ON EQUIVARIANT COMPLETE INVARIANCE PROPERTY

Dania Masood and Pooja Singh

Received February 22, 2013

Abstract. Equivariant version of uniform flow is introduced in order to study the notions of the complete invariance property, and also that of the complete invariance property with respect to a homeomorphism, in the category of topological transformation groups.

1. Introduction

The converse of the celebrated result by Brouwer which states that every continuous map from a closed disc to itself, has atleast one fixed point has been considered and dealt with by Robbins [4], followed by many others like Martin, Schirmer and Ward [3, 5, 6]. Indeed, spaces with the property that every nonempty closed set is a fixed point set of some continuous selfmap has been termed as spaces possessing the *complete invariance property*, abbreviated as CIP. In case, the continuous map can be chosen to be a homeomorphism, the space is termed to possess the *complete invariance property with respect to a homeomorphism*, abbreviated as CIPH.

To identify spaces possessing the CIP and also those possessing the CIPH, various concepts have been developed with the passage of time, resulting into obtaining several specific classes of spaces which possess these properties. Among these notions, the property W and that of the uniform flow are some which made considerable impact in this direction of work.

Topological transformation groups and equivariant maps between them constitute an important category which has been extensively studied with reference to different aspects in the field of mathematics. The purpose of the present work is to introduce the equivariant version of uniform flow and to study the properties of the CIP and the CIPH in this setting.

2. Prerequisites

By a *flow* φ on a topological space X, we mean a continuous right action of the additive topological group of real numbers $\mathbb{R}$ on X.

By a *G-space* (or, a *topological transformation group*) we mean a topological space X on which a topological group G acts continuously. If a subset A of X remains unaltered under the group action, then it is called an *invariant subset* of X. A map f from a G-space X to a G-space Y is said to be *equivariant* if f preserves the action of the group. From now on, $g.x$ will denote the image of (g, x) under the group action.

On a metric space X, the notion of uniform flow is defined in [3] as follows:

Definition 2.1. A flow $\varphi : X \times \mathbb{R} \to X$, where X denotes a metric space (X, d), is called *uniform* if

2010 *Mathematics Subject Classification.* 54H15, 54H25.
Key words and phrases. Complete invariance property, G-space, uniform flow.

(i) $d(x, \varphi(x,t)) \leq C|t|$, for some $C > 0$ and $x \in X$, $t \in \mathbb{R}$.

(ii) There is a real number $p \geq 0$, such that for $x \in X$ and $t \in \mathbb{R}$, $\varphi(x,t)=x$ if and only if $t \in p\mathbb{Z}$.

Using the notion of uniform flow on a metric space, the following result is established in [3]:

'Let (M, d) be a compact metric space with a uniform flow φ, then every nonempty closed subset of M is the fixed point set of an orbit preserving autohomeomorphism of M. In particular*, M has the CIPH.'*

In the same paper, the authors have modified the metric on the product of two metric spaces (X_1, d_1) and (X_2, d_2) such that the uniform flow on one of them induces a uniform flow on the product $X_1 \times X_2$. Using this technique, they obtain the following result:

'If (M, d) is a compact metric space with a uniform flow and X is a metrizable space, then $M \times X$ has the CIPH.'

Further, they dealt with compact metrizable groups, wherein, requiring the existence of one-parameter subgroup with certain conditions they obtained the following result among several others:

'Every compact metrizable group has the CIP.'

3. Equivariant complete invariance property

In this Section, we introduce the notion of complete invariance property of a G-space in its equivariant version. Also, some other notions have been introduced in their equivariant version in order to have earlier findings by Martin and others [3] in this setting.

Definition 3.1 [1]. A topological transformation group (X, G, θ) is said to possess the *equivariant complete invariance property* (abbreviated, ECIP) if for each nonempty invariant closed set F of X, there is an equivariant continuous selfmap f such that Fix $f = F$, where Fix f denotes the set of all fixed points of f.

In case, a homeomorphism can be chosen in place of the map f, we say that the topological transformation group (X, G, θ) possesses the *equivariant complete invariance property with respect to a homeomorphism* (abbreviated, ECIPH).

Remark 3.2. If a topological transformation group (X, G, θ) possesses the ECIP (ECIPH), then the orbit space X/G possesses the CIP (CIPH).

Definition 3.3 [1]. A uniform flow φ on a topological transformation group (X, G, θ) is said to be an *equivariant uniform flow* if the following compatibility condition holds:

$$\varphi(g.x, t) = g.\varphi(x, t)$$

where $g \in G$, $x \in X$, $t \in \mathbb{R}$.

Example 3.4. Consider the topological transformation group (S^1, S^1, θ), where S^1 is the unit circle with arc length distance, and $\theta : S^1 \times S^1 \to S^1$ is defined by

$$\theta(e^{2\pi i\alpha}, e^{2\pi i\beta}) = e^{2\pi i(\alpha+\beta)}, \qquad \alpha, \beta \in [0, 2\pi).$$

Then the flow $\varphi : S^1 \times \mathbb{R} \to S^1$ defined by

$$\varphi(e^{2\pi i\alpha}, t) = e^{2\pi i(\alpha+t)}, \qquad \alpha \in [0, 2\pi),\ t \in \mathbb{R},$$

describes an equivariant uniform flow on (S^1, S^1, θ) with C as 1 and p as 2π.

Example 3.5. Consider the topological transformation group $(\mathbb{R}^2, S^1, \theta)$ where $\theta : S^1 \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\theta(e^{2\pi i\alpha}, re^{2\pi i\beta}) = re^{2\pi i(\alpha+\beta)}, \qquad \alpha, \beta \in [0, 2\pi),\ r \in \mathbb{R}.$$

Then the flow $\varphi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ defined by

$$\varphi((x, y), t) = (x + t, y), \qquad (x, y) \in \mathbb{R}^2,\ t \in \mathbb{R},$$

is a uniform flow on $\mathbb{R}^2$ with C as 1 and p as 0, but fails to be an equivariant uniform flow on $(\mathbb{R}^2, S^1, \theta)$.

