h) &= \sigma_{A\times B}((f,g)) \cap \{z \in \mathbb{C} : |z| = \|(f,g)\|\} \\ &= [\sigma_A(f) \cup \sigma_B(g)] \cap \{z \in \mathbb{C} : |z| = \|(f,g)\|\}\end{aligned}$$

Then $\lambda \in \sigma_A(f)$ or $\lambda \in \sigma_B(g)$. Also $|\lambda| = \|f\|$ or $|\lambda| = \|g\|$ or $|\lambda| = \|f\| = \|g\|$. Suppose $\lambda \in \sigma_A(f)$ and $\|(f,g)\| = \|f\|$. Then clearly $\lambda \in \sigma_{\pi_A}(f)$. If $\|(f,g)\| = \|g\|$, then $|\lambda| = \|g\| \geq \|f\|$ and as $\lambda \in \sigma_A(f)$, $|\lambda| \leq \|f\|$. So $|\lambda| = \|f\|$. So $\lambda \in \sigma_{\pi_A}(f)$. Thus whenever $\lambda \in \sigma_A(f)$, $\lambda \in \sigma_{\pi_A}(f)$.

Similarly, if $\lambda \in \sigma_B(g)$, then $\lambda \in \sigma_{\pi_B}(g)$.

Thus $\lambda \in \sigma_{\pi_A}(f) \cup \sigma_{\pi_B}(g)$. Hence $\sigma_{\pi_{A\times B}}(h) \subset \sigma_{\pi_A}(f) \cup \sigma_{\pi_B}(g)$.

(b) Now assume that $\sigma_{\pi_{A\times B}}(h) = \sigma_{\pi_A}(f) \cup \sigma_{\pi_B}(g)$. If $\lambda \in \sigma_{\pi_A}(f)$, then $|\lambda| = \|f\|$ and also $|\lambda| = \|h\|$, as $\lambda \in \sigma_{\pi_{A\times B}}(h)$. So $\|f\| = \|h\|$. Similarly, if $\lambda \in \sigma_{\pi_B}(g)$, we get $\|g\| = \|h\|$. Thus $\|f\| = \|g\|$.

Conversely, suppose $\|f\| = \|g\|$. Then clearly $\|f\| = \|g\| = \|h\|$. Now if $\lambda \in \sigma_{\pi_A}(f)$, then $\lambda \in \sigma_A(f)$ and $|\lambda| = \|f\|$. But then $\lambda \in \sigma_{A\times B}(h)$ with $|\lambda| = \|h\|$. So $\lambda \in \sigma_{\pi_{A\times B}}(h)$. Hence $\sigma_{\pi_A}(f) \subset \sigma_{\pi_{A\times B}}(h)$.

Similarly, we get $\sigma_{\pi_B}(g) \subset \sigma_{\pi_{A\times B}}(h)$. Thus $\sigma_{\pi_A}(f) \cup \sigma_{\pi_B}(g) \subset \sigma_{\pi_{A\times B}}(h)$. Combining with (a), we get $\sigma_{\pi_{A\times B}}(h) = \sigma_{\pi_A}(f) \cup \sigma_{\pi_B}(g)$.

(c) Suppose $\|f\| > \|g\|$. Then $\|h\| = \|f\| > \|g\|$. So $\Gamma_{\|f\|} = \Gamma_{\|h\|} = \Gamma$ (say). Now

$$\begin{aligned}\sigma_{\pi_{A\times B}}(h) &= \sigma_{A\times B}(h) \cap \Gamma \\ &= [\sigma_A(f) \cup \sigma_B(g)] \cap \Gamma \\ &= [\sigma_A(f) \cap \Gamma] \cup [\sigma_B(g) \cap \Gamma] \\ &= \sigma_{\pi_A}(f)\end{aligned}$$

as $\sigma_B(g) \cap \Gamma = \emptyset$, because $\|g\| < \|f\| = \|h\|$.

Similarly, $\sigma_{\pi_{A\times B}}(h) = \sigma_{\pi_B}(g)$, if $\|f\| < \|g\|$.

Definition 2.2 *[1] Let A be a function algebra on a compact Hausdorff space X. Then for $f \in A$, the peripheral range, $Ran_{\pi_A}(f)$ is defined as,*

$$\begin{aligned}Ran_{\pi_A}(f) &= f(X) \cap \{z \in \mathbb{C} : |z| = \|f\|\} \\ &= f(X) \cap \Gamma_{\|f\|},\end{aligned}$$

where $f(X)$ is the range of f.

Remarks 2.3 *(1) Since $\sigma_{\pi_A}(f) \subset bd\sigma_A(f) \subset \widehat{f}(\partial_A) = f(\partial_A) \subset f(X)$ for a function algebra A on X, we have $\sigma_{\pi_A}(f) = Ran_{\pi_A}(f)$, $\forall f \in A$ [1], where ∂_A is the Šilov boundary for A.*
(2) Suppose A and B are function algebras on X with $A \subset B$. Then for $f \in A$, $\sigma_B(f) \subset \sigma_A(f)$ and the inclusion may be proper. However, by (1) above, $\sigma_{\pi_B}(f) = \sigma_{\pi_A}(f)$, $\forall f \in A$.

Definition 2.4 *[4] Let A be a function algebra on X and $f \in A$. The set of all x in X at which f attains its maximum modulus is called the maximum modulus set and is denoted by $E(f)$, i.e.,*

$$E(f) = \{x \in X : |f(x)| = \|f\|\}.$$

Remark 2.5 *It is clear from the Definitions 1.1 and 2.4 that $E(f) = f^{-1}(\sigma_{\pi_A}(f))$, for $f \in A$.*

Theorem 2.6 *For $h=(f,g)\in A\times B$,*
(a) $E(h)\subset E(f)\cup E(g)$
(b) $E(h)=E(f)\cup E(g)$ iff $\|f\|=\|g\|$
(c) $E(h)=\begin{cases} E(f), & \text{if } \|f\|>\|g\|;\\ E(g), & \text{if } \|f\|<\|g\|.\end{cases}$

Proof. (a) Let $z_0\in E(h)=\{z\in X+Y:|h(z)|=\|h\|\}$. Then $|h(z_0)|=\|h\|$. If $z_0\in X$, then $h(z_0)=f(z_0)$. Therefore $|f(z_0)|=|h(z_0)|=\|h\|\le\|f\|\le\|h\|$. Therefore we must have $|f(z_0)|=\|f\|$. So $z_0\in E(f)$.

Similarly, if $z_0\in Y$, then $z_0\in E(g)$. Thus $E(h)\subset E(f)\cup E(g)$.
(b) Suppose that $E(h)=E(f)\cup E(g)$. Also assume that $\|f\|>\|g\|$. Then $\|h\|=\|f\|$. Let $y\in E(g)$. Then $|h(y)|=|g(y)|=\|g\|<\|h\|$, i.e., $y\notin E(h)$ which is not possible. Therefore we must have $\|f\|=\|g\|=\|h\|$.

Conversely, suppose that $\|f\|=\|g\|=\|h\|$ and let $z_0\in E(f)\cup E(g)$. If $z_0\in E(f)$, then $z_0\in X\subset X+Y$ and $|h(z_0)|=|f(z_0)|=\|f\|=\|h\|$, i.e., $z_0\in E(h)$.

Similarly, if $z_0\in E(g)$, then $z_0\in E(h)$. Thus $E(f)\cup E(g)\subset E(h)$. Combining with (a), we get $E(h)=E(f)\cup E(g)$.
(c) Suppose $\|f\|>\|g\|$. Then $\|h\|=\|f\|$. Let $z_0\in E(h)$. Then if $z_0\in Y$, we get $\|h\|=|h(z_0)|=|g(z_0)|\le\|g\|<\|f\|$ which is a contradiction. So we must have $z_0\in X$. So $z_0\in E(f)$. Thus $E(h)\subset E(f)$.

Conversely, let $z_0\in E(f)$. Then as above, we get $E(f)\subset E(h)$. Hence $E(f)=E(h)$. Thus $E(f)=E(h)$, if $\|f\|>\|g\|$.

Similarly, $E(h)=E(g)$, if $\|f\|<\|g\|$.

Remark 2.7 *Since $E(f)=f^{-1}(\sigma_{\pi_A}(f))$, we can prove Theorem 2.6 using Theorem 2.1, directly also.*

Definition 2.8 *[4] Let A a function algebra on X. For $x\in X$ define,*

$$\mathcal{E}_x(A)=\{f\in A:|f(x)|=\|f\|\}=\{f\in A:x\in E(f)\}.$$

For a fixed $f\in A$ and $g\in B$ we define, $A_g=\{f\in A:\|f\|\le\|g\|\}$ and $B_f=\{g\in B:\|g\|\le\|f\|\}$.

Theorem 2.9 *For $z\in X+Y$, $\mathcal{E}_z(A\times B)=\begin{cases} \bigcup\{(f,g):f\in\mathcal{E}_z(A),\ g\in B_f\}, & \text{if } z\in X;\\ \bigcup\{(f,g):g\in\mathcal{E}_z(B),\ f\in A_g\}, & \text{if } z\in Y.\end{cases}$*

Proof. Let $h=(f,g)\in\mathcal{E}_z(A\times B)$. Then $|h(z)|=\|h\|$. If $z\in X$, then $h(z)=f(z)$. So $|f(z)|=|h(z)|=\|h\|=\|f\|$. Thus $|f(z)|=\|f\|$. So $f\in\mathcal{E}_z(A)$ and $\|h\|=\|f\|\ge\|g\|$, i.e., $g\in B_f$. Thus $\mathcal{E}_z(A\times B)\subset\bigcup\{(f,g):f\in\mathcal{E}_z(A),\ g\in B_f\}$.

Conversely, suppose that $h=(f,g)\in\bigcup\{(f,g):f\in\mathcal{E}_z(A),\ g\in B_f\}$. Then $|f(z)|=\|f\|$ and $\|f\|\ge\|g\|$. Now $|h(z)|=|f(z)|=\|f\|=\|h\|$. So $h\in\mathcal{E}_z(A\times B)$. Thus $\mathcal{E}_z(A\times B)=\bigcup\{(f,g):f\in\mathcal{E}_z(A),\ g\in B_f\}$.

Similarly, if $z\in Y$, then $\mathcal{E}_z(A\times B)=\bigcup\{(f,g):g\in\mathcal{E}_z(B),\ f\in A_g\}$.

Next we relate peaking function of A and B with that of $A\times B$.

Definition 2.10 *[1] Let A be a function algebra on X. An element $f\in A$ is called a peaking function for A if $\sigma_{\pi_A}(f)=\{1\}$, i.e., $\|f\|=1$ and $|f(x)|<1$ whenever $f(x)\neq 1$.*

In this case, $E(f)=\{x\in X:f(x)=1\}=f^{-1}\{1\}$ is called the peak set of f.

The set of all peaking functions in A is denoted by $\mathscr{P}(A)$.

In general, $\mathscr{P}(A{\times}B)\neq\mathscr{P}(A){\times}\mathscr{P}(B)$, as the following example shows. Let $A=B=(\mathbb{C},|.|)$ and $h=(f,g)=(1,\frac{1}{2})\in A\times B$. Then $\sigma_{\pi_{A\times B}}(h)=\{1\}$. So $h\in\mathscr{P}(A\times B)$ and $\sigma_{\pi_A}(f)=\{1\},\sigma_{\pi_B}(g)=\{\frac{1}{2}\}$. Hence $f\in\mathscr{P}(A)$ but $g\notin\mathscr{P}(B)$. So $h\notin\mathscr{P}(A)\times\mathscr{P}(B)$.

Hence $\mathscr{P}(A\times B)\neq\mathscr{P}(A)\times\mathscr{P}(B)$.

Thus $\mathscr{P}(A\times B)\not\subset\mathscr{P}(A)\times\mathscr{P}(B)$. However, we get $\mathscr{P}(A\times B)\supset\mathscr{P}(A)\times\mathscr{P}(B)$ from the following result.

Now let us denote $U_A=\{f\in A:\|f\|\leq 1\}$, $S_A=\{f\in A:\|f\|=1\}$, $U_B=\{g\in B:\|g\|\leq 1\}$, $S_B=\{g\in B:\|g\|=1\}$.

Theorem 2.11 $\mathscr{P}(A\times B)=[\mathscr{P}(A)\times(U_B\setminus S_B)]\cup[(U_A\setminus S_A)\times\mathscr{P}(B)]\cup[\mathscr{P}(A)\times\mathscr{P}(B)]$.

Proof. Let $h=(f,g)\in\mathscr{P}(A\times B)$. Then $\sigma_{\pi_{A\times B}}(h)=\{1\}$, i.e., $\|h\|=1$. If $\|f\|>\|g\|$, then $\|f\|=\|h\|=1$ and $\|g\|<1$. Therefore $g\in U_B\setminus S_B$ and $\sigma_{\pi_A}(f)=\{1\}$, by Theorem 2.1 (c), i.e., $f\in\mathscr{P}(A)$. Thus $h\in\mathscr{P}(A)\times(U_B\setminus S_B)$.

If $\|f\|<\|g\|$, then by similar argument we get $h\in(U_A\setminus S_A)\times\mathscr{P}(B)$.

If $\|f\|=\|g\|$, then also by similar argument we get $h\in\mathscr{P}(A)\times\mathscr{P}(B)$.

Thus $\mathscr{P}(A\times B)\subset[\mathscr{P}(A)\times(U_B\setminus S_B)]\cup[(U_A\setminus S_A)\times\mathscr{P}(B)]\cup[\mathscr{P}(A)\times\mathscr{P}(B)]$.

Conversely, let $h\in[\mathscr{P}(A)\times(U_B\setminus S_B)]\cup[(U_A\setminus S_A)\times\mathscr{P}(B)]\cup[\mathscr{P}(A)\times\mathscr{P}(B)]$. Suppose $h=(f,g)\in\mathscr{P}(A)\times(U_B\setminus S_B)$ with $f\in\mathscr{P}(A)$ and $g\in(U_B\setminus S_B)$. Then $\sigma_{\pi_A}(f)=\{1\}$, i.e., $\|f\|=1$ and $\|g\|<1$. So $\|h\|=1$. Thus $\|f\|>\|g\|$. Then by Theorem 2.1 (c), $\sigma_{\pi_{A\times B}}(h)=\{1\}$. So $h\in\mathscr{P}(A\times B)$.

Similarly, if $h\in(U_A\setminus S_A)\times\mathscr{P}(B)$, then also $h\in\mathscr{P}(A\times B)$.

Suppose $h=(f,g)\in\mathscr{P}(A)\times\mathscr{P}(B)$. Then $\sigma_{\pi_A}(f)=\{1\}=\sigma_{\pi_B}(g)$, i.e., $\|f\|=\|g\|=1$. Hence $\|h\|=1$. Then by Theorem 2.1 (b), $\sigma_{\pi_{A\times B}}(h)=\{1\}$, i.e., $h\in\mathscr{P}(A\times B)$. Hence $\mathscr{P}(A\times B)\supset[\mathscr{P}(A)\times(U_B\setminus S_B)]\cup[(U_A\setminus S_A)\times\mathscr{P}(B)]\cup[\mathscr{P}(A)\times\mathscr{P}(B)]$. Hence the result.

Note that in above result the sets on right hand side are mutually disjoint.

Acknowledgments. The research is supported by the SAP programme to the department by UGC. The Third author is sponsored by the UGC BSR Research Fellowship.

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Scientiae Mathematicae Japonicae 78, No.2(2015)(139-147)

A SEARCH GAME WITH A NON-ADDITIVE COST

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Received February 21,2014;revised June 27,2014

Abstract. We treat a zero-sum two-person game, what is called, a search game between the hider and the seeker, in which there is a cost for searching a region. If the seeker searches two regions, it is usual that the total cost for two regions is the sum of each cost for a region. However, there may be a saving of the setup cost for the second region when the seeker decides in advance two regions efficiently, and plans to change from one region to another region efficiently. If we take into mind this kind of saving, the cost may not be non-additive. In this paper, we analyze a search game when the cost is not necessarily additive.

1 Introduction In this paper we treat a zero-sum two-person game, what is called, a search game between the hider and the seeker. In a search game, there is a cost for searching a region. If the seeker searches two regions, it is usual that the total cost for two regions is the sum of each cost for a region. In this sense the cost is additive. It is possible to consider that each search cost for a region includes a setup cost. It is likely, however, that there is a saving of the setup cost for the second region when the seeker decides in advance two regions efficiently, and plans to change from one region to another region efficiently. If we take into mind this kind of saving, the cost may not be non-additive. In this paper, we analyze a search game when the cost is not necessarily additive. [4] considers an additive search cost but multiple objects. [2] constructs and analyzes another search game with non-additive costs. There exists an extensive literature on search games. For example, see [1] and [3].

2 Model and properties. Let $N = \{1, \ldots, n\}$ be the set of boxes. Define a search game on N. The hider chooses a box $i \in N$ and hides an (immobile) object in that box. Without knowing the hider's choice, the seeker chooses an ordered partition $\mathcal{S} = \{S_1, \ldots, S_k\}$ of N, first inspects the set of boxes S_1 , and he finds an object if i is in S_1. If i is not in S_1, then he does not find and he inspects the set of boxes S_2, and so on. We assume he finds an object certainly (with probability 1) if he examines the right set of boxes. Associated with an inspection of $S \subseteq N$ is the inspection cost $c(S)$. An interpretation of an inspection of a set of boxes is as follows. The cost $c(\{i\})$ for $i \in N$ may include some setup cost for beginning the search of the box i. If the searcher can save this setup cost by considering a set of boxes and by devising the method of search, then the cost for a set of boxes could be defined. Under this kind of consideration, it is reasonable to assume $c(\emptyset) = 0$ and to assume

$$\begin{aligned} c(S) + c(T) &\geq c(S \cup T), \quad \forall S, T \subseteq N, S, T \neq \emptyset, S \cap T = \emptyset, \\ c(S) &\geq c(T) \geq 0, \quad \forall T \subseteq S \subseteq N, T \neq \emptyset. \end{aligned} \tag{1}$$

The first inequality in (1) says that there may be some saving in cost by considering a search for the sets S and T simultaneously. The second is very usual.

2010 *Mathematics Subject Classification.* Primary 90B40,91A05,91A12.

Key words and phrases. Search theory, two-person zero-sum game, search game.

The set of all ordered partitions of N is denoted by Σ which is the set of all strategies for the seeker. The set of all strategies for the hider is N. When the hider and the seeker use strategies $i \in N$ and $\mathcal{S} = \{S_1, \ldots, S_k\} \in \Sigma$, and if $i \in S_j, 1 \le j \le k$, the cost for the seeker is $f(i, \mathcal{S}) = \sum_{\ell=1}^{j} c(S_\ell)$. The hider wishes to maximize it and the seeker wishes to minimize it by choosing $i \in N$ and $\mathcal{S} \in \Sigma$ respectively. We have a two-person zero-sum game $\Gamma(N, c)$ which can be expressed by a finite matrix. A mixed strategy for the hider is $p = (p_1, \ldots, p_n)$ which is a probability distribution over N where $\sum_{i \in N} p_i = 1, p_i \ge 0$ for all $i \in N$. We use the notation $p(S) \equiv \sum_{i \in S} p_i$ and $p|_S \equiv \{p_i/p(S)\}$ for all $S \subseteq N$. We let $p(\emptyset) = 0$. A mixed strategy for the seeker is a probability distribution over Σ, that is, $q = \{q(\mathcal{S})\}_{\mathcal{S} \in \Sigma}$ where $\sum_{\mathcal{S} \in \Sigma} q(\mathcal{S}) = 1$ and $q(\mathcal{S}) \ge 0$ for all $\mathcal{S} \in \Sigma$. When the hider and the seeker use strategies p, q, the expected cost is expressed as $f(p, q)$.

Example 1.

In this example, for simplicity we restrict strategies for the seeker to ordered partitions $\mathcal{S} = \{S_1, \ldots, S_k\}$ such that

$$i \in S_\alpha, j \in S_\beta, \alpha < \beta \Longrightarrow i < j. \tag{2}$$

We assume that the hider knows this. Let $n = 2$. From (2) the strategies for the seeker are[1] $\mathcal{S}_1 = \{\overline{1}, \overline{2}\}, \mathcal{S}_2 = \{\overline{12}\}$. The payoff matrix for the hider is

Table 1

	$\mathcal{S}_1$	$\mathcal{S}_2$
1	$c(\overline{1})$	$c(\overline{12})$
2	$c(\overline{1}) + c(\overline{2})$	$c(\overline{12})$

By (1), a pair of optimal strategies is $(2, \mathcal{S}_2)$ and the value is $c(\overline{12})$.

Let $n = 3$. The strategies for the seeker are $\mathcal{S}_1 = \{\overline{1}, \overline{2}, \overline{3}\}, \mathcal{S}_2 = \{\overline{12}, \overline{3}\}, \mathcal{S}_3 = \{\overline{1}, \overline{23}\}, \mathcal{S}_4 = \{\overline{123}\}$. The payoff matrix for the hider is

Table 2

	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$	$\mathcal{S}_4$
1	$c(\overline{1})$	$c(\overline{12})$	$c(\overline{1})$	$c(\overline{123})$
2	$c(\overline{1}) + c(\overline{2})$	$c(\overline{12})$	$c(\overline{1}) + c(\overline{23})$	$c(\overline{123})$
3	$c(\overline{1}) + c(\overline{2}) + c(\overline{3})$	$c(\overline{12}) + c(\overline{3})$	$c(\overline{1}) + c(\overline{23})$	$c(\overline{123})$

By (1), a pair of optimal strategies is $(3, \mathcal{S}_4)$ and the value is $c(\overline{123})$.

From the observation on $n = 2, 3$ in Example 1 we see easily a solution for $n \ge 2$ as follows.

Proposition 2.1. Let $n \ge 2$. Restrict the strategies for the seeker to ordered partitions which satisfy (2). A pair of optimal strategies is $(n, \{N\})$ and the value is $c(N)$. If $f(n, \mathcal{S}) > c(N)$ for all $\mathcal{S} \in \Sigma \setminus \{\{N\}\}$ then it is unique.

Proof: For any $\mathcal{S} = \{S_1, \ldots, S_k\}$ which satisfies (2),

$$f(n, \mathcal{S}) = \sum_{j=1}^{k} c(S_j) \ge c(N), \tag{3}$$

by (1). On the other hand, for any $i \in N$, it holds $f(i, \{N\}) = c(N)$. So a strategy pair $(n, \{N\})$ is a saddle point in the payoff matrix for the hider. Next assume that $f(n, \mathcal{S}) >$

[1] For simplicity, we write $\{1\}, \{1, 2\}$, etc. as $\overline{1}, \overline{12}$, etc.

$c(N)$ for all $\mathcal{S} \in \Sigma \setminus \{\{N\}\}$. Suppose $(i, \mathcal{T})$ is another saddle point. If $i = n$ then $\mathcal{T} \neq \{N\}$ and $f(n, \mathcal{T}) > c(N) = f(n, \{N\})$ which contradicts the fact that $(i, \mathcal{T})$ is a saddle point. If $i \neq n$ then $f(i, \mathcal{T}) \geq f(n, \mathcal{T}) > f(n, \{N\}) = f(i, \{N\})$, which implies $f(i, \mathcal{T}) > f(i, \{N\})$, contradicting the fact that $(i, \mathcal{T})$ is a saddle point. □

We can see the solution when the inspection cost is additive. This is an extreme case of the cost function.

Proposition 2.2. Assume the inspection cost is additive, that is, it satisfies

$$c(S) = \sum_{i \in S} c(i), \text{ for all } S \subseteq N. \tag{4}$$

An optimal strategy for the hider is

$$p_i = \frac{c(i)}{\sum_{j \in N} c(j)}, \quad \forall i \in N. \tag{5}$$

An optimal strategy for the seeker is to choose at random an ordered partition from the set $\Sigma^1 \equiv \{\{\{\pi(1)\}, \ldots, \{\pi(n)\}\} : \pi \text{ is a permutation on } N\}$. The value of the game is

$$\frac{1}{\sum_{j \in N} c(j)} \sum_{i=1}^{n} \sum_{j=1}^{i} c(i)c(j). \tag{6}$$

Proof: For any $\mathcal{S} = \{S_1, \ldots, S_k\}$, let $S_j = \{i_1^j, \ldots, i_{s_j}^j\}, i_1^j < \ldots < i_{s_j}^j$ for all $j = 1, \ldots, k$. Define $\mathcal{S}'$ by $\mathcal{S}' = \{\{i_1^1\}, \ldots, \{i_{s_1}^1\}, \ldots, \{i_1^j\}, \ldots, \{i_{s_j}^j\}, \ldots, \{i_1^k\}, \ldots, \{i_{s_k}^k\}\}$. For any $i \in N$, if $i = i_t^j \in S_j$, then

$$\begin{aligned} f(i, \mathcal{S}) &= c(S_1) + \ldots + c(S_{j-1}) + c(S_j) \\ &= c(S_1) + \ldots + c(S_{j-1}) + \sum_{\ell=1}^{s_j} c(i_\ell^j) \\ &\geq c(S_1) + \ldots + c(S_{j-1}) + \sum_{\ell=1}^{t} c(i_\ell^j) = f(i, \mathcal{S}'). \end{aligned} \tag{7}$$

This implies that $\mathcal{S}$ is dominated by $\mathcal{S}'$. So the seeker chooses from the set Σ^1. The hider knows this, and if he takes p defined by (5), then $f(p, \mathcal{S})$ equals to the quantity given in (6). If the seeker takes q which means that he chooses from the set Σ^1 at random, the expected inspection cost $f(i, q)$ is equals to the quantity given in (6) for all $i \in N$. □

In general, the inspection cost is not always additive. Suppose that p is a strategy for the hider. Suppose the seeker can guess this strategy. Let

$$F_p(N) \equiv \min\{f(p, \mathcal{S}') : \mathcal{S}' \in \Sigma\}. \tag{8}$$

with $F_p(\emptyset) = 0$. By the theory of dynamic programming, we have

$$F_p(N) = \min\{c(S) + p(N \setminus S) F_{p|_{N \setminus S}}(N \setminus S) : S \subseteq N, S \neq \emptyset\}, \tag{9}$$

where $p|_{N \setminus S}$ is a posteriori probability distribution on $N \setminus S$ after the seeker searches S. As an initial condition we have

$$F_p(\{i\}) = c(\{i\}), \forall i \in N. \tag{10}$$

If we can guess an optimal strategy for the hider, then by (9) and (10), we could calculate a best reply of the seeker, as in the next example.

Example 2. Assume that the inspection cost depends on the number of boxes in the set,that is,

$$c(S) = C(|S|), \quad \forall S \subseteq N, \tag{11}$$

where $C(\bullet)$ is a function on $\{0, 1, \dots, n\}$. From (1), $C(\bullet)$ satisfies

$$\begin{aligned} &C(s) + C(t) \geq C(s+t), \quad \forall s, t : s + t \leq n, \\ &C(s) \geq C(t), \quad \forall s, t : n \geq s \geq t \geq 0. \end{aligned} \tag{12}$$

Then an optimal strategy for the hider is $p^e \equiv (\frac{1}{n}, \dots, \frac{1}{n})$. We write as $F(s) \equiv F_{p^e}(S)$ for all $S \subseteq N$ such that $|S| = s$, since F depends only on the number of elements in S for every $S \subseteq N$. The equations (9) and (10) become

$$\begin{aligned} &F(n) = \min\{C(s) + \frac{n-s}{n} F(n-s) : 1 \leq s \leq n\}, \\ &F(1) = C(1), F(0) = 0. \end{aligned} \tag{13}$$

Let $G(s) \equiv sF(s)$ for $1 \leq s \leq n$. Then (13) becomes

$$G(n) = \min\{nC(s) + G(n-s) : 1 \leq s \leq n\}, \quad G(1) = C(1), G(0) = 0. \tag{14}$$

Case 1. $C(s) = \sqrt{s}$. By (14), we see

$$G(n) = \begin{cases} n\sqrt{n}, & \text{if } 1 \leq n \leq 3; \ (s = n) \\ n\sqrt{n-1} + 1, & \text{if } 4 \leq n \leq 11. \ (s = n-1) \end{cases} \tag{15}$$

For $n \geq 12$ we could calculate sequentially by (14).

Case 2. $C(s) = \log(s+1)$. By (14), $G(n) = n \log n + \log 2$ for $1 \leq n \leq 5$, by $s = n - 1$.

3 A search game with strictly monotonic cost function.

In this section we analyze an optimal strategy for the hider when the costs are strictly monotonic with respect to inclusion relation.

Proposition 3.1. For $i \in N$, assume the inspection cost satisfies

$$c(S) > c(S \setminus \{i\}), \text{ for all } S \text{ such that } i \in S. \tag{16}$$

Let p be an optimal strategy for the hider. Then $p_i > 0$.

Proof: Assume that the inspection cost satisfies (16) but $p_i = 0$. Let $\mathcal{S} = \{S_1, \dots, S_k\}$ be a best reply to p. Suppose $i \in S_j$. Let $\mathcal{S}' = \{S_1, \dots, S_{j-1}, S_j \setminus \{i\}, S_{j+1}, \dots, S_k, \{i\}\}$. Since $\mathcal{S}$ is a best reply, we have

$$0 \geq f(p, \mathcal{S}) - f(p, \mathcal{S}') = [c(S_j) - c(S_j \setminus \{i\})][p(S_j \setminus \{i\}) + \sum_{\ell=j+1}^{k} p(S_\ell)], \tag{17}$$

while, by (16), $c(S_j) - c(S_j \setminus \{i\}) > 0$. This and (17) imply $p(S_j \setminus \{i\}) + \sum_{\ell=j+1}^{k} p(S_\ell) = 0$. This implies $p(S_j) = p(S_{j+1}) = \ldots = p(S_k) = 0$. Define p' by

$$p'_x = \begin{cases} p_x - \varepsilon, & \text{if } p_x > 0; \\ p_x, & \text{if } x \neq i, p_x = 0; \\ \kappa\varepsilon, & \text{if } x = i, \end{cases} \tag{18}$$

where $\kappa \equiv |X|$ and $X \equiv \{x : p_x > 0\}$. We note $X \subset S_1 \cup \cdots \cup S_{j-1}$, $p'(S_j) = \kappa\varepsilon$ and $p'(S_{j+1}) = \ldots = p'(S_k) = 0$.

$$\begin{aligned} f(p', \mathcal{S}) &= \sum_{\ell=1}^{k} p'(S_\ell) \sum_{m=1}^{\ell} c(S_m) \\ &= \sum_{\ell=1}^{j-1} [p(S_\ell) - |S_\ell \cap X|\varepsilon] \sum_{m=1}^{\ell} c(S_m) + \kappa\varepsilon[\sum_{m=1}^{j-1} c(S_m) + c(S_j)] \\ &= \sum_{\ell=1}^{j-1} p(S_\ell) \sum_{m=1}^{\ell} c(S_m) \\ &\quad + \varepsilon[\kappa \sum_{m=1}^{j-1} c(S_m) + \kappa c(S_j) - \sum_{\ell=1}^{j-1} |S_\ell \cap X| \sum_{m=1}^{\ell} c(S_m)] \\ &= \sum_{\ell=1}^{j-1} p(S_\ell) \sum_{m=1}^{\ell} c(S_m) \\ &\quad + \varepsilon[\kappa \sum_{m=1}^{j-1} c(S_m) + \kappa c(S_j) - \sum_{m=1}^{j-1} c(S_m) \sum_{\ell=m}^{j-1} |S_\ell \cap X|] \\ &= \sum_{\ell=1}^{j-1} p(S_\ell) \sum_{m=1}^{\ell} c(S_m) \\ &\quad + \varepsilon[\sum_{m=1}^{j-1} c(S_m)[\kappa - \sum_{\ell=m}^{j-1} |S_\ell \cap X|] + \kappa c(S_j)] \\ &> \sum_{\ell=1}^{j-1} p(S_\ell) \sum_{m=1}^{\ell} c(S_m) = f(p, \mathcal{S}). \end{aligned} \tag{19}$$

Let $\mathcal{T} = \{T_1, \ldots, T_b\}$ be any pure strategy for the seeker. If $f(p, \mathcal{T}) > f(p, \mathcal{S})$, then $f(p', \mathcal{T}) > f(p, \mathcal{S})$ by making $\varepsilon > 0$ sufficiently small. If $f(p, \mathcal{T}) = f(p, \mathcal{S})$ then $\mathcal{T}$ is a best reply to p, and there is a such that $i \in T_a$. For the same reason as in $\mathcal{S}$, we have $p(T_a) = \ldots = p(T_b) = 0$. Changing j, k to a, b and $\mathcal{S}$ to $\mathcal{T}$ in (19), we can see $f(p', \mathcal{T}) > f(p, \mathcal{T}) = f(p, \mathcal{S})$. Since the number of pure strategies for the seeker is finite, we obtain a better strategy p' for the hider. This contradicts the optimality of p. Hence, $p_i > 0$. $\square$

It is easy to see that (16) holds for every $i \in N$ if and only if the inspection cost is strictly monotonic, that is, $c(S) > c(T)$ for all S, T such that $T \subset S$ and $S \neq T$. In practice, it is very likely that the inspection cost is strictly monotonic since it costs by all means if the seeker behaves. Let $q = \{q(\mathcal{S})\}_{\mathcal{S} \in \Sigma}$ be a mixed strategy for the seeker, where $q(\mathcal{S})$ is the probability that he chooses $\mathcal{S}$. By the complementary slackness theorem in linear programming and Proposition 3.1, we obtain

Corollary 3.2. Assume the inspection cost is strictly monotonic. Let p be an optimal strategy for the hider. Then $p_i > 0$ for all $i \in N$. Let q be an optimal strategy for the seeker. Then $f(i, q) = v(N)$ for all $i \in N$, where $v(N)$ is the value of the game.

4 A search game with mass-effective cost function For $S \subseteq N$, let Σ^S be the set of all ordered partitions of S. A restricted game $\Gamma(S, c)^-$ on S is defined as follows. The set of strategies for the hider is S, the set of strategies for the seeker is $\Sigma^S \setminus \{\{S\}\}$, and the cost for a strategy pair $(i, \mathcal{S}), i \in S, \mathcal{S} \in \Sigma^S \setminus \{\{S\}\}$, is $f(i, \mathcal{S})$. The value of the game $\Gamma(S, c)^-$ is denoted by $v(S)^-$. In the restricted game on S, the strategy $\{S\}$ for the seeker is excluded from the strategy set in the original game $\Gamma(S, c)$ on S.

Proposition 4.1. Assume $c(N) < v(N)^-$. An optimal strategy for the seeker is $\{N\}$. An optimal strategy for the hider is p which is an optimal strategy for the hider in the restricted game $\Gamma(N, c)^-$. The value of the game is $c(N)$.

Proof: For every $i \in N$, we have $f(i, \{N\}) = c(N)$. Let p be an optimal strategy for the hider in the restricted game $\Gamma(N, c)^-$. Then $f(p, \mathcal{S}) \geq v(N)^- > c(N)$ for all $\mathcal{S} \in \Sigma^N \setminus \{\{N\}\}$. Furthermore, $f(p, \{N\}) = c(N)$. This completes the proof. □

The discussion in Proposition 4.1 could be extended to the restricted game on every $S \subseteq N$ if an optimal strategy for the hider has some property. For a mixed strategy p for the hider, we define a mixed strategy p^S for the hider on the game on $S \subseteq N$ by $p^S = p|_S$, which is a projection of p on the strategy space of the game on S.

Proposition 4.2. Suppose p is an optimal strategy for the hider in the game on N. Assume that p^S is an optimal strategy for the hider in the game on $S \subset N$. Assume $c(S) < v(S)^-$. Then the seeker can exclude from the consideration a strategy such as $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2\}$ where $\mathcal{S}_2$ is an ordered partition of $N \setminus S$ and $\mathcal{S}_1$ is an ordered partition of S and $\mathcal{S}_1 \neq \{S\}$.

Proof: Let $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2\}$ be a strategy for the seeker in the statement of Proposition 4.2.

$$\begin{aligned} f(p, \mathcal{S}) &= p(S) \sum_{i \in S} p_i^S f(i, \mathcal{S}) + \sum_{i \in N \setminus S} p_i f(i, \mathcal{S}) \\ &\geq p(S) v(S)^- + \sum_{i \in N \setminus S} p_i f(i, \mathcal{S}) \\ &> p(S) c(S) + \sum_{i \in N \setminus S} p_i f(i, \mathcal{S}) \\ &= f(p, \{\{S\}, \mathcal{S}_2\}). \end{aligned} \tag{20}$$

This implies $\mathcal{S}$ is not a best reply to p. □

5 A game on a star network. In practice, it may cost the absurdity for the change of the box when we consider whether the seeker searches for another box j after having searched for a box i. In this case, the seeker will search a box $k \neq j$ after he has searched for a box i. This kind of things could be expressed by a network where nodes are boxes. Edges express changeability between boxes. This situation is expressed as a game on a network. In this section we treat this model.

Let $G = (N, E)$ be an undirected graph with the node set N and the edge set E. A subgraph $(S, E(S))$ is an undirected graph where $S \subseteq N$ and $E(S) = \{(i, j) \in E, i, j \in$

$S\} \subseteq E$. A subset $S \subseteq N$ is called connected if the subgraph $(S, E(S))$ is connected. A subset $\Sigma^* \subset \Sigma$ is defined by the set of all ordered partitions of N such that every element of each ordered partition is connected. Hereafter, we assume that the strategy space for the seeker is Σ^* when the game is on a network. If G is a complete graph, then $\Sigma^* = \Sigma$. Let G is a linear graph, that is, $E = \{(i, i+1) : 1 \le i \le n-1\}$. This model is the same as Example 1 in Section 1.

In this section we treat the case where the graph G is a tree in which $E = \{(1, i) : 2 \le i \le n\}$. It is easy to see that a subset S is connected if and only if $1 \in S$. It is possible to analyze if we assume a symmetry in cost as follows.

Assumption 1.

$$c(S) = c(S') \text{ and } c(S \cup \{1\}) = c(S' \cup \{1\}) \text{ for all } S, S' \subseteq N \setminus \{1\} : |S| = |S'|. \tag{21}$$

Since nodes in $N \setminus \{1\}$ are symmetric both in inspection cost and in position in the tree, it is easy to see that there is an optimal strategy $p = (p_1, \ldots, p_n)$ for the hider such that

$$x \equiv p_2 = \ldots = p_n \text{ and } y \equiv p_1 = 1 - (n-1)x, 0 \le x \le \frac{1}{n-1}. \tag{22}$$

This strategy is expressed as $p = p(x)$. A pure strategy in Σ^* for the seeker is expressed as

$$\{\{i_1\}, \ldots, \{i_k\}, S, \{i_{k+1}\}, \ldots, \{i_{n-|S|}\}\} \tag{23}$$

where $1 \in S$ and $i_1, \ldots i_k, i_{k+1}, \ldots, i_{n-|S|}$ is a permutation on $N \setminus S$. By the same symmetry as for the hider, it is easy to see that there is an optimal strategy $q = \{q(\mathcal{S})\}_{\mathcal{S} \in \Sigma^*}$ such that $q(\mathcal{S}) = q(\mathcal{S}')$ if $\mathcal{S}$ is obtained from $\mathcal{S}'$ by a permutation on $N \setminus S$. From this observation, it suffices to restrict our attention to pure strategies $\mathcal{S}$ where $S = \{1, \ldots, s\}, s \equiv |S|$ and

$$\mathcal{S} = \mathcal{S}(s, k) \equiv \{\overline{s+1}, \ldots, \overline{s+k}, S, \overline{s+k+1}, \ldots, \overline{n}\}, \tag{24}$$

for $k = 0, \ldots, n-s$ and for $s = 1, \ldots, n$.

Lemma 5.1. For each $p = p(x)$,

$$f(p, \mathcal{S}(s, k)) = k[c - xc(S) - (n-s)cx] + c(S) + \frac{(n-s)(n-s+1)}{2}cx, \tag{25}$$

where $c(i) = c, \forall i \notin S$.

Proof: For $p = (p_1, \ldots, p_n)$,

$$\begin{aligned} f(p, \mathcal{S}(s, k)) = {} & p_{s+1}c(s+1) + \cdots + p_{s+k}[c(s+1) + \cdots + c(s+k)] \\ & + p(S)[c(s+1) + \cdots + c(s+k) + c(S)] \\ & + p_{s+k+1}[c(s+1) + \cdots + c(s+k) + c(S) + c(s+k+1)] \\ & + \cdots \\ & + p_n[c(s+1) + \cdots + c(s+k) + c(S) + c(s+k+1) + \cdots + c(n)]. \end{aligned} \tag{26}$$

From (21) and (22), we have

$$\begin{aligned} f(p, \mathcal{S}(s, k)) = {} & xc + \cdots + kcx + p(S)[kc + c(S)] \\ & + x[kc + c(S) + c] \\ & + \cdots \\ & + x[kc + c(S) + (n-s-k)c]. \end{aligned} \tag{27}$$

Since $p(S) = y + (s-1)x = 1 - (n-s)x$, from (27), we have (25). □

The hider will consider that the seeker may choose (s,k) so that it minimizes $f(p, \mathcal{S}(s,k))$, given by (25).

Lemma 5.2. For each $p = p(x)$ and $1 \le s \le n$,

$$\min_{0 \le k \le n-s} \{f(p, \mathcal{S}(s,k))\} = \min\{f(p, \mathcal{S}(s,0)), f(p, \mathcal{S}(s,n-s))\}. \tag{28}$$

Proof: From (25), if $c - xc(S) - (n-s)cx > 0$, then $k = 0$ minimizes (25). If $c - xc(S) - (n-s)cx < 0$, then $k = n-s$ minimizes (25). □

For $p = p(x)$, let

$$a(x) \equiv \min_{1 \le s \le n} \{\min\{f(p, \mathcal{S}(s,0)), f(p, \mathcal{S}(s,n-s))\}\}. \tag{29}$$

The hider will choose x so that it maximizes $a(x)$. Here we note that $f(p, \mathcal{S}(s,0))$ is increasing in x and $f(p, \mathcal{S}(s,n-s))$ is decreasing in x as follows:

$$\begin{aligned} f(p, \mathcal{S}(s,0)) &= c(S) + \frac{(n-s)(n-s+1)}{2}cx, \\ f(p, \mathcal{S}(s,n-s)) &= (n-s)c + c(S) - x[(n-s)c(S) + \frac{(n-s)(n-s-1)}{2}c]. \end{aligned} \tag{30}$$

Suppose x^* maximizes $a(x)$. There are $s_1 \le \ldots \le s_\alpha$ and $t_1 \le \ldots \le t_\beta$ such that

$$\begin{aligned} a(x^*) &= f(p(x^*), \mathcal{S}(s_1,0)) = \cdots = f(p(x^*), \mathcal{S}(s_\alpha,0)) \\ &= f(p(x^*), \mathcal{S}(t_1,n-t_1)) = \cdots = f(p(x^*), \mathcal{S}(t_\beta,n-t_\beta)). \end{aligned} \tag{31}$$

By the complementary slackness theorem, there is q such that, for all $j \in N$,

$$\begin{aligned} &\sum_{i=1}^{\alpha} q(\mathcal{S}(s_i,0)) f(j, \mathcal{S}(s_i,0)) + \sum_{i=1}^{\beta} q(\mathcal{S}(t_i,n-t_i)) f(j, \mathcal{S}(t_i,n-t_i)) = a(x^*), \\ &q(\mathcal{S}) = 0, \text{for other } \mathcal{S}. \end{aligned} \tag{32}$$

In summary we have

Proposition 5.3. Under Assumption 1, an optimal strategy for the hider is $p(x^*)$ which is defined by (22) and (31). An optimal strategy for the seeker is q defined by (32). The value of the game is $a(x^*)$.

Let's illustrate the above argument by an example.

Example 3. Let $n = 3$ and $c(\overline{2}) = c(\overline{3}) = 2, c(\overline{12}) = c(\overline{13}) = 3, c(\overline{123}) = 4$. By (1), we have $1 \le c(\overline{1}) \le 3$. By (30),

$$\begin{aligned} &f(p, \mathcal{S}(1,0)) = c(\overline{1}) + 6x, f(p, \mathcal{S}(1,2)) = 4 + c(\overline{1}) - 2[c(\overline{1}) + 1]x, \\ &f(p, \mathcal{S}(2,0)) = 3 + 2x, f(p, \mathcal{S}(2,1)) = 5 - 3x, f(p, \mathcal{S}(3,0)) = 4. \end{aligned} \tag{33}$$

Suppose $c(\overline{1}) = 2$. By drawing a diagram, we see that $x^* = \frac{3}{8}$ maximizes $a(x)$ where $f(p, \mathcal{S}(2,0))$ and $f(p, \mathcal{S}(1,2))$ intersect. $a(\frac{3}{8}) = \frac{15}{4}$. The first in (32) becomes

$$\begin{aligned} f(1,q) &= q(\overline{2},\overline{3},\overline{1}) \times 6 + q(\overline{3},\overline{2},\overline{1}) \times 6 + q(\overline{12},\overline{3}) \times 3 + q(\overline{13},\overline{2}) \times 3 \\ f(2,q) &= q(\overline{2},\overline{3},\overline{1}) \times 2 + q(\overline{3},\overline{2},\overline{1}) \times 4 + q(\overline{12},\overline{3}) \times 3 + q(\overline{13},\overline{2}) \times 5 \\ f(3,q) &= q(\overline{2},\overline{3},\overline{1}) \times 4 + q(\overline{3},\overline{2},\overline{1}) \times 2 + q(\overline{12},\overline{3}) \times 5 + q(\overline{13},\overline{2}) \times 3. \end{aligned} \tag{34}$$

We find q so that these three are equal to $a(\frac{3}{8}) = \frac{15}{4}$:

$$q(\overline{2},\overline{3},\overline{1}) = q(\overline{3},\overline{2},\overline{1}) = \frac{1}{8}, \quad q(\overline{12},\overline{3}) = q(\overline{13},\overline{2}) = \frac{3}{8}. \tag{35}$$

Then $f(1,q) = f(2,q) = f(3,q) = \frac{15}{4} = a(x^*)$.

Suppose $c(\overline{1}) = 1$. In the same way as above, we have $x^* = \frac{2}{5}$ and $a(\frac{2}{5}) = \frac{17}{5}$. The intersection of lines $f(p,\mathcal{S}(1,0))$ and $f(p,\mathcal{S}(1,2))$ is critical. So the seeker will choose

$$q(\overline{2},\overline{3},\overline{1}) = q(\overline{3},\overline{2},\overline{1}) = \frac{3}{10}, \quad q(\overline{1},\overline{2},\overline{3}) = q(\overline{1},\overline{3},\overline{2}) = \frac{1}{5}. \tag{36}$$

Then $f(1,q) = f(2,q) = f(3,q) = \frac{17}{5} = a(x^*)$.

Suppose $c(\overline{1}) = 3$. We have $x^* = \frac{2}{5}$ and $a(\frac{2}{5}) = \frac{19}{5}$. The intersection of lines $f(p,\mathcal{S}(2,0))$, $f(p,\mathcal{S}(2,1))$ and $f(p,\mathcal{S}(1,2))$ is critical. So the seeker will choose

$$\begin{aligned} q(\overline{2},\overline{3},\overline{1}) &= q(\overline{3},\overline{2},\overline{1}), \quad q(\overline{3},\overline{12}) = q(\overline{2},\overline{13}), q(\overline{12},\overline{3}) = q(\overline{13},\overline{2}), \\ q(\overline{12},\overline{3}) &= \frac{3}{2}q(\overline{3},\overline{12}) + 4q(\overline{2},\overline{3},\overline{1}), \\ 10q(\overline{2},\overline{3},\overline{1}) &+ 5q(\overline{3},\overline{12}) = 1. \end{aligned} \tag{37}$$

Then $f(1,q) = f(2,q) = f(3,q) = \frac{19}{5} = a(x^*)$.

Acknowledgments We thank the referee for his/her useful suggestions and comments. David Ramsey was supported partially by Dutch Science Foundation(NWO) research grant 040.11.401. Kensaku Kikuta was partially supported by JSPS KAKENHI Grant Number 23510177.

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Communicated by *Ryusuke Hohzaki*

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Scientiae Mathematicae Japonicae 78, No.2(2015)(149-169)

SIMPLICITY OF THE C^*-ALGEBRAS OF SKEW PRODUCT k-GRAPHS

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Received March 21,2014;Revised May 19,2014

Abstract. We consider conditions on a k-graph Λ, a semigroup S and a functor $\eta : \Lambda \to S$ that ensure that the C^*-algebra of the skew-product graph $\Lambda \times_\eta S$ is simple. Our results give some necessary and sufficient conditions for the AF-core of a k-graph C^*-algebra to be simple.

1 Introduction In [24] Robertson and Steger investigated C^*-algebras that they considered to be higher-rank versions of the Cuntz-Krieger algebras. Subsequently in [9] Kumjian and Pask introduced higher-rank graphs, or k-graphs, as a graphical means to provide combinatorial models for the Cuntz-Krieger algebras of Robertson and Steger. They showed how to construct a C^*-algebra that is associated to a k-graph. Since then k-graphs and their C^*-algebras have attracted a lot of attention from many authors (see [1, 3–5, 9, 12–14, 17–19, 21, 23]).

Roughly speaking, a k-graph is a category Λ together with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying a certain factorisation property. A 1-graph is then the path category of a directed graph. Given a functor $\eta : \Lambda \to S$, where S is a semigroup with identity, we may form the skew product k-graph $\Lambda \times_\eta S$. Skew product graphs play an important part in the development of k-graph C^*-algebras. For example [9, Corollary 5.3] shows that $C^*(\Lambda \times_d \mathbb{Z}^k)$ is isomorphic to $C^*(\Lambda) \times_\gamma \mathbb{T}^k$ where $\gamma : \mathbb{T}^k \to \operatorname{Aut} C^*(\Lambda)$ is the canonical gauge action. Skew product graphs feature in nonabelian duality: In [13] it is shown that if a right-reversible semigroup (Ore semigroup) S acts freely on a k-graph Λ then the crossed product $C^*(\Lambda) \times S$ is stably isomorphic to $C^*(\Lambda/S)$. On the other hand if S is a group G then $C^*(\Lambda \times_\eta G)$ is isomorphic to the crossed product $C^*(\Lambda) \times_{\delta_\eta} G$ where δ_η is the coaction of G on $C^*(\Lambda)$ induced by η.

The main purpose of this paper is to investigate necessary and sufficient conditions for the C^*-algebra of a skew product k-graph to be simple. We will be particularly interested in the specific case when $S = \mathbb{N}^k$ and $\eta = d$. It can be shown that simplicity of $C^*(\Lambda \times_d \mathbb{N}^k)$ is equivalent to simplicity of the fixed point algebra (AF core) $C^*(\Lambda)^\gamma$. This is important as many results in

2010 *Mathematics Subject Classification.* Primary 46L05.
Key words and phrases. C^*-algebra; Graph algebra; k-graph.

the literature apply particularly when AF core is simple; see [8, Proposition 3.8] for example.

We begin by introducing some basic facts we will need during this paper.

2 Background

2.1 Basic facts about k-graphs All semigroups in this paper will be countable, cancellative and have an identity, hence any semigroup may be considered as a category with a single object. The semigroup $\mathbb{N}^k$ is freely generated by $\{e_1, \ldots, e_k\}$ and comes with the usual order structure: if $n = \sum_{i=1}^k n_i e_i$ and $m = \sum_{i=1}^k m_i e_i$ then $m > n$ (resp. $m \geq n$) if $m_i > n_i$ (resp. $m_i \geq n_i$) for all i. For $m, n \in \mathbb{N}^k$ we define $m \vee n \in \mathbb{N}^k$ by $(m \vee n)_i = \max\{m_i, n_i\}$ for $i = 1, \ldots, k$.

A directed graph E is a quadruple (E^0, E^1, r, s) where E^0, E^1 are countable sets of vertices and edges. The direction of an edge $e \in E^1$ is given by the maps $r, s : E^1 \to E^0$. A path λ of length $n \geq 1$ is a sequence $\lambda = \lambda_1 \cdots \lambda_n$ of edges such that $s(\lambda_i) = r(\lambda_{i+1})$ for $i = 1, \ldots, n-1$. The set of paths of length $n \geq 1$ is denoted E^n. We may extend r, s to E^n for $n \geq 1$ by $r(\lambda) = r(\lambda_1)$ and $s(\lambda) = s(\lambda_n)$ and to E^0 by $r(v) = v = s(v)$.

A *higher-rank graph* or *k-graph* is a combinatorial structure, and is a k-dimensional analogue of a directed graph. A k-graph consists of a countable category Λ together with a functor $d : \Lambda \to \mathbb{N}^k$, known as the degree map, with the following factorisation property: for every morphism $\lambda \in \Lambda$ and every decomposition $d(\lambda) = m+n$, there exist unique morphisms $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$.

For $n \in \mathbb{N}^k$ we define $\Lambda^n := d^{-1}(n)$ to be those morphisms in Λ of degree n. Then by the factorisation property Λ^0 may be identified with the objects of Λ, and are called vertices. For $u, v \in \Lambda^0$ and $X \subseteq \Lambda$ we set

$$uX = \{\lambda \in X : r(\lambda) = u\} \quad Xv = \{\lambda \in X : s(\lambda) = v\} \quad uXv = uX \cap Xv.$$

A k-graph Λ is visualised by a k-coloured directed graph E_Λ with vertices Λ^0 and edges $\sqcup_{i=1}^k \Lambda^{e_i}$ together with range and source maps inherited from Λ called its 1-skeleton. The 1-skeleton is provided with square relations $\mathcal{C}_\Lambda$ between the edges in E_Λ, called *factorisation rules*, which come from factorisations of morphisms in Λ of degree $e_i + e_j$ where $i \neq j$. By convention the edges of degree e_1 are drawn blue (solid) and the edges of degree e_2 are drawn red (dashed). For more details about the 1-skeleton of a k-graph see [21]. On the other hand, if G is a k-coloured directed graph with a complete, associative collection of square relations $\mathcal{C}$ completely determines a k-graph Λ such that $E_\Lambda = G$ and $\mathcal{C}_\Lambda = \mathcal{C}$ (see [6]).

A k-graph Λ is *row-finite* if for every $v \in \Lambda^0$ and every $n \in \mathbb{N}^k$, $v\Lambda^n$ is finite. A k-graph has *no sources* if $v\Lambda^n \neq \emptyset$ for all $v \in \Lambda^0$ and nonzero $n \in \mathbb{N}^k$. A k-graph has *no sinks* is $\Lambda^n v \neq \emptyset$ for all $v \in \Lambda^0$ and nonzero $n \in \mathbb{N}^k$.

For $\lambda \in \Lambda$ and $m \leq n \leq d(\lambda)$, we define $\lambda(m,n)$ to be the unique path in Λ^{n-m} obtained from the k-graph factorisation property such that $\lambda = \lambda'(\lambda(m,n))\lambda''$ for some $\lambda' \in \Lambda^m$ and $\lambda'' \in \Lambda^{d(\lambda)-n}$.

Examples 2.1. (a) In [9, Example 1.3] it is shown that the path category $E^* = \cup_{i\geq 0} E^i$ of a directed graph E is a 1-graph, and vice versa. For this reason we shall move seamlessly between 1-graphs and directed graphs.

(b) For $k \geq 1$ let T_k be the category with a single object v and generated by k commuting morphisms $\{f_1, \ldots, f_k\}$. Define $d : T_k \to \mathbb{N}^k$ by $d(f_1^{n_1} \ldots f_k^{n_k}) = (n_1, \ldots, n_k)$ then it is straightforward to check that T_k is a k-graph. We frequently identify T_k with $\mathbb{N}^k$ via the map $f_1^{n_1} \cdots f_k^{n_k} \mapsto (n_1, \ldots, n_k)$.

(c) For $k \geq 1$ define a category Δ_k as follows: Let $\operatorname{Mor} \Delta_k = \{(m,n) \in \mathbb{Z}^k \times \mathbb{Z}^k : m \leq n\}$ and $\operatorname{Obj} \Delta_k = \mathbb{Z}^k$; structure maps $r(m,n) = m$, $s(m,n) = n$, and composition $(m,n)(n,p) = (m,p)$. Define $d : \Delta_k \to \mathbb{N}^k$ by $d(m,n) = n - m$, then one checks that (Δ_k, d) is a row-finite k-graph. We identify $\operatorname{Obj} \Delta_k$ with $\{(m,m) : m \in \mathbb{Z}^k\} \subset \operatorname{Mor} \Delta_k$.

(d) For $n \geq 1$ let $\underline{n} = \{1, \ldots, n\}$. For $m, n \geq 1$ let $\theta : \underline{m} \times \underline{n} \to \underline{m} \times \underline{n}$ a bijection. Let $\mathbb{F}_\theta^2$ be the 2-graph which has 1-skeleton which consists of with single vertex v and edges $f_1, \ldots, f_m, g_1, \ldots, g_n$, such that f_i have the same colour (blue) for $i \in \underline{m}$ and g_j have the same colour (red) for $j \in \underline{n}$ together with complete associative square relations $f_i g_j = g_{j'} f_{i'}$ where $\theta(i,j) = (i', j')$ for $(i,j) \in \underline{m} \times \underline{n}$ (for more details see [3, 4, 19]).

2.2 Skew product k-graphs Let Λ be a k-graph and $\eta : \Lambda \to S$ a functor into a semigroup S. We can make the cartesian product $\Lambda \times S$ into a k-graph $\Lambda \times_\eta S$ by taking $(\Lambda \times_\eta S)^0 = \Lambda^0 \times S$, defining $r, s : \Lambda \times_\eta S \to (\Lambda \times_\eta S)^0$ by

$$r(\lambda, t) = (r(\lambda), t) \ \text{ and } \ s(\lambda, t) = (s(\lambda), t\eta(\lambda)), \tag{1}$$

defining the composition by

$$(\lambda, t)(\mu, u) = (\lambda\mu, t) \ \text{ when } s(\lambda, t) = r(\mu, u) \text{ (so that } u = t\eta(\lambda)\text{)},$$

and defining $d : \Lambda \times_\eta S \to \mathbb{N}^k$ by $d(\lambda, t) = d(\lambda)$. As in [13] it is straightforward to show that this defines a k-graph.

Remark 2.2. If Λ is row-finite with no sources and $\eta : \Lambda \to S$ a functor then $\Lambda \times_\eta S$ is row-finite with no sources.

A k-graph morphism is a degree preserving functor between two k-graphs. If a k-graph morphism is bijective, then it is called an isomorphism.

Examples 2.3. (i) Let Λ be a k-graph and $\eta : \Lambda \to S$ a functor, where S is a semigroup and $\Lambda \times_\eta S$ the associated skew product graph. Then the map $\pi : \Lambda \times_\eta S \to \Lambda$ given by $\pi(\lambda, s) = \lambda$ is a surjective k-graph morphism.

(ii) For $\ell \geq 1$ the map $(\ell, m) \mapsto (m, \ell + m)$ gives an isomorphism from $T_k \times_d \mathbb{Z}^k$ to Δ_k.

Definition 2.4. Let Λ, Γ be row-finite k-graphs. A surjective k-graph morphism $p : \Lambda \to \Gamma$ has *r-path lifting* if for all $v \in \Lambda^0$ and $\lambda \in p(v)\Gamma$ there is $\lambda' \in v\Lambda$ such that $p(\lambda') = \lambda$. If λ' is the unique element with this property then p has *unique r-path lifting.*

Example 2.5. Let Λ be a row-finite k-graph and $\eta : \Lambda \to S$ a functor where S is a semigroup, and $\Lambda \times_\eta S$ the associated skew product graph. The map $\pi : \Lambda \times_\eta S \to \Lambda$ described in Examples 2.3(i) has unique r-path lifting.

2.3 Connectivity A k-graph Λ is *connected* if the equivalence relation on Λ^0 generated by the relation $\{(u, v) : u\Lambda v \neq \emptyset\}$ is $\Lambda^0 \times \Lambda^0$. The k-graph Λ is *strongly connected* if for all $u, v \in \Lambda^0$ there is $N > 0$ such that $u\Lambda^N v \neq \emptyset$. If Λ is strongly connected, then it is connected and has no sinks or sources. The k-graph Λ is *primitive* if there is $N > 0$ such that $u\Lambda^N v \neq \emptyset$ for all $u, v \in \Lambda^0$. If Λ is primitive then it is strongly connected.

Examples 2.6. The graphs T_k and $\mathbb{F}^2_\theta$ in Examples 2.1 are primitive as they have one vertex.

The connectivity of a k-graph may also be described in terms of its component matrices as defined in [9, §6]: Given a k-graph Λ, for $1 \leq i \leq k$ and $u, v \in \Lambda^0$, we define k non-negative $\Lambda^0 \times \Lambda^0$ matrices M_i with entries $M_i(u, v) = |u\Lambda^{e_i} v|$. Using the k-graph factorisation property, we have that $|u\Lambda^{e_i+e_j} v| = |u\Lambda^{e_j+e_i} v|$ for all $u, v \in \Lambda^0$, and so $M_i M_j = M_j M_i$. For $m = (m_1, \ldots, m_k) \in \mathbb{N}^k$ and $u, v \in \Lambda^0$, we have $|u\Lambda^m v| = (M_1^{m_1} \cdots M_k^{m_k})(u, v) = M^m(u, v)$, using multiindex notation. The following lemma follows directly from the above definitions.

Lemma 2.7. *Let Λ be a row-finite k-graph with no sources.*

(a) Then Λ is strongly connected if and only if for all pairs $u, v \in \Lambda^0$ there is $N \in \mathbb{N}^k$ such that $M^N(u, v) > 0$.

(b) Then Λ is primitive if and only if there is $N > 0$ such that $M^N(u, v) > 0$ for all pairs $u, v \in \Lambda^0$.

Remarks 2.8. Following [18, §4], a primitive 1-graph Λ is strongly connected with period 1; that is, the greatest common divisor of all n such that $v\Lambda^n v \neq \emptyset$ for some $v \in \Lambda^0$ is 1.

Lemma 2.9. *Let Λ be a k-graph with no sinks, and Λ^0 finite. Then for all $v \in \Lambda^0$, there exists $w \in \Lambda^0$ and $\alpha \in w\Lambda w$ such that $d(\alpha) > 0$ and $w\Lambda v \neq \emptyset$.*

Proof. Let $p = (1, \ldots, 1) \in \mathbb{N}^k$. Since v is not a sink, there exists $\beta_1 \in \Lambda^p v$. Since $r(\beta_1)$ is not a sink, there exists $\beta_2 \in \Lambda^p r(\beta_1)$. Inductively, there exist infinitely many β_i such that $d(\beta_i) = p$ and $r(\beta_i) = s(\beta_{i+1})$. Since Λ^0 is finite, there exists $w \in \Lambda^0$ such that $r(\beta_i) = w$ for infinitely many i. Suppose $r(\beta_n) = w = r(\beta_m)$ with $m > n$. Then $\alpha = \beta_m \ldots \beta_{n+1}$ has the requisite properties, and $w\Lambda v \neq \emptyset$, since $\beta_n \ldots \beta_1 \in w\Lambda v$. □

2.4 The graph C^*-algebra Let Λ be a row-finite k-graph with no sources, then following [9], a *Cuntz-Krieger Λ-family* in a C^*-algebra B consists of partial isometries $\{S_\lambda : \lambda \in \Lambda\}$ in B satisfying the *Cuntz-Krieger relations*:

(CK1) $\{S_v : v \in \Lambda^0\}$ are mutually orthogonal projections;

(CK2) $S_\lambda S_\mu = S_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;

(CK3) $S_\lambda^* S_\lambda = S_{s(\lambda)}$ for every $\lambda \in \Lambda$;

(CK4) $S_v = \sum_{\{\lambda \in v\Lambda^n\}} S_\lambda S_\lambda^*$ for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

The k-graph C^*-algebra $C^*(\Lambda)$ is generated by a universal Cuntz-Krieger Λ-family $\{s_\lambda\}$. By [9, Proposition 2.11] there exists a Cuntz-Krieger Λ-family such that each vertex projection S_v (and hence by (CK3) each S_λ) is nonzero and so there exists a nonzero universal k-graph C^*-algebra for a Cuntz-Krieger Λ-family. Moreover,

$$C^*(\Lambda) = \overline{\operatorname{span}}\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda, s(\lambda) = s(\mu)\} \quad \text{(see [9, Lemma 3.1])}.$$

We will use [23, Theorem 3.1] by Robertson and Sims when considering the simplicity of graph C^*-algebras:

Theorem 2.10 (Robertson-Sims)**.** *Suppose Λ is a row-finite k-graph with no sources. Then $C^*(\Lambda)$ is simple if and only if Λ is cofinal and aperiodic.*

We now focus on the two key properties involved in the simplicity criterion of Theorem 2.10, namely aperiodicity and cofinality. Our attention will be directed towards applying these conditions on skew product graphs.

3 Aperiodicity Our definition of aperiodicity is taken from Robertson-Sims, [23, Theorem 3.2].

Definitions 3.1. A row-finite k-graph Λ with no sources has *no local periodicity* at $v \in \Lambda^0$ if for all $m \neq n \in \mathbb{N}^k$ there exists a path $\lambda \in v\Lambda$ such that $d(\lambda) \geq m \vee n$ and

$$\lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n)).$$

Λ is called *aperiodic* if every $v \in \Lambda^0$ has no local periodicity.

Examples 3.2. (a) The k-graph Δ_k is aperiodic for all $k \geq 1$. First observe that there is no local periodicity at $v = (0, 0)$. Given $m \neq n \in \mathbb{N}^k$, let $N \geq m \vee n$; then $\lambda = (0, N)$ is the only element of $v\Delta_k$. Then $\lambda(m, m) = (m, m) \neq (n, n) = \lambda(n, n)$. A similar argument applies for any other vertex $w = (n, n)$ in Δ_k and so there is no local periodicity at w for all $w \in \Delta_k^0$.

(b) The k-graph T_k is not aperiodic for all $k \geq 1$. For all $n \in \mathbb{N}^k$ one checks that $f_1^{n_1} \cdots f_k^{n_k}$ is the only element of vT_k^n. Hence given $m \neq n \in \mathbb{N}^k$ it follows that for all $\lambda \in v\Lambda^N$ with $N \geq m \vee n$ we have

$$\lambda(m, m + (m \vee n)) = \lambda(n, n + (m \vee n)).$$

Since the map $\pi : \Lambda \times_\eta S \to \Lambda$ has unique r-path lifting, we wish to know if we can deduce the aperiodicity of $\Lambda \times_\eta S$ from that of Λ. A corollary of our main result Theorem 3.3, shows that this is true.

Theorem 3.3. *Let Λ, Γ be row-finite k-graphs with no sources and $p : \Lambda \to \Gamma$ have r-path lifting. If Γ is aperiodic, then Λ is aperiodic.*

Proof. Suppose that Γ is aperiodic. Let $v \in \Lambda^0$ and $m \neq n \in \mathbb{N}^k$. Since Γ is aperiodic, there exists $\lambda \in p(v)\Gamma$ with $d(\lambda) \geq m \vee n$ such that $\lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n))$. By r-path lifting there is $\lambda' \in v\Lambda$ with $p(\lambda') = \lambda$ such that $d(\lambda') \geq m \vee n$ and

$$\lambda'(m, m + d(\lambda) - (m \vee n)) \neq \lambda'(n, n + d(\lambda) - (m \vee n)),$$

and so Λ is aperiodic. □

The converse of Theorem 3.3 is false:

Example 3.4. The surjective k-graph morphism $p : \Delta_k \to T_k$ given by $p(m, m + e_i) = f_i$ for all $m \in \mathbb{Z}^k$ and $i = 1, \ldots, k$ has r-path lifting. However by Examples 3.2 we see that $\Delta_k \cong T_k \times_d \mathbb{Z}^k$ is aperiodic but T_k is not.

Corollary 3.5. *Let Λ be a row-finite k-graph with no sources, $\eta : \Lambda \to S$ a functor where S is a semigroup and $\Lambda \times_\eta S$ the associated skew product graph. If Λ is aperiodic then $\Lambda \times_\eta S$ is aperiodic.*

Proof. Follows from Theorem 3.3 and Example 2.5. □

In some cases the aperiodicity of a skew product graph $\Lambda \times_\eta S$ can be deduced directly from properties of η.

Proposition 3.6. *Suppose S is a semigroup, Λ is a row-finite k-graph with no sources, $\eta : \Lambda \to S$ is a functor, and there exists a map $\phi : S \to \mathbb{Z}^k$ such that $d = \phi \circ \eta$. Then $\Lambda \times_\eta S$ is aperiodic.*

Proof. Fix $(v, s) \in (\Lambda \times_\eta S)^0$ and $m \neq n \in \mathbb{N}^k$. Let $\lambda \in (v, s)(\Lambda \times_\eta S)$ be such that $d(\lambda) \geq m \vee n$. Observe that $\lambda(m, m) = s(\lambda(0, m))$, $\lambda(m, m)$ is of the form $(w, s\eta(\lambda(0, m)))$ for some $w \in \Lambda^0$. Similarly, $\lambda(n, n)$ is of the form $(w', s\eta(\lambda(0, n)))$ for some $w' \in \Lambda^0$.

We claim $\lambda(m, m) \neq \lambda(n, n)$: Suppose, by hypothesis, $\eta(\lambda(0, n)) = \eta(\lambda(0, m))$. Then $n = \phi \circ \eta(\lambda(0, n)) = \phi \circ \eta(\lambda(0, m)) = m$, which provides a contradiction, and $m \neq n$. Then $\eta(\lambda(0, m)) \neq \eta(\lambda(0, n))$, and so $\lambda(m, m) \neq \lambda(n, n)$, and hence $\lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n))$. □

Corollary 3.7. *Suppose Λ is a row-finite k-graph with no sources. Then $\Lambda \times_d \mathbb{N}^k$ and $\Lambda \times_d \mathbb{Z}^k$ are aperiodic.*

Proof. Apply Proposition 3.6 with $\eta = d$ and $S = \mathbb{N}^k, \mathbb{Z}^k$ respectively. □

4 Cofinality We will use the Lewin-Sims definition of cofinality, [12, Remark A.3]. By [12, Appendix A] this definition is equivalent to the other standard definitions of cofinality:

Definition 4.1. A row-finite, k-graph Λ with no sources is *cofinal* if for all pairs $v, w \in \Lambda^0$ there exists $N \in \mathbb{N}^k$ such that $v\Lambda s(\alpha) \neq \emptyset$ for every $\alpha \in w\Lambda^N$.

Lemma 4.2. *Let Λ be a row-finite k-graph with no sources.*

(a) If Λ is cofinal then Λ is connected.

(b) Suppose that for all pairs $v, w \in \Lambda^0$ there exists $N \in \mathbb{N}^k$ such that $v\Lambda s(\alpha) \neq \emptyset$ for every $\alpha \in w\Lambda^N$. Then for $n \geq N$ we have $v\Lambda s(\alpha) \neq \emptyset$ for every $\alpha \in w\Lambda^n$.

Proof. Fix $v, w \in \Lambda^0$. If Λ is cofinal it follows that there is $\alpha \in w\Lambda$ such that $w\Lambda s(\alpha)$ and $v\Lambda s(\alpha)$ are non-empty. It then follows that (v, w) belongs to the equivalence relation described in Section 2.3. Since v, w were arbitrary it follows that the equivalence relation is $\Lambda^0 \times \Lambda^0$ and so Λ is connected.

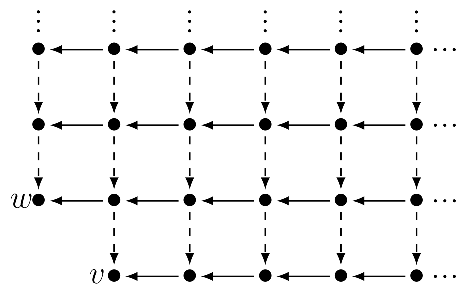
Fix $v, w \in \Lambda^0$, then there is $N \in \mathbb{N}^k$ such that $v\Lambda s(\alpha) \neq \emptyset$ for every $\alpha \in w\Lambda^N$. Let $n \geq N$ and consider $\beta \in w\Lambda^n$ then $\beta' = \beta(0, N) \in w\Lambda^N$ and so by hypothesis there is $\lambda \in v\Lambda s(\beta')$. Then $\lambda\beta(N, n) \in v\Lambda s(\beta)$ and the result follows. □

Lemma 4.3. *Let Λ be a row-finite k-graph with no sources with skeleton E_Λ. If E_Λ is cofinal then Λ is cofinal. Furthermore, Λ is strongly connected if and only if E_Λ strongly connected*

Proof. Fix $v, w \in \Lambda^0 = E_\Lambda^0$. As E_Λ is cofinal there is $n \in \mathbb{N}$ with $vE_\Lambda s(\alpha) \neq \emptyset$ for all $\alpha \in wE_\Lambda^n$. Let $N \in \mathbb{N}^k$ be such that $\sum_{i=1}^k N_i = n$. Then for all $\alpha' \in w\Lambda^N$ we have $\alpha' \in E_\Lambda^n$ and so $v\Lambda^N s(\alpha') \neq \emptyset$.

Suppose that Λ is strongly connected and $v, w \in E_\Lambda^0 = \Lambda^0$. As Λ is strongly connected there is $\alpha \in v\Lambda w$ with $d(\alpha) > 0$. Let $n = \sum_{i=1}^n d(\alpha)_i$ then $n > 0$ and $vE_\Lambda w \neq \emptyset$, so E_Λ is strongly connected. Suppose that E_Λ is strongly connected, and $v, w \in \Lambda^0 = E_\Lambda^0$. As Λ has no sources, there is $\alpha \in vE_\Lambda^k$ which uses an edge of each of the k-colours. Let $u = s(\alpha)$. Since E_Λ is strongly connected there is $\beta \in uE_\Lambda^n w$. Let λ be the element of Λ which may be represented by $\alpha\beta \in E_\Lambda$. Then $\lambda \in v\Lambda w$ and $d(\lambda) > 0$ and so Λ is strongly connected. □

Remark 4.4. The converse to the first part of Lemma 4.3 is not true: Let Λ be the 2-graph which is completely determined by its 1-skeleton as shown:



Then Λ is cofinal: For example for v, w as shown, $N = (1, 0)$ will suffice. However E_Λ is not cofinal: For example for v, w as shown, for any $n \geq 0$ the vertex which is the source of the vertical path of length n with range w does not connect to v.

The following result establishes a link between cofinality and strongly connectivity for a row-finite k-graph.

Proposition 4.5. *Suppose Λ is a row-finite k-graph with no sources.*

1. *If Λ is strongly connected then Λ is cofinal.*
2. *If Λ is cofinal, has no sinks and Λ^0 finite then Λ is strongly connected.*

Proof. Suppose Λ is strongly connected. Fix $v, w \in \Lambda^0$ then for $N = e_1$ we have $v\Lambda s(\alpha) \neq \emptyset$ for all $\alpha \in w\Lambda^N$ since Λ is strongly connected, and so Λ is cofinal.

Suppose Λ is cofinal. Fix $u, v \in \Lambda^0$. Then by Lemma 2.9, there exists $w \in \Lambda^0$ and $\alpha \in w\Lambda w$ such that $d(\alpha) > 0$ and $w\Lambda v \neq \emptyset$. Let $\alpha' \in w\Lambda v$. Given $u, w \in \Lambda^0$, since Λ is cofinal and has no sources, by Lemma 4.2(ii) there exists $N \in \mathbb{N}^k$ such that for all $n \geq N$ and all $\alpha'' \in w\Lambda^n$, there exists $\beta \in u\Lambda s(\alpha'')$. Since $d(\alpha) > 0$ we may choose $t \in \mathbb{N}$ such that $td(\alpha) > N$. Then $\alpha^t \in w\Lambda^n$ where $n > N$, and so by cofinality of Λ exists $\beta \in u\Lambda s(\alpha^t) = u\Lambda w$. Hence $\beta\alpha\alpha' \in u\Lambda v$ with $d(\beta\alpha\alpha') > d(\alpha) > 0$ and so Λ is strongly connected. □

Example 4.6. The condition that Λ^0 is finite in Proposition 4.5(2) is essential: For instance Δ_k is cofinal by Lemma 4.3 since its skeleton is cofinal; however it is not strongly connected by Lemma 4.3 since its skeleton is not strongly connected.

Since the map $\pi : \Lambda \times_\eta S \to \Lambda$ has unique r-path lifting, we wish to know if we can deduce the cofinality of $\Lambda \times_\eta S$ from that of Λ. By Theorem 4.7 the image of a cofinal k-graph under a map with r-path lifting is cofinal, however Example 4.9 shows that the converse is not true. For a cofinal k-graph Λ, we must then seek additional conditions on the functor η which guarantees that $\Lambda \times_\eta S$ is cofinal. In Definition 4.10 we introduce the notion of (Λ, S, η) cofinality to address this problem.

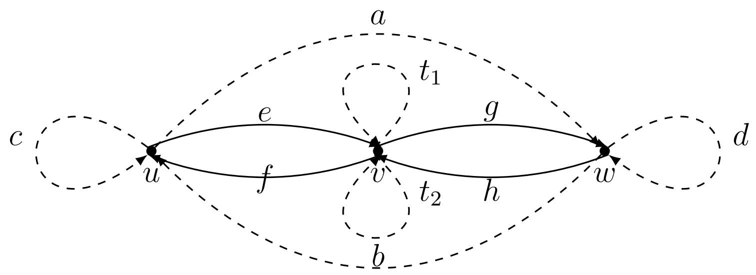
Theorem 4.7. *Suppose Λ, Γ be row-finite k-graphs with no sources and $p : \Lambda \to \Gamma$ have r-path lifting. If Λ is cofinal then Γ is cofinal.*

Proof. Suppose that Λ is cofinal. Fix $v, w \in \Gamma^0$. Let $v', w' \in \Lambda^0$ be such that $p(v') = v$ and $p(w') = w$. As Λ is cofinal there is an N such that for all $\alpha' \in w'\Lambda^N$ there is $\beta' \in v'\Lambda s(\alpha')$. Then for $\alpha \in v\Gamma^N$ there is $\alpha' \in v'\Lambda^N$ with $p(\alpha') = \alpha$. By hypothesis there is $\beta' \in v'\Lambda s(\alpha')$, and so $\beta = p(\beta')$ satisfies $s(\beta) = s(\alpha)$ and $r(\beta) = v$, hence $v\Lambda s(\alpha) \neq \emptyset$ as required. □

Corollary 4.8. *Let Λ be a row-finite k-graph with no sources, $\eta : \Lambda \to S$ a functor where S is a semigroup and $\Lambda \times_\eta S$ the associated skew product graph. If $\Lambda \times_\eta S$ is cofinal then Λ is cofinal.*

The converse of Theorem 4.7 is false:

Example 4.9. Consider the following 2-graph Λ with 1-skeleton



and factorisation rules: $ec = t_1 e$ and $ha = t_2 e$ for paths from u to v; $cf = ft_1$ and $bg = ft_2$ for paths from v to u. Also $hd = t_1 h$ and $eb = t_2 h$ for paths from w to v; $dg = gt_1$ and $af = gt_2$ for paths from v to w. By Lemma 4.3 Λ is strongly connected as its skeleton is strongly connected. Note there are no paths of degree $e_1 + e_2$ from a vertex to itself.