Example 3.6. Consider the Euclidean n-space $\mathbb{R}^n$ on which $GL(n, \mathbb{R})$ acts through the action θ, sending (A, x) to $A \cdot x$, where $A \in GL(n, \mathbb{R})$ and $x \in \mathbb{R}^n$, treated as a column matrix. Then the uniform flow φ on $\mathbb{R}^n$ defined by

$$\varphi((x_1, x_2, \ldots, x_n), t) = (x_1 + t, x_2, \ldots, x_n)$$

is not an equivariant uniform flow on $(\mathbb{R}^n, GL(n, \mathbb{R}), \theta)$, whereas the restriction of the above action on $H \times \mathbb{R}^n$, where H is the topological subgroup of $GL(n, \mathbb{R})$ given by

$$H = \left\{ \begin{pmatrix} 1 & a_{0j} \\ 0 & a_{ij} \end{pmatrix} : (a_{ij})_{1\le i,\ j\le(n-1)} \in GL((n-1), \mathbb{R}); a_{0j} \in \mathbb{R}, j = 1, ..., (n-1) \right\},$$

makes φ an equivariant uniform flow on $(\mathbb{R}^n, H, \theta)$.

Theorem 3.7. Let (X, G, θ) be a topological transformation group, where X is a compact space possessing a metric d satisfying

$$d(x, y) = d(g.x, g.y)$$

for every $x, y \in X$ and $g \in G$. If (X, G, θ) has an equivariant uniform flow φ, then it possesses the ECIPH.

Proof. Let F be a nonempty closed invariant set of X. Define $h_F : X \to X$ by

$$h_F(x) = \varphi\left(x, \frac{1}{2C} d(x, F)\right),$$

where C is the positive number for which $d(x, \varphi(x, t)) \le C|t|$, where $x \in X$, $t \in \mathbb{R}$. That h_F is an autohomeomorphism with Fix $h_F = F$, follows from [3, Proposition 3.2], and the equivariance of h_F is straightforward.

Theorem 3.8. Suppose (X_1, d_1) and (X_2, d_2) are G_1 and G_2 - metric spaces, respectively, and φ is an equivariant uniform flow on (X_2, d_2), then the map

$$D : (X_1 \times X_2) \times (X_1 \times X_2) \to X_1 \times X_2,$$

defined by

$$D((x_1, x_2), (y_1, y_2)) = max\{d_1(x_1, y_1), d_2(x_2, y_2)\},$$

where $x_1,\ y_1 \in X_1$, and $x_2,\ y_2 \in X_2$, defines a metric on $X_1 \times X_2$, and the map $\Phi : (X_1 \times X_2) \times \mathbb{R} \to X_1 \times X_2$ given by

$$\Phi((x_1, x_2), t) = (x_1, \varphi(x_2, t)),$$

where $(x_1, x_2) \in X_1 \times X_2$ and $t \in \mathbb{R}$, describes an equivariant uniform flow on $(X_1 \times X_2, D)$.

Proof. It is left to the reader.

Remark 3.9. Arguing as above, we obtain that a countable product $\prod_{i\in\mathbb{N}} X_i$ of G_i -spaces X_i, in which one of the factors has equivariant uniform flow, possesses equivariant uniform flow.

Theorem 3.10. *Let (M, G, θ) be a topological transformation group, where M is a compact space with a metric d satisfying $d(x, y) = d(g.x, g.y)$ for every $x, y \in X$ and $g \in G$, and an equivariant uniform flow. Then for a metrizable space X, $M \times X$ as a G-space enjoys the ECIPH.*

Proof. It follows on the lines of the proof of Proposition 3.4 in [3].

Recalling that a metric d on a topological group G satisfying $d(gxh, gyh) = d(x, y)$, is completely characterized by the function $g \mapsto ||g|| : G \to \mathbb{R}$ defined by $||g|| = d(1, g)$ called an *invariant norm* on G [2, 3], we have

Theorem 3.11. *Assume that $(G, \circ)$ is a topological group with an invariant norm on which a topological group H acts continuously with the action θ and there is a one parameter subgroup $\alpha : \mathbb{R} \to G$ such that $\|\alpha(t)\| \leq C\,|t|$, for some positive real number C. Let N be a closed normal subgroup of G such that $\alpha(\mathbb{R})$ is not contained in N. Then, under the metric D induced by the norm on G/N given by*

$$\|Ng\| = inf\{\|ng\| : n \in N\},$$

the map $\varphi : G/N \times \mathbb{R} \to G/N$ defined by

$$\varphi(Ng, t) = Ng \circ \alpha(t),$$

defines an equivariant uniform flow on $(G/N,\ D)$, provided for $h \in H, g \in G$, $\theta(h, g \circ \alpha(t)) \circ (\theta(h, g) \circ \alpha(t))^{-1}$ lies in N.

Proof. That φ is a uniform flow on G/N follows from [3, Lemma 3.5]. The action θ of H on G induces an action θ' of H on G/N given by

$$\theta'(h, Ng) = N\theta(h, g),$$

where $h \in H, g \in G$. For φ to be an equivariant uniform flow, we should have

$$\varphi(\theta'(h, Ng), t) = \theta'(h, \varphi(Ng, t)),$$

where $Ng \in G/N, t \in \mathbb{R}$ and $h \in H$. Since

$$\varphi(\theta'(h, Ng), t) = \varphi(N\theta(h, g), t) = N(\theta(h, g) \circ \alpha(t)),$$

and
$$\theta'(h, \varphi(Ng, t)) = \theta'(h, N(g \circ \alpha(t)) = N\theta(h, g \circ \alpha(t)),$$
φ is an equivariant uniform flow provided
$$N\theta(h, g \circ \alpha(t)) = N(\theta(h, g) \circ \alpha(t))$$
or
$$\theta(h, g \circ \alpha(t)) \circ (\theta(h, g) \circ \alpha(t))^{-1} \in N,$$
where $h \in H, g \in G$.

Remark 3.12. By choosing H=$\{e\}$, the result [3, Lemma 3.5] follows.

Example 3.13. Consider the additive topological group $\mathbb{R}$ of real numbers on which the additive group of integers $\mathbb{Z}$ acts through the action θ defined by
$$\theta(n, t) = n + t,$$
where $n \in \mathbb{Z}$, $t \in \mathbb{R}$. Taking $N = \mathbb{Z}$ and a one parameter subgroup $\alpha(\mathbb{R})$, where $\alpha : \mathbb{R} \to \mathbb{R}$ is defined, for a nonzero real number λ, by
$$\alpha(t) = \lambda \cdot t, \quad t \in \mathbb{R},$$
the flow φ on $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$ defined by
$$\varphi(\mathbb{Z}t_1, t_2) = \mathbb{Z}(t_1 + \lambda t_2), \quad t_1, t_2 \in \mathbb{R},$$
is an equivariant uniform flow on $(\mathbb{S}^1, \mathbb{Z}, \theta)$.

Example 3.14. Consider the topological subgroup $G = \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \right\}$, of $GL(2, \mathbb{R})$, acting on $\mathbb{R}^2$ by the action θ given by
$$\theta\left(\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} x_1 + ax_2 \\ bx_2 \end{pmatrix},$$
where $a, b \in \mathbb{R}$ and $(x_1, x_2) \in \mathbb{R}^2$. Taking $N = \{0\} \times \mathbb{R}$ and a one parameter subgroup $\alpha(\mathbb{R})$, where $\alpha : \mathbb{R} \to \mathbb{R}^2$ is defined by
$$\alpha(t) = (t, 0), \quad t \in \mathbb{R},$$
the flow φ on $\mathbb{R}^2/N$ defined by
$$\varphi(N(x_1, x_2), t) = N(x_1 + t, x_2)$$
is an equivariant uniform flow on $(\mathbb{R}^2/N, G, \theta)$.

Example 3.15. Let the additive group $\mathbb{Z}_2$ act on the additive topological group $\mathbb{R}$ by the action θ defined by
$$\theta(k, t) = (-1)^k \cdot t, \quad k \in \mathbb{Z}_2, t \in \mathbb{R}.$$
Taking $N = \mathbb{Z}$ and a one parameter subgroup $\alpha(\mathbb{R})$, where $\alpha : \mathbb{R} \to \mathbb{R}$ is defined, for an irrational number λ, by
$$\alpha(t) = \lambda \cdot t, \quad t \in \mathbb{R},$$

we have, for $k \in \mathbb{Z}_2$ and $r,\, t \in \mathbb{R}$,

$$(-1)^k \cdot (r + \lambda t) - ((-1)^k r + \lambda t) = (-1)^k \lambda t - \lambda t,$$

may fail to lie in $\mathbb{Z}$. Thus the flow φ on $\mathbb{R}/\mathbb{Z}$ defined by

$$\varphi(\mathbb{Z}t, r) = \mathbb{Z}(t + \lambda r),$$

where $t, r \in \mathbb{R}$ is not an equivariant uniform flow on $(\mathbb{S}^1, \mathbb{Z}_2, \theta)$.

Acknowledgments

The authors thank their supervisor Professor K.K. Azad for his valuable guidance and encouragement.

References

[**1**] K. K. Azad, *On Complete Invariance Property*, Sixth International Conference on Dynamic Systems and Applications, Morehouse College, Atlanta, Georgia, USA, May 25-28, 2011.

[**2**] N. Bourbaki, *Topologie générale*, Herman, Paris, 1974.

[**3**] A. Chigogidze, K. H. Hofmann and J. R. Martin, *Compact groups and fixed point sets*, Trans. Amer. Math. Soc., 349 (1997), 4537-4554.

[**4**] H. Robbins, *Some complements to Brouwer's fixed point theorem,* Israel J. Math., 5 (1967), 225-226.

[**5**] H. Schirmer, *Fixed point sets of continuous selfmaps,* Fixed Point Theory Proc. (Sherbrooke, 1980), Lecture Notes in Math., 886, Springer-Verlag, Berlin, 1981, 417-428.

[**6**] L. E. Ward, Jr., *Fixed point sets,* Pacific J. Math., 47 (1973), 553-565.

Communicated by *Ioan A. Rus*

Department of Mathematics
University of Allahabad
Allahabad, 211 002 India
E-mail: dpub@pphmj.com

A NEW CHARACTERIZATION OF ACG^*-FUNCTIONS

Surinder Pal Singh and Inder. K. Rana

Received November 23, 2012; revised April 19, 2013

Abstract. In this paper, we propose an equivalent form of the restricted generalized absolute continuity, which characterizes the Denjoy integral of real valued functions.

1 Introduction The notion of absolute continuity and its relation with generalized forms of integrable functions has been studied by various authors, see e.g. [3, 4]. An extension of absolute continuity is the restricted generalized absolute continuity, denoted by ACG^*.

We characterize the class of ACG^*-functions in a much simpler manner, waiving off the continuity hypothesis and the oscillations therein and call them ACG_V-functions. The continuity will be intrinsically involved in ACG_V-functions, giving more insights to the fact that ACG^*-functions belong to the class of ACG_δ-functions. This further gives an alternative proof of the fact that every real valued Denjoy integrable function is Henstock-Kurzweil integrable.

We shall also discuss some questions arising from the standard Radon-Nikodym theorem, reflecting the essence of these extensions of absolute continuity.

2 Preliminaries Let $I = [a, b]$ be a compact real interval and μ be the Lebesgue measure on I. For a subinterval $J \subset I$, let $Sub(J)$ denotes the collection of subintervals of J and $\mathcal{F}(J)$ be the algebra generated by $Sub(J)$. Finally, let $\mathcal{F}$ denotes the algebra $\mathcal{F}(I)$.

Given a function $F : I \to \mathbb{R}$, we define the corresponding finitely additive set function on $\mathcal{F}$, still denoted by F, such that

$$F(\cup_{i=1}^{p}[c_i, d_i]) = \sum_{i=1}^{p}(F(d_i) - F(c_i))$$

holds for each non-overlapping finite collection $\{[c_i, d_i] : i = 1, 2, \dots, p\}$ of subintervals in $[a, b]$. Similarly, given a set function F on $\mathcal{F}$, we define the corresponding point function $F(x) = F([a, x])$, for each $x \in [a, b]$.

Throughout this paper, F will denote a set function on $\mathcal{F}$ as well as the corresponding point function on I and vice versa. Whenever we would have to deal only with the point functions we shall denote them by small letters, mostly by f. The oscillation of F on an interval $J \subset I$ is defined as

$$\omega(F, J) = sup\{|F(K)| : K \in Sub(J)\}.$$

Definition 1 (ACG^***-functions**).

(i) A function $F : I \to \mathbb{R}$ is called AC^* over a set $E \subset I$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\sum_{i=1}^{p}\omega(F, J_i) < \varepsilon$ holds for each non-overlapping collection $\{J_i : i = 1, 2, \dots, p\}$ of intervals in I such that both end points of each J_i belong to the set E and $\sum_{i=1}^{p}\mu(J_i) < \delta$.

2010 *Mathematics Subject Classification.* 26A39, 28B05.

Key words and phrases. Henstock-Kurzweil integral, charges and generalized absolute continuity .