Since $M_1 = \begin{pmatrix} 0&1&0\\1&0&1\\0&1&0 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 1&0&1\\0&2&0\\1&0&1 \end{pmatrix}$, we calculate that $M^{(2j_1,j_2)} = 2^{j_1+j_2-1}M_2$ and $M^{(2j_1+1,j_2)} = 2^{j_1+j_2+1}M_1$. Hence $M^{(2j-1,2j-1)} = \begin{pmatrix} 0&4j&0\\4j&0&4j\\0&4j&0 \end{pmatrix}$ and $M^{(2j,2j)} = \begin{pmatrix} 4j&0&4j\\0&8j&0\\4j&0&4j \end{pmatrix}$. In particular by Lemma 2.7 (b) Λ is not primitive, even though it is strongly connected.

We claim that the skew product graph $\Lambda \times_d \mathbb{Z}^2$ is not cofinal. Consider $v_1 = (v,(m,n))$ and $v_2 = (v,(m+1,n))$ in $(\Lambda \times_d \mathbb{Z}^2)^0$. We claim that for all $N \in \mathbb{N}^2$, for all $\alpha \in v_1(\Lambda \times_d \mathbb{Z}^2)^N$, we have $v_2(\Lambda \times_d \mathbb{Z}^2)s(\alpha) \neq \emptyset$. Let $N = (N_1, N_2)$. Suppose N_1 is even. Then for all $\alpha \in v_1(\Lambda \times_d \mathbb{Z}^2)^N$, $s(\alpha) = (v,(m+N_1,n+N_2))$. In order for this vertex to connect to $(v,(m+1,n))$, we have $M^{(N_1-1,N_2)}(v,v) \neq 0$. But $N_1 - 1$ is odd, and this matrix entry is zero. If N_1 is odd, then $s(\alpha) = (u,(m+N_1,n+N_2))$ or $s(\alpha) = (w,(m+N_1,n+N_2))$. In order for either of these vertices to connect to $(v,(m+1,n))$, we must have $M^{(N_1-1,N_2)}(u,v) \neq 0$, or $M^{(N_1-1,N_2)}(w,v) \neq 0$. But $N_1 - 1$ is even, and so both of these matrix entries are zero. Hence $\Lambda \times_d \mathbb{Z}^2$ is not cofinal, even though Λ is cofinal.

To establish a sufficient condition for $\Lambda \times_\eta S$ to be cofinal, we need Λ to be cofinal and an additional condition on η.

Definition 4.10. Let Λ be a row-finite k-graph with no sources and $\eta : \Lambda \to S$ a functor, where S is a semigroup. The system (Λ, S, η) is *cofinal* if for all $v, w \in \Lambda^0$, $a, b \in S$, there exists $N \in \mathbb{N}^k$ such that for all $\alpha \in w\Lambda^N$, there exists $\beta \in v\Lambda s(\alpha)$ such that $a\eta(\beta) = b\eta(\alpha)$.

Proposition 4.11. *Let Λ be a row-finite k-graph with no sources and $\eta : \Lambda \to S$ a functor, where S is a semigroup and $\Lambda \times_\eta S$ the associated skew product graph. Then the system (Λ, S, η) is cofinal if and only if $\Lambda \times_\eta S$ is cofinal.*

Proof. Suppose $\Lambda \times_\eta S$ is cofinal. Fix $a, b \in S$ and $v, w \in \Lambda^0$. By hypothesis there is $N \in \mathbb{N}^k$ such that $(v, a)(\Lambda \times_\eta S)s(\alpha, b)$ is non-empty for every $(\alpha, b) \in (w, b)(\Lambda \times_\eta S)^N$. In particular for all $\alpha \in w\Lambda^N$ there exists $\beta \in w\Lambda^N$ such that $a\eta(\beta) = b\eta(\alpha)$, and so (Λ, S, η) is cofinal.

Now suppose (Λ, S, η) is cofinal. Fix $(v, a), (w, b) \in (\Lambda \times_\eta S)^0$. By hypothesis there exists $N \in \mathbb{N}^k$ such that for all $\alpha \in w\Lambda^N$, there exists $\beta \in v\Lambda s(\alpha)$ with $a\eta(\beta) = b\eta(\alpha)$. In particular for all $(\alpha, b) \in (w, b)(\Lambda \times_\eta S)^N$ there is $(\beta, a) \in (v, a)\Lambda s(\alpha, b)$, and so $\Lambda \times_\eta S$ is cofinal. □

Theorem 4.12. *Let Λ be an aperiodic row-finite k-graph with no sources, $\eta : \Lambda \to S$ a functor, where S is a semigroup and $\Lambda \times_\eta S$ the associated skew product graph. Then $C^*(\Lambda \times_\eta S)$ is simple if and only if the system (Λ, S, η) is cofinal.*

Proof. If the system (Λ, S, η) is cofinal, then by Proposition 4.11, $\Lambda \times_\eta S$ is cofinal. By Corollary 3.5, $\Lambda \times_\eta S$ is aperiodic and so by [23, Theorem 3.1], $C^*(\Lambda \times_\eta S)$ is simple.

Now suppose that $C^*(\Lambda \times_\eta S)$ is simple. Then by [23, Theorem 3.1], $\Lambda \times_\eta S$ is cofinal. By Proposition 4.11 this implies that (Λ, S, η) is cofinal. □

The condition of (Λ, S, η) cofinality is difficult to check in practice. For 1-graphs it was shown in [18, Proposition 5.13] that $\Lambda \times_d \mathbb{Z}^k$ is cofinal if Λ is primitive[1]. We seek an equivalent condition for k-graphs which guarantees (Λ, S, η) cofinality.

5 Primitivity and left-reversible semigroups

A semigroup S is said to be *left-reversible* if for all $s, t \in S$ we have $sS \cap tS \neq \emptyset$. It is more common to work with right-reversible semigroups, which are then called Ore semigroups (see [13]). In analogy with the results of Dubriel it can be shown that a left-reversible semigroup has an enveloping group Γ such that $\Gamma = SS^{-1}$.

In equation (1) we see that functor $\eta : \Lambda \to S$ multiplies on the right in the semigroup coordinate in the definition of the source map in a skew product graph $\Lambda \times_\eta S$. This forces us to consider left-reversible semigroups here. In order to avoid confusion we have decided not to call them Ore.

Examples 5.1. (i) Any abelian semigroup is automatically right- and left-reversible. Moreover, any group is a both a right- and left-reversible semigroup.

(ii) Let $\mathbb{N}$ denote the semigroup of natural numbers under addition and $\mathbb{N}^\times$ denote the semigroup of nonzero natural numbers under multiplication. Let $S = \mathbb{N} \times \mathbb{N}^\times$ be gifted with the associative binary operation $\star$ given

[1] Actually strongly connected with period 1 which is equivalent to primitive

by $(m_1, n_1) \star (m_2, n_2) = (m_1 n_2 + m_2, n_1 n_2)$, then one checks that S is a nonabelian left-reversible semigroup. It is not right-reversible; for example, $S(m,n) \cap S(p,q) = \emptyset$ when $n = q = 0$ and $m \neq p$.

(iii) The free semigroup $\mathbb{F}_n^+$ on $n \geq 2$ generators is not an left-reversible semigroup since for all $s, t \in \mathbb{F}_n^+$ with $s \neq t$ we have $s\mathbb{F}_n^+ \cap t\mathbb{F}_n^+ = \emptyset$ as there is no cancellation, and so not only the left-reversibility but also the right-reversibility conditions cannot be satisfied.

A *preorder* is a reflexive, transitive relation $\leq$ on a set X. A preordered set $(X, \leq)$ is *directed* if the following condition holds: for every $x, y \in X$, there exists $z \in X$ such that $x \leq z$ and $y \leq z$. A subset Y of X is *cofinal* if for each $x \in X$ there exists $y \in Y$ such that $x \leq y$. We say that sets $X \leq Y$ if $x \leq y$ for all $x \in X$ and for all $y \in Y$. We say that $t \in S$ is *strictly positive* if $\{t^n : n \geq 0\}$ is a cofinal set in S.

The following result appears as [15, Lemma 2.2] for right-reversible semigroups.

Lemma 5.2. *Let S be a left-reversible semigroup with enveloping group Γ, and define $\geq_l$ on Γ by $h \geq_l g$ if and only if $g^{-1}h \in S$. Then $\geq_l$ is a left-invariant preorder that directs Γ, and for any $t \in S$, tS is cofinal in S.*

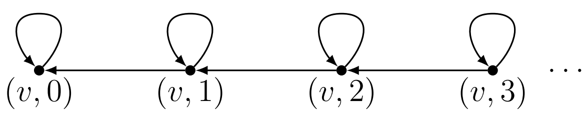
Our first attempt at a condition on η which guarantees cofinality of (Λ, S, η) is one which ensures that η takes arbitrarily large values on paths which terminate a given vertex.

Definition 5.3. Let Λ be a k-graph with no sources and $\eta : \Lambda \to S$ be a functor where S is a left-reversible semigroup. We will say that η is *upper dense* if for all $w \in \Lambda^0$ and $a, b \in S$ there exists $N \in \mathbb{N}^k$ such that $b\eta(w\Lambda^N) \geq_l a$.

Lemma 5.4. *Let (Λ, d) be a row-finite k-graph with no sources then d is upper dense for Λ.*

Proof. Since Λ has no sources it is immediate that $w\Lambda^N \neq \emptyset$ for all $w \in \Lambda^0$ and $N \in \mathbb{N}^k$. For any $b, a \in \mathbb{N}^k$ we have $b + d(w\Lambda^N) = b + N \geq a$ provided $N \geq a$. $\square$

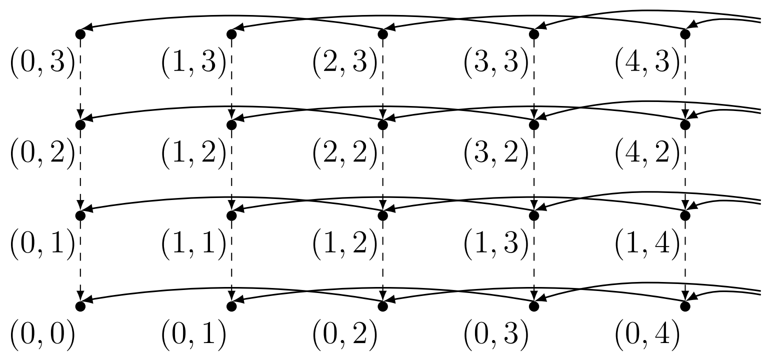
Examples 5.5. (i) Let B_2 be the 1-graph which is the path category of the directed graph with a single vertex v and two edges e, f. Define a functor $\eta : B_2 \to \mathbb{N}$ by $\eta(e) = 1$ and $\eta(f) = 0$. We may form the skew product $B_2 \times_\eta \mathbb{N}$ with 1-skeleton:



Fix $a, b \in \mathbb{N}$, then since $n \in \eta(vB_2^n)$ for all $n \in \mathbb{N}$ it follows that if we choose $N = a$, then $b + \eta(vB_2^N) \geq a$ and so η is upper dense. However $(B_2, \mathbb{N}, \eta)$ is not cofinal: Choose $a = 1$, $b = 0$, then for all $N \geq 0$ there is $f^N \in vB_2^N$ is such that

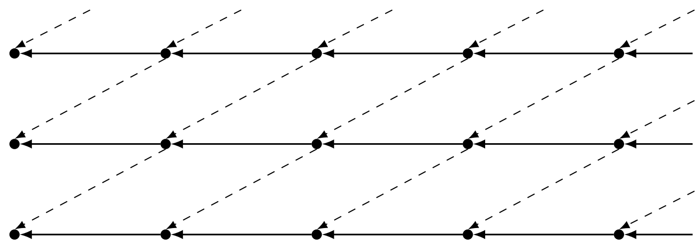
$$b + \eta(f^N) = 0 \neq 1 + \eta(\beta) \text{ for all } \beta \in B_2 v.$$

(ii) Define a functor η from T_2 to $\mathbb{N}^2$ such that $\eta(f_1) = (2, 0)$, and $\eta(f_2) = (0, 1)$. We may form the skew product $T_2 \times_\eta \mathbb{N}^2$ with the following 1-skeleton:



We claim that the functor η is not upper dense: Fix $b = (b_1, b_2)$ and $a = (a_1, a_2)$ in $\mathbb{N}^2$. Let N_1 be such that $b_1 + 2N_1 \geq a_1$ and N_2 be such that $b_2 + N_2 \geq a_2$ then $b\eta(vT_2^N) \geq_l u$ where $N = (N_1, N_2)$. Moreover $(T_2, \mathbb{N}^2, \eta)$ is not cofinal: Let $b = (0, 0)$ and $a = (1, 0)$ then since $\eta(f_1^{N_1} f_2^{N_2}) = (2N_1, N_2)$ it follows that there cannot be $N = (N_1, N_2) \in \mathbb{N}^2$ such that for $\alpha \in vT_2^N$ there is $\beta \in vT_2 v$ with $b\eta(\alpha) = a\eta(\beta)$.

(iii) Taking T_2 again, we define a functor $\eta : T_2 \to \mathbb{N}^2$ by $\eta(f_1) = (1, 0)$ and $\eta(f_2) = (1, 1)$. The skew product graph has 1-skeleton:



We claim that η is upper dense: Fix $b = (b_2, b_2)$ and $a = (a_1, a_2)$ in $\mathbb{N}^2$ then there is N_1 such that $b_1 + N_1 \geq a_1$ and N_2 such that $b_2 + N_1 + N_2 \geq a_2$. Then with $N = (N_1, N_2)$ for all $\alpha \in vT_2^N$ we have $b\eta(\alpha) \geq_l a$. In this case $(T_2, \mathbb{N}^2, \eta)$ is cofinal: Fix $b = (b_1, b_2)$ and $a = (a_1, a_2)$ in $\mathbb{N}^2$. Then there is N_1 such that $b_1 + N_1 = a_1 + m_1$ for some $m_1 \in \mathbb{N}$ and N_2 such that $b_2 + N_1 + N_2 = a_2 + m_2$ for some $m_2 \in \mathbb{N}$. Hence for all $\alpha \in vT_2^N$ where $N = (N_1, N_2)$ there is $\beta = (f_1^{m_1}, f_2^{m_2}) \in vT_2v$ such that $b\eta(\alpha) = a\eta(\beta)$.

The last two examples show that η being upper dense is not sufficient to guarantee cofinality of (Λ, S, η). The following definition allows for the interaction of the values of η at different vertices of Λ and the following result gives us the required extra condition.

Definition 5.6. Let Λ be a k-graph and $\eta : \Lambda \to S$ be a functor where S is a left-reversible semigroup. We say that η is *S-primitive* for Λ if there is a strictly positive $t \in S$ such that for all $v, w \in \Lambda^0$ we have $v\eta^{-1}(s)w \neq \emptyset$ for all $s \in S$ such that $s \geq_l t$.

Remarks 5.7. (i) The condition that t is strictly positive in the above definition guarantees that $\eta(v\Lambda w)$ is cofinal in S for all $v, w \in \Lambda^0$.

(ii) If $\eta : \Lambda \to S$ is S-primitive for Λ where S is a left-reversible semigroup, then if we extend η to $\Gamma = SS^{-1}$ then η is Γ-primitive for Λ.

Examples 5.8. (i) Let Λ be a k-graph. Then the degree functor $d : \Lambda \to \mathbb{N}^k$ is $\mathbb{N}^k$–primitive for Λ if and only if Λ is primitive as defined in Section 2.3. For this reason we will say that Λ is primitive if d is $\mathbb{N}^k$ primitive for Λ.

(ii) As in Examples 5.5 (i) let $\eta : B_2 \to \mathbb{N}$ be defined by $\eta(e) = 1$, $\eta(f) = 0$. Then the functor η is $\mathbb{N}$-primitive since $\eta^{-1}(n)$ is nonempty for all $n \in \mathbb{N}$. Hence $\mathbb{N}$-primitivity does not, by itself, guarantee cofinality.

(iii) As in Examples 5.5 (ii) let η be the functor from T_2 to $\mathbb{N}^2$ such that $\eta(f_1) = (2, 0)$, and $\eta(f_2) = (0, 1)$. Then the functor η is not $\mathbb{N}^2$-primitive for T_2: Take $t = (2m, n) \geq 0$ then if $s = (2m+1, n)$ we have $v\eta^{-1}(s)v = \emptyset$ and $s \geq_l t$. Similarly if $t = (2m + 1) \geq 0$ then if $s = (2m + 2, n)$ we have $v\eta^{-1}(s)v = \emptyset$ and $s \geq_l t$.

(iv) As in Examples 5.5 (iii) let $\eta : T_2 \to \mathbb{N}^2$ be defined by $\eta(f_1) = (1, 0)$ and $\eta(f_2) = (0, 1)$. Then η is not $\mathbb{N}^2$-primitive for T_2 as $v\eta^{-1}(m, n)v = \emptyset$ whenever $n > m$.

The last two examples above illustrate that upper density and primitivity are unrelated conditions on a k-graph. Together they provide a necessary condition for cofinality.

Proposition 5.9. *Let Λ be a k-graph with no sources and $\eta : \Lambda \to S$ be a functor where S is a left-reversible semigroup. If (Λ, S, η) is cofinal then η is upper dense. If η is S-primitive for Λ and upper dense then (Λ, S, η) is cofinal.*

Proof. Suppose that (Λ, S, η) is cofinal. Fix $w \in \Lambda^0$ and $a, b \in S$ and let v be any vertex of Λ. By cofinality of (Λ, S, η) there exists $N \in \mathbb{N}^k$ such that for all $\alpha \in w\Lambda^N$ there is $\beta \in v\Lambda s(\alpha)$ such that $a\eta(\beta) = b\eta(\alpha)$. Then any element of $b\eta(w\Lambda^N)$ is of the form

$$b\eta(\alpha) = a\eta(\beta) \geq_l a.$$

Suppose η is S-primitive and upper dense for Λ. Since η is S–primitive for Λ there exists $t \in S$ such that for all $v, w \in \Lambda^0$ we have $v\eta^{-1}(s)w \neq \emptyset$ for all $s \geq_l t$. Fix $v, w \in \Lambda^0$ and $a, b \in S$. Since η is upper dense there exists $N \in \mathbb{N}^k$ such that $b\eta(\alpha) \geq_l at$ for all $\alpha \in w\Lambda^N$. Since S is left-reversible, it is directed, and so by definition $b\eta(\alpha) = atu$ for some $u \in S$. But $tu \geq_l t$ and so since η is S–primitive there exists $\beta \in v\Lambda s(\alpha)$ such that $\eta(\beta) = tu$ and hence $b\eta(\alpha) = a\eta(\beta)$. □

Corollary 5.10. *Let Λ be a row-finite k-graph such that d is $\mathbb{N}^k$ primitive for Λ then $(\Lambda, \mathbb{N}^k, d)$ is cofinal.*

Proof. Since d is $\mathbb{N}^k$ primitive for Λ it follows that Λ has no sources. The result then follows from Lemma 5.4 and Proposition 5.9. □

Example 5.11. Let $\eta : T_2 \to S$ be any functor, then $\eta(S)$ is a subsemigroup of S since T_2 has a single vertex; moreover η is $\eta(S)$–primitive for T_2. Hence if η is upper dense for T_2, it follows that $(T_2, \eta(S), \eta)$ is cofinal. In particular, in Example5.5 (ii) one checks that $(T_2, \eta(\mathbb{N}^2, \eta)$ is cofinal.

Theorem 5.12. *Let Λ be an aperiodic k-graph, $\eta : \Lambda \to S$ be a functor into a left-reversible semigroup, and η be S–primitive for Λ. Then $C^*(\Lambda \times_\eta S)$ is simple if and only if η is upper dense.*

Proof. If η is upper dense then the result follows from Proposition 5.9. On the other hand if $C^*(\Lambda \times_\eta S)$ is simple then the result follows from Theorem 4.12 and Corollary 3.5. □

6 Skew products by a group Let Λ be a row-finite k-graph. A functor $\eta : \Lambda \to G$ defines a coaction δ_η on $C^*(\Lambda)$ determined by $\delta_\eta(s_\lambda) = s_\lambda \otimes \eta(\lambda)$. It is shown in [14, Theorem 7.1] that $C^*(\Lambda \times_\eta G)$ is isomorphic to $C^*(\Lambda) \times_{\delta_\eta} G$. Hence we may relate the simplicity of the C^*-algebra of a skew product graph to the simplicity of the associated crossed product. This can be done by using the results of [20].

Following [14, Lemma 7.9], for $g \in G$ the spectral subspace $C^*(\Lambda)_g$ of the coaction δ_η is given by

$$C^*(\Lambda)_g = \overline{\operatorname{span}}\{s_\lambda s_\mu^* : \eta(\lambda)\eta(\mu)^{-1} = g\}.$$

We define $\mathrm{sp}(\delta_\eta) = \{g \in G : C^*(\Lambda)_g \neq \emptyset\}$, to be the collection of non-empty spectral subspaces. The *fixed point algebra*, $C^*(\Lambda)^{\delta_\eta}$ of the coaction is defined to be $C^*(\Lambda)_{1_G}$. For more details on the coactions of discrete groups on k-graph algebras, see [14, §7] and [20].

We give necessary and sufficient conditions for the skew product graph C^*-algebra to be simple in terms of the fixed-point algebra as our main result in Theorem 6.3. We are particularly interested in the case when η is the degree functor.

Definition 6.1. Let Λ be a row-finite k-graph, G be a discrete group and $\eta : \Lambda \to G$ a functor, then we define

$$\Gamma(\eta) = \{g \in G : g = \eta(\lambda)\eta(\mu)^{-1} \text{ for some } \lambda, \mu \in \Lambda \text{ with } s(\lambda) = s(\mu)\}.$$

Lemma 6.2. *Let Λ be a row-finite graph with no sources and $\eta : \Lambda \to G$ a functor, where G is a discrete group.*

(a) If (Λ, G, η) is cofinal then $\Gamma(\eta) = G$.

(b) $\mathrm{sp}(\delta_\eta) = G$ if and only if $\Gamma(\eta) = G$.

Proof. Fix $g \in G$ and write $g = b^{-1}a$ for some $a, b \in G$. Now fix $v, w \in \Lambda^0$; since (Λ, G, η) is cofinal there exist $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$ such that $a\eta(\mu) = b\eta(\lambda)$. Hence $b^{-1}a = \eta(\lambda)\eta(\mu)^{-1}$ and so $g \in \Gamma(\eta)$. Since g was arbitrary the result follows.

The second statement follows by definition. □

Theorem 6.3. *Let Λ be an aperiodic row-finite k-graph with no sources, $\eta : \Lambda \to G$ a functor and δ_η the associated coaction of G on $C^*(\Lambda)$. Then $C^*(\Lambda \times_\eta G)$ is simple if and only if $C^*(\Lambda)^{\delta_\eta}$ is simple and $\Gamma(\eta) = G$.*

Proof. By [14, Theorem 7.1] it follows that $C^*(\Lambda \times_\eta G)$ is isomorphic to $C^*(\Lambda) \times_{\delta_\eta} G$. Then by [20, Theorem 2.10] $C^*(\Lambda) \times_{\delta_\eta} G$ is simple if and only if $C^*(\Lambda)^{\delta_\eta}$ is simple and $\mathrm{sp}(\delta_\eta) = G$. The result now follows from Lemma 6.2. □

Example 6.4. Let Λ be a row-finite k-graph with no sources and $d : \Lambda \to \mathbb{N}^k$ be the degree functor. We claim that $\Gamma(d) = \mathbb{Z}^k$. Fix $p \in \mathbb{Z}^k$, and write $p = m - n$ where $m, n \in \mathbb{N}^k$. Since Λ has no sources, for every $v \in \Lambda^0$ there is $\lambda \in \Lambda^m v$ and $\mu \in \Lambda^n v$. Then

$$d(\lambda) - d(\mu) = m - n = p \in \Gamma(d),$$

and so $\Gamma(d) = \mathbb{Z}^k$. Since $\Gamma(d) = \mathbb{Z}^k$, and $(\Lambda, \mathbb{Z}^k, d)$ is aperiodic, we have that $C^*(\Lambda)^{\delta_d}$ is simple if and only $(\Lambda, \mathbb{N}^k, d)$ is cofinal.

We seek conditions on Λ that will guarantee $(\Lambda, \mathbb{N}^k, d)$ is cofinal.

7 The gauge coaction The coaction δ_d of $\mathbb{Z}^k$ on $C^*(\Lambda)$ defined in Section 6 is such that the fixed point algebra $C^*(\Lambda)^{\delta_d}$ is precisely the fixed point algebra $C^*(\Lambda)^\gamma$ for the canonical gauge action of $\mathbb{T}^k$ on $C^*(\Lambda)$ by the Fourier transform (cf. [2, Corollary 4.9].

By [9, Lemma 3.3] the fixed point algebra $C^*(\Lambda)^\gamma$ is AF, and is usually referred to as the AF core. In Theorem 7.2 we use the results of the last two sections to give necessary and sufficient conditions for the AF core $C^*(\Lambda)^\gamma$ to be simple when Λ^0 is finite. When there are infinitely many vertices we show, in Theorem 7.8 that in many cases the AF core is not simple.

The AF core of a k-graph algebra plays a significant role in the development of crossed products by endomorphisms. Results of Takehana and Katayama [8] show that when Λ is a finite 1-graph such that the core $C^*(\Lambda)$ is simple, then every nontrivial automorphism of $C^*(\Lambda)$ is outer (see [17, Proposition 3.4]).

We saw in Example 4.9 that a k-graph being strongly connected is not enough to guarantee that $\Lambda \times_d \mathbb{Z}^k$ is cofinal, and hence by [23, Theorem 3.1] $C^*(\Lambda \times_d \mathbb{Z}^k)$ is not simple and then by Theorem 6.3 the AF core is not simple. Another condition is required to guarantee that $\Lambda \times_d \mathbb{Z}^k$ is cofinal, which is suggested by [18] and was introduced in Section 5:

Theorem 7.1. *Let Λ be a row-finite k-graph with no sinks and sources and Λ^0 finite. If $(\Lambda, d, \mathbb{Z}^k)$ is cofinal then Λ is primitive.*

Proof. We claim that for $v \in \Lambda^0$ there is $N(v) \in \mathbb{N}^k$ such that for all $n \geq N(v)$ we have $v\Lambda^n v \neq \emptyset$. Fix $(v, 0) \in (\Lambda \times_d \mathbb{Z}^k)^0$ then for each $w \in \Lambda^0$, when we apply the cofinality condition to $(w, 0) \in (\Lambda \times_d \mathbb{Z}^k)^0$ we obtain $N_w \in \mathbb{N}^k$ such that $(v, 0)(\Lambda \times_d \mathbb{Z}^k)s(\alpha, 0) \neq \emptyset$ for all $(\alpha, 0) \in (w, 0)(\Lambda \times_d \mathbb{Z}^k)^{N_w}$. Define $N = \max_{w\in\Lambda^0}\{N_w\}$, which is finite since Λ^0 is finite.

By Proposition 4.5 it follows that Λ is strongly connected, hence there exists $\alpha \in v\Lambda v$ with $d(\alpha) = r > 0$. Hence, there exists $t \geq 1$ such that $tr \geq N$. Let $N(v) = tr$.

Let $m = n - tr \geq 0$. Since Λ has no sources, $v\Lambda^m \neq \emptyset$; hence there exists $\gamma \in v\Lambda^m$. Let $w = s(\gamma)$. For $(v, 0), (w, 0) \in (\Lambda \times_d \mathbb{Z}^k)^0$, we have $(\alpha^t, 0) \in (v, 0)(\Lambda \times_d \mathbb{Z}^k)^{tr}$ where $tr \geq N \geq N_w$. By cofinality and Lemma 4.2 (b), there exists $(\beta, 0) \in (w, 0)(\Lambda \times_d \mathbb{Z}^k)(v, tr)$ as $s(\alpha^t, 0) = (v, tr)$. As $\beta \in w\Lambda^{tr}v$ it follows that $\gamma\beta \in v\Lambda^n v$, which proves the claim. □

The following result generalises results from [18]:

Theorem 7.2. *Let (Λ, d) be a row-finite k-graph with no sinks or sources, and Λ^0 finite. Then $C^*(\Lambda)^{\delta_d}$ is simple if and only if Λ is primitive.*

Proof. Suppose that Λ is primitive. Then $(\Lambda, \mathbb{Z}^k, d)$ is strongly connected and cofinal by Remarks 2.8. Hence $C^*(\Lambda \times_d \mathbb{Z}^k)$ is simple and so $C^*(\Lambda)^{\delta_d}$ is simple by Theorem 6.3.

Suppose that $C^*(\Lambda)^{\delta_d}$ is simple. Recall from Example 6.4 that since Λ has no sources then $\Gamma(d) = \mathbb{Z}^k$. Then by Theorem 6.3, $C^*(\Lambda \times_d \mathbb{Z}^k)$ is simple, and hence $(\Lambda, d, \mathbb{Z}^k)$ is cofinal by [23, Theorem 3.1] and Proposition 4.11. By Theorem 7.1 this implies that Λ is primitive. $\square$

Example 7.3. Since it has a single vertex it is easy to see that the 2-graph $\mathbb{F}^2_\theta$ defined in Examples 2.1 (d) is primitive. Hence by Theorem 7.2 we see that $C^*(\mathbb{F}^2_\theta)^\gamma$ is simple for all θ. Indeed in [4, §2.1] it is shown that $C^*(\mathbb{F}^2_\theta)^\gamma \cong \mathrm{UHF}(mn)^\infty$.

We now turn our attention to the case when Λ^0 is infinite. We adapt the technique used in [18] to show that, in many cases the AF core is not simple.

Definition 7.4. Let Λ be a row-finite k-graph with no sources. For $v \in \Lambda^0$, $n \in \mathbb{N}^k$ let

$$V(n,v) = \{s(\lambda) : \lambda \in v\Lambda^m, m \leq n\}$$
$$FV(n,v) = V(n,v) \setminus \cup_{i=1}^k V(n-e_i, v).$$

Remarks 7.5. For $v \in \Lambda^0$, $m \leq n \in \mathbb{N}^k$ we have, by definition, that $V(m,v) \subseteq V(n,v)$.

For $v \in \Lambda^0$, $n \in \mathbb{N}^k$ the set $FV(n,v)$ denotes those vertices which connect to v with a path of degree n and there is no path from that vertex to v with degree less than n.

Lemma 7.6. *Let Λ be a row-finite k-graph with no sources. For $v \in \Lambda^0$, $n \in \mathbb{N}^k$ then $V(n,v)$ is finite and if $V(n) = V(n-e_i)$ for some $1 \leq i \leq k$ then $V(n+re_i) = V(n-e_i)$ for all $r \geq 0$.*

Proof. Fix, $v \in \Lambda^0$, $n \in \mathbb{N}^k$, since Λ row-finite it follows that $\cup_{m \leq n} v\Lambda^m$ is finite and hence so is $V(n,v)$.

Suppose, without loss of generality that $V(n) = V(n-e_1)$. Let $w \in V(n+e_1)$, then there is $\lambda \in v\Lambda^{n+e_1}w$. Now $\lambda(0,n) \in v\Lambda^n$ and so $s(\lambda(0,n)) \in V(n) = V(n-e_1)$. Hence there is $\mu \in v\Lambda^m s(\lambda(0,n))$ for some $m \leq n-e_1$ and so $\mu\lambda(n, n+e_1) \in v\Lambda^{m+e_1}$. Since $s(\mu\lambda(n, e+e_1)) = s(\lambda) = w$ and $m+e_1 \leq n$ it follows that $w \in V(n)$. As w was an arbitrary element of $V(n+e_1)$ it follows that $V(n+e_1) \subseteq V(n) = V(n-e_1)$. By Remarks 7.5 we have $V(n-e_1) \subseteq V(n+e_1)$ and so $V(n+e_1) = V(n-e_1)$. It then follows that $V(n+re_1) = V(n-e_1)$ for $r \geq 0$ by an elementary induction argument. $\square$

We adopt the following notation, used in [11]: Let Λ be a k-graph for $1 \leq i \leq k$ we set $\Lambda^{\mathbb{N}e_i} = \cup_{r \geq 0} \Lambda^{re_i}$.

Proposition 7.7. *Let Λ be a row-finite k-graph with no sources such that for all $w \in \Lambda^0$ and for $1 \leq i \leq k$, the set $s^{-1}\left(w\Lambda^{\mathbb{N}e_i}\right)$ is infinite. Then for all $n \in \mathbb{N}^k$, $v \in \Lambda^0$ we have $FV(n,v) \neq \emptyset$.*

Proof. Suppose, for contradiction, that $FV(n,v) = \emptyset$ for some $n \in \mathbb{N}^k$ and $v \in \Lambda^0$. Then, without loss of generality we may assume that $V(n) = V(e-e_1)$.

Let $\lambda \in v\Lambda^n$, then $s(\lambda) \in V(n) = V(n-e_1)$. Fix $r \geq 0$, then since Λ has no sources there is $\mu \in s(\lambda)\Lambda^{re_1}$. Then $\lambda\mu \in v\Lambda^{n+re_1}$ and so $s(\lambda\mu) = s(\mu) \in V(n+re_1, v)$. By Lemma 7.6 it follows that $V(n+re_1) = V(n-e_1)$ and so for any $\mu \in s(\lambda)\Lambda^{\mathbb{N}e_1}$ we have $s(\mu) \in V(n-e_1)$. By Remarks 7.5 $V(n-e_1)$ is finite and so we have contradicted the hypothesis that $s^{-1}\left(w\Lambda^{\mathbb{N}e_1}\right)$ is infinite. □

Note that k-graphs satisfying the hypothesis of Proposition 7.7 must have infinitely many vertices. The following result generalises results from [18]:

Theorem 7.8. *Let Λ be a row-finite k-graph with no sources such that for all $w \in \Lambda^0$ and for $1 \leq i \leq k$, the set $s^{-1}\left(w\Lambda^{\mathbb{N}e_i}\right)$ is infinite. Then $\Lambda \times_d \mathbb{Z}^k$ is not cofinal.*

Proof. Suppose, for contradiction, that $\Lambda \times_d \mathbb{Z}^k$ is cofinal.

Fix $v \in \Lambda^0$ then since Λ is row-finite and has no sources $W = s^{-1}(v\Lambda^{e_1})$ is finite and nonempty. Without loss of generality let $W = \{w_1, \ldots, w_n\}$.

Since $\Lambda \times_d \mathbb{Z}^k$ is cofinal, for $1 \leq i \leq n$ if we consider $(w_i, 0)$ and $(v, 0) \in \Lambda^0 \times \mathbb{Z}^k$ then there is $N_i \in \mathbb{N}^k$ such that for all $(\alpha, 0) \in (w_i, 0)\left(\Lambda \times_d \mathbb{Z}^k\right)^{N_i}$ we have $(v,0)\left(\Lambda \times_d \mathbb{Z}^k\right)(s(\alpha), N_i) \neq \emptyset$. Let $N = \max\{N_1, \ldots, N_n\}$. By Proposition 7.7 $FV(N+e_1, v) \neq \emptyset$, hence there is $\lambda \in v\Lambda^{N+e_1}$ such that there is no path of degree less than $N+e_1$ from $s(\lambda)$ to v. Without loss of generality $s(\lambda(0,e_1)) = w_1$, and so $(\lambda(e_1, N+e_1), 0) \in (w_1, 0)\left(\Lambda \times_d \mathbb{Z}^k\right)^N$. Since $N \geq N_1$ and Λ has no sources, by Lemma 4.2(ii) there is $(\alpha, 0) \in (v,0)\left(\Lambda \times_d \mathbb{Z}^k\right)(s(\lambda), N)$ which implies that $\alpha \in v\Lambda^N s(\lambda)$, contradicting the defining property of $\lambda \in v\Lambda^{N+e_1}$. □

Examples 7.9.

1. Let Λ be a strongly connected k-graph with Λ^0 infinite, then Λ has no sources and for all $w \in \Lambda^0$ we have $s^{-1}\left(w\Lambda^{\mathbb{N}e_i}\right)$ is infinite for $1 \leq i \leq k$. Hence by Theorem 7.8 it follows that $\Lambda \times_d \mathbb{Z}^k$ is not cofinal.

2. Let Λ be a k-graph with Λ^0 infinite, no sources and no paths with the same source and range. Then for all $w \in \Lambda^0$ we have $s^{-1}\left(w\Lambda^{\mathbb{N}e_i}\right)$ is infinite for $1 \leq i \leq k$. Hence by Theorem 7.8 it follows that $\Lambda \times_d \mathbb{Z}^k$ is not cofinal.

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Communicated by *Moto O'uchi*

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MORE ON DECOMPOSITIONS OF A FUZZY SET IN FUZZY TOPOLOGICAL SPACES

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Received April 7, 2014

ABSTRACT. Using new properties (Theorem B in Section 2) of the concept of fuzzy points in the sense of Pu Pao-Ming and Liu Ying-Ming (Definition 2.1), we first prove that every fuzzy set $\lambda \neq 0$ is decomposed by two fuzzy sets $\lambda_{\mathcal{O}(X,\sigma^f)}$ and $\lambda^*_{\mathcal{PC}(X,\sigma^f)}$ (Theorem A;cf. Theorem 2.5(ii)), where (X, σ^f) is a specified Chang's fuzzy space (Definition 1.2, Remarks 1.3,1.4). Namely, $\lambda = \lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda^*_{\mathcal{PC}(X,\sigma^f)}$ and $\lambda_{\mathcal{O}(X,\sigma^f)} \wedge \lambda^*_{\mathcal{PC}(X,\sigma^f)} = 0$ hold, and the fuzzy set $\lambda_{\mathcal{O}(X,\sigma^f)}$ is fuzzy open in (X, σ^f) (Theorem 2.5(iii)). Finally, these results are applied to the case where $X = \mathbb{Z}^n (n > 0)$ and $\sigma^f = (\kappa^n)^f$ (Theorem 3.3 and Theorem 3.5), where the topological space (X, σ) is the digital n-space $(\mathbb{Z}^n, \kappa^n)$ (cf. Section 3).