(ii) *The function F is said to be generalized absolutely continuous in the restricted sense (namely ACG^*) over some set $E \subset I$ if F is continuous and E can be written as a countable union of sets E_n on each of which F is AC^*.*

It is easy to see that if a function F is absolutely continuous over some interval $J \subset I$, then F is AC^* over J. Thus every absolutely continuous function is ACG^*. Before we define another important generalization of absolute continuity, we need to fix certain notions.

A collection $\{(t_i, I_i) : i = 1, 2, \ldots, p\}$ of point-interval pairs is said to be a *partial division* in I if I_i's are mutually disjoint compact intervals in I and $t_i \in I_i$, for each i. If further, $\cup_{i=1}^{p} I_i = I$, it is called a *division* of I.

A positive valued function $\delta : I \to (0, \infty)$ is called *a gauge on* I and a partial division $\{(t_i, I_i) : i = 1, 2, \ldots, p\}$ of I is called δ-*fine* if for every i, we have $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$.

Now we present the class of ACG_δ-functions and the Henstock-Kurzweil integral, as follows:

Definition 2 (ACG_δ**-functions).**

(i) *A function F is said to be AC_δ over a set $E \subset I$ if for every $\varepsilon > 0$ there exists a positive number $\eta > 0$ and a gauge $\delta : E \to (0, \infty)$ such that $\sum_{i=1}^{p} |F(J_i)| < \varepsilon$ holds for each δ-fine partial division $\{(t_i, J_i) : i = 1, 2, \ldots, p\}$ in I such that each tag $t_i \in E$ and $\sum_{i=1}^{p} \mu(J_i) < \eta$.*

(ii) *The function F is said to be ACG_δ over some set $E \subset I$ if E can be written as a union of countable collection of sets $\{E_n : n \in \mathbb{N}\}$ on each of which F is AC_δ.*

Definition 3 (The Henstock-Kurzweil integral). *A function $f : I \to \mathbb{R}$ is said to be Henstock-Kurzweil integrable (or simply HK-integrable), with $A \in \mathbb{R}$ as its integral, if for every $\varepsilon > 0$ there is a gauge $\delta : I \to (0, \infty)$ such that the inequality*

$$\left| \sum_{i=1}^{p} f(t_i)\mu(I_i) - A \right| < \varepsilon$$

is satisfied for all δ-fine divisions $\{(t_i, I_i) : i = 1, \ldots, p\}$ of I.

The Henstock-Kurzweil integral of f over I is denoted by $(HK) \int_I f d\mu$. A function $F : \mathcal{F} \to \mathbb{R}$ is called the *primitive* of f if $F(J) = (HK) \int_J f d\mu$, for each $J \in \mathcal{F}$.

It is well known that the Henstock-Kurzweil integral on real line generalizes the notions of Riemann, Lebesgue and improper integrals. By keeping the tags independent of the subintervals, the McShane integral is defined which is equivalent to the Lebesgue integral.

Further, by replacing the subintervals in HK-integration with measurable sets or closed sets or by making the tags free, we are restricted to the McShane integration. For more details about such integrals, see [2, 5].

3 Absolute Continuity of Charges and the HK-integral .

Let $f : I \to R$ be a Henstock-Kuzweil integrable function with primitive F, that is

$$F(A) = (HK) \int_A f d\mu \text{ for all } A \in \mathcal{F}. \tag{1}$$

Then F is finitely additive on $\mathcal{F}$ and is called the *charge associated with* f.

The notion of absolute continuity for charges can be defined similar to that of measures (abbreviated as $F \ll \mu$). Let Ω denote the Lebesgue σ-algebra on I. Recall the Radon-Nikodym theorem for measures.

Theorem 4 (Radon-Nikodym)**.** *A measure $F : \Omega \longrightarrow \mathbb{R}$ is absolutely continuous with respect to μ (abbreviated as $F \ll \mu$) if and only if there exists a Lebesgue integrable function $f : I \to \mathbb{R}$ satisfying*

$$F(E) = \int_E f d\mu \text{ for all } E \in \Omega.$$

A corresponding result for charges is not known, see [1] for more details. Since the primitive of an HK-integrable function need not be countably additive, consider the following questions:

(i) Let F be a charge such that $F \ll \mu$. Does there exist an HK-integrable function f such that (1) holds true?

(ii) Let $F : \mathcal{F} \longrightarrow R$ be the charge associated with an HK-integrable function $f : I \longrightarrow R$. Is $F \ll \mu$?

The first question is answered affirmatively and the second one negatively, even for functions on metric measure spaces, in [7]. Further, some Radon-Nikodym type properties for the Henstock-Kurzweil integral have been proved in [6].

Thus the primitives of Henstock-Kurzweil integrable functions may not be absolutely continuous. However, these primitives happen to be the class of ACG^*-functions, see [2, Theorem 11.4].

Now we define some other extensions of the absolute continuity and prove them to be equivalent to the restricted generalized absolute continuity, ACG^*.

Definition 5 (ACG_V^***-functions**)**.**

(i) A function F is is said to be AC_V^* over a set $E \subset I$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ satisfying $\sum_{i=1}^{p} \omega(F, J_i) < \varepsilon$ for each non-overlapping finite collection of intervals $\{J_i : i = 1, 2, \ldots, p\}$ in I such that at least one point of each J_i belongs to the set E and $\sum_{i=1}^{p} \mu(J_i) < \delta$.

(ii) The function F is said to be ACG_V^* over some set $E \subset I$ if E can be written as a countable union of sets E_n on each of which F is AC_V^*.

Definition 6 (ACG_V**-functions**)**.**

(i) A function F is said to be AC_V over a set $E \subset I$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ satisfying $\sum_{i=1}^{p} |F(J_i)| < \varepsilon$ for each non-overlapping finite collection of intervals $\{J_i : i = 1, 2, \ldots, p\}$ in I such that at least one point of each J_i belong to the set E and $\sum_{i=1}^{p} \mu(J_i) < \delta$.

(ii) The function F is said to be ACG_V over some set $E \subset I$ if E can be written as a countable union of sets E_n on each of which F is AC_V.

It should be noted that we don't assume the continuity of F in both of the above definitions. Also, as earlier, it is easy to see that if F is absolutely continuous on I then it is ACG_V as well as ACG_V^* over I.