1 Introduction and preliminaries In 1965, Zadeh [26] introduced the fundamental concept of fuzzy sets, which formed the backbone of fuzzy mathematics. After his works, Chang [4] used them to introduce the concept of a fuzzy topology. Throughout the present paper, the symbol I will denote the unit interval [0, 1] and Y a nonempty set. A *fuzzy set* on Y ([26]) is a function with domain Y and values in I, i.e., an element of I^Y.

We recall some concepts and properties as follows. Let (Y, τ_Y) be a Chang's fuzzy topological space [4].

Definition 1.1 (C.L. Chang [4, Definition 2.2]) A *Chang's fuzzy topological space* is a pair (Y, τ_Y), where Y is a non-emptyset and τ_Y is a *Chang's fuzzy topology* on it, where $\tau_Y \subset I^Y$, i.e., a family τ_Y of fuzzy sets satisfying the following three axioms:
(1) $0, 1 \in \tau_Y$;
(2) if $\lambda \in \tau_Y$ and $\mu \in \tau_Y$, then $\lambda \wedge \mu \in \tau_Y$;
(3) let J be an index set. If $\lambda_j \in \tau_Y$ for each $j \in J$, then $\bigvee\{\lambda_j | j \in J\} \in \tau_Y$.
The elements of τ_Y are called *fuzzy open sets* of (X, τ_Y). A fuzzy set μ is called a *fuzzy closed set* of (Y, τ_Y) if the complement $\mu^c \in \tau_Y$.

For a Chang's fuzzy topological space (Y, τ_Y), a fuzzy set μ on Y is said to be *fuzzy preopen* [23] if $\mu \leq$Int(Cl(μ)) holds in (Y, τ_Y). The fuzzy complement of a fuzzy preopen set is said to be *fuzzy preclosed*. Namely, a fuzzy set λ is fuzzy preclosed in (Y, τ_Y) if and only if Cl(Int(λ)) $\leq \lambda$ holds in (Y, τ_Y). A fuzzy set λ is said to be *fuzzy semi-open* [1] in (Y, τ_Y) if there exists a fuzzy open set ν on Y such that $\nu \leq \lambda \leq$Cl(ν) holds in (Y, τ_Y). It is well known that a fuzzy set λ is *fuzzy semi-open* if and only if $\lambda \leq$Cl(Int(λ)). For a subset A of X, χ_A denotes the characteristic function of A, i.e., $\chi_A(y) := 1$ if $y \in A$ and $\chi_A(y) := 0$ if $y \notin A$. The concept of the *ordinary preopen sets* (resp. *ordinary semi-open sets*) was introduced by [21] (resp. [17], [10]).

Definition 1.2 (e.g., [19, Example II, p.244], [8, p.161]) Let (X, σ^f) be a *fuzzy topological space induced by a topological space* (X, σ), where X is a nonempty set and $\sigma^f := \{\chi_U | U \in \sigma\}$; (X, σ^f) is an example of a Chang's fuzzy topological space [4] (cf. Definition 1.1 above).

*2010 Math. Subject classification–:54A40.
Key words and phrases —Topology; Chang's fuzzy topological spaces; Fuzzy points; Fuzzy preclosed sets; Fuzzy open sets; Decompositions of fuzzy sets; The digital n-spaces.

There is a bijection, say f, between σ and σ^f which is defined by $f(U) = \chi_U$ for every $U \in \sigma$, because an ordinary subset U is open in (X, σ) (i.e., $U \in \sigma$) if and only if the characteristic function χ_U is fuzzy open in (X, σ^f)(i.e., $\chi_U \in \sigma^f$). However, the below Remark 1.3 and Remark 1.4 show that the fuzzy topology σ^f has some interesting and distinct properties comparing the given ordinary topology σ.

Let $SO(X, \sigma)$ (resp. $FSO(X, \sigma^f)$) denote the family of all ordinary semi-open sets (resp. fuzzy semi-open sets) in (X, σ) (resp. (X, σ^f)); then $\sigma \subset SO(X, \sigma)$ and $\sigma^f \subset FSO(X, \sigma^f)$ hold. An extension of $f : \sigma \to \sigma^f$ to $SO(X, \sigma)$, say $f_s : SO(X, \sigma) \to FSO(X, \sigma^f)$, is well defined by $f_s(A) := \chi_A$ for every $A \in SO(X, \sigma)$. The following Remark 1.3 shows that $f_s : SO(X, \sigma) \to FSO(X, \sigma^f)$ is not onto.

Remark 1.3 For the following topological space (X, σ), the correspondence $f_s : SO(X, \sigma) \to FSO(X, \sigma^f)$ is not onto, where $f_s(V) := \chi_V$ for every set $V \in SO(X, \sigma)$. Let $X := \{a, b, c\}$ and $\sigma := \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then, we have $SO(X, \sigma) = \sigma \cup \{\{a, c\}, \{b, c\}\}$; and $\{\chi_U | U \in SO(X, \sigma)\} = f_s(SO(X, \sigma))$. Let λ_c be a fuzzy set on X defined by $\lambda_c(a) = 0, \lambda_c(b) = 1, \lambda_c(c) = t$, where t is a real number with $0 < t < 1$. Then, we see that λ_c is fuzzy semi-open in (X, σ^f), i.e., $\lambda_c \in FSO(X, \sigma^f)$). Indeed, there exists a fuzzy open set $\chi_{\{b\}}$ such that $\chi_{\{b\}} \leq \lambda_c \leq \mathrm{Cl}(\chi_{\{b\}})$ hold in (X, σ^f), because $\mathrm{Cl}(\chi_{\{b\}}) = \chi_{Cl(\{b\})} = \chi_{\{b,c\}}$ hold. Since $\lambda_c(c) = t$ and $0 < t < 1$, we see that $\lambda_c \neq \chi_A$ for any set $A \subset X$; and so $\lambda_c \notin f_s(SO(X, \sigma))$. Namely, $f_s : SO(X, \sigma) \to FSO(X, \sigma^f)$ is not onto.

We find an alternative example in [19, (3.5),(III-11)] which is shown on the digital plane $(X, \sigma) = (\mathbb{Z}^2, \kappa^2)$. And, by Remark 3.6 in Section 3, it's general version for the digital n-space $(\mathbb{Z}^n, \kappa^n)$ is given.

The below Remark 1.4 shows that a property for a topological space (X, σ) does not be hereditary to (X, σ^f). In order to explain it, we recall some definitions and properties $(*1)$-$(*3)$ as follows.

In 1970, the concept of $T_{1/2}$-spaces (cf. $(*3)$ below) was studied initiately by Levine [18] by introducing the concept of *generalized closed sets* for a topological space. The work on generalized closed sets and their related works are developing by many authors until now. A subset A of (X, σ) is said to be *generalized closed* [18, Definition 2.1] in (X, σ), if $\mathrm{Cl}(A) \subset O$ holds in (X, σ) whenever $A \subset O$ and O is open in (X, σ). The complement of a generalized closed set of (X, σ) is called *generalized open* [18, Definition 4.1] in (X, σ). It is well known that:

$(*1)$ ([18, Theorem 2.4]) the union of two "generalized closed sets" is "generalized closed"; and

$(*2)$ ([18, Example 2.5]) the intersection of two "generalized closed sets" is generally not "generalized closed". Moreover, it is well known that every closed set is generalized closed.

$(*3)$ A topological space (X, σ) is said to be $T_{1/2}$ [18, Definition 5.1] if every "generalized closed set" of (X, σ) is closed in (X, σ). By Dunham [6], it was proved that a topological space (X, σ) is $T_{1/2}$ if and only if, for each point $x \in X, \{x\}$ is open or closed ([6, Theorem 2.5]).

In 1970, E. Khalimsky [11] studied initiately the concept of the digital line $(\mathbb{Z}, \kappa)$ and it is also called the *Khalimsky line* (e.g., Section 3 below; cf. [13] and references there, [12], [14, p.905, line -5],[15, p.175]; e.g., [7]). The digital line $(\mathbb{Z}, \kappa)$ is an interesting and importante example of the $T_{1/2}$-topological space ([5, Example 4.6]) and, moreover, $(\mathbb{Z}, \kappa)$ is a $T_{3/4}$-space ([5, Definition 4, Theorem 4.1]).

Remark 1.4 The digital line $(\mathbb{Z}, \kappa)$ is a $T_{1/2}$-topological space ([5, Example 4.6]); however the induced fuzzy topological space $(\mathbb{Z}, \kappa^f)$ from $(\mathbb{Z}, \kappa)$ is not fuzzy $T_{1/2}$ ([8, Example 4.8]). Here, a fuzzy topological space (Y, τ_Y) is said to be fuzzy $T_{1/2}$ [2] if every fuzzy generalied closed set is fuzzy closed. The above property shows that the property on such separation axiom for a topological space (X, σ) does not be hereditary to the corresponding fuzzy separation axiom for (X, σ^f) even if there is a bijectin $f : \sigma \to \sigma^f$.

One of the purposes in the present paper is to prove the following Theorem A using some properties on (X, σ^f) in Section 2 below. Roughly speaking, when a fuzzy set on X, say λ, is given, then we can consider a decomposition such that $\lambda = \lambda_1 \vee \lambda_2 (\lambda_1 \wedge \lambda_2 = 0)$ and λ_1 and λ_2 are two fuzzy sets characterized from an induced and specified fuzzy topological space (X, σ^f), where σ is a topology of X. And so, let $\lambda \in I^X$ be a given fuzzy set on X; when we choice many topologies on X, say σ, σ',...., we can get many decompositions of the fuzzy set λ , which are characterized from the induced and specified fuzzy topologies on X, say σ^f, $(\sigma')^f$,...., respectively. Some analogous decomposition properties of a fuzzy set are investigated by [19, Theorem 3.1, Corollary 3.7] and [9, Corollary 2.9, Theorem 3.6].

Theorem A (Theorem 2.5 (ii) in Section 2 below) *Let $\lambda \in I^X$ be a fuzzy set such that $\lambda \neq 0$. Let (X, σ^f) be a fuzzy topological space induced by (X, σ). Then, we have the following decomposition of λ:*
$\lambda = \lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda^*_{\mathcal{PC}(X,\sigma^f)}$ *and* $\lambda_{\mathcal{O}(X,\sigma^f)} \wedge \lambda^*_{\mathcal{PC}(X,\sigma^f)} = 0$.

In Section 3 we have the explicite form of $\lambda_{\mathcal{O}(\mathbb{Z}^n,(\kappa^n)^f)}$ and $\lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)}$ for the case where $(X, \sigma) = (\mathbb{Z}^n, \kappa^n)$ and $(X, \sigma^f) = (\mathbb{Z}^n, (\kappa^n)^f)$ (cf. Corollary 3.1, Theorem 3.5 below).

2 Proof of Theorem A

In the present section we prove Theorem A. We need the concept of fuzzy points in the sense of Pu Pao-Ming and Liu Ying-Ming (Definition 2.1 below), the following notations (Notation I below) and a result (Theorem B below).

In the present paper, for the concept of fuzzy points, we adopt Pu's definition of a fuzzy point in the sense of ([22]).

Definition 2.1 (Pu Pao-Ming and Liu Ying-Ming [22, Definition 2.1], e.g., [19, Definition 1.3]) A fuzzy set on a set Y is said to be *fuzzy point* if it takes the value 0 for all point $y \in Y$ except one point, say $x \in Y$. If it value at x is a $(0 < a \leq 1)$, we denote this fuzzy point by x_a. We note that $\mathrm{supp}(x_a) = \{a\}$ holds and $0 < a \leq 1$. Namely, for a point $x \in Y$ and a real number $a \in I$ such that $0 < a \leq 1$,
- a *fuzzy point* $x_a \in I^Y$ is a fuzzy set defined as, for any point $y \in Y, x_a(y) := a$ if $y = x; x_a(y) := 0$ if $y \neq x$.

Notation I. For a Chang's fuzzy topological space (Y, τ_Y),
(i) $FPO(Y, \tau_Y) := \{\lambda \in I^Y | \ \lambda$ is fuzzy preopen in $(Y, \tau_Y)\}$,
$FPC(Y, \tau_Y) := \{\lambda \in I^Y | \ \lambda$ is fuzzy preclosed in $(Y, \tau_Y)\}$.
Namely, by definition, $FPO(Y, \tau_Y) = \{\lambda \in I^Y \ |\lambda \leq Int(Cl(\lambda))$ holds in $(Y, \tau_Y)\}$ and $FPC(Y, \tau_Y) = \{\lambda \in I^Y | \ Cl(Int(\lambda)) \leq \lambda$ holds in $(Y, \tau_Y)\}$.
(ii) For a fuzzy set $\lambda \in I^Y$ such that $\lambda \neq 0$ (i.e., $\mathrm{supp}(\lambda) := \{x \in Y | \lambda(x) \neq 0\} \neq \emptyset$),
$O(\lambda) := \{y \in \mathrm{supp}(\lambda) | \ y_{\lambda(y)} \in \tau_Y\}$,
$PC(\lambda) := \{y \in \mathrm{supp}(\lambda) | \ y_{\lambda(y)} \in FPC(Y, \tau_Y)\}$,
$PC^*(\lambda) := \{y \in \mathrm{supp}(\lambda) | \ y_{\lambda(y)} \in FPC(Y, \tau_Y)$ and $y_{\lambda(y)} \notin \tau_Y\}$.

In the category of fuzzy topological spaces (X, σ^f) induced by topological spaces (X, σ), we know the following theorem [19], say Theorem B in the present paper:
Theorem B (i) ([19, (3.6)(i)]) *Every fuzzy point x_a is fuzzy open or fuzzy preclosed in (X, σ^f). Namely, for every fuzzy point x_a, we have $x_a \in \sigma^f \cup FPC(X, \sigma^f)$.*

(ii) ([19, (3.6)(ii)]) *A fuzzy point x_a is fuzzy open in (X, σ^f) if and only if $a = 1$ and $\{x\}$ is open in (X, σ).*

(iii) ([19, (3.2)]) *For a fuzzy set λ on X, $Cl(\lambda) = \chi_{Cl(supp(\lambda))}$ holds in (X, σ^f); and $Int(\lambda) = \chi_{Int(\lambda^{-1}(\{1\}))}$ holds in (X, σ^f).* □

Theorem B (i) above is a fuzzy version of the following property:([3, Lemma 2.4]) *for a topological space (X, σ), every singleton $\{x\}$ is open or preclosed in (X, σ).*

For a fuzzy set λ on Y and a fuzzy topological space (Y,τ_Y), we define three fuzzy sets $\lambda_{\mathcal{O}(Y,\tau_Y)}, \lambda_{\mathcal{PC}(Y,\tau_Y)}$ and $\lambda^*_{\mathcal{PC}(Y,\tau_Y)}$ as follows.

Definition 2.2 Let $\lambda \in I^Y$ be a fuzzy set such that $\lambda \neq 0$ and (Y,τ_Y) a Chang's fuzzy topological space. The following fuzzy sets are well defined: for λ above,

(i) $\lambda_{\mathcal{O}(Y,\tau_Y)} := \bigvee\{x_{\lambda(x)} \in I^Y |\ x_{\lambda(x)} \in \tau_Y\}$ if $O(\lambda) \neq \emptyset$; $\lambda_{\mathcal{O}(Y,\tau_Y)} := 0$ if $O(\lambda) = \emptyset$;

(ii) $\lambda_{\mathcal{PC}(Y,\tau_Y)} := \bigvee\{x_{\lambda(x)} \in I^Y |\ x_{\lambda(x)} \in FPC(Y,\tau_Y)\}$ if $PC(\lambda) \neq \emptyset$; $\lambda_{\mathcal{PC}(Y,\tau_Y)} := 0$ if $PC(\lambda) = \emptyset$,

(iii) $\lambda^*_{\mathcal{PC}(Y,\tau_Y)} := \bigvee\{x_{\lambda(x)} \in I^Y |\ x_{\lambda(x)} \in FPC(Y,\tau_Y)$ and $x_{\lambda(x)} \notin \tau_Y\}$ if $PC^*(\lambda) \neq \emptyset$; $\lambda^*_{\mathcal{PC}(Y,\tau_Y)} := 0$ if $PC^*(\lambda) = \emptyset$.

Lemma 2.3 *Let λ be a fuzzy set in Y such that $\lambda \neq 0$, i.e., supp(λ) $\neq \emptyset$ and (Y,τ_Y) a Chang's fuzzy topological space. Then, we have the following properties:*

(i) $\lambda_{\mathcal{O}(Y,\tau_Y)} = 0$ *holds if and only if* $x_{\lambda(x)} \notin \tau_Y$ *for each point* $x \in$*supp*(λ) (i.e., $O(\lambda) = \emptyset$).

(ii) $\lambda^*_{\mathcal{PC}(Y,\tau_Y)} = 0$ *if and only if* $x_{\lambda(x)} \notin FPC(Y,\tau_Y)$ *or* $x_{\lambda(x)} \in \tau_Y$ *for each point* $x \in$*supp*(λ) (i.e., $PC^*(\lambda) = \emptyset$).

(iii) (a) *If* $O(\lambda) \neq \emptyset$, *then* $\lambda_{\mathcal{O}(Y,\tau_Y)} = \bigvee\{x_{\lambda(x)} |\ x \in O(\lambda)\}$.

(b) *If* $PC(\lambda) \neq \emptyset$, *then* $\lambda_{\mathcal{PC}(Y,\tau_Y)} = \bigvee\{x_{\lambda(x)} |\ x \in PC(\lambda)\}$.

(c) *If* $PC^*(\lambda) \neq \emptyset$, *then* $\lambda^*_{\mathcal{PC}(Y,\tau_Y)} = \bigvee\{x_{\lambda(x)} |\ x \in PC^*(\lambda)\}$.

(iv) $\lambda^*_{\mathcal{PC}(Y,\tau_Y)} \leq \lambda_{\mathcal{PC}(Y,\tau_Y)} \leq \lambda$ *hold.*

Proof. **(i)** (**Necessity**) Suppose that there exists a point $z \in$supp(λ) such that $z_{\lambda(z)} \in \tau_Y$. Then, $O(\lambda) \neq \emptyset$. For the point z we set $\mathcal{A}_z := \{x_{\lambda(x)}(z) \in I | x_{\lambda(x)} \in \tau_Y\}$; and so $\mathcal{A}_z \neq \emptyset$. Then, by Definition 2.2 (i), $(\lambda_{\mathcal{O}(Y,\tau_Y)})(z) = \sup \mathcal{A}_z$ and so $\lambda_{\mathcal{O}(Y,\tau_Y)}(z) = \sup\{\lambda(z), 0\} = \lambda(z)$. Indeed, $x_{\lambda(x)}(z) = \lambda(z)$ or 0. Thus we have $\lambda_{\mathcal{O}(Y,\tau_Y)} \neq 0$; this contradicts the assumption. (**Sufficiency**) The proof is obtained by Definition 2.2 (i). **(ii)** The sufficiency is obtained by Definition 2.2 (iii). (**Necessity**) Suppose that there exists a point $z \in$supp(λ) such that $z_{\lambda(z)} \in FPC(Y,\tau_Y)$ and $z_{\lambda(z)} \notin \tau_Y$. Then, $PC^*(\lambda) \neq \emptyset$. For the point z, we set $\mathcal{B}^*_z :=\{x_{\lambda(x)}(z) \in I | x_{\lambda(x)} \in FPC(Y,\tau_Y)$ and $x_{\lambda(x)} \notin \tau_Y\}$ and note $\mathcal{B}^*_z \neq \emptyset$. Then $\lambda^*_{\mathcal{PC}(Y,\tau_Y)}(z) = \sup \mathcal{B}^*_z$. Since $x_{\lambda(x)}(z) = \lambda(z)$ or 0 and $z \in$supp(λ) we have $\lambda^*_{\mathcal{PC}(Y,\tau_Y)}(z) = \sup\{\lambda(z), 0\} = \lambda(z)$ and hence $\lambda^*_{\mathcal{PC}(Y,\tau_Y)}(z) > 0$ for the point z. Namely, we have $\lambda^*_{\mathcal{PC}(Y,\tau_Y)} \neq 0$; this contradicts the assumption. **(iii)** By using definitions (cf. Notation I, Definition 2.2), it is shown that $\{x_{\lambda(x)} |\ x_{\lambda(x)} \in \tau_Y\} = \{x_{\lambda(x)} |\ x \in O(\lambda)\}$, $\{x_{\lambda(x)} |\ x_{\lambda(x)} \in FPC(Y,\tau_Y)\}$=$\{x_{\lambda(x)} |\ x \in PC(\lambda)\}$ and $\{x_{\lambda(x)} |\ x_{\lambda(x)} \in FPC(Y,\tau_Y), x_{\lambda(x)} \notin \tau_Y\}$=$\{x_{\lambda(x)} |\ x \in PC^*(\lambda)\}$ hold. Thus we have the required equalities. **(iv)** It is obvious that supp$(\lambda) \supset PC(\lambda) \supset PC^*(\lambda)$ (cf. Notation above). Therefore, we have that $\lambda \geq \lambda_{\mathcal{PC}(Y,\tau_Y)} \geq \lambda^*_{\mathcal{PC}(Y,\tau_Y)}$, because $\lambda = \bigvee\{x_{\lambda(x)} |\ x \in$supp$(\lambda)\}$ holds ([22, Definition 2.2]; e.g., [16, Lemma 2.1], [19, Lemma 2.5(i)]) and the equalities (b) and (c) hold in (iii) above. □

Theorem 2.4 *Let $\lambda \in I^X$ be a fuzzy set such that $\lambda \neq 0$. For a fuzzy topological space (X,σ^f) induced by a topological space (X,σ), $\lambda_{\mathcal{O}(X,\sigma^f)} = 0$ if and only if $\lambda = \lambda^*_{\mathcal{PC}(X,\sigma^f)} = \lambda_{\mathcal{PC}(X,\sigma^f)}$ hold.*

Proof. (**Necessity**) It follows from assumption and Lemma 2.3(i) that $x_{\lambda(x)} \notin \sigma^f$ for every point $x \in$supp(λ). Thus, by Theorem B(i) above, it is shown that, for every point $x \in$supp$(\lambda), x_{\lambda(x)}$ is fuzzy preclosed in (X,σ^f). Thus, we have $\lambda = \bigvee\{x_{\lambda(x)} |\ x \in$supp$(\lambda)\} = \bigvee\{x_{\lambda(x)} |\ x_{\lambda(x)} \in FPC(X,\sigma^f)$ and $x_{\lambda(x)} \notin \sigma^f\} = \lambda^*_{\mathcal{PC}(X,\sigma^f)}$. Therefore, using Lemma 2.3(iv), we conclude that $\lambda = \lambda^*_{\mathcal{PC}(X,\sigma^f)} = \lambda_{\mathcal{PC}(X,\sigma^f)}$ hold. (**Sufficiency**) Assume that $\lambda = \lambda_{\mathcal{PC}(X,\sigma^f)}$=$\lambda^*_{\mathcal{PC}(X,\sigma^f)}$ hold. We recall that $\lambda^*_{\mathcal{PC}(X,\sigma^f)}$= $\bigvee\{x_{\lambda(x)} | x_{\lambda(x)} \in FPC(X,\sigma^f)$ and $x_{\lambda(x)} \notin \sigma^f\}$= $\bigvee\{x_{\lambda(x)} |\ x \in PC^*(\lambda)\}$ (cf. Lemma 2.3 (iii)). Suppose $PC^*(\lambda) = \emptyset$.

Then, $\lambda^*_{\mathcal{PC}(X,\sigma^f)} = 0$ (cf. Definition 2.2(iii)); and so we have $\lambda = 0$; this contradicts the assumption on λ (i.e., $\mathrm{supp}(\lambda) \neq \emptyset$). Thus, we consider the case where $PC^*(\lambda) \neq \emptyset$ for λ. We claime that $\mathrm{supp}(\lambda) \subset PC^*(\lambda)$. Indeed, let w be any point such that $w \notin PC^*(\lambda)$. Then, for each point $x \in PC^*(\lambda)$, we have $x_{\lambda(x)}(w) = 0$, because of $w \neq x$. Here, we put $\mathcal{B}^*_w := \{x_{\lambda(x)}(w) \in I | x \in PC^*(\lambda)\}$; then $\mathcal{B}^*_w = \{0\}$; and so we have $(\lambda^*_{\mathcal{PC}(X,\sigma^f)})(w)$ =sup $\mathcal{B}^*_w = 0$. By using the assumption of the present Sufficiency, it is shown that $\lambda(w) = 0$ and so $w \not\in \mathrm{supp}(\lambda)$. Therefore, we show $\mathrm{supp}(\lambda) \subset PC^*(\lambda)$. Therefore, we have $x_{\lambda(x)} \notin \sigma^f$ for every point $x \in \mathrm{supp}(\lambda)$, because of $x \in PC^*(\lambda)$. By Lemma 2.3(i), it is obtained that $\lambda_{\mathcal{O}(X,\sigma^f)} = 0$. □

We shall prove Theorem A as follows; Theorem A is included in Theorem 2.5 below (i.e., Theorem 2.5 (ii)). First we recall the following notation:

Notation II: for a topological space (X, σ) and a subset E of X,
let $X_\sigma := \{x \in X | \ \{x\} \in \sigma\}$; and $E_\sigma := E \cap X_\sigma$. It is obvious that E_σ is open in (X, σ) for any subset $E \subset X$.

Notation III : for a fuzzy set λ on X and a topological space (X, σ),
(i) $\lambda^{-1}(\{1\}) := \{y \in X | \ \lambda(y) = 1\}$; then $\lambda^{-1}(\{1\})$ is a subset of X, because $\lambda \in I^X$;
(ii) $(\lambda^{-1}(\{1\}))_\sigma := \lambda^{-1}(\{1\}) \cap X_\sigma$ (i.e., $(\lambda^{-1}(\{1\}))_\sigma = \{y | \ y \in \lambda^{-1}(\{1\}), \{y\}$ is open in $(X, \sigma)\}$).

Theorem 2.5 *Let $\lambda \in I^X$ be a fuzzy set such that $\lambda \neq 0$. Let (X, σ) be a topological space and (X, σ^f) a fuzzy topological space induced by (X, σ). Then, we have the following properties of λ:*

(i) $\lambda = \lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda_{\mathcal{PC}(X,\sigma^f)}$.

(ii) $\lambda = \lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda^*_{\mathcal{PC}(X,\sigma^f)}$ *and* $\lambda_{\mathcal{O}(X,\sigma^f)} \wedge \lambda^*_{\mathcal{PC}(X,\sigma^f)} = 0$.

(iii) $\lambda_{\mathcal{O}(X,\sigma^f)} = \chi_E$, *where* $E := X_\sigma \cap \lambda^{-1}(\{1\}) = (\lambda^{-1}(\{1\}))_\sigma$; $\lambda_{\mathcal{O}(X,\sigma^f)}$ *is fuzzy open in* (X, σ^f).

Proof. We first recall the following $(*^1)$ with Notation I and we claim the following properties $(*^2)$ and $(*^3)$:
$(*^1)$ $\mathrm{supp}(\lambda) \supset PC(\lambda) \supset PC^*(\lambda)$ and $\mathrm{supp}(\lambda) \supset O(\lambda)$ hold in (X, σ) (cf. Notation I);
$(*^2)$ $\mathrm{supp}(\lambda) = O(\lambda) \cup PC(\lambda)$ holds in (X, σ);
$(*^3)$ $\mathrm{supp}(\lambda) = O(\lambda) \cup PC^*(\lambda)$ and $O(\lambda) \cap PC^*(\lambda) = \emptyset$ hold in (X, σ).

Proof of $(*^2)$. By Theorem B, it is shown that, for a point $x \in \mathrm{supp}(\lambda)$, the fuzzy point $x_{\lambda(x)}$ is fuzzy open or fuzzy preclosed in (X, σ^f), i.e., $x_{\lambda(x)} \in \sigma^f$ or $x_{\lambda(x)} \in FPC(\lambda)$. Thus, for a point $x \in$supp $(\lambda), x \in O(\lambda)$ or $x \in PC(\lambda)$; and so we have $\mathrm{supp}(\lambda) \subset O(\lambda) \cup PC(\lambda)$. Since $O(\lambda) \subset \mathrm{supp}(\lambda)$ and $PC(\lambda) \subset \mathrm{supp}(\lambda)$, we have the required equality $(*^2)$. $(\diamond)$

Proof of $(*^3)$. By definition, it is easily shown that $PC^*(\lambda) \subset PC(\lambda)$. And, we have $PC^*(\lambda) = \{y \in \mathrm{supp}(\lambda) | \ y_{\lambda(y)} \in FPC(X, \sigma^f)\} \cap \{y \in \mathrm{supp}(\lambda) | \ y_{\lambda(y)} \notin \sigma^f\} = PC(\lambda) \cap [\mathrm{supp}(\lambda) \setminus O(\lambda)]$; and so $PC^*(\lambda) = PC(\lambda) \cap [\mathrm{supp}\ (\lambda) \setminus O(\lambda)]$. Thus, we have $PC^*(\lambda) \cup O(\lambda) = [PC(\lambda) \cap (\mathrm{supp}(\lambda) \setminus O(\lambda)] \cup O(\lambda) = \mathrm{supp}(\lambda)$ (cf. $(*^2)$) and $PC^*(\lambda) \cap O(\lambda) \subset PC(\lambda) \cap [X \setminus O(\lambda)] \cap O(\lambda) = \emptyset$. $\diamond$

In the finnal stage, we prove (i), (ii) and (iii) as follows.

(i). For the proof of (i) we consider the following three cases. And it is well known that $\lambda = \bigvee\{x_{\lambda(x)} | \ x \in \mathrm{supp}(\lambda)\}$ holds (cf. [22, Definition 2.2], e.g., [16, lemma 2.2],[19, Lemma 2.5(i)]).

Case 1. $O(\lambda) \neq \emptyset, PC(\lambda) \neq \emptyset$: for this case, using $(*^2)$ above and Lemma 2.3 (iii), we have $\lambda = \bigvee\{x_{\lambda(x)} | \ x \in \mathrm{supp}(\lambda)\} = (\bigvee\{x_{\lambda(x)} | \ x \in O(\lambda)\}) \vee (\bigvee\{x_{\lambda(x)} | \ x \in PC(\lambda)\} = \lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda_{\mathcal{PC}(X,\sigma^f)}$.

Case 2. $O(\lambda) \neq \emptyset, PC(\lambda) = \emptyset$: for this case, we have $\lambda_{\mathcal{PC}(X,\sigma^f)} = 0$ (cf. Definition 2.2(ii)) and $\mathrm{supp}(\lambda) = O(\lambda)$ (cf. $(*^2)$ above). Thus, we have $\lambda = \bigvee\{x_{\lambda(x)} | \ x \in \mathrm{supp}(\lambda)\} = \bigvee\{x_{\lambda(x)} | \ x \in O(\lambda)\} = \lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda_{\mathcal{PC}(X,\sigma^f)}$, because $\lambda_{\mathcal{PC}(X,\sigma^f)} = 0$.

Case 3. $O(\lambda) = \emptyset$: for this case, by $(*^2)$ above and Lemma 2.3(i), it is shown that $\lambda_{\mathcal{O}(X,\sigma^f)} = 0$ and $\mathrm{supp}(\lambda) = PC(\lambda)$; and so $PC(\lambda) \neq \emptyset$, because of $\lambda \neq 0$. Thus, we have $\lambda = \bigvee\{x_{\lambda(x)} |\ x \in \mathrm{supp}(\lambda)\} = 0 \vee (\bigvee\{x_{\lambda(x)} |\ x \in PC(\lambda)\} = \lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda_{\mathcal{PC}(X,\sigma^f)}$.
Therefore, we show that the equality (i) holds for all cases.

(ii). Since $\mathrm{supp}(\lambda) = O(\lambda) \cup PC^*(\lambda)$ (cf. $(*^3)$), we are able to conclude that (ii-1) $\lambda = \lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda^*_{\mathcal{PC}(X,\sigma^f)}$; and (ii-2) $\lambda_{\mathcal{O}(X,\sigma^f)} \wedge \lambda^*_{\mathcal{PC}(X,\sigma^f)} = 0$.

Proof of (ii-1). We consider the following three cases for the proof.

Case 1. $O(\lambda) \neq \emptyset, PC^*(\lambda) \neq \emptyset$: for this case, using $(*^3)$ above and Lemma 2.3 (iii), we have $\lambda = \{x_{\lambda(x)} |\ x \in \mathrm{supp}(\lambda)\} = (\bigvee\{x_{\lambda(x)} |\ x \in O(\lambda)\}) \vee (\bigvee\{x_{\lambda(x)} |\ x \in PC^*(\lambda)\} = \lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda^*_{\mathcal{PC}(X,\sigma^f)}$.

Case 2. $O(\lambda) \neq \emptyset, PC^*(\lambda) = \emptyset$: for this case, we have $\lambda^*_{\mathcal{PC}(X,\sigma^f)} = 0$ (cf. Definition 2.2(iii)) and $\mathrm{supp}(\lambda) = O(\lambda)$ (cf. $(*^3)$ above). Thus, we have $\lambda = \bigvee\{x_{\lambda(x)} |\ x \in \mathrm{supp}(\lambda)\} = \bigvee\{x_{\lambda(x)} |\ x \in O(\lambda)\} = \lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda^*_{\mathcal{PC}(X,\sigma^f)}$, because $\lambda^*_{\mathcal{PC}(X,\sigma^f)} = 0$.

Case 3. $O(\lambda) = \emptyset$: for this case, we have $\lambda_{\mathcal{O}(X,\sigma^f)} = 0$ (cf. Definition 2.2(i)). By $(*^3)$, it is shown that $\mathrm{supp}(\lambda) = PC^*(\lambda)$; and so $PC^*(\lambda) \neq \emptyset$, because of $\lambda \neq 0$. Thus, we have $\lambda = \bigvee\{x_{\lambda(x)} |\ x \in \mathrm{supp}(\lambda)\} = 0 \vee (\bigvee\{x_{\lambda(x)} | x \in PC^*(\lambda)\} = \lambda_{\mathcal{O}(X,\sigma^f)} \vee \lambda^*_{\mathcal{PO}(X,\sigma^f)}$. $(\diamond)$

Proof of (ii-2). For a point $y \in X$, we claim that $(\lambda_{\mathcal{O}(X,\sigma^f)} \wedge \lambda^*_{\mathcal{PC}(X,\sigma^f)})(y) = 0$; i.e., $\mathrm{Min}\{\lambda_{\mathcal{O}(X,\sigma^f)}(y), \lambda^*_{\mathcal{PC}(X,\sigma^f)}(y)\} = 0$. For the point y, we consider the following two cases.

Case 1. $y \in O(\lambda)$: for this point y, we have $y \notin PC^*(\lambda)$ (cf. $(*^3)$ before the proof of (i) above). Then, we have that $y \neq x$ for each $x \in PC^*(\lambda)$, i.e., $x_{\lambda(x)}(y) = 0$ for each $x \in PC^*(\lambda)$. Thus, if $PC^*(\lambda) \neq \emptyset$, then $\lambda^*_{\mathcal{PC}(X,\sigma^f)}(y) = (\bigvee\{x_{\lambda(x)} |\ x \in PC^*(\lambda)\})(y)$ $= \sup\{x_{\lambda(x)}(y) |\ x \in PC^*(\lambda)\} = \sup\{0\} = 0$ (cf. Lemma 2.3(iii)(c)). And, if $PC^*(\lambda) = \emptyset$, then $\lambda^*_{\mathcal{PC}(X,\sigma^f)}(y) = 0$ (cf. Definition 2.2(iii)). Thus, for this Case 1, we show that $\mathrm{Min}\{\lambda_{\mathcal{O}(X,\sigma^f)}(y), \lambda^*_{\mathcal{PC}(X,\sigma^f)}(y)\} = 0$.

Case 2. $y \notin O(\lambda)$: for the point y, we have that $x \neq y$ for each point $x \in O(\lambda)$; and so $x_{\lambda(x)}(y) = 0$ for each point $x \in O(\lambda)$. Thus, if $O(\lambda) \neq \emptyset$, then $\lambda_{\mathcal{O}(X,\sigma^f)}(y) = (\bigvee\{x_{\lambda(x)} |\ x \in O(\lambda)\})(y) = \sup\{x_{\lambda(x)}(y) |\ x \in O(\lambda)\} = \sup\{0\} = 0$ (cf. Lemma 2.3(iii)(a)). And, if $O(\lambda) = \emptyset$, then $\lambda_{\mathcal{O}(X,\sigma^f)}(y) = 0$ (cf. Definition 2.2(i)). Thus, for this Case 2, we show that $\mathrm{Min}\{\lambda_{\mathcal{O}(X,\sigma^f)}(y), \lambda^*_{\mathcal{PC}(X,\sigma^f)}(y)\} = 0$.
Therefore we prove $\lambda_{\mathcal{O}(X,\sigma^f)} \wedge \lambda^*_{\mathcal{PC}(X,\sigma^f)} = 0$.

(iii). By Theorem B(ii) in the top of the present section, it is well known that a fuzzy point x_a is fuzzy open in (X, σ^f) if and only if $a = 1$ and $\{x\}$ is open in (X, σ). For a point $x \in \mathrm{supp}(\lambda), \lambda(x) > 0$ and so a fuzzy point $x_{\lambda(x)}$ is well defined. Thus, we have that $x_{\lambda(x)}$ is fuzzy open in (X, σ^f) (i.e., $x_{\lambda(x)} \in \sigma^f$) if and only if $\lambda(x) = 1$ and $\{x\}$ is open in (X, σ) (i.e., $x \in E := \lambda^{-1}(\{1\}) \cap X_\sigma$, cf. Notation II, Notation III). Therefore, if $E \neq \emptyset$, then we have that $\lambda_{\mathcal{O}(X,\sigma^f)} = \bigvee\{x_{\lambda(x)} |\ x_{\lambda(x)} \in \sigma^f\} = \bigvee\{x_{\lambda(x)} |\ x \in \lambda^{-1}(\{1\}) \cap X_\sigma\} = \bigvee\{x_1 |\ x \in E\}$ $= \bigvee\{\chi_{\{x\}} |\ x \in E\} = \chi_F = \chi_E$, where $F = \bigcup\{\{x\} |\ x \in E\}$, and hence $\lambda_{\mathcal{O}(X,\sigma^f)} = \chi_E$.
If $E = \emptyset$, then $O(\lambda) := \{y \in \mathrm{supp}(\lambda) |\ y_{\lambda(y)} \in \sigma^f\} = \{y \in \mathrm{supp}(\lambda) |\ \lambda(y) = 1$ and $\{y\} \in \sigma\} = \{y \in \mathrm{supp}(\lambda) | y \in E = \emptyset\} = \emptyset$ and so $\lambda_{\mathcal{O}(X,\sigma^f)} = 0 = \chi_\emptyset$. Therefore, we prove $\lambda_{\mathcal{O}(X,\sigma^f)} = \chi_E$. For the proof of $\lambda_{\mathcal{O}(X,\sigma^f)} \in \sigma^f$, it is evident from the openness of $E := \lambda^{-1}(\{1\}) \cap X_\sigma = (\lambda^{-1}(1))_\sigma$ and the definition of σ^f. □

3 Decompositions of fuzzy sets on $(\mathbb{Z}^n, (\kappa^n)^f)$ Let $(\mathbb{Z}^n, \kappa^n)$ be the digital n-space and $(\mathbb{Z}^n, (\kappa^n)^f)$ a Chang's fuzzy topological space induced from $(\mathbb{Z}^n, \kappa^n)$ (cf. Definition 1.2). In the present section, we have the following decomposition theorem (Corollary 3.1) of a fuzzy set λ on $\mathbb{Z}^n$ by two fuzzy sets χ_E and $\lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)}$ with fuzzy topological properties in $(\mathbb{Z}^n, (\kappa^n)^f)$ and the precise form of $\lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)}$ (Theorem 3.5).

We recall that:

• the *digital n-space* $(\mathbb{Z}^n, \kappa^n)$ (e.g., [15, Definition 4],[7]) is the topological product of n-copies of the *digital line* $(\mathbb{Z}, \kappa)$ (cf. this is called the *Khalimsky line* in the contents between

Remark 1.4 and (∗3) in Section 1), where n is an integer with $n \geq 2$. The *digital line* $(\mathbb{Z}, \kappa)$ is the set of the integers, $\mathbb{Z}$, equipped with the topology κ having $\{\{2m-1, 2m, 2m+1\} | \ m \in \mathbb{Z}\}$ as a subbace (e.g., [15, p.175]). Some joint papers by the one of the present authors include a short survey or frequently used properties on $(\mathbb{Z}^n, \kappa^n)$ where $n \geq 1$ (cf. [20, Section 3], [25], [7]). It is well known that a singleton $\{2m\}$ is closed and not open and $\{2m+1\}$ is open and not closed in $(\mathbb{Z}, \kappa)$, where $m \in \mathbb{Z}$; moreover $\mathrm{Cl}(\{2s+1\}) = \{2s, 2s+1, 2s+2\}$ holds and $\mathrm{Int}(\{2s\}) = \emptyset$ holds in $(\mathbb{Z}, \kappa)$, where $s \in \mathbb{Z}$. We use the following notation (cf. [7, Section 6], [24, Section 2], [25, Definition 2.1], [20, Definition 3.11]): for $n \geq 1$,

- $(\mathbb{Z}^n)_{\kappa^n} := \{(y_1, y_2, ..., y_n) \in \mathbb{Z}^n | \ y_i$ is odd for each integer i with $1 \leq i \leq n\}$; for any element x of $(\mathbb{Z}^n)_{\kappa^n}$, $\{x\}$ is an open singleton of $(\mathbb{Z}^n, \kappa^n)$ (cf. Notation II in Section 2 for $X := \mathbb{Z}^n$ and $\sigma := \kappa^n$);
- $(\mathbb{Z}^n)_{\mathcal{F}^n} := \{(y_1, y_2, ..., y_n) \in \mathbb{Z}^n | \ y_i$ is even for each integer i with $1 \leq i \leq n\}$; for any element x of $(\mathbb{Z}^n)_{\mathcal{F}^n}$, $\{x\}$ is a closed singleton of $(\mathbb{Z}^n, \kappa^n)$;
- $(\mathbb{Z}^n)_{mix(r)} := \{(y_1, y_2, ..., y_n) \in \mathbb{Z}^n | \ r = \#\{i \in \{1, 2, ..., n\} | \ y_i$ is even$\}\}$, where $1 \leq r \leq n$ and $\#A$ denotes the cardinality of a set A. Especially, for the case where $r = n$, we note $(\mathbb{Z}^n)_{mix(n)} = (\mathbb{Z}^n)_{\mathcal{F}^n}$.
- For a nonempty subset E of $(\mathbb{Z}^n, \kappa^n)$, the following subsets $E_{\kappa^n}, E_{\mathcal{F}^n}$ and $E_{mix(r)}$ are well defined as follows: $E_{\kappa^n} := E \cap (\mathbb{Z}^n)_{\kappa^n}$, $E_{\mathcal{F}^n} := E \cap (\mathbb{Z}^n)_{\mathcal{F}^n}$, $E_{mix(r)} := E \cap (\mathbb{Z}^n)_{mix(r)}$ $(1 \leq r \leq n)$. Namely, we have that $E_{\kappa^n} := \{x \in E | \ \{x\}$ is open in $(\mathbb{Z}^n, \kappa^n)\} \subset E$ and $E_{\mathcal{F}^n} := \{x \in E | \ \{x\}$ is closed in $(\mathbb{Z}^n, \kappa^n)\} \subset E$; and E_{κ^n} is an open subset of $(\mathbb{Z}^n, \kappa^n)$.

First we apply Theorem 2.5 to the digital n-space $(\mathbb{Z}^n, \kappa^n)$; then we have the following corollary of Theorem 2.5.

Corollary 3.1 *Let $\lambda \in I^{\mathbb{Z}^n}$ be a fuzzy set on $\mathbb{Z}^n$ such that $\lambda \neq 0$. Then, we have the following properties.*

(i) $\lambda_{\mathcal{O}(\mathbb{Z}^n,(\kappa^n)^f)} = \chi_E$, *where* $E := (\lambda^{-1}(\{1\}))_{\kappa^n}$.

(ii) *Any fuzzy set λ has a decomposition:* $\lambda = \chi_E \vee \lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)}$ *and* $\chi_E \wedge \lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} = 0$, *where* $E := (\lambda^{-1}(\{1\}))_{\kappa^n}$.

Proof. **(i)** (resp. **(ii)**) By Theorem 2.5(iii) (resp. Theorem 2.5(ii)) for $(X, \sigma) = (\mathbb{Z}^n, \kappa^n)$, (i) (resp. (ii)) is obtained. □

In the below, we shall show an exlicite expression of the fuzzy set $\lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)}$ above (cf. Theorem 3.5).

Theorem 3.2 *For a fuzzy topological space $(\mathbb{Z}^n, (\kappa^n)^f)$ induced by the digital n-space $(\mathbb{Z}^n, \kappa^n)$, where $n \geq 1$, and a fuzzy point x_a in $\mathbb{Z}^n$, where $x \in \mathbb{Z}^n$ and $0 < a \leq 1$, we have the following properties.*

(i) (i-1) *Let* $x \in (\mathbb{Z}^n)_{\kappa^n}$ (i.e., $x = (2m_1+1, 2m_2+1, ..., 2m_n+1)$, *where* $m_i \in \mathbb{Z}(1 \leq i \leq n)$). *Then,*

$\mathrm{Cl}(x_a) = \chi_{E^o_x}$, *where* $E^o_x := \prod_{i=1}^n \{2m_i, 2m_i+1, 2m_i+2\}$.

(i-2) *Let* $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ (i.e, $x = (y_1, y_2, ..., y_n)$ *for some even integers* $y_i (1 \leq i \leq n)$). *Then,*

$\mathrm{Cl}(x_a) = \chi_{\{x\}}$.

(i-3) *Suppose that* $n \geq 2$. *Let* $x := (y_1, y_2, ..., y_n) \in (\mathbb{Z}^n)_{mix(r)} (1 \leq r \leq n-1)$ *and* $E^m(y_i) = \{y_i\}$, *if y_i is even in* $\mathbb{Z}(1 \leq i \leq n)$; $E^m(y_i) = \{y_i - 1, y_i, y_i + 1\}$, *if y_i is odd in* $\mathbb{Z}(1 \leq i \leq n)$. *Then,*

$\mathrm{Cl}(x_a) = \chi_{E^m_x}$, *where* $E^m_x := \prod_{i=1}^n E^m(y_i)$.

(ii) (ii-1) *If* $x \in (\mathbb{Z}^n)_{\kappa^n}$ *and* $a = 1$, *then* $\mathrm{Int}(x_a) = \chi_{\{x\}} = x_a$ *holds.*

(ii-2) *If* $x \in (\mathbb{Z}^n)_{\kappa^n}$ *and* $a \neq 1$, *then* $\mathrm{Int}(x_a) = 0$ *holds.*

(ii-3) *If* $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$, *then* $\mathrm{Int}(x_a) = 0$ *holds.*

(ii-4) *If* $x \in (\mathbb{Z}^n)_{mix(r)}$ *with* $1 \leq r \leq n-1$, *then* $\mathrm{Int}(x_a) = 0$ *holds.*

Proof. **(i) (i-1)** It is well known that $\{x\}$ is an open singleton in $(\mathbb{Z}^n, \kappa^n)$ and $\mathrm{Cl}(\{x\}) = \prod_{i=1}^n \mathrm{Cl}(\{2m_i + 1\}) = \prod_{i=1}^n \{2m_i, 2m_i + 1, 2m_i + 2\} = E_x^o$ in $(\mathbb{Z}^n, \kappa^n)$. Thus, we have $\mathrm{Cl}(x_a) = \chi_{\mathrm{Cl}(\{x\})} = \chi_{E_x^o}$ in $(\mathbb{Z}^n, (\kappa^n)^f)$ for a point $x \in (\mathbb{Z}^n)_{\kappa^n}$, because $\mathrm{supp}(x_a) = \{x\}$ (cf. Theorem B (iii)).
(i-2) We have $\mathrm{Cl}(x_a) = \chi_{\mathrm{Cl}(\{x\})} = \chi_{\{x\}}$ in $(\mathbb{Z}^n, (\kappa^n)^f)$ (cf. Theorem B (iii)) for a point $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ (i.e., $\{x\}$ is a closed singleton of $(\mathbb{Z}^n, \kappa^n)$).
(i-3) Let $x = (y_1, y_2, ..., y_n) \in (\mathbb{Z}^n)_{mix(r)} (1 \leq r \leq n-1)$ (i.e., $r = \#\{i|\ y_i$ is even $\}$). Since $\mathrm{Cl}(\{x\}) = \prod_{i=1}^n \mathrm{Cl}(y_i) = \prod_{i=1}^n E^m(y_i) = E_x^m$ in $(\mathbb{Z}^n, \kappa^n)$, it is shown that $\mathrm{Cl}(x_a) = \chi_{E_x^m}$ in $(\mathbb{Z}^n, (\kappa^n)^f)$ (cf. Theorem B(iii)).

(ii) (ii-1) Since $a = 1$, we have $x_a = \chi_{\{x\}}$ and $(x_a)^{-1}(\{1\}) = \{x\}$. And, since $\{x\}$ is an open singleton of $(\mathbb{Z}^n, \kappa^n)$, it is shown that $\mathrm{Int}(x_a) = \chi_{\mathrm{Int}(x_1)^{-1}(\{1\})} = \chi_{\mathrm{Int}(\{x\})}$ (cf. Theorem B (iii)).
(ii-2) For this fuzzy point x_a, where $a \neq 1$, we have $(x_a)^{-1}(\{1\}) = \emptyset$ and so $\mathrm{Int}(x_a) = \chi_{\mathrm{Int}(\emptyset)} = 0$ in $(\mathbb{Z}^n, (\kappa^n)^f)$ (cf. Theorem B (iii)).
(ii-3) For this fuzzy point x_a, we have $(*)$ $\mathrm{Int}(x_a) = \chi_{\mathrm{Int}((x_a)^{-1}(\{1\}))} = \chi_{\mathrm{Int}(\{x\})}$ if $a = 1$; $\mathrm{Int}(x_a) = \chi_{\mathrm{Int}((x_a)^{-1}(\{1\}))} = \chi_\emptyset = 0$ if $a \neq 1$ (cf. Theorem B (iii)).
Thus, we show (ii-3) for the case where $a = 1$ only. Since $\mathrm{Int}(\{x\}) = \emptyset$ in $(\mathbb{Z}^n, \kappa^n)$ for this point x. we have $\mathrm{Int}(x_1) = \chi_{\mathrm{Int}(\{x\})} = \chi_\emptyset = 0$ (cf. Theorem B (iii)).
(ii-4) For this point x, say $x = (y_1, y_2, ..., y_n)$, there exists even integers, say $y_{i(e)} (1 \leq e \leq r)$, where $\{i(1), i(2), ..., i(r)\} \subset \{1, 2, ..., n\}$, because $1 \leq r \leq n-1$ and $r = \#\{i|1 \leq i \leq n, y_i$ is even$\}$; and $\mathrm{Int}(\{y_{i(e)}\}) = \emptyset$ for each e with $1 \leq e \leq r$ in $(\mathbb{Z}, \kappa)$. Then, we have $\mathrm{Int}(\{x\}) = \prod_{j=1}^n \mathrm{Int}(y_j) = \emptyset$ in $(\mathbb{Z}^n, \kappa^n)$. Thus, if $a = 1$, then $\mathrm{supp}(x_a) = (x_1)^{-1}(\{1\}) = \{x\}$ and so $\mathrm{Int}(x_a) = \chi_{\mathrm{Int}(\mathrm{supp}(x_1))} = \chi_{\mathrm{Int}(\{x\})} = \chi_\emptyset = 0$ in $(\mathbb{Z}^n, (\kappa^n)^f)$; if $a \neq 1$, then $\mathrm{supp}(x_a) = (x_a)^{-1}(\{1\}) = \emptyset$ and so $\mathrm{Int}(x_a) = \chi_{\mathrm{Int}(\mathrm{supp}(x_a))} = \chi_\emptyset = 0$ in $(\mathbb{Z}^n, (\kappa^n)^f)$ (cf. Theorem B (iii)). Therefore, for this fuzzy point x_a, we show $\mathrm{Int}(x_a) = 0$. □

Theorem 3.3 *A fuzzy point x_a is fuzzy open, otherwise x_a is fuzzy preclosed in $(\mathbb{Z}^n, (\kappa^n)^f)$.*

Proof. In general, by Theorem B(i) in Section 2, every fuzzy point is fuzzy open or fuzzy preclosed in (X, σ^f), where (X, σ) is a topological space. Then we prove only that non-existence of fuzzy point x_a which is fuzzy open and fuzzy preclosed in $(\mathbb{Z}^n, (\kappa^n)^f)$. Suppose that there exists a fuzzy point x_a such that $x_a \in FPC(\mathbb{Z}^n, (\kappa^n)^f)$ and $x_a \in (\kappa^n)^f$. Since x_a is fuzzy open in $(\mathbb{Z}^n, (\kappa^n)^f)$, we have $a = 1$ and $\{x\}$ is open in $(\mathbb{Z}^n, \kappa^n)$ (cf. Theorem B(ii) in Section 2). Thus, we can put $x := (2m_1 + 1, 2m_2 + 1, ..., 2m_n + 1) \in (\mathbb{Z}^n)_{\kappa^n}$. For this point x and fuzzy singleton x_a, where $a = 1$, by Theorem 3.2, $\mathrm{Cl}(\mathrm{Int}(x_a))$=$\mathrm{Cl}(x_a) = \chi_{E_x^O}$, where $E_x^O := \prod_{i=1}^n \{2m_i, 2m_i + 1, 2m_i + 2\}$ in $(\mathbb{Z}^n, (\kappa^n)^f)$. Put $x^+ := (2m_1 + 2, 2m_2 + 2, ..., 2m_n + 2)$. Then, we have $x \neq x^+$ and so $\mathrm{Cl}(\mathrm{Int}(x_1))(x^+) = \chi_{E_x^O}(x^+) = 1 \not\leq x_1(x^+) = 0$; this contradicts $x_a \in FPC(\mathbb{Z}^n, (\kappa^n)^f)$ (cf. Notation I in Section 2). □

Since $\mathbb{Z}^n = (\mathbb{Z}^n)_{\kappa^n} \cup (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)} | 1 \leq r \leq n-1\})$ (disjoint union), we see obviously that $\mathbb{Z}^n \setminus (\mathbb{Z}^n)_{\kappa^n} = (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)} | 1 \leq r \leq n-1\})$ holds in the digital n-space $(\mathbb{Z}^n, \kappa^n)$, where $n \geq 2$. And, we see $\mathbb{Z} \setminus \mathbb{Z}_\kappa = \mathbb{Z}_{\mathcal{F}}$ hold in the digital line $(\mathbb{Z}, \kappa)$.

Corollary 3.4 *Let x_a be a fuzzy point on $\mathbb{Z}^n$, where $0 < a \leq 1$. The following properties are equivalent:*

(1) *$x_a \in FPC(\mathbb{Z}^n, (\kappa^n)^f)$;*
(2) *$x \in E$ or $0 < a < 1$, where $E := \mathbb{Z}^n \setminus (\mathbb{Z}^n)_{\kappa^n}$;*
(2)' *$x \notin (\mathbb{Z}^n)_{\kappa^n}$ or $a \neq 1$;*
(3) *$x_a \notin (\kappa^n)^f$* (i.e., *x_a is not fuzzy open in $(\mathbb{Z}^n, (\kappa^n)^f)$*).

Proof. **(1)⇒(2)** Suppose that $x \in (\mathbb{Z}^n)_{\kappa^n}$ and $a = 1$. Then, by Theorem B(ii) in Section 2, x_a is fuzzy open; and hence by Theorem 3.3, x_a is not fuzzy preclosed in $(\mathbb{Z}^n, (\kappa^n)^f)$; this

contradicts the assumption (1). Therefore, we showed that $x \in E$ or $0 < a < 1$. **(2)⇔(2)'** It is obvious.
(2)⇒(3) By Theorem B(ii) in Section 2 for $(X, \sigma) = (\mathbb{Z}^n, \kappa^n), x_a$ is not fuzzy open in $(\mathbb{Z}^n, (\kappa^n)^f)$. **(3)⇒(1)** It is proved by Theorem 3.3. □

Finally we show some explicite forms of $\lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)}$.

Theorem 3.5 *Let λ be a fuzzy set on $\mathbb{Z}^n$ with $\lambda \neq 0$. Then, we have the following properties:*
(i) $\lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} = \lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)}$ *holds.*
(ii) *If* $\mathrm{supp}(\lambda) \cap (\mathbb{Z}^n \setminus (\mathbb{Z}^n)_{\kappa^n}) \neq \emptyset$, *then*
(ii-1) $\lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} \neq 0$;
(ii-2) $\lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} = \bigvee\{x_{\lambda(x)} \in I^{\mathbb{Z}^n} |\ x \in \mathrm{supp}(\lambda) \setminus (\lambda^{-1}(\{1\}))_{\kappa^n}\}$; *and*
(ii-3) $\lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} = \mathcal{A}(\lambda)_0 \vee (\bigvee\{\mathcal{A}(\lambda)_r |\ 1 \leq r \leq n\})$, *where*
$\mathcal{A}(\lambda)_0 := \bigvee\{x_{\lambda(x)} |\ x \in (\ \mathrm{supp}(\lambda) \setminus \lambda^{-1}(\{1\}))_{\kappa^n}\}$ *and* $\mathcal{A}(\lambda)_r := \bigvee\{x_{\lambda(x)} |\ x \in (\mathrm{supp}(\lambda))_{mix(r)}\}$ *for each integer r with $1 \leq r \leq n$.*

Proof. **(i)** We consider the following two cases for the proof.
Case 1. $PC^*(\lambda) \neq \emptyset$: by Definition 2.2(iii) and Corollary 3.4(1)⇔(3), it is obtained that $\lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} := \bigvee\{x_{\lambda(x)} |\ x_{\lambda(x)} \in FPC(\mathbb{Z}^n, (\kappa^n)^f)$ and $x_{\lambda(x)} \notin (\kappa^n)^f\} = \bigvee\{x_{\lambda(x)} |\ x_{\lambda(x)} \in FPC(\mathbb{Z}^n, (\kappa^n)^f)\}$. And so, we have $\lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} = \lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)}$, because $PC^*(\lambda) \subset PC(\lambda)$ and $PC(\lambda) \neq \emptyset$ hold.
Case 2. $PC^*(\lambda) = \emptyset$: for this case, $\lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} := 0$ (cf. Notation I in Section 2, Definition 2.2(iii)). We claim that $PC(\lambda) = \emptyset$ holds under the assumption of Case 2 (i.e.,$PC^*(\lambda) = \emptyset$). Suppose that $PC(\lambda) \neq \emptyset$ (cf. Notation I in Section 2, Definition 2.2(ii)). Then, there exists a point of $\mathbb{Z}^n$, say $z \in PC(\lambda)$, and so $z_{\lambda(z)} \in PC(\mathbb{Z}^n, (\kappa^n)^f)$ and, by Theorem 3.3, $z_{\lambda(z)} \notin (\kappa^n)^f$. The above result shows that $z_{\lambda(z)} \in PC^*(\mathbb{Z}^n, (\kappa^n)^f)$ holds, i.e., $z \in PC^*(\lambda)$ (cf. Notation I in Section 2); this contradicts the assumption of Case 2 (i.e., $PC^*(\lambda) = \emptyset$). Thus, we claimed that if $PC^*(\lambda) = \emptyset$ then $PC(\lambda) = \emptyset$. And, under the assumption of Case 2, we show that $\lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} := 0 = \lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)}$ hold.

Therefore, by Case 1 and Case 2, it is proved that $\lambda^*_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} = \lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)}$ holds.

(ii) **(ii-1)** It follows from the assumption of (ii) that there exists a point $z \in \mathrm{supp}(\lambda)$ (i.e., $\lambda(z) > 0$) and $z \notin (\mathbb{Z}^n)_{\kappa^n}$. By Corollary 3.4(2)'⇔ (1), it is obtained that $z_{\lambda(z)} \in FPC(\mathbb{Z}^n, (\kappa^n)^f)$ and so $z \in PC(\lambda) \neq \emptyset$ (cf. Notation I). We have that $\lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} = \bigvee\{x_{\lambda(x)} |\ x_{\lambda(x)} \in FPC(\mathbb{Z}^n, (\kappa^n)^f)\}$ (cf. Definition 2.2(ii)) and $\lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)}(z) \neq 0$ for the point z, i.e., $\lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} \neq 0$.

(ii-2) For a fuzzy point $x_{\lambda(x)}$, we have that $\lambda(x) > 0$, i.e., $x \in \mathrm{supp}(\lambda)$. Then, by using definitions and Corollary 3.4 (1)⇔(2)', it is shown that: $x_{\lambda(x)} \in FPC(\mathbb{Z}^n, (\kappa^n)^f)$ if and only if $x \in \mathrm{supp}(\lambda) \setminus (\lambda^{-1}(1))_{\kappa^n}$. By (ii-1) and Definition 2.2(ii), it is shown that: $PC(\lambda) \neq \emptyset$ and so $\lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} = \bigvee\{x_{\lambda(x)} |\ x \in \mathrm{supp}(\lambda) \setminus (\lambda^{-1}(\{1\}))_{\kappa^n}\}$.

(ii-3) We use the well known decomposition of $\mathbb{Z}$: $\mathbb{Z}^n = (\mathbb{Z}^n)_{\kappa^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)} |\ 1 \leq r \leq n\})$(disjoint union) and $(\mathbb{Z}^n)_{mix(n)} = (\mathbb{Z}^n)_{\mathcal{F}^n}$. It follows from assumption that $\mathrm{supp}(\lambda) \neq \emptyset$. We consider the decomposition of $\mathrm{supp}(\lambda)$ in $(\mathbb{Z}^n, (\kappa^n)^f)$:
$\mathrm{supp}(\lambda) = (\mathrm{supp}(\lambda))_{\kappa^n} \cup (\bigcup\{(\mathrm{supp}(\lambda))_{mix(r)} |\ 1 \leq r \leq n\})$; then, we have the following equality in $(\mathbb{Z}^n, (\kappa^n)^f)$ (cf. the right hand side equality in the end of the proof of (ii-2)):

(•) $\mathrm{supp}(\lambda) \setminus (\lambda^{-1}(\{1\}))_{\kappa^n} = (\mathrm{supp}(\lambda) \setminus \lambda^{-1}(1))_{\kappa^n} \cup (\bigcup\{(\mathrm{supp}(\lambda))_{mix(r)})|\ 1 \leq r \leq n\}$.

Then, using (ii-2), the equality (•) above and a property of fuzzy union of fuzzy points (e.g. [19, Lemma 2.5(ii)]), we have that:
$\lambda_{\mathcal{PC}(\mathbb{Z}^n,(\kappa^n)^f)} = \bigvee\{x_{\lambda(x)} |\ x \in \mathrm{supp}(\lambda) \setminus (\lambda^{-1}(\{1\}))_{\kappa^n}\}$
$= [\bigvee\{x_{\lambda(x)} |\ x \in (\ \mathrm{supp}(\lambda) \setminus \lambda^{-1}(\{1\}))_{\kappa^n}] \vee [\bigvee\{x_{\lambda(x)} |\ x \in (\ \mathrm{supp}(\lambda))_{mix(r)} |\ 1 \leq r \leq n\}]$
$= \mathcal{A}(\lambda)_0 \vee (\bigvee\{\mathcal{A}(\lambda)_r) |\ 1 \leq r \leq n\})$; and hence (ii-3) is proved. □

The following remark is pre-announced in Remark 1.3.

Remark 3.6 (cf. Remark 1.3, [19, (III-12) in Section 3]) The following example also shows that the correspondence $f_s : SO(\mathbb{Z}^n, \kappa^n) \to FSO(\mathbb{Z}^n, (\kappa^n)^f)$ is not onto, even if $f : \kappa^n \to (\kappa^n)^f$ is bijective, where $f_s(U) := \chi_U$ and $f(V) := \chi_V$ for every $U \in SO(\mathbb{Z}^n, \kappa^n)$ and every $V \in \kappa^n$. We choice the follwing subset A as follows:

$A := \{y^{(1)}, y^{(2)}\} \subset \mathbb{Z}^n$, where $y^{(1)} := (2m_1, 2m_2, ..., 2m_n)$ and $y^{(2)} = (2m_1 + 1, 2m_2 + 1, ..., 2m_n + 1)$ for some integers $m_i (1 \leq i \leq n)$; and so $y^{(1)} \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ and $y^{(2)} \in (\mathbb{Z}^n)_{\kappa^n}$. Using the subset A, we define the fuzzy set $\lambda_A \in I^{\mathbb{Z}^n}$ as follows:

$\lambda_A(y^{(2)}) := 1, \lambda_A(y^{(1)}) := 1/2$ and $\lambda_A(y) := 0$ for every point $y \in \mathbb{Z}^n$ with $y \notin A$.

Then, we have that $\lambda_A \in FSO(\mathbb{Z}^n, (\kappa^n)^f)$; indeed, $\mathrm{Cl}(\mathrm{Int}(\lambda_A)) = \chi_{Cl(\{y^{(2)}\})} \geq \lambda_A$ hold (cf. Theorem B(iii)). However, $\lambda_A \notin f_s(SO(\mathbb{Z}^n, \kappa^n))$; indeed, it follows from the definition of f_s that $f_s(SO(\mathbb{Z}^n, \kappa^n)) = \{\chi_U | U \in SO(\mathbb{Z}^n, \kappa^n)\}$ and $\lambda_A \neq \chi_U$ for each $U \in SO(\mathbb{Z}^n, \kappa^n)$.

Remark to [19, Definition 1.2 (i)]: the authors of the present paper have this opportunity of taking notice the following typographical correction in [19, Definition 1.2 (i)].

($\bullet$) line +3 from the top of the text of [19, Definition 1.2]:
" if $\lambda \leq$Int(Cl(τ_Y)) " should be replaced by " if $\lambda \leq$Int(Cl(λ)) ".

Acknowlegements The authors thank the referee for making several suggestions which improved the quality of this paper.

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Communicated by *Hiroaki Ishii*

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ON RELATIVE EXTREME AMENABILITY

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Received November 9, 2013

Abstract. The purpose of this paper is to study the notion of *relative extreme amenability* for pairs of topological groups. We give a characterization by a fixed point property on universal spaces. In addition we introduce the concepts of an *extremely amenable interpolant* as well as *maximally relatively extremely amenable* pairs and give examples. It is shown that relative extreme amenability does not imply the existence of an extremely amenable interpolant. The theory is applied to generalize results of [KPT05] relating to the application of Fraïssé theory to theory of Dynamical Systems. In particular, new conditions enabling to characterize universal minimal spaces of automorphism groups of Fraïssé structures are given.

1 Introduction The goal of this paper is to study the notion of *relative extreme amenability*: a pair of topological groups $H \subset G$ is called relatively extremely amenable if whenever G acts continuously on a compact space, there is an H-fixed point. This notion was isolated by the second author while investigating transfer properties between Fraïssé theory and dynamical systems along the lines of [KPT05], and the corresponding results appears in [NVT13]. We now provide a short description of the contents of the present article and some of the results. Section 2 contains notation. Subsection 3.1 recalls the notion of *universal spaces.* In subsection *3.2* it is shown that (G, H) is relatively extremely amenable if and only if there exists a universal G-space with a H-fixed point. In subsection 3.3 the notion of *extremely amenable interpolant* is introduced and an example of a non trivial interpolant is given. Subsection 3.4 contains technical lemmas. In subsection *3.5* the notions of *maximal relative extreme amenability* and *maximal extreme amenability* are introduced and illustrated. It is also shown that relative extreme amenability does not imply the existence of an extremely amenable interpolant and that $Aut(\mathbb{Q}, <)$ is maximally extremely amenable in S_∞. Subsections 3.6 and 3.7 deal with applications to a beautiful theory developed in [KPT05] - the application of Fraïssé theory to the theory of Dynamical Systems. In subsection 3.6 the following theorem is shown (see subsection for the definitions of the various terms appearing in the statement):

Theorem 1. *Let $\{<\} \subset L$,$L_0 = L \setminus \{<\}$ be signatures, K_0 a Fraïssé class in L_0, K an order Fraïssé expansion of K in L, $F_0 = Flim(K_0)$, $F = Flim(K)$. Let $G_0 = Aut(F_0)$ and $G = Aut(F)$. Denote $<^F = <_0$ and $X_K = \overline{G_0 <_0}$. (G_0, G) is relatively extremely amenable and $Fix_{X_K}(G)$ is transitive w.r.t X_K* if and only if X_K is the universal minimal space of G_0.

In subsection 3.7 the *weak ordering property* is introduced and it is proven that if (G_0, G) is relatively extremely amenable then the weak ordering property implies the ordering property. Finally in subsection 3.8 a question is formulated.

2010 *Mathematics Subject Classification.* Primary: 54H20. Secondary: 05D10, 22F05, 37B05.
Key words and phrases. Topological groups actions, extreme amenability, universal minimal space, Kechris-Pestov-Todorcevic correspondance, Fraïssé theory.

Ackowledgements: This project began while both of us were attending the thematic program on Asymptotic and Geometric Analysis taking place in Fall 2010 at the Fields Institute in Toronto. We would therefore like to ackowledge the support of the Fields Institute, and thank the organizers Vitali Milman, Vladimir Pestov and Nicole Tomczak-Jaegermann for having made this work possible. We would also like to thank Todor Tsankov for mentioning Peter Cameron's article [Cam76] regarding Theorem 3.5.9.

2 Preliminaries We denote by (G, X) a topological dynamical system (t.d.s), where G is a (Hausdorff) topological group and X is a compact (Hausdorff) topological space. We may also refer to X as a G-space. If it is desired to distinguish a specific point $x_0 \in X$, we write (G, X, x_0). Given a continuous action (G, X) and $x \in X$, denote by $Stab_G(x) = Stab(x) = \{g \in G \,|\, gx = x\} \subset G$, the subgroup of elements of G fixing x, and for $H \subset G$ denote by $Fix_X(H) = Fix(H) = \{x \in X \,|\, \forall h \in H\, hx = x\} \subset X$, the set of elements of X, fixed by H. Note that $Fix_X(H)$ is a closed set. Given a linear order $<$ on a set D, we denote by $<^*$ the linear ordering defined on D by $a <^* b \Leftrightarrow b < a$ for all $a, b \in D$.

3 Results

3.1 Universal spaces. Let G be a topological group. The topological dynamical system (t.d.s.) (G, X) is said to be **minimal** if X and $\emptyset$ are the only G-invariant closed subsets of X. By Zorn's lemma each G-space contains a minimal G-subspace. (G, X) is said to be **universal** if any minimal G-space Y is a G-factor of X. One can show there exists a minimal and universal G-space U_G unique up to isomorphism. (G, U_G) is called the **universal minimal space of** G (for existence and uniqueness see for example [Usp02] , or the more recent [GL13]). (G, X, x_0) is said to be **transitive** if $\overline{Gx_0} = X$. One can show there exists a transitive t.d.s (G, A_G, a_0), unique up to isomorphism, such that for any transitive t.d.s (G, Y, y_0), there exists a G-equivariant mapping $\phi_Y : (G, A_G, a_0) \to (G, Y, y_0)$ such that $\phi(a_0) = y_0$. (G, A_G, a_0) is called the **greatest ambit**. Because any minimal subspace of A_G is isomorphic to the universal minimal space, A_G is universal. Note that if A_G is not minimal (e.g., this is the case if A_G is not distal see [dV93] IV(4.35)), then it is an example of a *non-minimal* universal space.

3.2 A Characterization of Relative Extreme Amenability Recall the following classical definition (originating in [Mit66]):

Definition 3.2.1. *Let G be a topological group. G is called* ***extremely amenable*** *if any t.d.s (G, X) has a G-fixed point, i.e. there exists $x_0 \in X$, such that for every $g \in G$, $gx_0 = x_0$.*

It is easy to see that for G to be extremely amenable is equivalent to $U_G = \{*\}$. Here is a generalization of the previous definition which appears in [NVT13]:

Definition 3.2.2. *Let G be a topological group and $H \subset G$, a subgroup. The pair (G, H) is called* ***relatively extremely amenable*** *if any t.d.s (G, X) has a H-fixed point, i.e. there exists $x_0 \in X$, such that for every $h \in H$, $hx_0 = x_0$.*

Proposition 3.2.3. *Let G be a topological group and $H \subset G$, a subgroup. The following conditions are equivalent:*

1. The pair (G, H) is relatively extremely amenable.
2. U_G has a H-fixed point.
3. There exists a universal G-space T_G and $t_0 \in T_G$ which is H-fixed.

Proof. (1)⇒(2). If (G, H) is relatively extremely amenable, then by definition (G, U_G) has a H-fixed point.

(2)⇒(3). Trivial.

(3)⇒(1). Let X be a minimal G-space. By universality of T_G, there exists a surjective G-equivariant mapping $\phi : (G, T_G) \to (G, X)$. Denote $x = \phi(t_0)$. Clearly for every $h \in H$, $hx = h\phi(t_0) = \phi(ht_0) = \phi(t_0) = x$ □

It is well-known that a non-compact locally compact group cannot be extremely amenable. Here is a strengthening of this fact:

Proposition 3.2.4. *Let G be a non-compact locally compact group and $\{e\} \subsetneq H \subset G$, a subgroup. The pair (G, H) is not relatively extremely amenable.*

Proof. By Veech's Theorem ([Vee77]) G acts freely on U_G. Now use Proposition 3.2.3(2). □

3.3 Extremely Amenable Interpolants

Definition 3.3.1. *Let G be a topological group and $H \subset G$, a subgroup. An extremely amenable group E is called an* ***extremely amenable interpolant*** *for the pair (G, H) if $H \subset E \subset G$.*

The following lemma is trivial:

Lemma 3.3.2. *Let G be a topological group and $H \subset G$, a subgroup. If there exists an extremely amenable interpolant for the pair (G, H), then (G, H) is relatively extremely amenable.*

Here is an example of a non trivial extremely amenable interpolant E for a pair (G, H), in the sense that neither $E = G$, nor $E = H$:

Example 3.3.3. *Let Q be the Hilbert cube. Recall that by a result of Uspenskij (Theorem 9.18 of [Kec95]), $Homeo(Q)$, equipped with the compact-open topology, is a universal Polish group, in the sense that any Polish group embeds inside it through a homomorphism. Let $Homeo_+(I)$ be the group of increasing homeomorphisms of the interval I, equipped with the compact-open topology. By a result of Pestov (see [Pes98]) $Homeo_+(I)$ is extremely amenable. Let $\phi : Homeo_+(I) \hookrightarrow Homeo(Q)$ be an embedding through a homomorphism. Let $f : I \to I$ given by $f(x) = x^2$. Notice $f \in Homeo_+(I)$. Denote $G = Homeo(Q)$, $E = \phi(Homeo_+(I))$ and $H = \phi(\{f^n \,|\, n \in \mathbb{Z}\})$. Notice $H \subsetneq E \subsetneq G$. E is clearly an extremely amenable interpolant for (G, H), but G (which acts homogeneously on Q) and H (which is isomorphic to $\mathbb{Z}$) are not extremely amenable.*

A natural question is if any relatively extremely amenable pair has an extremely amenable interpolant. Theorem 3.5.8 in the next subsection answers the question in the negative.

3.4 Order fixing groups Let S_∞ be the permutation group of the integers $\mathbb{Z}$, equipped with the pointwise convergence topology. Let F be an infinite countable set and fix a bijection $F \simeq \mathbb{Z}$. Let $LO(F) \subset \{0,1\}^{F \times F}$, be the space of linear orderings on F, equipped with the pointwise convergence topology. Under the above mentioned bijection $LO(F)$ becomes an S_∞-space. By Theorem 8.1 of [KPT05] $U_{S_\infty} = LO(F)$. Notice that we consider F as a set and not a topological space. In this subsection we will use $F = \mathbb{Z}$ and $F = \mathbb{Q}$, considered as infinitely countable sets with convenient enumerations (bijections) and the corresponding dynamical systems $(S_\infty, LO(\mathbb{Z}))$ and $(S_\infty, LO(\mathbb{Q}))$.

Lemma 3.4.1. *Let $<\in LO(\mathbb{Z})$ be the usual linear order on $\mathbb{Z}$, i.e. the order for which $n < n+1$ for every $n \in \mathbb{Z}$. Then*

1. $Stab_{\mathbb{Z}}(<) = \{T_a | \, a \in \mathbb{Z}\}$, *where* $T_a : \mathbb{Z} \to \mathbb{Z}$ *is given by* $T_a(x) = x + a$.

2. $Fix_{LO(\mathbb{Z})}(Stab_{\mathbb{Z}}(<)) = \{<, <^*\}$.

Proof. (1) Let $T \in Stab(<)$. Denote $a = T(0)$. Notice that for all $x > 1$, $T(x) > T(1) > a$ and for all $x < 0$, $T(x) < a$. As T is onto we must have $T(1) = a + 1$. Similarly for all $x \in \mathbb{Z}$, $T(x) = x + a$, which implies $T = T_a$.

(2) Let $\prec \in Fix_{LO(\mathbb{Z})}(Stab(<))$. We claim that $\prec = <$ or $\prec = <^*$. Indeed $0 \prec 1$ or $1 \prec 0$. In the first case applying $T_a \in Stab(<)$, we have for all $a \in \mathbb{Z}$, $a \prec a + 1$. This implies $\prec = <$. Similarly in the second case for all $a \in \mathbb{Z}$, $a + 1 \prec a$ which implies $\prec = <^*$. □

Let $< \in LO(\mathbb{Q})$ be the usual order on $\mathbb{Q}$. In the following lemma, we follow the standard convention and write $Aut(\mathbb{Q}, <)$ instead of $Stab_{S_\infty}(<) \subset S_\infty$.