The main motivation behind these generalizations was the following result from [3, page 27]:

Lemma 7. *Let F be a continuous function on I and $A \subset I$ be a closed set then F is AC^* over A if and only if F is AC_V over A.*

A close observation to the proof of this lemma shows that the analogous result holds true even if AC_V is replaced with AC_V^*. Now we prove the main result of this paper.

Theorem 8. *For any function $F : I \to \mathbb{R}$, the following are equivalent:*

(i) F *is* ACG^* *over* I

(ii) F *is* ACG_V^* *over* I

(iii) F *is* ACG_V *over* I

(iv) F *is* ACG_δ *over* I*.*

Proof. Let F be an ACG^* function. Then F is continuous and I can be written as a countable union of sets E_n on each of which F is AC^*. Using [3, Theorem 6.7], we may assume that each E_n is a closed set. Further an application of Lemma 7 implies that F is AC_V^* and AC_V over each E_n. This proves that F is ACG_V^* as well as ACG_V.

It is easy to observe that if F is an ACG_V^* function over I then F is an ACG_V function over I.

Now let F be an ACG_V function. Let E_n be the countable collection of sets on each of which F is AC_V and $\cup_n E_n = I$. Fix some $n \in \mathbb{N}$ and let $\varepsilon > 0$ be given. Choose $\delta > 0$ corresponding to $\varepsilon/2$, as per the definition of F being AC_V over E_n. Then we choose a gauge $\eta : I \to (0, \infty)$ such that $\eta(t) = b - a$ for all $t \in I$. Let $\{(t_i, [c_i, d_i] : 1, \ldots, p\}$ be a η-fine partial division in I which is tagged in E_n and $\sum_i (d_i - c_i) < \delta$. We refine the partitions $\{[c_i, d_i]\}$ by further splitting the intervals $[c_i, d_i]$ into further two subintervals $[c_i, t_i]$ and $[t_i, d_i]$. The resulting two partitions in I are suitable for the AC_V case with E_n. Thus we have

$$\sum_{i=1}^{p} |F(d_i) - F(c_i)| \leq \sum_{i=1}^{p} |F(d_i) - F(t_i)| + \sum_{i=1}^{p} |F(t_i) - F(c_i)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This proves that if F is ACG_V over I then F is ACG_δ over I.

Finally, for the proof of the fact ACG_δ-functions are ACG^*, see [2, page 339]. □

Thus we see that the continuity of F is intrinsically assumed in ACG_V^*, ACG_V and ACG_δ, as these function classes are equivalent to the class of ACG^*-functions. The class of ACG_V-functions seems simplest to handle since it is relieved from continuity, oscillations, gauges and their corresponding fine partitions.

Remarks: There are various extensions to the classes of ACG^*-functions over finite dimensional Euclidean spaces. Lee Tuo-Yeong has proved some of them to be equivalent to the primitives of HK-integrable functions, see [4].

If we directly extend the notion of ACG_δ-functions over $\mathbb{R}^m$, the following question remains open:

What is the relation between the class of real valued ACG_δ-functions over finite dimensional Euclidean spaces and the primitives of Henstock-Kurzweil integrable functions?

References

[1] C. Fefferman, *A Radon-Nikodym theorem for finitely additive set functions*, Pacific J. Math. 23 (1967), 35–45.

[2] R. A. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, Amer. Math. Soc., Providence, RI, 1994.

[3] P. Y. Lee, *Lanzhou Lectures on Henstock Integration*, World Scientific, (1989).

[4] Tuo-Yeong Lee, *Some full descriptive characterizations of the Henstock-Kurzweil integral in the Euclidean space*, Czechoslovak Math. J. 55(130) (2005), no. 3, 625–637.

[5] Surinder Pal Singh and Inder K. Rana *Some alternative approaches to the McShane integral*, Real Anal. Exchange 35 (2009/2010) no.1, 229-233.

[6] Surinder Pal Singh and Inder K. Rana *The Hake's theorem and variational measures.*, Real Analysis Exchange Vol 37 No 2, (2011/2012), 477-488.

[7] Surinder Pal Singh and Inder K. Rana *Absolute continuity of charges on metric measure spaces.*, *communicated*.

Communicated by *Eiichi Nakai*

Surinder Pal Singh, Department of Mathematics, Guru Nanak Dev University, Amritsar, India, email: spsingh.math@gmail.com
Inder K. Rana, Department of Mathematics, IIT Bombay, India, email: ikr@math.iitb.ac.in

Scientiae Mathematicae Japonicae 77,No.1 (2014) 13-26

EXISTENCE AND MEAN APPROXIMATION OF FIXED POINTS OF GENERALIZED HYBRID NON-SELF MAPPINGS IN HILBERT SPACES

Toshiharu Kawasaki and Tetsuo Kobayashi

Received April 19, 2013; revised August 27, 2013

Abstract. In this paper, we prove a fixed point theorem for widely more generalized hybrid non-self mappings in Hilbert spaces. Furthermore, we prove mean convergence theorems of Baillon's type for widely more generalized hybrid non-self mappings in a Hilbert space.

1 Introduction Let H be a real Hilbert space and let C be a non-empty subset of H. In 2010, Kocourek, Takahashi and Yao [13] defined a class of nonlinear mappings in a Hilbert space. A mapping T from C into H is said to be generalized hybrid if there exist real numbers α and β such that

$$\alpha\|Tx-Ty\|^2+(1-\alpha)\|x-Ty\|^2 \le \beta\|Tx-y\|^2+(1-\beta)\|x-y\|^2$$

for any $x, y \in C$. We call such a mapping an (α, β)-generalized hybrid mapping. We observe that the class of the mappings covers the classes of well-known mappings. For example, an (α, β)-generalized hybrid mapping is nonexpansive [18] for $\alpha = 1$ and $\beta = 0$, i.e., $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. It is nonspreading [15] for $\alpha = 2$ and $\beta = 1$, i.e., $2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2$ for all $x, y \in C$. It is also hybrid [19] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e., $3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$ for all $x, y \in C$. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [14] and Iemoto and Takahashi [9]. Moreover, they proved the following nonlinear ergodic theorem.

Theorem 1.1 ([13]). *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H, let T be a generalized hybrid mapping from C into itself which has a fixed point, and let P be the metric projection from H onto the set of fixed points of T. Then for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

is weakly convergent to a fixed point p of T, where $p = \lim_{n\to\infty} PT^n x$.