Lemma 3.4.2. *Let* $< \in LO(\mathbb{Q})$ *be the usual linear order on* $\mathbb{Q}$, *then*

$$Fix_{LO(\mathbb{Q})}(Aut(\mathbb{Q}, < \;)) = \{<, <^*\}.$$

Proof. Let $\prec \in Fix_{LO(\mathbb{Q})}(Aut(\mathbb{Q}, <))$. Note that $0 \prec 1$ or $1 \prec 0$. In the first case, let $q_1, q_2 \in \mathbb{Q}$ with $q_1 < q_2$ and define $T : \mathbb{Q} \to \mathbb{Q}$ with $Tx = (q_2 - q_1)x + q_1$. Note that $T \in Aut(\mathbb{Q}, <)$. Hence, $q_1 = T(0) \prec T(1) = q_2$. As the argument works for any $q_1' < q_2'$ we have $\prec = <$. The second case is similar and implies $\prec = <^*$. □

3.5 Maximally Relatively Extremely Amenable Pairs

Proposition 3.5.1. *Let* G *be a topological group, then there exists a subgroup* $H \subset G$, *such that* (G, H) *is relatively extremely amenable and there exists no subgroup* $H \subset H' \subset G$, *such that* (G, H') *is relatively extremely amenable.*

Proof. By Zorn's lemma it is enough to show that any chain w.r.t. inclusion $\{G_\alpha\}_{\alpha \in A}$ such that (G, G_α) is relatively extremely amenable, has a maximal element. Note that if $G_\alpha \subset G_{\alpha'}$, then $Fix_{U_G}(G_{\alpha'}) \subset Fix_{U_G}(G_\alpha)$. In particular for any finite collection $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$, we have $\bigcap_{i=1}^n Fix_{U_G}(G_{\alpha_i}) \neq \emptyset$, which implies by a standard compactness argument $\bigcap_{\alpha \in A} Fix_{U_G}(G_\alpha) \neq \emptyset$. This in turn implies that $Fix_{U_G}(\bigcup_{\alpha \in A} G_\alpha) \neq \emptyset$, which finally implies $(G, \bigcup_{\alpha \in A} G_\alpha)$ is relatively extremely amenable by Proposition 3.2.3(2). □

Definition 3.5.2. *A pair* (G, H) *as in Proposition 3.5.1 is called* ***maximally relatively extremely amenable****.*

Similarly to the previous theorem and definition we have:

Proposition 3.5.3. *Let* G *be a topological group, then there exists a subgroup* $H \subset G$, *such that* H *is extremely amenable and there exists no subgroup* $H \subset H' \subset G$, *such that* H' *is extremely amenable.*

Proof. By Zorn's lemma it is enough to show that any chain w.r.t. inclusion $\{G_\alpha\}_{\alpha \in A}$ such that $G_\alpha \subset G$ and G_α is extremely amenable, has a maximal element. Let $(\bigcup_{\alpha \in A} G_\alpha, X)$ be a dynamical system. By assumption for any $\alpha \in A$, $Fix_X(G_\alpha) \neq \emptyset$. In addition if $G_\alpha \subset G_{\alpha'}$, then $Fix_X(G_{\alpha'}) \subset Fix_X(G_\alpha)$. We now continue as in the proof of Theorem 3.5.1 to conclude $\bigcup_{\alpha \in A} G_\alpha$ is extremely amenable. □

Definition 3.5.4. *A subgroup* $H \subset G$ *as in Proposition 3.5.3 is called* ***maximally extremely amenable in*** G.

Remark 3.5.5. *It was pointed out in [Pes02] that if H is second countable (Hausdorff) group then there always exists an extremely amenable group G such that $H \subset G$. Indeed by [Usp90] $H \subset Iso(\mathbb{U})$ the group of isometries of Urysohn's universal complete separable metric space $\mathbb{U}$, equipped with the compact-open topology, and by [Pes02], $Iso(\mathbb{U})$ is extremely amenable.*

Theorem 3.5.6. *Let $G = S_\infty$ be the permutation group of the integers, equipped with the pointwise convergence topology. Let $<$ be the usual order on $\mathbb{Z}$ and $H = Stab_{\mathbb{Z}}(<) \subset G$. The pair (G, H) is maximally relatively extremely amenable.*

Proof. By Theorem 8.1 of [KPT05] $U_G = LO(\mathbb{Z})$, the space of linear orderings on $\mathbb{Z}$. By Proposition 3.2.3(2) (G, H) is relatively extremely amenable. Assume that there exists a subgroup E, with $H \subset E \subset G$ such that (G, E) is a relatively extremely amenable. Evoking again Proposition 3.2.3(2), there exists $\prec \in U_G$, so that $E \subset Stab(\prec)$. As $H \subset E \subset Stab(\prec)$, conclude by Lemma 3.4.1(2) that $\prec \in \{<, <^*\}$. As $H = Stab(<) = Stab(<^*)$, we conclude in both cases $E = H$. □

Lemma 3.5.7. *If (G, H) is maximally relatively extremely amenable and neither G nor H are extremely amenable, then (G, H) does not admit an extremely amenable interpolant.*

Proof. Assume for a contradiction that there exists an extremely amenable subgroup E, with $H \subset E \subset G$. Notice that (G, E) is relatively extremely amenable which constitutes a contradiction with the fact that (G, H) is maximally relatively extremely amenable. □

Theorem 3.5.8. *There exists a relatively extremely amenable pair (G, H) which which does not admit an extremely amenable interpolant.*

Proof. Let $G = S_\infty$ be the permutation group of the integers, equipped with the pointwise convergence topology. Let $<$ be the usual order on $\mathbb{Z}$ and $H = Stab(<) \subset G$. By Theorem 3.5.6 (G, H) is maximally relatively extremely amenable. Clearly G is not extremely amenable as $U_G \neq \{*\}$. By Lemma 3.4.1(1) $H = \{T_a | \, a \in \mathbb{Z}\} \cong \mathbb{Z}$, where the second equivalence is as topological groups. This implies H is not extremely amenable. Now invoke Lemma 3.5.7. □

Theorem 3.5.9. *$Aut(\mathbb{Q}, <)$ is maximally extremely amenable in S_∞.*

Proof. By [Pes98] $Aut(\mathbb{Q}, <)$ is extremely amenable. Now we can proceed as in the proof of Theorem 3.5.8 using Lemma 3.4.2. □

Remark 3.5.10. *Even though the previous result never appeared in print, Todor Tsankov pointed out that it can be derived from an earlier result by Cameron. Indeed, the article [Cam76] allows a complete description of the closed subgroups G of S_∞ containing $Aut(\mathbb{Q})$ (essentially, there are only five of them, see [BP11] for an explicit description) and it can be verified that among those, only $Aut(\mathbb{Q})$ is extremely amenable.*

3.6 Applications in Fraïssé Theory The following two sections deal with applications Fraïssé Theory. Two general references for this theory are [Fra00] and [Hod93]. We follow the exposition and notation of [KPT05].

Let $\{<\} \subset L, L_0 = L \setminus \{<\}$ be signatures, K_0 a Fraïssé class in L_0, K an order Fraïssé expansion of K_0 in L, $F = Flim(K)$ the Fraïssé limit of K. By Theorem 5.2$(ii) \Rightarrow (i)$ of [KPT05], if we denote $F_0 = Flim(K_0)$ then $F_0 = F|L_0$. Let $G_0 = Aut(F_0)$ and $G = Aut(F)$. Denote $<^F = <_0$, i.e. $<_0$ is the linear order corresponding to the symbol $<$ in F, and let $X_K = \overline{G_0 <_0}$ (X_K is called set of K-*admissible* linear orderings of F in [KPT05]). In [KPT05], two combinatorial properties for K have considerable importance in order to compute universal minimal spaces. Those are called *ordering property* and *Ramsey property*:

Definition 3.6.1. *Let $\{<\} \subset L$ be a signature, $L_0 = L \setminus \{<\}$, K_0 a Fraïssé class in L_0, K an order Fraïssé expansion of K_0 in L, $F = Flim(K)$ the Fraïssé limit of K. We say that K satisfies the **ordering property** (relative to K_0) if for every $A_0 \in K_0$, there is $B_0 \in K_0$, such that for every linear ordering $\prec$ on A_0 and linear ordering $\prec'$ on B_0, if $A = \langle A_0, \prec\rangle \in K$ and $B = \langle B_0, \prec'\rangle \in K$, then there is an embedding $A \hookrightarrow B$.*

Definition 3.6.2. *Let $\{<\} \subset L$ be a signature and K be an order Fraïssé class in L. We say that K satisfies the **Ramsey property** if, for every positive $k \in \mathbb{N}$, every $A \in K$ and every $B \in K$, there exists $C \in K$ such that for every k-coloring of the substructures of C which are isomorphic to A, there is a substructure $\tilde{B}$ of C which is isomorphic to B and such that all substructures of $\tilde{B}$ which are isomorphic to A receive the same color.*

Those two properties are relevant because they capture dynamical properties of X_K. For example, Theorem 7.4 of [KPT05] states that the minimality of X_K is equivalent to K having the ordering property, and Theorem 10.8 of [KPT05] states that X_K being universal and minimal is equivalent to K having the ordering and Ramsey properties. Those results naturally led the authors of [KPT05] to ask whether X_K being universal is equivalent to K having the Ramsey property. This question is precisely the reason for which the concept of relative extremely amenability was introduced. Recall that by Theorem 4.7 of [KPT05], the Ramsey property of K is equivalent to G being extremely amenable. In [NVT13], it is shown that the universality of X_K is equivalent to (G_0, G) being relatively extremely amenable. However, it is still unknown whether (G_0, G) being relatively extremely amenable is really weaker than G being extremely amenable (see Section 3.8 for more about this aspect).

Remark 3.6.3. *The reason for which only* order *expansions (i.e. $\{<\} \subset L$,$L_0 = L \setminus \{<\}$, and $<$ is interpreted as a linear order) were considered in [KPT05] is that, at the time where the article was written, expanding the signature by such a symbol was sufficient in order to obtain Ramsey property and ordering property in all known practical cases. However, we know now that there are some cases where expanding the language with more symbols is necessary (E.g.* circular tournaments *and* boron tree structures*, whose Ramsey-type properties have been respectiveley analyzed by Laflamme, Nguyen Van Thé and Sauer in [LNVTS10], and by Jasiński in [Jas13]). The description of the corresponding universal minimal spaces is very similar to what is obtained in [KPT05] and will appear in a forthcoming paper. For the sake of clarity, we will only treat here the case of order expansions, which extends to the general case without difficulty.*

3.7 The weak ordering property. Theorem 10.8 of [KPT05] states that K has the ordering and Ramsey properties if and only if X_K is the universal minimal space of G_0. The purpose of this section is to show that the combinatorial assumptions made on K can actually be slightly weakened. We start with a generalization of the notion of transitivity mentioned in subsection 3.1.

Definition 3.7.1. *Let G be a topological group and X a G-space. $Y \subset X$ is said to be **transitive w.r.t** X if and only if for any $y \in Y$, $\overline{Gy} = X$.*

Proposition 3.7.2. *Let G_0 be a topological group and let T_{G_0} be G_0-universal. Let $x \in T_{G_0}$ and let $G = Stab_{G_0}(x) \subset G_0$. T_{G_0} is minimal if and only if $Fix_{T_{G_0}}(G)$ is transitive w.r.t T_{G_0}.*

Proof. If T_{G_0} is minimal then T_{G_0} is transitive w.r.t itself and trivially $Fix_{T_{G_0}}(G) \subset T_{G_0}$ is transitive w.r.t T_{G_0}. To prove the inverse direction, let $M \subset T_{G_0}$ be a G_0-minimal space. By Proposition 3.2.3(3), (G_0, G) is relatively extremely amenable and therefore there exists $t_0 \in M \cap Fix_{T_{G_0}}(G)$. As $Fix_{T_{G_0}}(G)$ is transitive w.r.t T_{G_0}, conclude $T_{G_0} = \overline{G_0 t_0} \subset M$, so $T_{G_0} = M$ is minimal. □

The previous proposition enables us to prove the following equivalence:

Theorem 3.7.3. *(G_0, G) is relatively extremely amenable and $Fix_{X_K}(G)$ is transitive w.r.t X_K if and only if X_K is the universal minimal space of G_0.*

Proof. As indicated previously, the universality of X_K is equivalent to the fact that (G_0, G) is relatively extremely amenable. By Proposition 3.7.2, given that X_K is universal, the minimality of X_K is equivalent to the fact that $Fix_{X_K}(G)$ is transitive w.r.t X_K .

Remark 3.7.4. *By Theorem 3.2.3(3) $(S_\infty, Aut(\mathbb{Q}, <))$ is relatively extremely amenable. By Lemma 3.4.2 $Fix_{LO(\mathbb{Q})}(Aut(\mathbb{Q}, <)) = \{<, <^*\}$. As $LO(\mathbb{Q}) = \overline{S_\infty <} = \overline{S_\infty <^*}$, we have that $Fix_{LO(\mathbb{Q})}(Aut(\mathbb{Q}, <))$ is transitive w.r.t $LO(\mathbb{Q})$. By Theorem 3.7.3, it follows that $Aut(\mathbb{Q}, <)$ is extremely amenable. It should be noted that in [KPT05], one obtains the same results but in reverse order: one concludes $LO(\mathbb{Q})$ is the universal minimal space of G, using the fact that G_0 is extremely amenable.*

□

We are now going to show how to reformulate Theorem 3.7.3 in terms of combinatorics.

Definition 3.7.5. *Let $\{<\} \subset L$ be a signature, $L_0 = L \setminus \{<\}$, K_0 a Fraïssé class in L_0, K an order Fraïssé expansion of K_0 in L. We say that (K_0, K) has the **relative Ramsey property** if for every positive $k \in \mathbb{N}$, every $A_0 \in K_0$ and every $B \in K$, there exists $C \in K_0$ such that for every k-coloring of the substructures of C_0 isomorphic to A_0, there is an embedding $\phi : B|L_0 \hookrightarrow C_0$ such that for any two substructures $\tilde{A}, \tilde{A}'$ of B_0 isomorphic to A_0, $\phi(\tilde{A})$ and $\phi(\tilde{A}')$ receive the same color whenever $\tilde{A}$ and $\tilde{A}'$ support isomorphic structures in B.*

In what follows, the relative Ramsey property will appear naturally because of the following fact (see [NVT13]):

Claim 3.7.6. *(G_0, G) is relatively extremely amenable iff (K_0, K) has the relative Ramsey property.*

We will also need the following variant of the notion of ordering property:

Definition 3.7.7. *Let $\{<\} \subset L$ be a signature, $L_0 = L \setminus \{<\}$, K_0 a Fraïssé class in L_0, K an order Fraïssé expansion of K_0 in L. We say that K satisfies the **weak ordering property** relative to K_0 if for every $A_0 \in K_0$, there is $B_0 \in K_0$, such that for every linear ordering $\prec$ on A_0 with $A = \langle A, \prec \rangle \in K$ and linear ordering $\prec' \in Fix_{X_K}(G)$ we have $A \hookrightarrow \langle B_0, \prec' |B_0 \rangle$.*

The following claim appears in the proof of Theorem 7.4 of [KPT05]:

Claim 3.7.8. *Let $<$ be a linear ordering on F_0. Then $<_0 \in \overline{G_0 <}$ if and only if for every $A \in K$ there is a finite substructure C_0 of F_0 such that $C = \langle C_0, < |C_0 \rangle \cong A$.*

Proposition 3.7.9. *Assume K satisfies the weak ordering property relative to K_0, and that (K_0, K) has the relative Ramsey property. Then K satisfies the ordering property.*

Proof. Again, the universality of X_K is equivalent to the fact that (G_0, G) is relatively extremely amenable, which is in turn equivalent to (K_0, K) having the relative Ramsey property. By Theorem 7.4 of [KPT05] the minimality of X_K is equivalent to the ordering property of K (relative to K_0). By Proposition 3.7.2 in order to establish X_K is minimal, it is enough to show that $Fix_{X_K}(G)$ is transitive w.r.t X_K. Let $<\in Fix_{X_K}(G)$. It is enough to show $<_0 \in \overline{G <}$. Fix $A \in K$. As K satisfies the weak ordering property, there is B_0 as in Definition 3.7.7 such that $A \hookrightarrow \langle B_0, < |B_0\rangle$. Using the same argument as in the proof of Theorem 7.4 of [KPT05], we notice that there is a substructure C of B isomorphic to A. Denote $C_0 = C|L_0$ and notice $C = \langle C_0, < |C_0\rangle \cong A$. We now use Claim 3.7.8. □

Theorem 3.7.10. *K has the weak ordering property and (K_0, K) has the relative Ramsey property if and only if X_K is the universal minimal space of G_0.*

Proof. By Theorem 10.8 of [KPT05], if X_K is the universal minimal space of G_0 then K satisfies the ordering property, a fortiori, K satisfies the weak ordering property. In addition K satisfies the Ramsey property which implies (K_0, K) has the relative Ramsey property. The reverse direction follows from Proposition 3.7.9. □

3.8 A question. We mentioned previously that the concept of relative extreme amenability was introduced in order to know whether X_K being universal is equivalent to K having the Ramsey property. By Theorem 4.7 of [KPT05], the Ramsey property of K is equivalent to G being extremely amenable. We still do not know the answer to the following question from [KPT05]:

Question 3.8.1. *Let $\{<\} \subset L$ be a signature, $L_0 = L \setminus \{<\}$, K_0 a Fraïssé class in L_0, K an order Fraïssé expansion of K_0 in L. Does universality for X_K imply that G is extremely amenable (equivalently, that K has the Ramsey property)?*

Moreover, in view of the notions we introduced previously, we ask:

Question 3.8.2. *Assume the previous question has a negative answer. Does there exist an extremely amenable interpolant for the pair (G_0, G)?*

As a final comment, and in view of Remark 3.6.3, it should be mentioned that Question 3.8.1 has a negative answer when K is not an order expansion of K_0, see [NVT13].

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Communicated by *Vladimir Pestov*

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SHRINKAGE ESTIMATION FOR THE AUTOCOVARIANCE MATRIX OF VECTOR-VALUED GAUSSIAN STATIONARY PROCESSES

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Received July 3, 2014 ; revised September 12, 2014

Abstract. We discuss the problem of shrinkage estimation for the autocovariance matrix of a Gaussian stationary vector-valued process to improve on the usual sample autocovariance matrix with respect to the mean squares error. We propose a kind of empirical Bayes estimators when the mean of the stochastic process is zero and non-zero. We show that the shrinkage estimators dominate the usual estimators, and the asymptotic risk differences are similar to that of scalar-valued Gaussian stationary processes. This result seems to be useful for the autocovariance estimation with vector-valued dependent observations.

1 Introduction There have been many discussions on shrinkage estimation to improve on the sample mean and the sample covariance of independent observations. Stein [6] showed the inadmissibility of the sample mean for k-dimensional independent normal observations when $k \geq 3$. James and Stein [5] suggested a shrinkage estimator which dominates the sample mean with respect to the mean squares error when $k \geq 3$. Furthermore, in the univariate case, Stein [7] proposed a truncated estimator and showed the estimator improves on the usual sample variance. Also in the multivariate case, Haff [2] proposed an empirical Bayes estimator for the normal covariance matrix and showed the estimator improves on the sample covariance matrix.

All mentioned above are the discussions for independent normal observations. However, it is natural that the actual data are dependent. Therefore, it is important to consider the shrinkage estimators which dominate the usual sample mean and the autocovariance when the observations are dependent. For a vector-valued Gaussian process, Taniguchi and Hirukawa [8] gave a sufficient condition for James-Stein type estimator to dominate the sample mean. Furthermore, for the scalar-valued Gaussian stationary process, Taniguchi *et al.* [9] suggested an empirical Bayes estimator motivated by Haff [2] and discussed on the improvement by the estimator.

Since it is useful to represent the actual time series data by dependent and multivariate statistical models, in this paper, we consider improved autocovariance estimation for vector-valued Gaussian stationary processes motivated by Taniguchi *et al.* [9]. We propose shrinked autocovariance estimators, and show that the estimators dominate the usual autocovariance estimators in case of vector-valued Gaussian stationary processes.

This paper is organized as follows. In Section 2, we introduce empirical Bayes estimators in view of Taniguchi *et al.* [9] when the mean of the stochastic process is zero and non-zero. Then we evaluate the asymptotic risk differences by the mean squares error between the shrinkage estimator and the usual sample autocovariance matrix. The improvements by the shrinkage estimators are expressed in terms of the spectral density of the process. Section 3 provides the proofs of theorems in Section 2.

2000 *Mathematics Subject Classification.* 60G10, 60G15, 62H12.

Key words and phrases. shrinkage estimation, Gaussian stationary process, autocovariance, spectral density, mean squares error.

Throughout this paper, $\boldsymbol{Z}$ denotes the set of all integers, and $\otimes$ denotes the Kronecker product of matrices, and $\circ$ denotes the Hadamard product (entrywise product) of matrices.

2 Shrinkage estimators for autocovariance matrix Let $\{\boldsymbol{X}(t), t \in \boldsymbol{Z}\}$ be an m-dimensional Gaussian stationary process with mean $E(\boldsymbol{X}(t)) = \boldsymbol{\mu}$ and autocovariance matrix $\boldsymbol{\gamma}(s) = E[(\boldsymbol{X}(t) - \boldsymbol{\mu})(\boldsymbol{X}(t+s) - \boldsymbol{\mu})']$ for $s \in \boldsymbol{Z}$ and all $t \in \boldsymbol{Z}$. We assume that $\boldsymbol{\gamma}(s)$'s satisfy

Assumption 1.

$$\sum_{s=-\infty}^{\infty} |s| \cdot \|\boldsymbol{\gamma}(s)\| < \infty,$$

where $\|\cdot\|$ is the Euclidean norm. Then the spectral density matrix of the process is given by

$$\boldsymbol{f}(\lambda) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} \boldsymbol{\gamma}(s) e^{-\boldsymbol{i}s\lambda}. \tag{1}$$

Here we consider to estimate the autocovariance matrix

$$\Gamma = \begin{pmatrix} \boldsymbol{\gamma}(0) & \boldsymbol{\gamma}(-1) & \dots & \boldsymbol{\gamma}(1-p) \\ \boldsymbol{\gamma}(1) & \boldsymbol{\gamma}(0) & \dots & \boldsymbol{\gamma}(2-p) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\gamma}(p-1) & \boldsymbol{\gamma}(p-2) & \dots & \boldsymbol{\gamma}(0) \end{pmatrix}. \tag{2}$$

for positive integer p. Since $\boldsymbol{\gamma}(-s) = \boldsymbol{\gamma}(s)'$, Γ is symmetric. Suppose that an observed stretch $\{\boldsymbol{X}(1), \dots, \boldsymbol{X}(n)\}$ of the process $\{\boldsymbol{X}(t)\}$ is available. When $\boldsymbol{\mu} = \boldsymbol{0}$, the usual estimator for Γ is

$$\hat{\Gamma}_0 = \frac{1}{n-k} S_n, \tag{3}$$

where

$$S_n = \sum_{t=p}^{n} \boldsymbol{Y}(t)\boldsymbol{Y}(t)', \quad \boldsymbol{Y}(t) = (\boldsymbol{X}(t)', \dots, \boldsymbol{X}(t-p+1)')',$$

and $k = 0$ or $p-1$. When $\boldsymbol{\mu} \neq \boldsymbol{0}$, the usual estimator for Γ is

$$\tilde{\Gamma}_0 = \frac{1}{n-k} \tilde{S}_n, \tag{4}$$

where

$$\tilde{S}_n = \sum_{t=p}^{n} \tilde{\boldsymbol{Y}}(t)\tilde{\boldsymbol{Y}}(t)', \quad \tilde{\boldsymbol{Y}}(t) = ((\boldsymbol{X}(t) - \bar{\boldsymbol{X}}_n)', \dots, (\boldsymbol{X}(t-p+1) - \bar{\boldsymbol{X}}_n)')',$$

with $\bar{\boldsymbol{X}}_n = n^{-1} \sum_{t=1}^{n} \boldsymbol{X}(t)$ and $k = 0$ or $p-1$. We measure the goodness of $\hat{\Gamma}_0$ by the following mean squares error loss function

$$L(\hat{\Gamma}_0, \Gamma) = \operatorname{tr}\{(\hat{\Gamma}_0 \Gamma^{-1} - \boldsymbol{I}_{mp})^2\} \quad (\boldsymbol{I}_{mp} \text{ is the } mp \times mp \text{ identity matrix}) \tag{5}$$

and the risk $R(\hat{\Gamma}_0, \Gamma) = E\{L(\hat{\Gamma}_0, \Gamma)\}$. Similarly, for $\tilde{\Gamma}_0$ we also define $L(\tilde{\Gamma}_0, \Gamma)$ and $R(\tilde{\Gamma}_0, \Gamma)$. Next, we consider to improve the estimators $\hat{\Gamma}_0$ and $\tilde{\Gamma}_0$ with respect to the risk $R(\cdot, \cdot)$. When

$\{X(t)\}$ is a scalar-valued process, Taniguchi *et al.* [9] introduced the following empirical Bayes estimators

$$\hat{\Gamma} = \frac{1}{n-k}\left(S_n + \frac{b}{n\ \mathrm{tr}(S_n^{-1}C)}C\right) \tag{6}$$

and

$$\tilde{\Gamma} = \frac{1}{n-k}\left(\tilde{S}_n + \frac{b}{n\ \mathrm{tr}(\tilde{S}_n^{-1}C)}C\right) \tag{7}$$

to improve $\hat{\Gamma}_0$ and $\tilde{\Gamma}_0$, respectively, where b is a constant and C is a positive definite matrix of the same size as Γ, and showed that $\hat{\Gamma}$ and $\tilde{\Gamma}$ dominate $\hat{\Gamma}_0$ and $\tilde{\Gamma}_0$, respectively, with respect to the risk. Similarly, when $\{\boldsymbol{X}(t)\}$ is a vector-valued process, we use the estimators in the form of (6) and (7), and show that $\hat{\Gamma}$ and $\tilde{\Gamma}$ dominate $\hat{\Gamma}_0$ and $\tilde{\Gamma}_0$, respectively. To evaluate the improvement of the estimator, we need the following assumption.

Assumption 2. *C is symmetric.*

The assumption seems to be natural because S_n in (6) and $\tilde{S}_n$ in (7) are symmetric and $\hat{\Gamma}$ in (6) and $\tilde{\Gamma}$ in (7) should be symmetric. Then, the following theorem holds.

Theorem 1. *When $\boldsymbol{\mu} = \boldsymbol{0}$, suppose that Assumptions 1 and 2 hold. Then the asymptotic risk difference for the estimator $\hat{\Gamma}_0$ and $\hat{\Gamma}$ is*

$$\lim_{n\to\infty} n^2[R(\hat{\Gamma}_0,\Gamma) - R(\hat{\Gamma},\Gamma)] = -b\frac{\mathrm{tr}\{(C\Gamma^{-1})^2\}}{\{\mathrm{tr}(\Gamma^{-1}C)\}^2}[b+B], \tag{8}$$

where

$$B = \begin{cases} 2(-p+1)\dfrac{\{\mathrm{tr}(\Gamma^{-1}C)\}^2}{\mathrm{tr}\{(C\Gamma^{-1})^2\}} \\ \quad +\dfrac{8\pi}{\mathrm{tr}\{(C\Gamma^{-1})^2\}}\displaystyle\int_{-\pi}^{\pi} \mathrm{tr}\{[\{\{(G(\lambda)\otimes \boldsymbol{I}_m)\Gamma^{-1}C\Gamma^{-1}\}\circ(\boldsymbol{I}_p\otimes U_m)\}(U_p\otimes \boldsymbol{f}(\lambda))]^2\}d\lambda, \\ \text{(if } k=0), \\ \dfrac{8\pi}{\mathrm{tr}\{(C\Gamma^{-1})^2\}}\displaystyle\int_{-\pi}^{\pi} \mathrm{tr}\{[\{\{(G(\lambda)\otimes \boldsymbol{I}_m)\Gamma^{-1}C\Gamma^{-1}\}\circ(\boldsymbol{I}_p\otimes U_m)\}(U_p\otimes \boldsymbol{f}(\lambda))]^2\}d\lambda, \\ \text{(if } k=p-1). \end{cases} \tag{9}$$

with $G(\lambda) = (e^{-\boldsymbol{i}(h-l)\lambda})_{h,l=1,\dots,p}$ ($p\times p$ matrix), *$U_m = \mathbf{1}_m\mathbf{1}_m'$ and $\mathbf{1}_m = (1,\dots,1)'$* ($m\times 1$ vector)*.*

We can see that this result includes Theorem 1 of Taniguchi *et al.* [9] as special case.

When $\boldsymbol{\mu} \neq \boldsymbol{0}$, we can show the following theorem for $\tilde{\Gamma}_0$ and $\tilde{\Gamma}$.

Theorem 2. *When $\boldsymbol{\mu} \neq \boldsymbol{0}$, suppose that Assumptions 1 and 2 hold. Then the asymptotic risk difference for the estimator $\tilde{\Gamma}_0$ and $\tilde{\Gamma}$ is*

$$\lim_{n\to\infty} n^2[R(\tilde{\Gamma}_0,\Gamma) - R(\tilde{\Gamma},\Gamma)] = -b\frac{\mathrm{tr}\{(C\Gamma^{-1})^2\}}{\{\mathrm{tr}(\Gamma^{-1}C)\}^2}[b+\tilde{B}], \tag{10}$$

where

$$\tilde{B} = B - 4\pi\frac{\mathrm{tr}\{(U_p\otimes \boldsymbol{f}(0))\Gamma^{-1}C\Gamma^{-1}\}\cdot \mathrm{tr}\{\Gamma^{-1}C\}}{\mathrm{tr}\{(C\Gamma^{-1})^2\}}. \tag{11}$$

We can see that this result includes Theorem 2 of Taniguchi *et al.* [9] as special case.

3 Proofs This section provides the proofs of the theorems. We need the following lemma to prove Theorem 1 (for the proofs, see Lemma A2.3 of Hosoya and Taniguchi [4] and Theorem 4.5.1 of Brillinger [1]).

Lemma 1. *Suppose that Assumption 1 holds.*
(a) *Denote the α-th component of $\boldsymbol{X}(t)$ by $X_\alpha(t)$, and denote the (α,β)-th component of $\boldsymbol{f}(\lambda)$ by $f_{\alpha\beta}(\lambda)$. If $\{\boldsymbol{X}(t)\}$ is Gaussian, Then*

$$\begin{aligned}&\lim_{n\to\infty} n\mathrm{Cov}\left\{\frac{1}{n}\sum_{t=p}^{n} X_{\alpha_1}(t-j_1)X_{\alpha_2}(t-j_2),\ \frac{1}{n}\sum_{t=p}^{n} X_{\alpha_3}(t-j_3)X_{\alpha_4}(t-j_4)\right\}\\ &= 2\pi\int_{-\pi}^{\pi}\{f_{\alpha_1\alpha_3}(\lambda)\overline{f_{\alpha_2\alpha_4}(\lambda)}e^{-\boldsymbol{i}(j_1-j_2+j_4-j_3)\lambda}+f_{\alpha_1\alpha_4}(\lambda)\overline{f_{\alpha_2\alpha_3}(\lambda)}e^{\boldsymbol{i}(j_2-j_1+j_4-j_3)\lambda}\}d\lambda\\ &= W^{\alpha_1,\dots,\alpha_4}_{j_1,\dots,j_4}\ (say)\ (0\le j_1,\dots,j_4\le p-1).\end{aligned}$$

(b) *Denote the (α,β)-th component of $\boldsymbol{\gamma}(s)$ by $\gamma_{\alpha\beta}(s)$. Then*

$$\frac{1}{\sqrt{n}}\sum_{t=p}^{n}\{X_\alpha(t-j_1)X_\beta(t-j_2)-\gamma_{\alpha\beta}(j_1-j_2)\}=O(\sqrt{\log n}),\quad a.s.$$

Proof of Theorem 1 We can calculate the asymptotic risk difference in the vector-valued case as same as (19) of Taniguchi *et al.* [9]. In the proof of Theorem 1 of [9], we can use the form of (23) of [9]. Therefore we only evaluate the numerator in the expectation of (23) of [9]. The numerator is given by

$$E\left[\left(\mathrm{tr}\left\{\sqrt{n}\left(\frac{1}{n-k}S_n-E\left(\frac{1}{n-k}S_n\right)\right)\Gamma^{-1}C\Gamma^{-1}\right\}\right)^2\right]. \tag{12}$$

Here we set $Z=\dfrac{1}{n-k}S_n-E\left(\dfrac{1}{n-k}S_n\right)$ and $V=\Gamma^{-1}C\Gamma^{-1}$. Then (12) is equal to

$$nE\left[\left\{\sum_{h=1}^{p}\sum_{l=1}^{p}\mathrm{tr}\left(Z^{hl}V^{lh}\right)\right\}^2\right], \tag{13}$$

where Z^{hl} and V^{hl} are the (h,l)-th $m\times m$ block matrices of Z and V, respectively. Denote the (i,j)-th component of Z^{hl} and V^{lh} by Z^{hl}_{ij} and V^{lh}_{ij}, respectively. Then (13) is equal to

$$\sum_{h,l,h',l'=1}^{p}\sum_{i,j,i',j'=1}^{m} nE[Z^{hl}_{ij}Z^{h'l'}_{i'j'}]V^{lh}_{ji}V^{l'h'}_{j'i'}. \tag{14}$$

Let S^{hl}_n be the (h,l)-th $m\times m$ block matrix of S_n. Since $Z^{hl}=\dfrac{1}{n-k}\{S^{hl}_n-E[S^{hl}_n]\}$ and $S^{hl}_n=\sum_{t=p}^{n}\boldsymbol{X}(t-h+1)\boldsymbol{X}(t-l+1)'$, (14) is equal to

$$\begin{aligned}&\sum_{h,l,h',l'=1}^{p}\sum_{i,j,i',j'=1}^{m}\frac{n^2}{(n-k)^2}V^{lh}_{ji}V^{l'h'}_{j'i'}\\ &\times n\mathrm{Cov}\left(\frac{1}{n}\sum_{t=p}^{n}X_i(t-h+1)X_j(t-l+1),\frac{1}{n}\sum_{t=p}^{n}X_{i'}(t-h'+1)X_{j'}(t-l'+1)\right).\end{aligned} \tag{15}$$

Using Lemma 1(a), as $n \to \infty$, (15) converges to

$$
\begin{aligned}
(16)\qquad & \sum_{h,l,h',l'=1}^{p} \sum_{i,j,i',j'=1}^{m} V_{ji}^{lh} V_{j'i'}^{l'h'} \\
& \times 2\pi \int_{-\pi}^{\pi} \{ f_{ii'}(\lambda) f_{j'j}(\lambda) e^{-\boldsymbol{i}(h-l+l'-h')\lambda} + f_{ij'}(\lambda) f_{i'j}(\lambda) e^{\boldsymbol{i}(l-h+l'-h')\lambda} \} d\lambda.
\end{aligned}
$$

Here, by Assumption 2, V is symmetric and then $(V^{lh})' = V^{hl}$. Therefore (16) is equal to

$$
(17)\qquad 4\pi \int_{-\pi}^{\pi} \operatorname{tr} \left\{ \sum_{h,l=1}^{p} e^{-(h-l)\lambda}\; V^{lh} \boldsymbol{f}(\lambda) \sum_{h',l'=1}^{p} e^{-(h'-l')\lambda}\; V^{l'h'} \boldsymbol{f}(\lambda) \right\} d\lambda.
$$

Therefore (17) can be expressed as

$$
(18)\qquad 4\pi \int_{-\pi}^{\pi} \operatorname{tr}\{[\{\{(G(\lambda) \otimes \boldsymbol{I}_m) V\} \circ (\boldsymbol{I}_p \otimes U_m)\} (U_p \otimes \boldsymbol{f}(\lambda))]^2\} d\lambda,
$$

which completes the proof of Theorem 1. $\square$

Next, we prove Theorem 2. To prove the theorem we need the following lemma.

Lemma 2. *Suppose that Assumption 1 holds. Then,*
(a)

$$
nE[(\bar{\boldsymbol{X}}_n - \boldsymbol{\mu})(\bar{\boldsymbol{X}}_n - \boldsymbol{\mu})'] = 2\pi \boldsymbol{f}(0) + o(1).
$$

(b)

$$
E\left(\frac{1}{n-k}\tilde{S}_n\right) = \left(1 + \frac{k-p+1}{n-k}\right)\Gamma - \frac{2\pi}{n-k}(U_p \otimes \boldsymbol{f}(0)) + o(n^{-1}).
$$

(c) *Denote the α-th component of $\bar{\boldsymbol{X}}_n$ by $\bar{X}_n^{\alpha}$. Then*

$$
\begin{aligned}
& \lim_{n\to\infty} n\mathrm{Cov}\left\{ \frac{1}{n}\sum_{t=p}^{n} (X_{\alpha_1}(t-j_1) - \bar{X}_n^{\alpha_1})(X_{\alpha_2}(t-j_2) - \bar{X}_n^{\alpha_2}), \right. \\
& \qquad\qquad \left. \frac{1}{n}\sum_{t=p}^{n} (X_{\alpha_3}(t-j_3) - \bar{X}_n^{\alpha_3})(X_{\alpha_4}(t-j_4) - \bar{X}_n^{\alpha_4}) \right\} \\
& \quad = W_{j_1,\ldots,j_4}^{\alpha_1,\ldots,\alpha_4}.
\end{aligned}
$$

(d)

$$
\frac{1}{\sqrt{n}}(\tilde{S}_n - n\Gamma) = O(\sqrt{\log n}),\ a.s.
$$

Proof of Lemma 2 (a) is due to [3] (p.208, Corollary 4).

(b) $\tilde{S}_n^{hl}$ denotes the (h,l)-th $m \times m$ block matrix of $\tilde{S}_n$. Then

$$
\begin{aligned}
(19)\qquad & \frac{1}{n-k}\tilde{S}_n^{hl} \\
&= \frac{1}{n-k}\sum_{t=p}^{n}(\boldsymbol{X}(t-h+1)-\bar{\boldsymbol{X}}_n)(\boldsymbol{X}(t-l+1)-\bar{\boldsymbol{X}}_n)' \\
&= \frac{1}{n-k}\sum_{t=p}^{n}(\boldsymbol{X}(t-h+1)-\boldsymbol{\mu}+\boldsymbol{\mu}-\bar{\boldsymbol{X}}_n)(\boldsymbol{X}(t-l+1)-\boldsymbol{\mu}+\boldsymbol{\mu}-\bar{\boldsymbol{X}}_n)' \\
&= \frac{1}{n-k}\sum_{t=p}^{n}(\boldsymbol{X}(t-h+1)-\boldsymbol{\mu})(\boldsymbol{X}(t-l+1)-\boldsymbol{\mu})' + \frac{n-p+1}{n-k}(\bar{\boldsymbol{X}}_n-\boldsymbol{\mu})(\bar{\boldsymbol{X}}_n-\boldsymbol{\mu})' \\
&\quad + \frac{1}{n-k}(\boldsymbol{\mu}-\bar{\boldsymbol{X}}_n)\sum_{t=p}^{n}(\boldsymbol{X}(t-l+1)-\boldsymbol{\mu})' + \frac{1}{n-k}\sum_{t=p}^{n}(\boldsymbol{X}(t-l+1)-\boldsymbol{\mu})(\boldsymbol{\mu}-\bar{\boldsymbol{X}}_n)' \\
&= \frac{1}{n-k}\sum_{t=p}^{n}(\boldsymbol{X}(t-h+1)-\boldsymbol{\mu})(\boldsymbol{X}(t-l+1)-\boldsymbol{\mu})' - \frac{n}{n-k}(\bar{\boldsymbol{X}}_n-\boldsymbol{\mu})(\bar{\boldsymbol{X}}_n-\boldsymbol{\mu})' \\
&\quad + \frac{1}{n-k}o_p(1).
\end{aligned}
$$

From (a), we obtain

$$
\begin{aligned}
E\left(\frac{1}{n-k}\tilde{S}_n^{hl}\right) &= \frac{n-p+1}{n-k}\boldsymbol{\gamma}(h-l) - \frac{1}{n-k}(2\pi\boldsymbol{f}(0)+o(1)) + \frac{1}{n-k}o(1) \\
&= \left(1+\frac{k-p+1}{n-k}\right)\boldsymbol{\gamma}(h-l) - \frac{2\pi}{n-k}\boldsymbol{f}(0) + o(n^{-1}).
\end{aligned}
$$

Then we get the result.