Furthermore, they defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings. A mapping T from C into H is said to be super hybrid if there exist real numbers α, β and γ such that

$$\begin{aligned}\alpha\|Tx-Ty\|^2&+(1-\alpha+\gamma)\|x-Ty\|^2\\ &\le (\beta+(\beta-\alpha)\gamma)\|Tx-y\|^2+(1-\beta-(\beta-\alpha-1)\gamma)\|x-y\|^2\\ &\quad+(\alpha-\beta)\gamma\|x-Tx\|^2+\gamma\|y-Ty\|^2\end{aligned}$$

2010 *Mathematics Subject Classification.* 47H10.
Key words and phrases. Fixed point theorem, mean convergence theorem, Hilbert space, non-self mapping, generalied hybrid mapping, widely generalized mapping, widely more generalized mapping.

for any $x, y \in C$. A generalized hybrid mapping with a fixed point is quasinonexpansive. However, a super hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point. Very recently, the authors [11] also defined a class of nonlinear mappings in a Hilbert space which covers the class of contractive mappings and the class of generalized hybrid mappings defined by Kocourek, Takahashi and Yao [13]. A mapping T from C into H is said to be widely generalized hybrid if there exist real numbers $\alpha, \beta, \gamma, \delta, \varepsilon$ and ζ such that

$$\begin{aligned}\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ + \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} \leq 0\end{aligned}$$

for any $x, y \in C$. Furthermore, the authors [12] defined a class of nonlinear mappings in a Hilbert space which covers the class of super hybrid mappings and the class of widely generalized hybrid mappings. A mapping T from C into H is said to be widely more generalized hybrid if there exist real numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ and η such that

$$\begin{aligned}\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0\end{aligned}$$

for any $x, y \in C$. Then we prove fixed point theorems for such new mappings in a Hilbert space. Furthermore, we prove nonlinear ergodic theorems of Baillon's type in a Hilbert space. It seems that the results are new and useful. For example, using our fixed point theorems, we can directly prove Browder and Petryshyn's fixed point theorem [5] for strictly pseudocontractive mappings and Kocourek, Takahashi and Yao's fixed point theorem [13] for super hybrid mappings. On the other hand, Hojo, Takahashi and Yao [8] defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings. A mapping T from C into H is said to be extended hybrid if there exist real numbers α, β and γ such that

$$\begin{aligned}\alpha(1 + \gamma)\|Tx - Ty\|^2 + (1 - \alpha(1 + \gamma))\|x - Ty\|^2 \\ \leq (\beta + \alpha\gamma)\|Tx - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\ -(\alpha - \beta)\gamma\|x - Tx\|^2 - \gamma\|y - Ty\|^2\end{aligned}$$

for any $x, y \in C$. Furthermore, they proved a fixed point theorem for generalized hybrid non-self mappings by using the extended hybrid mapping.

In this paper, using an idea of [8], we prove a fixed point theorem for widely more generalized hybrid non-self mappings in Hilbert spaces. Furthermore, we prove mean convergence theorems of Baillon's type for widely more generalized hybrid non-self mappings in a Hilbert space.

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let A be a non-empty subset of H. We denote by $\overline{co}A$ the closure of the convex hull of A. In a Hilbert space, it is known that

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \tag{2.1}$$

for any $x, y \in H$ and for any $\alpha \in \mathbb{R}$; see [18]. Furthermore, in a Hilbert space, we obtain that

$$2\langle x - y, z - w \rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2 \tag{2.2}$$

for any $x, y, z, w \in H$. Let C be a non-empty subset of H and let T be a mapping from C into H. We denote by $F(T)$ the set of fixed points of T. A mapping T from C into H with $F(T) \neq \emptyset$ is said to be quasi-nonexpansive if $\|x - Ty\| \leq \|x - y\|$ for any $x \in F(T)$ and for any $y \in C$. It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is closed and convex; see Ito and Takahashi [10]. It is not difficult to prove such a result in a Hilbert space; see, for instace, [21]. Let D be a non-empty closed convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in D$ such that $\|x - z\| = \inf_{y \in D} \|x - y\|$. We denote such a correspondence by $z = P_D x$. The mapping P_D is said to be the metric projection from H onto D. It is known that P_D is nonexpansive and

$$\langle x - P_D x, P_D x - u \rangle \geq 0$$

for any $x \in H$ and for any $u \in D$; see [18] for more details. For proving a mean convergence theorem in this paper, we also need the following lemma proved by Takahashi and Toyoda [20].

Lemma 2.1. *Let D be a non-empty closed convex subset of H. Let P be the metric projection from H onto D. Let $\{u_n\}$ be a sequence in H. If $\|u_{n+1} - u\| \leq \|u_n - u\|$ for any $u \in D$ and for any $n \in \mathbb{N}$, then $\{Pu_n\}$ converges strongly to some $u_0 \in D$.*

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \ldots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is said to be a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean μ is said to be a Banach limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \ldots) \in l^\infty$,

$$\liminf_{n \to \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \to \infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \ldots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we obtain $\mu(f) = \mu_n(x_n) = a$. See [17] for the proof of existence of a Banach limit and its other elementary properties. Using means and the Riesz theorem, we have the following result; see [16] and [17].

Lemma 2.2. *Let H be a Hilbert space, let $\{x_n\}$ be a bounded sequence in H and let μ be a mean on l^∞. Then there exists a unique point $z_0 \in \overline{co}\{x_n \mid n \in \mathbb{N}\}$ such that*

$$\mu_n \langle x_n, y \rangle = \langle z_0, y \rangle$$

for any $y \in H$.

Kawasaki and Takahashi [12] proved from Lemma 2.2 the following fixed point theorem.

Theorem 2.1. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following condition (1) or (2):*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ *and* $\zeta + \eta \geq 0$*;*

(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$ *and* $\varepsilon + \eta \geq 0$*.*

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \ldots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) and (2).

As a direct consequence of Theorem 2.1, we obtain the following.

Theorem 2.2. *Let H be a real Hilbert space, let C be a bounded closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta > 0$ *and* $\zeta+\eta \geq 0$*;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$ *and* $\varepsilon+\eta \geq 0$.

Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha+\beta+\gamma+\delta > 0$ on the conditions (1) *and* (2).