(c) From (19), Gaussianity of $\{\boldsymbol{X}_t\}$, and the properties of cumulant, we can show this lemma.

(d) Noting that Theorem 4.5.1 of Brillinger [1], we obtain

$$\sqrt{n}(\bar{\boldsymbol{X}}_n-\boldsymbol{\mu}) = O(\sqrt{\log n}) \quad a.s.$$

From Lemma 1 (b), we can see that (d) holds. □

Proof of Theorem 2 We can prove the theorem similarly to Theorem 1, except for the evaluation of

$$(20)\qquad -\frac{2n^2b}{n-k}\mathrm{tr}\left[\left\{E\left(\frac{1}{n-k}\tilde{S}_n\right)-\Gamma\right\}\Gamma^{-1}C\Gamma^{-1}\right]$$

in (21) of [9]. From Lemma 2 (b) it is seen that

$$(21)\qquad \lim_{n\to\infty}(20) = -2b[(k-p+1)\mathrm{tr}\{C\Gamma^{-1}\} - 2\pi\ \mathrm{tr}\{(U_p\otimes\boldsymbol{f}(0))\Gamma^{-1}C\Gamma^{-1}\}].$$

Therefore we obtain the Theorem 2. □

Acknowledgments The author thanks to Professor Masanobu Taniguchi for his comments and encouragements. The author also thanks to the referees for their understandings and constructive comments to improve this paper.

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A CHARACTERIZATION OF FREE LOCALLY CONVEX SPACES OVER METRIZABLE SPACES WHICH HAVE COUNTABLE TIGHTNESS

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Received July 8, 2014

Abstract. We prove that the free locally convex space $L(X)$ over a metrizable space X has countable tightness if and only if X is separable.

1 Introduction

A topological space X is called *first countable* if it has a countable open base at each point. Any first countable topological group is metrizable. Various topological properties generalizing first countability have been studied intensively by topologists and analysts, especially Fréchet-Urysohness, sequentiality, to be a k-space and countable tightness (see [5, 11]). It is well know that, metrizability $\Rightarrow$ Fréchet-Urysohness $\Rightarrow$ sequentiality $\Rightarrow$ countable tightness, and sequentiality $\Rightarrow$ to be a k-space. Although none of these implications is reversible, for many important classes of locally convex spaces (lcs for short) some of them can be reversed. Kąkol showed that for an (LM)-space (the inductive limit of a sequence of locally convex metrizable spaces), metrizability $\Leftrightarrow$ Fréchet-Urysohness. The Cascales and Orihuela result states that for an (LM)-space, sequentiality $\Leftrightarrow$ to be a k-space. Moreover, Kąkol and Saxon [12] proved the next structural theorem: An (LM)-space E is sequential (or a k-space) if and only if E is metrizable or is a Montel (DF)-space. Topological properties of a lcs E in the weak topology $\sigma(E, E')$ are of the importance and have been intensively studied from many years (see [11, 18]). Corson (1961) started a systematic study of certain topological properties of the weak topology of Banach spaces. If B is any infinite-dimensional Banach space, a classical result of Kaplansky states that $(E, \sigma(E, E'))$ has countable tightness (see [11]), but the weak dual $(E', \sigma(E', E))$ is not a k-space (see [12]). Note that there exists a (DF)-space with uncountable tightness whose weak topology has countable tightness [4]. We refer the reader to the book [11] for many references and facts.

In this paper we consider another class in the category **LCS** of locally convex spaces and continuous linear operators which is the most important from the categorical point of view, namely the class of free locally convex spaces over Tychonoff spaces introduced by Markov [15]. Recall that the *free locally convex space* $L(X)$ over a Tychonoff space X is a pair consisting of a locally convex space $L(X)$ and a continuous mapping $i : X \to L(X)$ such that every continuous mapping f from X to a locally convex space E gives rise to a unique continuous linear operator $\bar{f} : L(X) \to E$ with $f = \bar{f} \circ i$. The free locally convex space $L(X)$ always exists and is unique. The set X forms a Hamel basis for $L(X)$, and the mapping i is a topological embedding [19, 6, 7, 23]. It turns out (see [8]) that except for the trivial case when X is a countable discrete space, the free lcs $L(X)$ is never a k-space: For a Tychonoff space X, $L(X)$ is a k-space if and only if X is a countable discrete space.

The aforementioned results explain our interest to the following problem.

Question 1.1. *For which Tychonoff spaces X the free lcs $L(X)$ has countable tightness?*

2010 *Mathematics Subject Classification.* Primary 46A03; Secondary 54A25, 54D50.

Key words and phrases. Free locally convex space, free abelian topological group, countable Pytkeev network, the strong Pytkeev property, countable tightness.

We obtain a complete answer to Question 1.1 for the important case when X is metrizable. The following theorem is the main result of the article.

Theorem 1.2. *Let X be a metrizable space. Then the free lcs $L(X)$ has countable tightness if and only if X is separable.*

Below we prove even a stronger result (see Theorem 2.6).

2 Proof of Theorem 1.2 The free (resp. abelian) topological group $F(X)$ (resp. $A(X)$) over a Tychonoff space X were also introduced by Markov [15] and intensively studied over the last half-century (see [10, 13, 19, 21, 23]), we refer the reader to [2, Chapter 7] for basic definitions and results. We note that the topological groups $F(X)$ and $A(X)$ always exist and are essentially unique. Note also that the identity map $id_X : X \to X$ extends to a canonical homomorphism $id_{A(X)} : A(X) \to L(X)$ which is an embedding of topological groups [21, 24].

The space of all continuous functions on a topological space X endowed with the compact-open topology we denote by $C_c(X)$. It is well known that the space $L(X)$ admits a canonical continuous monomorphism $L(X) \to C_c(C_c(X))$. If X is a k-space, this monomorphism is an embedding of lcs [6, 7, 23]. So, for k-spaces, we obtain the next chain of topological embeddings:

$$A(X) \hookrightarrow L(X) \hookrightarrow C_c(C_c(X)). \tag{2.1}$$

Recall that a space X has *countable tightness* if whenever $x \in \overline{A}$ and $A \subseteq X$, then $x \in \overline{B}$ for some countable $B \subseteq A$. We use the following remarkable result of Arhangel'skii, Okunev and Pestov which shows that the topologies of $F(X)$ and $A(X)$ are rather complicated and unpleasant even for the simplest case of a metrizable space X.

Theorem 2.1 ([1]). *Let X be a metrizable space. Then:*

(i) *The tightness of $F(X)$ is countable if and only if X is separable or discrete.*

(ii) *The tightness of $A(X)$ is countable if and only if the set X' of all non-isolated points in X is separable.*

For the case X is discrete (hence metrizable) we have the following.

Theorem 2.2 ([8]). *For each uncountable discrete space D, the space $L(D)$ has uncountable tightness.*

Pytkeev [17] proved that every sequential space satisfies the property which is stronger than countable tightness. Following [14], we say that a topological space X has the *Pytkeev property at a point* $x \in X$ if for each $A \subseteq X$ with $x \in \overline{A} \setminus A$, there are infinite subsets $A_1, A_2, \dots$ of A such that each neighborhood of x contains some A_n. In [22] this property is strengthened as follows. A topological space X has the *strong Pytkeev property at a point* $x \in X$ if there exists a countable family $\mathcal{D}$ of subsets of X, which is called a *Pytkeev network at* x, such that for each neighborhood U of x and each $A \subseteq X$ with $x \in \overline{A} \setminus A$, there is $D \in \mathcal{D}$ such that $x \in D \subseteq U$ and $D \cap A$ is infinite. Following [3], a space X is called a *Pytkeev* $\aleph_0$*-space* if X is regular and has a countable family $\mathcal{D}$ which is a Pytkeev network at each point $x \in X$. The strong Pytkeev property for topological groups is thoroughly studied in [9], where, among others, it is proved that $A(X)$ and $L(X)$ have the strong Pytkeev property for each $\mathcal{MK}_\omega$-space X (i.e., X is the inductive limit of an increasing sequence of compact metrizable subspaces). Note also that in general (see [9]): Fréchet-Urysohness $\not\Leftarrow$ the strong Pytkeev property $\not\Leftarrow$ k-space.

Recall that a family $\mathcal{D}$ of subsets of a topological space X is called a *k-network* in X if, for every compact subset $K \subset X$ and each neighborhood U of K there exists a finite subfamily $\mathcal{F} \subset \mathcal{D}$ such that $K \subset \bigcup \mathcal{F} \subset U$. Following Michael [16], a topological space X is called an *$\aleph_0$-space* if it is regular and has a countable k-network. Every separable and metrizable space is a Pytkeev $\aleph_0$-space, and every Pytkeev $\aleph_0$-space is an $\aleph_0$-space [3]. We use the following strengthening of Michael's theorem [16] given by Banakh:

Theorem 2.3 ([3]). *If X is an $\aleph_0$-space, then $C_c(X)$ is a Pytkeev $\aleph_0$-space.*

The next theorem is an easy corollary of (2.1) and Theorem 2.3, the implication (iii)⇒(ii) was first observed by A. Leiderman (see [3, Theorem 3.12]).

Theorem 2.4. *For a k-space X the following assertions are equivalent:*

(i) *$A(X)$ is a Pytkeev $\aleph_0$-space.*

(ii) *$L(X)$ is a Pytkeev $\aleph_0$-space.*

(iii) *X is a Pytkeev $\aleph_0$-space.*

Proof. The implications (ii)⇒(i) and (i)⇒(iii) immediately follow from the fact that $A(X)$ is a subspace of $L(X)$ and X is a subspace of $A(X)$.

(iii)⇒(ii) If X is a Pytkeev $\aleph_0$-space, then $C_c(X)$ and $C_c(C_c(X))$ are Pytkeev $\aleph_0$-space by Theorem 2.3. As X is a k-space, $L(X)$ is a subspace of $C_c(C_c(X))$ by (2.1). So $L(X)$ is a Pytkeev $\aleph_0$-space. □

We need the next lemma (analogous results hold true also for $A(X)$ and $F(X)$).

Lemma 2.5. *If U is a clopen subset of a Tychonoff space X, then $L(U)$ embeds into $L(X)$ as a closed subspace.*

Proof. Denote by τ_U the topology of $L(U)$ and by $L(U)_a$ the underlying free vector space generated by U. Fix a point e belonging to U. Let $i : U \to X$ be the natural inclusion. By the definition of $L(U)$, i can be extended to a continuous inclusion $\widetilde{i} : L(U) \to L(X)$. So τ_U is stronger than the topology τ_U^X induced on $L(U)_a$ from $L(X)$. Define now $p : X \to U$ as follows: $p(x) = x$ if $x \in U$, and $p(x) = e$ if $x \in X \setminus U$. Clearly, p is continuous. By the definition of $L(X)$, p can be extended to a continuous linear mapping $\widetilde{p} : L(X) \to L(U)$. Since $p \circ i = \mathrm{id}_U$, we obtain $\widetilde{p} \circ \widetilde{i} = \mathrm{id}_{L(U)}$ and $\widetilde{p}$ is injective on $L(U)_a$. So τ_U^X is stronger than the topology τ_U. Thus $\tau_U^X = \tau_U$. Since U is a closed subset of X the subspace $L(U, X)$ of $L(X)$ generated by U is closed (we can repeat word for word the proof of Proposition 3.8 in [20]). Thus $\widetilde{i}$ is an embedding of $L(U)$ onto the closed subspace $L(U, X)$ of $L(X)$. □

Now the next theorem implies Theorem 1.2.

Theorem 2.6. *For a metrizable space X the following assertions are equivalent:*

(i) *$L(X)$ is a Pytkeev $\aleph_0$-space.*

(ii) *$L(X)$ has countable tightness.*

(iii) *X is separable.*

Proof. (i)⇒(ii) is clear. Let us prove (ii)⇒(iii). Since $A(X)$ is a subgroup of $L(X)$, we obtain that $A(X)$ also has countable tightness. Now Theorem 2.1 implies that the set X' of all non-isolated points of X is separable. So we have to show only that the set D of all isolated points of X is countable.

Suppose for a contradiction that D is uncountable. Then there is a positive number c and an uncountable subset D_0 of D such that $B_c(d) = \{d\}$ for every $d \in D_0$, where $B_c(d)$ is the c-ball centered at d. It is easy to see that D_0 is a clopen subset of X. So, by Lemma 2.5, $L(D_0)$ is a subspace of $L(X)$. Now Theorem 2.2 yields that $L(D_0)$ and hence $L(X)$ have uncountable tightness. This contradiction shows that D is countable. Thus X is separable.

(iii)⇒(i) immediately follows from Theorem 2.4. □

We do not know whether the assertions (i) and (ii) in Theorem 2.1 are equivalent respectively to the following: $F(X)$ is a Pytkeev $\aleph_0$-space for non-discrete X, and $A(X)$ has the strong Pytkeev property.

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A NOTE ON RATIONAL OPERATOR MONOTONE FUNCTIONS

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Received May 19, 2014 ; revised April 10, 2014

Abstract. Let f be oeprator monotone for some open interval I of $\mathbb{R}$. It is known that f has the analytic continuation on $\mathbb{H}_+ \cup I \cup \mathbb{H}_-$, where $\mathbb{H}_+$ (resp. $\mathbb{H}_-$) is the upper (resp. the lower) half plane of $\mathbb{C}$. In this note, we determine the form of rational operator monotone functions by using elementary argument, and prove the operator monotonicity of some meromorphic functions.

1 Introduction. We denote the set of all $n \times n$ matrices over $\mathbb{C}$ by M_n and set

$$H_n = \{A \in M_n \mid A^* = A\} \text{ and } H_n^+ = \{A \in H_n \mid A \geq 0\},$$

where $A \geq 0$ means that A is non-negative, that is, the value of inner product

$$(Ax, x) \geq 0 \quad \text{for all } x \in \mathbb{C}^n.$$

Let I be an open interval of the set $\mathbb{R}$ of real numbers. We also denote by $H_n(I)$ the set of $A \in H_n$ with its spectra $\mathrm{Sp}(A) \subset I$. A real continuous function f defined on the open interval I is said to be operator monotone if $A \leq B$ implies $f(A) \leq f(B)$ for any $n \in \mathbb{N}$ and $A, B \in H_n(I)$. In this note, we assume that an operator monotone function is not a constant function.

Let f be a real-valued continuous function on the interval I. We call f a Pick function if f has an analytic continuation on the upper half plane $\mathbb{H}_+ = \{z \in \mathbb{C} \mid \mathrm{Im}z > 0\}$ into itself. It also has an analytic continuation to the lower half plane $\mathbb{H}_-$, obtained by reflections across I.

We denote by $\mathbb{P}(I)$ the set of all Pick functions on I. It is well known that $f \in \mathbb{P}(I)$ is equivalent to that f is operator monotone on I ([1], [4], [5]).

We characterize the rational Pick function (rational operator monotone function) by an elementary method in Section 2 and give some examples using this characterization in Section 3.

2 Rational operator monotone functions. Let I be an open interval and $f(t) = \dfrac{at+b}{ct+d}$ $(a, b, c, d \in \mathbb{R},\ ad - bc \geq 0)$. It is well known that f is operator monotone on $(-\infty, -\frac{d}{c})$ or $(-\frac{d}{c}, +\infty)$ (see [1], [5]). So the following rational function is also operator monotone on I:

$$b_0 + a_0 t - \sum_{i=1}^{n} \frac{a_i}{t - \alpha_i},$$

where $b_0 \in \mathbb{R}$, $a_0, a_1, \ldots, a_n \geq 0$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R} \setminus I$.

2000 *Mathematics Subject Classification.* 47A63 .

Key words and phrases. Monotone matrix function, Operator monotone function, Pick function, Rational operator monotone function.

Let $g \in \mathbb{P}(I)$ be rational. Then there exists polynomials $p(t)$ and $q(t)$ with real coefficients such that

$$g(t) = \frac{p(t)}{q(t)} \qquad (t \in I),$$

where common devisors of $p(t)$ and $q(t)$ are only scalars and a coefficient of the highest degree term of $q(t)$ is 1. The polynomial $q(t)$ with real coefficients is represented as products of the following factors:

$$t-a, \quad t^2+at+b \qquad (a, b \in \mathbb{R}).$$

Since g has the analytic continuation to the upper half plane $\mathbb{H}_+$ and the lower half plane $\mathbb{H}_-$,

$$g(z) = \frac{p(z)}{q(z)} \qquad (z \in \mathbb{H}_+ \cup I \cup \mathbb{H}_+)$$

and g has no poles on $\mathbb{H}_+ \cup I$. So we may assume that $g(z)$ has the following form:

$$g(z) = \frac{p(z)}{(z-c_1)^{n(1)}(z-c_2)^{n(2)}\cdots(z-c_k)^{n(k)}},$$

where $c_1, c_2, \ldots, c_k \in \mathbb{R} \cap I^c$ and each $n(i)$ $(i = 1, 2, \ldots, k)$ is a positive integer with $n(1) + n(2) + \cdots + n(k) = \deg q(z)$. By the partially fractional decomposition of $g(z)$,

$$g(z) = r(z) + \sum_{i=1}^{k}\sum_{j=1}^{n(i)} \frac{b_{i,j}}{(z-c_i)^j},$$

where $r(z)$ is the remainder of $p(z)$ by $q(z)$ and $\{b_{i.j}\} \subset \mathbb{R}$.

Lemma 2.1. *In above setting, $g \in \mathbb{P}(I)$ satisfies the following conditions:*

(1) *There exist $r_0, r_1 \in \mathbb{R}$ such that $r_1 \geq 0$ and $r(z) = r_0 + r_1 z$.*

(2) *$n(i) = 1$ and $b_{i,1} \leq 0$ for all $i = 1, 2, \ldots, k$.*

Proof. (1) We set

$$r(z) = r_0 + r_1 z + \cdots + r_d z^d,$$

where $d = \deg r(z)$. Put

$$\theta = \begin{cases} \frac{3\pi}{2d} & \text{if } d \geq 2 \text{ and } r_d > 0 \\ \frac{\pi}{2d} & \text{if } d \geq 1 \text{ and } r_d < 0 \end{cases}.$$

For a sufficiently large $R > 0$ and $z = Re^{\theta\sqrt{-1}} \in \mathbb{H}_+$, we may assume that

$$|r_d|R^d = |r_d z^d| > |\sum_{i=0}^{d-1} r_i z^i + \sum_{i=1}^{k}\sum_{j=1}^{n(i)} \frac{b_{i,j}}{(z-c_i)^j}|.$$

Then we have

$$\begin{aligned}\mathrm{Im} g(z) &= \mathrm{Im}(-|r_d|R^d\sqrt{-1} + \sum_{i=0}^{d-1} r_i z^i + \sum_{i=1}^{k}\sum_{j=1}^{n(i)} \frac{b_{i,j}}{(z-c_i)^j}) \\ &\leq -|r_d|R^d + |\sum_{i=0}^{d-1} r_i z^i + \sum_{i=1}^{k}\sum_{j=1}^{n(i)} \frac{b_{i,j}}{(z-c_i)^j}| < 0.\end{aligned}$$

This contradicts to $g(z) \in \mathbb{H}_+$. So we have that $r(z) = r_0 + r_1 z$ and $r_1 \geq 0$.

(2) In a suitable neighborhood of c_i in $\mathbb{H}_+ \cup I \cup \mathbb{H}_-$, $g \in \mathbb{P}(I)$ has the form

$$g(z) = \frac{b_{i,1}}{z - c_i} + \cdots + \frac{b_{i,n(i)}}{(z - c_i)^{n(i)}} + h(z),$$

where $h(z)$ is holomorphic on the neighborhood of c_i. Put

$$\theta = \begin{cases} \dfrac{\pi}{2n(i)} & \text{if } n(i) \geq 2 \text{ and } b_{i,n(i)} > 0 \\ \dfrac{3\pi}{2n(i)} & \text{if } n(i) \geq 1 \text{ and } b_{i,n(i)} < 0 \end{cases}.$$

For a sufficiently small $r > 0$, $z = c_i + re^{\theta\sqrt{-1}} \in \mathbb{H}_+$ and we may assume that

$$\frac{|b_{i,n(i)}|}{r^{n(i)}} = |\frac{b_{i,n(i)}}{(z - c_i)^{n(i)}}| > |\sum_{j=1}^{n(i)-1} \frac{b_{i,j}}{(z - c_i)^j} + h(z)|.$$

Then we have

$$\begin{aligned} \mathrm{Im} g(z) &= \mathrm{Im}(-\frac{|b_{i,n(i)}|}{r^{n(i)}}\sqrt{-1} + \sum_{j=1}^{n(i)-1} \frac{b_{i,j}}{(z - c_i)^j} + h(z)) \\ &\leq -\frac{|b_{i,n(i)}|}{r^{n(i)}} + |\sum_{j=1}^{n(i)-1} \frac{b_{i,j}}{(z - c_i)^j} + h(z)| < 0. \end{aligned}$$

This contradicts to $g(z) \in \mathbb{H}_+$. So we have that $n(i) = 1$ and $b_{i,1} \leq 0$ for all $i = 1, 2, \ldots, k$. $\square$

We can now prove the following theorem:

Theorem 2.2. *The following are equivalent:*

(1) $f \in \mathbb{P}(I)$ *is rational.*

(2) *There exist* $b_0 \in \mathbb{R}$, *non-negative numbers* $a_0, a_1, \ldots, a_n$ *and real numbers* $\alpha_1, \alpha_2, \ldots, \alpha_n \notin I$ *such that*

$$f(t) = b_0 + a_0 t - \sum_{i=1}^{n} \frac{a_i}{t - \alpha_i}.$$

(3) *There exist* $a_0, c \geq 0$, $b_0 \in \mathbb{R}$, $\alpha_1, \alpha_2, \ldots, \alpha_n \notin I$ *and* $\beta_1, \beta_2, \ldots, \beta_{n-1} \in \mathbb{R}$ *satisfying that*

$$f(t) = b_0 + a_0 t - \frac{c(t - \beta_1)(t - \beta_2) \cdots (t - \beta_{n-1})}{(t - \alpha_1)(t - \alpha_2) \cdots (t - \alpha_n)}$$
$$\text{and } \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \beta_{n-1} < \alpha_n.$$

Proof. (1) $\Leftrightarrow$ (2) This is proved by Lemma 2.1.

(2) $\Rightarrow$ (3) We assume

$$f(t) = b_0 + a_0 t - \sum_{i=1}^{n} \frac{a_i}{t - \alpha_i},$$

where $b_0 \in \mathbb{R}$, $a_0, a_1, \ldots, a_n \geq 0$, $\alpha_1, \alpha_2, \ldots, \alpha_n \notin I$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. We define $g(t)$ as follows:

$$\sum_{i=1}^{n} \frac{a_i}{t-\alpha_i} = \frac{g(t)}{(t-\alpha_1)\cdots(t-\alpha_n)},$$

that is,

$$g(t) = \sum_{i=1}^{n} a_i(t-\alpha_1)\cdots(t-\alpha_{i-1})(t-\alpha_{i+1})\cdots(t-\alpha_n).$$

Since

$$g(\alpha_i) = (\alpha_i-\alpha_1)\cdots(\alpha_i-\alpha_{i-1})(\alpha_i-\alpha_{i+1})\cdots(\alpha_i-\alpha_n),$$

we have

$$\text{sign } g(\alpha_i) = (-1)^{n-i} \qquad (i = 1, 2, \ldots, n).$$

By the fact $\deg g(t) = n-1$ and the continuity of g, there exist a positive number c and $\beta_1, \beta_2, \ldots, \beta_{n-1}$ such that

$$g(t) = c(t-\beta_1)(t-\beta_2)\cdots(t-\beta_{n-1})$$

and $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \beta_{n-1} < \alpha_n$.

(3) $\Rightarrow$ (2) Set

$$g(t) = \frac{c(t-\beta_1)(t-\beta_2)\cdots(t-\beta_{n-1})}{(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_n)},$$

where $c > 0$, $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \beta_{n-1} < \alpha_n$. Then $g(t)$ has the following form:

$$g(t) = \sum_{i=1}^{n} \frac{b_i}{t-\alpha_i},$$

for some $b_i \in \mathbb{R} \setminus \{0\}$. It suffices to show that $b_i > 0$ for $i = 1, 2, \ldots, n-1$.

When we choose t such that $\beta_{i-1} < t < \alpha_i$ and $\alpha_i - t$ is sufficiently small, we have

$$\text{sign } g(t) = -\text{sign } b_i.$$

Because $\alpha_1 < \cdots < \beta_{i-1} < t < \alpha_i < \cdots < \alpha_n$,

$$\text{sign } g(t) = (-1)^{(n-1)-(i-1)+n-(i-1)} = -1.$$

So we have $b_i > 0$. $\square$

For a rational function $f(t)$, we can choose polynomials $p(t)$ and $q(t)$ such that

$$f(t) = \frac{p(t)}{q(t)}$$

and common devisors of $p(t)$ and $q(t)$ are only scalars. Then we call f of order n if

$$n = \max\{\deg p(t),\ \deg q(t)\}.$$

Corollary 2.3. *The followings are equivalent:*

(1) $f \in \mathbb{P}(I)$ *is rational of order* n.

(2) f *has one of the following forms:*

$$(a)\ f(t) = \frac{a(t-\beta_2)(t-\beta_3)\cdots(t-\beta_{n+1})}{(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_n)},$$
$$(b)\ f(t) = \frac{a(t-\beta_1)(t-\beta_2)\cdots(t-\beta_n)}{(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_{n-1})},$$
$$(c)\ f(t) = -\frac{a(t-\beta_1)(t-\beta_2)\cdots(t-\beta_n)}{(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_n)}$$

or

$$(d)\ f(t) = -\frac{a(t-\beta_2)(t-\beta_3)\cdots(t-\beta_n)}{(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_n)},$$

where $a > 0$, $\alpha_i \notin I$ *and*

$$\beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots < \alpha_n < \beta_{n+1}.$$

Proof. (1) $\Rightarrow$ (2) When $f(t)$ has the form

$$f(t) = b_0 + a_0 t - \sum_{i=1}^{n-1} \frac{a_i}{(t-\alpha_i)},$$

where $a_1, a_2, \dots, a_{n-1} > 0$. Since f is rational of order n, we have $a_0 > 0$. We set

$$\begin{aligned} g(t) &= (b_0 + a_0 t)(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_{n-1}) \\ &\quad - \sum_{i=1}^{n-1} a_i (t-\alpha_1)\cdots(t-\alpha_{i-1})(t-\alpha_{i+1})\cdots(t-\alpha_{n-1}), \end{aligned}$$

that is,

$$f(t) = \frac{g(t)}{(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_{n-1})}.$$

Then we have

$$\begin{gathered} \operatorname{sign}(\lim_{t\to\infty} g(t)) = 1, \quad \operatorname{sign} g(\alpha_{n-1}) = -1, \quad \operatorname{sign} g(\alpha_{n-2}) = 1, \\ \cdots, \quad \operatorname{sign} g(\alpha_1) = (-1)^{n-1}, \quad \operatorname{sign}(\lim_{t\to-\infty} g(t)) = (-1)^n. \end{gathered}$$

So f has the form (b).

When $f(t)$ has the form

$$f(t) = b_0 + a_0 t - \sum_{i=1}^{n} \frac{a_i}{(t-\alpha_i)},$$

where $a_1, a_2, \dots, a_n > 0$. Since f is rational of order n, we have $a_0 = 0$. We set

$$\begin{aligned} g(t) &= b_0(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_n) \\ &\quad - \sum_{i=1}^{n} a_i (t-\alpha_1)\cdots(t-\alpha_{i-1})(t-\alpha_{i+1})\cdots(t-\alpha_n), \end{aligned}$$

that is,

$$f(t) = \frac{g(t)}{(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_n)}.$$

Using the same argument as above, f has the form (d) if $b = 0$, the form (a) if $b > 0$ and the form (c) if $b < 0$.

(2) $\Rightarrow$ (1) When f has the form (a),(b),(c) or (d), f is rational of order n.

When f has the form (d), $f \in \mathbb{P}(I)$ by Theorem 2.2.

When f has the form (a), f is represented as the following form:

$$f(t) = \sum_{i=1}^{n} \frac{b_i}{t-\alpha_i} + a,$$

where $a > 0$ and some $b_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$. Since

$$\lim_{t\to\alpha_i+0} f(t) = \lim_{t\to\alpha_i+0} \frac{a(t-\beta_2)(t-\beta_3)\cdots(t-\beta_{n+1})}{(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_n)} = -\infty,$$

we get $b_i < 0$ from the fact

$$\lim_{t\to\alpha_i+0} \sum_{i=1}^{n} \frac{b_i}{t-\alpha_i} + a = -\infty.$$

So $f \in \mathbb{P}(I)$.

By the similar reason, $f \in \mathbb{P}(I)$ if f has the form (b) or (c). □

3 Examples. The following Example 3.1 has been announced by M. Uchiyama in many Conferences (cf. [7], [8]).

Example 3.1. *Let $\{p_n(x)\}$ be the orthogonal polynomials on a closed interval $[a, b]$ whose leading coefficient is positive. It is well known that the zeros $\{c_1, c_2, \ldots, c_n\}$ of $p_n(x)$ satisfies that*

$$a = c_0 < c_1 < c_2 \cdots < c_n < c_{n+1} = b,$$

and each interval (c_i, c_{i+1}) $(i = 0, 1, \ldots, n)$ contains exactly one zeros of $p_{n+1}(x)$ ([6]). So $p_{n+1}(x)/p_n(x)$ has the form (b) in Corollary 2.3. This means that $p_{n+1}(x)/p_n(x)$ is operator monotone on any interval which does not contain any zeros of $p_n(x)$.

Example 3.2. *Let $0 = a_0 < a_1 < a_2 < \cdots < a_{2n-1} < a_{2n} = \pi$. Then*

$$f(x) = \frac{\cos(x-a_1)\cos(x-a_3)\cdots\cos(x-a_{2n-1})}{\cos(x-a_0)\cos(x-a_2)\cdots\cos(x-a_{2n-2})}$$

is operator monotone on any interval I contained in $\mathbb{R} \setminus \{\frac{(2m+1)\pi}{2} + a_{2i} \mid m \in \mathbb{Z},\ i = 0, 1, \ldots, n-1\}$.

In particular, $\tan x$ is operator monotone on any interval contained in $\mathbb{R} \setminus \{m\pi - \frac{\pi}{2} \mid m \in \mathbb{Z}\}$ (when $n = 1$, $a_0 = 0$, $a_1 = \frac{\pi}{2}$).

Proof. The function $\cos x$ is represented by the infinite product as follows:

$$\cos x = \lim_{m\to\infty} f_m(x),$$

where

$$f_m(x) = \prod_{k=-m}^{m-1} (1 - \frac{2x}{(2k+1)\pi}).$$

Remarking the fact

$$f_m(x) = \frac{(-1)^m 2^{4m-2}((m-1)!)^2}{((2m-1)!)^2} \prod_{k=-m}^{m-1} (x - \frac{2k+1}{2}\pi),$$

we have that

$$\begin{aligned} g_m(x) &= \frac{f_m(x-a_1)f_m(x-a_3)\cdots f_m(x-a_{2n-1})}{f_m(x-a_0)f_m(x-a_2)\cdots f_m(x-a_{2n-2})} \\ &= \prod_{k=-m}^{m-1} \frac{(x-(\frac{(2k+1)\pi}{2}+a_1))(x-(\frac{(2k+1)\pi}{2}+a_3))\cdots(x-(\frac{(2k+1)\pi}{2}+a_{2n-1}))}{(x-(\frac{(2k+1)\pi}{2}+a_0))(x-(\frac{(2k+1)\pi}{2}+a_2))\cdots(x-(\frac{(2k+1)\pi}{2}+a_{2n-2}))} \end{aligned}$$

belongs to $\mathbb{P}(I)$ by Corollary 2.3. Since

$$f(x) = \lim_{m\to\infty} g_m(x),$$

$f(x)$ is operator monotone on I. □

Example 3.3. *Let $a_0 < a_1 < a_2 < \cdots < a_{2n-1} < a_0 + 1$ and $k(1), k(2), \ldots, k(n) \in \mathbb{Z}$. Then*

$$f(x) = \frac{\Gamma(x-a_0-k(1))\Gamma(x-a_2-k(2))\cdots\Gamma(x-a_{2n-2}-k(n))}{\Gamma(x-a_1-k(1))\Gamma(x-a_3-k(2))\cdots\Gamma(x-a_{2n-1}-k(n))}$$

is operator monotone on any interval I contained in $\mathbb{R}\backslash\{a_{2i-1}+k(i)-m \mid i=1,2,\ldots,n,\ m=0,1,2,\ldots\}$, where $\Gamma(x)$ is the Gamma function, i.e.,

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt \qquad (x>0).$$

Proof. We use Gauss's Formula of $\Gamma(x)$ as follows:

$$\Gamma(x) = \lim_{m\to\infty} g_m(x),$$

where $g_m(x) = \frac{m^x m!}{x(x+1)\cdots(x+m)}$ and the convergence is uniformly on any compact subset of $\mathbb{R}\setminus\{0,-1,-2,\ldots\}$ ([3]). For $a < b < a+1$,

$$\frac{g_m(x-a)}{g_m(x-b)} = m^{b-a}\frac{(x-b)(x-(b-1))\cdots(x-(b-m))}{(x-a)(x-(a-1))\cdots(x-(a-m))}$$

is operator monotone on any interval contained in $\mathbb{R}\setminus\{a, a-1, \ldots, a-m\}$ by Corollary 2.4. Then we have that

$$h_m(x) = \frac{g_m(x-a_0-k(1))g_m(x-a_2-k(2))\cdots g_m(x-a_{2n-2}-k(n))}{g_m(x-a_1-k(1))g_m(x-a_3-k(2))\cdots g_m(x-a_{2n-1}-k(n))}$$

also has the form (a) in Corollary 2.3, and is operator monotone on I. So is $f(x)$, because $f(x) = \lim_{m\to\infty} h_m(x)$. □

Acknowledgement. This work was partially supported by Grant-in-Aid for Scientific Research (C)22540220.

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Communicated by *Hiroyuki Osaka*

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ON RELATIONS BETWEEN OPERATOR VALUED α-DIVERGENCE AND RELATIVE OPERATOR ENTROPIES

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Received March 5, 2014 ; revised June 20, 2014

Abstract. Let A and B be two strictly positive operators, and $\alpha \in (0,1)$. The operator valued α-divergence is defined by

$$D_\alpha(A|B) \equiv \frac{1}{\alpha(1-\alpha)}\left(A \nabla_\alpha B - A \sharp_\alpha B\right),$$

where $A \nabla_\alpha B = (1-\alpha)A + \alpha B$ and $A \sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$. In this paper, firstly, we show some fundamental relations between operator valued α-divergence and relative operator entropies (relative operator entropy, Tsallis relative operator entropy etc.). Next, we introduce noncommutative ratio $(A \natural_{u+v} B)(A \natural_u B)^{-1}$ on the path $A \natural_w B$, and we discuss noncommutative ratio translation. Moreover, we discuss α-divergence for operator distributions.

1 Introduction. Throughout this paper, an operator means a bounded linear operator on a Hilbert space H. An operator T on H is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and an operator T is said to be strictly positive (denoted by $T > 0$) if T is invertible and positive.

A relative operator entropy is introduced by Fujii and Kamei [3] as follows: For strictly positive operators A and B,

$$S(A|B) \equiv A^{\frac{1}{2}} \log\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) A^{\frac{1}{2}}.$$

Moreover, for $u \in \mathbf{R}$, Furuta [8] introduced

$$S_u(A|B) \equiv A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^u \log\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) A^{\frac{1}{2}}$$

as an extension of $S(A|B)$, and Yanagi, Kuriyama and Furuichi [16] call it generalized relative operator entropy.

For $w \in \mathbf{R}$, we consider a path $A \natural_w B$ through A and B defined by [4], [5], [12] etc.:

$$A \natural_w B \equiv A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^w A^{\frac{1}{2}}.$$

A path through A and B is an extended notion of weighted geometric mean $A \sharp_\alpha B \equiv A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$ defined for $\alpha \in [0,1]$. $S_u(A|B)$ can be regarded as a tangent vector at u on the path, and from this viewpoint, we showed several relations between $S(A|B)$ and $S_u(A|B)$ in [9].

Yanagi, Kuriyama and Furuichi [16] introduced Tsallis relative operator entropy as follows:

2010 *Mathematics Subject Classification.* 47A63, 47A64 and 94A17.

Key words and phrases. operator divergence, operator valued α-divergence, relative operator entropy, Tsallis relative operator entropy, operator mean.

For strictly positive operators A and B,

$$T_\alpha(A|B) \equiv \frac{A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\alpha A^{\frac{1}{2}} - A}{\alpha} = \frac{A \sharp_\alpha B - A}{\alpha}, \ \alpha \in (0,1].$$

Since $\lim_{\alpha\to 0}\frac{x^\alpha - 1}{\alpha} = \log x$ holds for $x > 0$, we have $T_0(A|B) \equiv \lim_{\alpha\to 0} T_\alpha(A|B) = S(A|B)$. Tsallis relative operator entropy can be extended as the notion for $\alpha \in \mathbf{R}$. In [9], we showed the following essential relation between relative operator entropies: For strictly positive operators A and B, and for $\alpha \in (0,1)$,

$$(*) \qquad S(A|B) \le T_\alpha(A|B) \le S_\alpha(A|B) \le -T_{1-\alpha}(B|A) \le -S(B|A) = S_1(A|B).$$

In the information geometry, α-divergence defined by Amari [1] plays an important role as a notion to measure the difference between two probability distributions. Fujii [2] defined an operator version of α-divergence as follows: For strictly positive operators A and B, and for $\alpha \in (0,1)$,

$$D_\alpha(A|B) \equiv \frac{1}{\alpha(1-\alpha)}\left(A \nabla_\alpha B - A \sharp_\alpha B\right),$$

where $A \nabla_\alpha B \equiv (1-\alpha)A + \alpha B$ is weighted arithmetic mean. In section 2, we show some fundamental relations between operator valued α-divergences and relative operator entropies.