3 Fixed point theorem Let H be a real Hilbert space and let C be a non-empty subset of H. A mapping T from C into H was said to be widely more generalized hybrid if there exist $\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\|Tx-Ty\|^2+\beta\|x-Ty\|^2+\gamma\|Tx-y\|^2+\delta\|x-y\|^2 \\ &+\varepsilon\|x-Tx\|^2+\zeta\|y-Ty\|^2+\eta\|(x-Tx)-(y-Ty)\|^2 \leq 0 \end{aligned} \tag{3.1}$$

for any $x,y \in C$; see Introduction. Such a mapping T is said to be $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid; see [12]. An $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping is generalized hybrid in the sense of Kocourek, Takahashi and Yao [13] if $\alpha+\beta=-\gamma-\delta=1$ and $\varepsilon=\zeta=\eta=0$. Moreover it is an extension of widely generalized hybrid mappings in the sence of Kawasaki and Takahashi [11]. Using Theorem 2.2, we prove a fixed point theorem for widely more generalized hybrid non-self mappings in a Hilbert space.

Theorem 3.1. *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into H which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta > 0$*, and there exists $\lambda \in \mathbb{R}$ such that $\lambda \neq 1$ and* $(\alpha+\beta)\lambda+\zeta+\eta \geq 0$*;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$*, and there exists $\lambda \in \mathbb{R}$ such that $\lambda \neq 1$ and* $(\alpha+\gamma)\lambda+\varepsilon+\eta \geq 0$.

Suppose that for any $x \in C$, there exist $m \in \mathbb{R}$ and $y \in C$ such that $0 \leq (1-\lambda)m \leq 1$ and $Tx = x+m(y-x)$. Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha+\beta+\gamma+\delta > 0$ on the conditions (1) *and* (2).

Proof. Let $S=(1-\lambda)T+\lambda I$. Since

$$\begin{aligned} Sx &= (1-\lambda)Tx+\lambda x \\ &= (1-\lambda)(x+m(y-x))+\lambda x \\ &= (1-(1-\lambda)m)x+(1-\lambda)my \in C \end{aligned}$$

for any $x \in C$, S is a mapping from C into itself. Since $\lambda \neq 1$, we obtain that $F(S)=F(T)$. Moreover from (2.1) we obtain that

$$\alpha\left\|\left(\frac{1}{1-\lambda}Sx-\frac{\lambda}{1-\lambda}x\right)-\left(\frac{1}{1-\lambda}Sy-\frac{\lambda}{1-\lambda}y\right)\right\|^2$$

$$
\begin{aligned}
&+\beta\left\|x-\left(\frac{1}{1-\lambda}Sy-\frac{\lambda}{1-\lambda}y\right)\right\|^2+\gamma\left\|\left(\frac{1}{1-\lambda}Sx-\frac{\lambda}{1-\lambda}x\right)-y\right\|^2\\
&+\delta\|x-y\|^2\\
&+\varepsilon\left\|x-\left(\frac{1}{1-\lambda}Sx-\frac{\lambda}{1-\lambda}x\right)\right\|^2+\zeta\left\|y-\left(\frac{1}{1-\lambda}Sy-\frac{\lambda}{1-\lambda}y\right)\right\|^2\\
&+\eta\left\|\left(x-\left(\frac{1}{1-\lambda}Sx-\frac{\lambda}{1-\lambda}x\right)\right)-\left(y-\left(\frac{1}{1-\lambda}Sy-\frac{\lambda}{1-\lambda}y\right)\right)\right\|^2\\
=\,&\alpha\left\|\frac{1}{1-\lambda}(Sx-Sy)-\frac{\lambda}{1-\lambda}(x-y)\right\|^2\\
&+\beta\left\|\frac{1}{1-\lambda}(x-Sy)-\frac{\lambda}{1-\lambda}(x-y)\right\|^2\\
&+\gamma\left\|\frac{1}{1-\lambda}(Sx-y)-\frac{\lambda}{1-\lambda}(x-y)\right\|^2+\delta\|x-y\|^2\\
&+\varepsilon\left\|\frac{1}{1-\lambda}(x-Sx)\right\|^2+\zeta\left\|\frac{1}{1-\lambda}(y-Sy)\right\|^2\\
&+\eta\left\|\frac{1}{1-\lambda}(x-Sx)-\frac{1}{1-\lambda}(y-Sy)\right\|^2\\
=\,&\frac{\alpha}{1-\lambda}\|Sx-Sy\|^2+\frac{\beta}{1-\lambda}\|x-Sy\|^2\\
&+\frac{\gamma}{1-\lambda}\|Sx-y\|^2+\left(-\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta\right)\|x-y\|^2\\
&+\frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}\|x-Sx\|^2+\frac{\zeta+\beta\lambda}{(1-\lambda)^2}\|y-Sy\|^2\\
&+\frac{\eta+\alpha\lambda}{(1-\lambda)^2}\|(x-Sx)-(y-Sy)\|^2\le 0.
\end{aligned}
$$

Then S is an $\left(\frac{\alpha}{1-\lambda},\frac{\beta}{1-\lambda},\frac{\gamma}{1-\lambda},-\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta,\frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2},\frac{\zeta+\beta\lambda}{(1-\lambda)^2},\frac{\eta+\alpha\lambda}{(1-\lambda)^2}\right)$-widely more generalized hybrid mapping. Furthermore, we obtain that

$$
\begin{aligned}
\frac{\alpha}{1-\lambda}+\frac{\beta}{1-\lambda}+\frac{\gamma}{1-\lambda}-\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta &= \alpha+\beta+\gamma+\delta\ge 0,\\
\frac{\alpha}{1-\lambda}+\frac{\gamma}{1-\lambda}+\frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}+\frac{\eta+\alpha\lambda}{(1-\lambda)^2} &= \frac{\alpha+\gamma+\varepsilon+\eta}{(1-\lambda)^2}>0,\\
\frac{\zeta+\beta\lambda}{(1-\lambda)^2}+\frac{\eta+\alpha\lambda}{(1-\lambda)^2} &= \frac{(\alpha+\beta)\lambda+\zeta+\eta}{(1-\lambda)^2}\ge 0.
\end{aligned}
$$

Therefore by Theorem 2.2 we obtain $F(S)\neq\emptyset$.

Next suppose that $\alpha+\beta+\gamma+\delta>0$. Let p_1 and p_2 be fixed points of T. Then

$$
\begin{aligned}
&\alpha\|Tp_1-Tp_2\|^2+\beta\|p_1-Tp_2\|^2+\gamma\|Tp_1-p_2\|^2+\delta\|p_1-p_2\|^2\\
&\qquad+\varepsilon\|p_1-Tp_1\|^2+\zeta\|p_2-Tp_2\|^2+\eta\|(p_1-Tp_1)-(p_2-Tp_2)\|^2\\
&=(\alpha+\beta+\gamma+\delta)\|p_1-p_2\|^2\le 0
\end{aligned}
$$

and hence $p_1=p_2$. Therefore a fixed point of T is unique.