In section 3, we show the following equality for u, $v \in \mathbf{R}$:

$$(\diamondsuit) \qquad (A \natural_{u+v} B)(A \natural_u B)^{-1} S_u(A|B) = S_{u+v}(A|B).$$

We call $(A \natural_{u+v} B)(A \natural_u B)^{-1}$ *noncommutative ratio* on the path $A \natural_w B$, and show a preservation on this ratio. We call to multiply $S_u(A|B)$ by $(A \natural_{u+v} B)(A \natural_u B)^{-1}$ like the equality $(\diamondsuit)$ *noncommutative ratio translation* for generalized relative operator entropy. Applying noncommutative ratio translation to fundamental relations between operator valued α-divergences and relative operator entropies shown in section 2, we get similar results to the waving property in [9].

For discrete (positive) probability distributions $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n)$, Shannon inequality $0 \ge \sum_{i=1}^n p_i \log \dfrac{q_i}{p_i}$ holds. Furuta [8] showed operator Shannon inequality, that is, $0 \ge \sum_{i=1}^n S(A_i|B_i)$ for A_i, $B_i > 0$ $(1 \le i \le n)$ with $\sum_{i=1}^n A_i = \sum_{i=1}^n B_i = I$. We call an operator sequence $\mathbb{A} = (A_1, A_2, \cdots, A_n)$ *an operator distribution* if $A_i > 0$ $(1 \le i \le n)$ and $\sum_{i=1}^n A_i = I$, since it can be regarded as an operator version of discrete probability distribution.

Let $\mathbb{A} = (A_1, A_2, \cdots, A_n)$ and $\mathbb{B} = (B_1, B_2, \cdots, B_n)$ be operator distributions, and $\alpha \in (0,1)$. In [9] and [10], we introduced relative operator entropy for operator distributions $S(\mathbb{A}|\mathbb{B})$, Tsallis relative entropy for operator distributions $T_\alpha(\mathbb{A}|\mathbb{B})$, and generalized relative entropy for operator distributions $S_\alpha(\mathbb{A}|\mathbb{B})$ as follows:

$$S(\mathbb{A}|\mathbb{B}) = \sum_{i=1}^n S(A_i|B_i), \ T_\alpha(\mathbb{A}|\mathbb{B}) = \sum_{i=1}^n T_\alpha(A_i|B_i), \ S_\alpha(\mathbb{A}|\mathbb{B}) = \sum_{i=1}^n S_\alpha(A_i|B_i).$$

Yanagi, Kuriyama and Furuichi [16] improved the operator Shannon inequality:

$$0 \ge T_\alpha(\mathbb{A}|\mathbb{B}) \ge S(\mathbb{A}|\mathbb{B}), \ \alpha \in (0,1).$$

From the viewpoint of this improvement of Shannon inequality, in [9], we got

$$S(\mathbb{A}|\mathbb{B}) \le T_\alpha(\mathbb{A}|\mathbb{B}) \le S_\alpha(\mathbb{A}|\mathbb{B}) \le -T_{1-\alpha}(\mathbb{B}|\mathbb{A}) \le -S(\mathbb{B}|\mathbb{A}) = S_1(\mathbb{A}|\mathbb{B})$$

by $(*)$ and showed related inequalities. Moreover, in [10], we discussed generalizations of these inequalities. In section 4, we define α-divergence for operator distributions, and show its fundamental properties.

2 Operator valued α-divergence and fundamental properties. Amari [1] defined α-divergence as a notion to measure the difference between two probability distributions as follows: For two discrete probability distributions $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n)$, that is, $p_i, q_i > 0$ $(1 \le i \le n)$ and $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$, and for $\alpha \in \mathbf{R}$,

$$D_\alpha[p : q] \equiv \frac{4}{1-\alpha^2}\left(1 - \sum_{i=1}^n p_i^{\frac{1-\alpha}{2}} q_i^{\frac{1+\alpha}{2}}\right), \ \alpha \ne \pm 1.$$

If $\alpha = -1$, then $D_{-1}[p : q] \equiv \lim_{\alpha \to -1} D_\alpha[p : q] = \sum_{i=1}^n p_i \log \frac{p_i}{q_i}$, and if $\alpha = 1$, then $D_1[p : q] \equiv \lim_{\alpha \to 1} D_\alpha[p : q] = D_{-1}[q : p]$. We call this quantity $D_{-1}[p : q]$ the relative entropy (Kullback-Leibler divergence, Kullback-Leibler distance), and denote it by $D_{KL}(p\,|\,q)$ ([13], [14]). If we put $t = \frac{1+\alpha}{2}$, then α-divergence can be expressed as follows:

$$D_t(p\,|\,q) \equiv D_{2t-1}[p : q] = \frac{1}{t(1-t)} \sum_{i=1}^n \left\{(1-t)p_i + tq_i - p_i^{1-t} q_i^t\right\}, \ t \ne 0, 1.$$

Based on this expression, Fujii [2] defined an operator valued α-divergence as follows.

Definition 2.1. For strictly positive operators A and B, and for $\alpha \in (0,1)$, operator valued α-divergence is defined as follows ([2], [6], [7]):

$$D_\alpha(A|B) \equiv \frac{1}{\alpha(1-\alpha)}\left(A \nabla_\alpha B - A \sharp_\alpha B\right),$$

where $A \nabla_\alpha B \equiv (1-\alpha)A + \alpha B$ and $A \sharp_\alpha B \equiv A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$.

In this section, we show some fundamental properties of operator valued α-divergences. Petz [15] introduced the operator divergence $D_{FK}(A|B) \equiv B - A - S(A|B)$. Fujii et al. showed the following relation between $D_{FK}(A|B)$ and operator valued α-divergences at end points for interval $(0,1)$.

Proposition 2.2. (Fujii-Mićić-Pečarić-Seo, [6], [7]) *Let A and B be strictly positive operators. Then,*

(1) $$D_0(A|B) \equiv \lim_{\alpha \to 0} D_\alpha(A|B) = D_{FK}(A|B) = B - A - S(A|B),$$

(2) $$D_1(A|B) \equiv \lim_{\alpha \to 1} D_\alpha(A|B) = D_{FK}(B|A) = A - B + S_1(A|B)$$

hold.

The following (1) in Proposition 2.3 interpolates (1) and (2) in Proposition 2.2 since $T_0(A|B) = S(A|B)$ and $-S(B|A) = S_1(A|B)$ by $(*)$.

Proposition 2.3. *Let A and B be strictly positive operators. Then,*

(1) $$D_\alpha(A|B) = \frac{1}{1-\alpha}(B - A - T_\alpha(A|B)) = \frac{1}{\alpha}(A - B - T_{1-\alpha}(B|A)), \ \textit{for } \alpha \in (0,1),$$

(2) $$D_{1-\alpha}(B|A) = D_\alpha(A|B), \ \textit{for } \alpha \in [0,1]$$

hold.

Proof. **(1)** This can be shown as follows:

$$\begin{aligned}(1-\alpha)D_\alpha(A|B) &= \frac{A \nabla_\alpha B - A \sharp_\alpha B}{\alpha} = \frac{A \nabla_\alpha B - A}{\alpha} - \frac{A \sharp_\alpha B - A}{\alpha} \\ &= B - A - T_\alpha(A|B), \\ \alpha D_\alpha(A|B) &= \frac{A \nabla_\alpha B - A \sharp_\alpha B}{1-\alpha} = \frac{A \nabla_\alpha B - B}{1-\alpha} - \frac{A \sharp_\alpha B - B}{1-\alpha} \\ &= A - B - \frac{B \sharp_{1-\alpha} A - B}{1-\alpha} = A - B - T_{1-\alpha}(B|A).\end{aligned}$$

(2) For $\alpha \in (0,1)$,

$$D_{1-\alpha}(B|A) = \frac{B \nabla_{1-\alpha} A - B \sharp_{1-\alpha} A}{(1-\alpha)\{1-(1-\alpha)\}} = \frac{A \nabla_\alpha B - A \sharp_\alpha B}{(1-\alpha)\alpha} = D_\alpha(A|B)$$

holds. In case of $\alpha = 0$ or $\alpha = 1$, this can be obtained by Proposition 2.2 and the relation $-S(B|A) = S_1(A|B)$ in $(*)$.

□

The following result gives bounds of operator value $D_\alpha(A|B)$.

Theorem 2.4. *Let A and B be strictly positive operators. Then,*

$$0 \le D_\alpha(A|B) \le \frac{1}{1-\alpha} D_0(A|B), \tag{1}$$

$$0 \le D_\alpha(A|B) \le \frac{1}{\alpha} D_1(A|B) \tag{2}$$

hold for $\alpha \in (0,1)$.

Proof. Since $A \nabla_\alpha B \ge A \sharp_\alpha B$ for any $\alpha \in (0,1)$, $D_\alpha(A|B) \ge 0$ holds. Moreover, by $(*)$ and (1) in Proposition 2.3, we have

$$\begin{aligned}D_\alpha(A|B) &= \frac{1}{1-\alpha}(B - A - T_\alpha(A|B)) \le \frac{1}{1-\alpha}(B - A - S(A|B)) = \frac{1}{1-\alpha} D_0(A|B), \\ D_\alpha(A|B) &= \frac{1}{\alpha}(A - B - T_{1-\alpha}(B|A)) \le \frac{1}{\alpha}(A - B + S_1(A|B)) = \frac{1}{\alpha} D_1(A|B).\end{aligned}$$

□

By the following Theorem 2.5, it is shown that an operator value $D_\alpha(A|B)$ can be represented by the sum of two operator values for Tsallis entropies.

Theorem 2.5. *Let A and B be strictly positive operators. Then,*

$$D_\alpha(A|B) = -\{T_\alpha(A|B) + T_{1-\alpha}(B|A)\}$$

holds for $\alpha \in (0,1)$.

Proof. This theorem can be shown as follows:

$$\begin{aligned}D_\alpha(A|B) &= \frac{A \nabla_\alpha B - A \sharp_\alpha B}{\alpha(1-\alpha)} \\ &= \frac{\{(1-\alpha)A + \alpha B\} - \{(1-\alpha)(A \sharp_\alpha B) + \alpha(A \sharp_\alpha B)\}}{\alpha(1-\alpha)} \\ &= -\left\{\frac{(1-\alpha)(A \sharp_\alpha B) - (1-\alpha)A}{\alpha(1-\alpha)} + \frac{\alpha(B \sharp_{1-\alpha} A) - \alpha B}{\alpha(1-\alpha)}\right\} \\ &= -\left\{\frac{A \sharp_\alpha B - A}{\alpha} + \frac{B \sharp_{1-\alpha} A - B}{1-\alpha}\right\} = -\{T_\alpha(A|B) + T_{1-\alpha}(B|A)\}.\end{aligned}$$

□

Theorem 2.5 gives a new viewpoint for operator valued α-divergence. Tsallis relative operator entropy $T_\alpha(A|B)$ can be regarded as the slope of the line through points A and $A \sharp_\alpha B$. Since $-T_{1-\alpha}(B|A) = -\frac{B \sharp_{1-\alpha} A - B}{1-\alpha} = \frac{B - A \sharp_\alpha B}{1-\alpha}$, we can regard this operator value as the slope of the line through points $A \sharp_\alpha B$ and B. Therefore, we can regard $D_\alpha(A|B)$ as the difference between the slops of these two lines. We illustrate the quantity corresponding to $D_\alpha(A|B)$ by bold straight line in Figure 1.

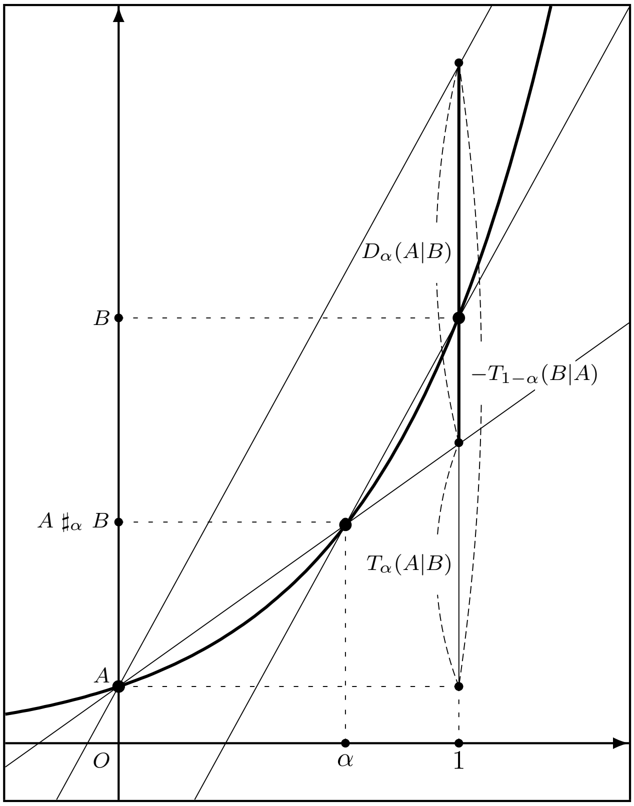

Figure 1: $D_\alpha(A|B) = -T_{1-\alpha}(B|A) - T_\alpha(A|B)$.

The following result can be shown by Theorem 2.5 and $(*)$ easily.

Corollary 2.6. *Let A and B be strictly positive operators. Then,*

$$D_\alpha(A|B) \leq S_1(A|B) - S(A|B)$$

holds for $\alpha \in (0, 1)$.

3 Noncommutative ratio translation on the path. First, we show the following result on translation of generalized relative operator entropies.

Proposition 3.1. *Let A and B be strictly positive operators. Then,*

$$(A \natural_{u+v} B)(A \natural_u B)^{-1} S_u(A|B) = S_{u+v}(A|B)$$

holds for u, $v \in \mathbf{R}$.

Proof. This can be shown as follows:

$$\begin{aligned}
& (A \natural_{u+v} B)(A \natural_u B)^{-1} S_u(A|B) \\
= \;& A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{u+v} A^{\frac{1}{2}} A^{-\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{-u} A^{-\frac{1}{2}} \\
& \qquad \times A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{u} \log \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}} \\
= \;& A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{u+v} \log \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}} \\
= \;& S_{u+v}(A|B).
\end{aligned}$$

□

We can regard $S_u(A|B)$ and $S_{u+v}(A|B)$ as tangent vectors at u and $u+v$ on the path $A \natural_w B$, respectively. Then, Proposition 3.1 means that $S_{u+v}(A|B)$ is parallelly transferring $S_u(A|B)$ by v along the path.

Here, we define the following noncommutative ratio on the path $A \natural_w B$, and give a new viewpoint for the equality in Proposition 3.1.

Definition 3.2. For strictly positive operators A and B, and for $u, v \in \mathbf{R}$, *noncommutative ratio* on the path $A \natural_w B$ is defined as follows:

$$\mathcal{R}(u, v; A, B) \equiv (A \natural_{u+v} B)(A \natural_u B)^{-1}.$$

We have the following property of noncommutative ratio.

Proposition 3.3. *Let A and B be strictly positive operators. Then,*

$$(A \natural_{u+v} B)(A \natural_u B)^{-1} = (A \natural_v B)A^{-1},$$

that is,

$$\mathcal{R}(u, v; A, B) = \mathcal{R}(0, v; A, B) = (A \natural_v B)A^{-1}$$

holds for u, $v \in \mathbf{R}$.

Proof. This can be shown as follows:

$$\begin{aligned}
\mathcal{R}(u, v; A, B) &= (A \natural_{u+v} B)(A \natural_u B)^{-1} \\
&= A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{u+v} A^{\frac{1}{2}} A^{-\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{-u} A^{-\frac{1}{2}} \\
&= A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{v} A^{-\frac{1}{2}} \\
&= (A \natural_v B)A^{-1} \\
&= \mathcal{R}(0, v; A, B).
\end{aligned}$$

□

By Proposition 3.3, $\mathcal{R}(u, v; A, B)$ does not depend on u. So, we denote $\mathcal{R}(u, v; A, B)$ by $\mathcal{R}(v; A, B)$, or simply $\mathcal{R}(v)$ in the rest of this section. We call multiplying by $\mathcal{R}(v)$ from the left side *noncommutative ratio translation*.

From Proposition 3.1 and Definition 3.2, we get the following immediately.

Corollary 3.4. *Let A and B be strictly positive operators. Then,*

$$\mathcal{R}(v)S_u(A|B) = S_{u+v}(A|B)$$

hold for $u,\ v \in \mathbf{R}$.

In particular, by putting $u = 0$ in Corollary 3.4 , we have

$$\mathcal{R}(v)S(A|B) = S_v(A|B).$$

Tsallis relative operator entropy can be extended as follows: For strictly positive operators A and B, and for $u \in \mathbf{R}$,

$$T_u(A|B) \equiv \frac{A \natural_u B - A}{u}.$$

From above definition and Proposition 3.3, we have

$$\mathcal{R}(v)T_u(A|B) = \frac{A \natural_{u+v} B - A \natural_v B}{u}.$$

Let n be an integer. Then, $\mathcal{R}(n) = (A \natural_n B)A^{-1} = (BA^{-1})^n$ holds. In [9], we showed a similar relation to $(*)$ as follows: For strictly positive operators A, B and $u \in (n, n+1)$,

$$(\star) \quad S_n(A|B) \le \frac{A \natural_u B - A \natural_n B}{u-n} \le S_u(A|B) \le \frac{A \natural_{n+1} B - A \natural_u B}{n+1-u} \le S_{n+1}(A|B),$$

or equivalently,

$$\begin{aligned}(\star\star) \qquad (BA^{-1})^n S(A|B) &\le (BA^{-1})^n T_{u-n}(A|B) \le (BA^{-1})^n S_{u-n}(A|B) \\ &\le -(BA^{-1})^n T_{n+1-u}(B|A) \le (BA^{-1})^n S_1(A|B).\end{aligned}$$

The relation $(\star)$ can be expressed by $(\star\star)$ which is the transferred form of $(*)$ by n along the path. We call this the waving property in [9].

The relation $(\star\star)$ can be generalized as follows:

Corollary 3.5. *Let A and B be strictly positive operators and $u \in (v, v+1)$. Then,*

$$\begin{aligned}S_v(A|B) = \mathcal{R}(v)S(A|B) &\le \mathcal{R}(v)T_{u-v}(A|B) \le S_u(A|B) \\ \le -\mathcal{R}(v)T_{v+1-u}(B|A) &= \mathcal{R}(u)T_{v+1-u}(A|B) \le S_{v+1}(A|B)\end{aligned}$$

hold for $u,\ v \in \mathbf{R}$.

Proof. We only show the relation $-\mathcal{R}(v)T_{v+1-u}(B|A) = \mathcal{R}(u)T_{v+1-u}(A|B)$ since the others can be obtained by the similar way to the proof in [9].

By Proposition 3.3, we have

$$\begin{aligned}\mathcal{R}(v)T_{v+1-u}(B|A) &= (A \natural_v B)A^{-1}T_{v+1-u}(B|A) = (A \natural_v B)A^{-1}\frac{B \natural_{v+1-u} A - B}{v+1-u} \\ &= \frac{(A \natural_v B)A^{-1}(A \natural_{u-v} B) - (A \natural_v B)A^{-1}B}{v+1-u} \\ &= \frac{A \natural_u B - A \natural_{v+1} B}{v+1-u} \\ &= (A \natural_u B)A^{-1}\frac{A - A \natural_{v+1-u} B}{v+1-u} \\ &= -\mathcal{R}(u)T_{v+1-u}(A|B).\end{aligned}$$

□

We apply noncommutative ratio translation to fundamental relations shown in section 2, and try to show the similar property to the waving property. To see this, we make some preparations.

Lemma 3.6. *Let A and B be strictly positive operators. Then,*

$$(A \natural_u B) \natural_w (A \natural_{u+v} B) = A \natural_{u+vw} B$$

holds for u, v, $w \in \mathbf{R}$.

Proof. By Lemma 4.2 in [11], $T^*(X \natural_u Y)T = (T^*XT) \natural_u (T^*YT)$ holds for any invertible operator T, for any positive invertible operators X, Y and for $u \in \mathbf{R}$. Therefore, we have

$$\begin{aligned}(A \natural_u B) \natural_w (A \natural_{u+v} B) &= \left\{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^u A^{\frac{1}{2}}\right\} \natural_w \left\{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{u+v} A^{\frac{1}{2}}\right\} \\ &= A^{\frac{1}{2}}\left\{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^u \natural_w (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{u+v}\right\}A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{u+vw} A^{\frac{1}{2}} \\ &= A \natural_{u+vw} B.\end{aligned}$$

□

In [12], Kamei showed some kind of the additivity for entropy

$$S(A|A \sharp_t B) = tS(A|B)$$

for $t \in [0, 1]$. The following is an extension of this result.

Proposition 3.7. *Let A and B be strictly positive operators. Then,*

$$S_u(A \natural_v B|A \natural_{v+w} B) = wS_{v+uw}(A|B)$$

holds for u, v, $w \in \mathbf{R}$.

Proof. Since $\lim_{t\to 0}\frac{x^{u+t}-x^u}{t} = x^u \lim_{t\to 0}\frac{x^t-1}{t} = x^u \log x$ holds for $x > 0$, we have

$$\lim_{t\to 0}\frac{X \natural_{u+t} Y - X \natural_u Y}{t} = S_u(X|Y)$$

for strictly positive operators X, Y and $u \in \mathbf{R}$. Therefore, by Lemma 3.6, we get

$$\begin{aligned}S_u(A \natural_v B|A \natural_{v+w} B) &= \lim_{t\to 0}\frac{(A \natural_v B) \natural_{u+t} (A \natural_{v+w} B) - (A \natural_v B) \natural_u (A \natural_{v+w} B)}{t} \\ &= w \lim_{wt\to 0}\frac{A \natural_{v+uw+wt} B - A \natural_{v+uw} B}{wt} \quad = \quad wS_{v+uw}(A|B).\end{aligned}$$

□

We give the special cases of Proposition 3.7 which are useful in our calculations.

Corollary 3.8. *Let A and B be strictly positive operators. Then,*

$$\begin{array}{llcl} (1) & S(A \natural_v B | A \natural_{v+w} B) & = & wS_v(A|B), \\ (2) & S_u(A \natural_v B | A \natural_{v+1} B) & = & S_{u+v}(A|B) \end{array}$$

hold for v, $w \in \mathbf{R}$.

For the following two operator values which appear in $(\star)$,

$$\begin{aligned} \frac{A \natural_u B - A \natural_n B}{u-n} &= \frac{(A \natural_n B) \sharp_{u-n} (A \natural_{n+1} B) - A \natural_n B}{u-n} \\ &= T_{u-n}(A \natural_n B | A \natural_{n+1} B), \\ \frac{A \natural_{u+1} B - A \natural_u B}{n+1-u} &= -\frac{(A \natural_n B) \sharp_{u-n} (A \natural_{n+1} B) - A \natural_{n+1} B}{n+1-u} \\ &= -\frac{(A \natural_{n+1} B) \sharp_{1-(u-n)} (A \natural_n B) - A \natural_{n+1} B}{1-(u-n)} \\ &= -T_{1-(u-n)}(A \natural_{n+1} B | A \natural_n B) \end{aligned}$$

hold.

From these facts and (2) in Corollary 3.8, the relation $(\star)$ is equivalent to the following:

$$\begin{aligned} S(A \natural_n B | A \natural_{n+1} B) \leq T_{u-n}(A \natural_n B | A \natural_{n+1} B) \leq S_{u-n}(A \natural_n B | A \natural_{n+1} B) \\ \leq -T_{1-(u-n)}(A \natural_n B | A \natural_{n+1} B) \leq S_1(A \natural_n B | A \natural_{n+1} B). \end{aligned}$$

We show the similar phenomena for each operator value $S_u(A|B)$, $T_u(A|B)$, and $D_\alpha(A|B)$.

Theorem 3.9. *Let A and B be strictly positive operators. Then,*

$$\begin{array}{ll} (1) & \mathcal{R}(v)S_u(A|B) = S_u(A \natural_v B | A \natural_{v+1} B), \\ (2) & \mathcal{R}(v)T_u(A|B) = T_u(A \natural_v B | A \natural_{v+1} B) \end{array}$$

hold for u, $v \in \mathbf{R}$.

In particular, by putting $u = 0$ in Theorem 3.9, we have

$$\mathcal{R}(v)S(A|B) = S(A \natural_v B | A \natural_{v+1} B).$$

Proof. **(1)** By Corollary 3.4 and (2) in Corollary 3.8, we have

$$\mathcal{R}(v)S_u(A|B) = S_{u+v}(A|B) = S_u(A \natural_v B | A \natural_{v+1} B).$$

(2) By Proposition 3.3 and Lemma 3.6, we get

$$\begin{aligned} \mathcal{R}(v)T_u(A|B) &= (A \natural_v B)A^{-1}T_u(A|B) \\ &= \frac{(A \natural_v B)A^{-1}(A \natural_u B) - (A \natural_v B)A^{-1}A}{u} \\ &= \frac{A \natural_{u+v} B - A \natural_v B}{u} \\ &= \frac{(A \natural_v B) \natural_u (A \natural_{v+1} B) - A \natural_v B}{u} \\ &= T_u(A \natural_v B | A \natural_{v+1} B). \end{aligned}$$

$\square$

Theorem 3.10. *Let A and B be strictly positive operators. Then,*

$$\mathcal{R}(v)D_\alpha(A|B) = D_\alpha(A \natural_v B|A \natural_{v+1} B)$$

holds for $\alpha \in (0,1)$ and $v \in \mathbf{R}$.

Proof. By Proposition 3.3 and Lemma 3.6, we have

$$\begin{aligned}
\mathcal{R}(v)D_\alpha(A|B) &= (A \natural_v B)A^{-1}D_\alpha(A|B) \\
&= (A \natural_v B)A^{-1}\frac{A \nabla_\alpha B - A \sharp_\alpha B}{\alpha(1-\alpha)} \\
&= \frac{(1-\alpha)(A \natural_v B)A^{-1}A + \alpha(A \natural_v B)A^{-1}B - (A \natural_v B)A^{-1}(A \sharp_\alpha B)}{\alpha(1-\alpha)} \\
&= \frac{(1-\alpha)(A \natural_v B) + \alpha(A \natural_{v+1} B) - A \natural_{v+\alpha} B}{\alpha(1-\alpha)} \\
&= \frac{(A \natural_v B) \nabla_\alpha (A \natural_{v+1} B) - (A \natural_v B) \sharp_\alpha (A \natural_{v+1} B)}{\alpha(1-\alpha)} \\
&= D_\alpha(A \natural_v B|A \natural_{v+1} B).
\end{aligned}$$

□

Theorem 3.9 can be generalized as follows.

Theorem 3.11. *Let A and B be strictly positive operators. Then,*

(1) $$w\mathcal{R}(v)S_{uw}(A|B) = S_u(A \natural_v B|A \natural_{v+w} B),$$

(2) $$w\mathcal{R}(v)T_{uw}(A|B) = T_u(A \natural_v B|A \natural_{v+w} B)$$

hold for u, v, $w \in \mathbf{R}$.

Proof. **(1)** By Corollary 3.4 and Proposition 3.7, we have

$$w\mathcal{R}(v)S_{uw}(A|B) = wS_{v+uw}(A|B) = S_u(A \natural_v B|A \natural_{v+w} B).$$

(2) By Proposition 3.3 and Lemma 3.6, we get

$$\begin{aligned}
w\mathcal{R}(v)T_{uw}(A|B) &= w(A \natural_v B)A^{-1}T_{uw}(A|B) \\
&= w\frac{(A \natural_v B)A^{-1}(A \natural_{uw} B) - (A \natural_v B)A^{-1}A}{uw} \\
&= \frac{A \natural_{v+uw} B - A \natural_v B}{u} \\
&= \frac{(A \natural_v B) \natural_u (A \natural_{v+w} B) - A \natural_v B}{u} \\
&= T_u(A \natural_v B|A \natural_{v+w} B).
\end{aligned}$$

□

By using Theorem 3.10, we get the following properties by applying noncommutative ratio translation to fundamental relations between operator valued α-divergences and relative operator entropies shown in section 2.

Theorem 3.12. *Let A and B be strictly positive operators. Then,*

(1-a) $\mathcal{R}(v)D_0(A|B) = D_0(A \natural_v B|A \natural_{v+1} B)$,

(1-b) $\mathcal{R}(v)D_1(A|B) = D_1(A \natural_v B|A \natural_{v+1} B)$,

(2) $\mathcal{R}(v)D_\alpha(A|B) = -\mathcal{R}(v)\{T_\alpha(A|B) + T_{1-\alpha}(B|A)\}$,

(3-a) $0 \leq \mathcal{R}(v)D_\alpha(A|B) \leq \dfrac{1}{1-\alpha}\mathcal{R}(v)D_0(A|B)$,

(3-b) $0 \leq \mathcal{R}(v)D_\alpha(A|B) \leq \dfrac{1}{\alpha}\mathcal{R}(v)D_1(A|B)$,

(4) $\mathcal{R}(v)D_\alpha(A|B) \leq \mathcal{R}(v)\{S_1(A|B) - S(A|B)\}$

hold for $\alpha \in (0,1)$ and $v \in \mathbf{R}$.

Proof. These can be obtained by applying Theorem 3.9 and Theorem 3.10 to Proposition 2.2, Theorem 2.5, Theorem 2.4, and Corollary 2.6. □

Remark 1. Although noncommutative ratio translation has been defined as multiplying each operator value by noncommutative ratio $\mathcal{R}(v)$ from the left side, this is equivalent to multiplying the operator value by $\mathcal{R}(v)^*$ from the right side. For instance, in Theorem 3.9,

(1) $\mathcal{R}(v)S_u(A|B) = S_u(A|B)\mathcal{R}(v)^*$,

(2) $\mathcal{R}(v)T_u(A|B) = T_u(A|B)\mathcal{R}(v)^*$

hold for u, $v \in \mathbf{R}$.

Remark 2. In [9], we introduced $D_r(A, B) \equiv A \natural_{r+1} B - A \natural_r B - S_r(A|B)$ for $r \in \mathbf{R}$ as a generalization of $D_{FK}(A|B) = D_0(A|B)$. We remark that $D_v(A, B) = \mathcal{R}(v)D_0(A|B)$ holds for $v \in \mathbf{R}$ by (2) in Corollary 3.8 and (1-a) in Theorem 3.12.

4 α-divergence for operator distributions. On operator entropies for operator distributions $\mathbb{A} = (A_1, \cdots, A_n)$ and $\mathbb{B} = (B_1, \cdots, B_n)$, in [9], we obtained that the relations

$$S(\mathbb{A}|\mathbb{B}) \leq I_\alpha(\mathbb{A}|\mathbb{B}) \leq T_\alpha(\mathbb{A}|\mathbb{B}) \leq 0,$$

$$0 \leq -T_{1-\alpha}(\mathbb{B}|\mathbb{A}) \leq -I_{1-\alpha}(\mathbb{B}|\mathbb{A}) \leq S_1(\mathbb{A}|\mathbb{B})$$

and

$$T_\alpha(\mathbb{A}|\mathbb{B}) \leq S_\alpha(\mathbb{A}|\mathbb{B}) \leq -T_{1-\alpha}(\mathbb{B}|\mathbb{A})$$

hold for $0 < \alpha < 1$, where $I_\alpha(\mathbb{A}|\mathbb{B}) \equiv \dfrac{1}{\alpha} \log \sum_{i=1}^{n} A_i \sharp_\alpha B_i$ is Rényi relative operator entropy for operator distributions. By these inequalities and Corollary 3.5, we have

$$S(\mathbb{A}|\mathbb{B}) \leq T_\alpha(\mathbb{A}|\mathbb{B}) \leq S_\alpha(\mathbb{A}|\mathbb{B}) \leq -T_{1-\alpha}(\mathbb{B}|\mathbb{A}) = T^{\alpha}_{1-\alpha}(\mathbb{A}|\mathbb{B}) \leq S_1(\mathbb{A}|\mathbb{B})$$

for $0 < \alpha < 1$, where $T^v_\alpha(\mathbb{A}|\mathbb{B}) \equiv \sum_{i=1}^{n} \mathcal{R}_i(v)T_\alpha(A_i|B_i)$ for $v \in \mathbf{R}$ and $\mathcal{R}_i(v) = \mathcal{R}(v; A_i, B_i)$, as used in section 3. In this section, we investigate fundamental properties and relations between α-divergences and relative operator entropies for operator distributions.

Here, we define α-divergence for operator distributions.

Definition 4.1. For operator distributions $\mathbb{A} = (A_1, A_2, \cdots, A_n)$ and $\mathbb{B} = (B_1, B_2, \cdots, B_n)$, and for $\alpha \in (0,1)$, α-divergence for operator distributions is defined as follows:

$$D_\alpha(\mathbb{A}|\mathbb{B}) \equiv \sum_{i=1}^{n} D_\alpha(A_i|B_i) = \sum_{i=1}^{n} \frac{A_i \nabla_\alpha B_i - A_i \sharp_\alpha B_i}{\alpha(1-\alpha)}.$$

As in section 2, we show fundamental properties of α-divergences for operator distributions.

Proposition 4.2. *Let $\mathbb{A} = (A_1, A_2, \cdots, A_n)$ and $\mathbb{B} = (B_1, B_2, \cdots, B_n)$ be operator distributions. Then,*

$$D_0(\mathbb{A}|\mathbb{B}) \equiv \lim_{\alpha\to 0} D_\alpha(\mathbb{A}|\mathbb{B}) = -S(\mathbb{A}|\mathbb{B}), \tag{1}$$

$$D_1(\mathbb{A}|\mathbb{B}) \equiv \lim_{\alpha\to 1} D_\alpha(\mathbb{A}|\mathbb{B}) = S_1(\mathbb{A}|\mathbb{B}) \tag{2}$$

hold.

Proof. We only show the proof of equality (1) since the equality (2) can be shown similarly. By Proposition 2.2, we have

$$D_0(\mathbb{A}|\mathbb{B}) = \sum_{i=1}^{n} D_0(A_i|B_i) = \sum_{i=1}^{n} \{B_i - A_i - S(A_i|B_i)\} = -S(\mathbb{A}|\mathbb{B}).$$

□

By Proposition 2.3, Theorem 2.4, Theorem 2.5 and Corollary 2.6, we get the following Proposition 4.3, Theorem 4.4, Theorem 4.5 and Corollary 4.6, respectively.

Proposition 4.3. *Let $\mathbb{A} = (A_1, A_2, \cdots, A_n)$ and $\mathbb{B} = (B_1, B_2, \cdots, B_n)$ be operator distributions. Then,*

$$D_\alpha(\mathbb{A}|\mathbb{B}) = -\frac{1}{1-\alpha} T_\alpha(\mathbb{A}|\mathbb{B}) = -\frac{1}{\alpha} T_{1-\alpha}(\mathbb{B}|\mathbb{A}), \text{ for } \alpha \in (0,1), \tag{1}$$

$$D_{1-\alpha}(\mathbb{B}|\mathbb{A}) = D_\alpha(\mathbb{A}|\mathbb{B}), \text{ for } \alpha \in [0,1] \tag{2}$$

hold.

Theorem 4.4. *Let $\mathbb{A} = (A_1, A_2, \cdots, A_n)$ and $\mathbb{B} = (B_1, B_2, \cdots, B_n)$ be operator distributions. Then,*

$$0 \le D_\alpha(\mathbb{A}|\mathbb{B}) \le \frac{1}{1-\alpha} D_0(\mathbb{A}|\mathbb{B}), \tag{1}$$

$$0 \le D_\alpha(\mathbb{A}|\mathbb{B}) \le \frac{1}{\alpha} D_1(\mathbb{A}|\mathbb{B}) \tag{2}$$

hold for $\alpha \in (0,1)$.

Theorem 4.5. *Let $\mathbb{A} = (A_1, A_2, \cdots, A_n)$ and $\mathbb{B} = (B_1, B_2, \cdots, B_n)$ be operator distributions. Then,*

$$D_\alpha(\mathbb{A}|\mathbb{B}) = -\{T_\alpha(\mathbb{A}|\mathbb{B}) + T_{1-\alpha}(\mathbb{B}|\mathbb{A})\}$$

holds for $\alpha \in (0,1)$.

Corollary 4.6. *Let $\mathbb{A} = (A_1, A_2, \cdots, A_n)$ and $\mathbb{B} = (B_1, B_2, \cdots, B_n)$ be operator distributions. Then,*

$$D_\alpha(\mathbb{A}|\mathbb{B}) \leq S_1(\mathbb{A}|\mathbb{B}) - S(\mathbb{A}|\mathbb{B})$$

holds for $\alpha \in (0,1)$.

From above discussion, we remark that the relations

$$\alpha T_\alpha(\mathbb{A}|\mathbb{B}) = (1-\alpha)T_{1-\alpha}(\mathbb{B}|\mathbb{A})$$

and

$$T_\alpha(\mathbb{A}|\mathbb{B}) \geq -(1-\alpha)\{S_1(\mathbb{A}|\mathbb{B}) - S(\mathbb{A}|\mathbb{B})\}$$

hold for $\alpha \in (0,1)$.

Finally, we apply noncommutative ratio translation to α-divergence for operator distributions by the following notation:

Definition 4.7. Let $\mathbb{A} = (A_1, A_2, \cdots, A_n)$ and $\mathbb{B} = (B_1, B_2, \cdots, B_n)$ be operator distributions. For $v \in \mathbf{R}$ and $\alpha \in (0,1)$, we define $D_\alpha^v(\mathbb{A}|\mathbb{B})$ as follows:

$$D_\alpha^v(\mathbb{A}|\mathbb{B}) \equiv \sum_{i=1}^n \mathcal{R}_i(v) D_\alpha(A_i|B_i).$$

Then, we get the following from Theorem 3.12.

Corollary 4.8. *Let $\mathbb{A} = (A_1, A_2, \cdots, A_n)$ and $\mathbb{B} = (B_1, B_2, \cdots, B_n)$ be operator distributions. Then,*

(1-a) $$D_0^v(\mathbb{A}|\mathbb{B}) = \sum_{i=1}^n \mathcal{R}_i(v) D_0(A_i|B_i),$$

(1-b) $$D_1^v(\mathbb{A}|\mathbb{B}) = \sum_{i=1}^n \mathcal{R}_i(v) D_1(A_i|B_i),$$

(2) $$D_\alpha^v(\mathbb{A}|\mathbb{B}) = -\sum_{i=1}^n \mathcal{R}_i(v)\{T_\alpha(A_i|B_i) + T_{1-\alpha}(B_i|A_i)\},$$

(3-a) $$0 \leq D_\alpha^v(\mathbb{A}|\mathbb{B}) \leq \frac{1}{1-\alpha} D_0^v(\mathbb{A}|\mathbb{B}),$$

(3-b) $$0 \leq D_\alpha^v(\mathbb{A}|\mathbb{B}) \leq \frac{1}{\alpha} D_1^v(\mathbb{A}|\mathbb{B}),$$

(4) $$D_\alpha^v(\mathbb{A}|\mathbb{B}) \leq \sum_{i=1}^n \mathcal{R}_i(v)\{S_1(A_i|B_i) - S(A_i|B_i)\}$$

hold for $\alpha \in (0,1)$ and $v \in \mathbf{R}$.

Acknowledgements. We would like to express our very great appreciation for fruitful discussions in Nakamura seminar of Osaka Kyoiku University.

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