In the case of the condition (2), we can obtain the result by replacing the variables x and y. □

Example 3.1. Let $H = \mathbb{R}$, let $C = \left[0, \frac{\pi}{2}\right]$, let $Tx = (1+2x)\cos x - 2x^2$ and let $\alpha = 1$, $\beta = \gamma = 11$, $\delta = -22$, $\varepsilon = \zeta = -12$ and $\eta = 1$. Then T is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H, $\alpha+\beta+\gamma+\delta = 1 \geq 0$ and $\alpha+\gamma+\varepsilon+\eta = 1 > 0$. Let $\lambda = \frac{2+3\pi}{3(1+\pi)}$ and let $m = 1+\pi$. Then $0 \leq (1-\lambda)m = \frac{1}{3} < 1$ and $(\alpha+\beta)\lambda+\zeta+\eta = \frac{\pi-3}{1+\pi} \geq 0$. Let $y = x + \frac{(1+2x)(\cos x - x)}{1+\pi}$ for any $x \in C$. Then $Tx = x + m(y-x)$ and $y \in C$. Therefore by Theorem 3.1 T has a unique fixed point.

4 Nonlinear ergodic theorems In this section, using the technique developed by Takahashi [16], we prove mean convergence theorems of Baillon's type in a Hilbert space. Before proving the results, we need the following lemmas.

Lemma 4.1. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition:*

$$\alpha + \gamma + \varepsilon + \eta > 0, \text{ or } \alpha + \beta + \zeta + \eta > 0.$$

Then $F(T)$ is closed.

Proof. Suppose that $\{x_n \mid n = 1, 2, \ldots\} \subset F(T)$ is convergent to $x \in H$. We show $x \in F(T)$. Putting $y = x_n$ in (3.1), we obtain that

$$\begin{aligned}\alpha\|Tx - Tx_n\|^2 + \beta\|x - Tx_n\|^2 + \gamma\|Tx - x_n\|^2 + \delta\|x - x_n\|^2 \\ +\varepsilon\|x - Tx\|^2 + \zeta\|x_n - Tx_n\|^2 + \eta\|(x - Tx) - (x_n - Tx_n)\}^2 \leq 0\end{aligned}$$

and hence

$$(\alpha + \gamma)\|Tx - x_n\|^2 + (\beta + \delta)\|x - x_n\|^2 + (\varepsilon + \eta)\|x - Tx\|^2 \leq 0. \tag{4.1}$$

Letting $n \to \infty$, we obtain that

$$(\alpha + \gamma + \varepsilon + \eta)\|Tx - x\|^2 \leq 0. \tag{4.2}$$

Since $\alpha + \gamma + \varepsilon + \eta > 0$, from (4.2) we obtain that $x \in F(T)$. Therefore $F(T)$ is closed. Similarly, we can obtain the desired result for the case of $\alpha + \beta + \zeta + \eta > 0$. □

Lemma 4.2. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$ *and* $\alpha + \gamma + \varepsilon + \eta > 0$*;*

(2) $\alpha + \beta + \gamma + \delta \geq 0$ *and* $\alpha + \beta + \zeta + \eta > 0$.

Then $F(T)$ is convex.

Proof. For $x_1, x_2 \in F(T)$ and $\lambda \in \mathbb{R}$ with $0 \leq \lambda \leq 1$, put $x = (1-\lambda)x_1 + \lambda x_2$. We show that $x \in F(T)$. Putting $y = x_1$ in (3.1), we obtain that

$$\begin{aligned}\alpha\|Tx - Tx_1\|^2 + \beta\|x - Tx_1\|^2 + \gamma\|Tx - x_1\|^2 + \delta\|x - x_1\|^2 \\ +\varepsilon\|x - Tx\|^2 + \zeta\|x_1 - Tx_1\|^2 + \eta\|(x - Tx) - (x_1 - Tx_1)\|^2 \leq 0\end{aligned}$$

and hence

$$(\alpha + \gamma)\|Tx - x_1\|^2 + (\beta + \delta)\lambda^2\|x_1 - x_2\|^2 + (\varepsilon + \eta)\|x - Tx\|^2 \leq 0. \tag{4.3}$$

Similarly, putting $y = x_2$ in (3.1), we obtain that

$$(\alpha+\gamma)\|Tx-x_2\|^2+(\beta+\delta)(1-\lambda)^2\|x_1-x_2\|^2+(\varepsilon+\eta)\|x-Tx\|^2 \leq 0. \tag{4.4}$$

Therefore from (4.3) we obtain that

$$\begin{aligned}&(\alpha+\gamma)\|Tx-x_1\|^2+(\beta+\delta)\lambda^2\|x_1-x_2\|^2\\ &\quad+(\varepsilon+\eta)(\|Tx-x_1\|^2+\lambda^2\|x_1-x_2\|^2+2\lambda\langle Tx-x_1,x_1-x_2\rangle) \leq 0.\end{aligned}$$

Thus we obtain that

$$\begin{aligned}&(\alpha+\gamma+\varepsilon+\eta)\|Tx-x_1\|^2+(\beta+\delta+\varepsilon+\eta)\lambda^2\|x_1-x_2\|^2\\ &\quad+2(\varepsilon+\eta)\lambda\langle Tx-x_1,x_1-x_2\rangle) \leq 0.\end{aligned} \tag{4.5}$$

Similarly, from (4.4) we obtain that

$$\begin{aligned}&(\alpha+\gamma+\varepsilon+\eta)\|Tx-x_2\|^2+(\beta+\delta+\varepsilon+\eta)(1-\lambda)^2\|x_1-x_2\|^2\\ &\quad-2(\varepsilon+\eta)(1-\lambda)\langle Tx-x_2,x_1-x_2\rangle) \leq 0.\end{aligned} \tag{4.6}$$

Using (2.1), (4.5), (4.6), $\alpha+\gamma+\varepsilon+\eta > 0$ and $\alpha+\beta+\gamma+\delta \geq 0$, we obtain that

$$\begin{aligned}&\|Tx-x\|^2\\ &= \|Tx-((1-\lambda)x_1+\lambda x_2)\|^2\\ &= (1-\lambda)\|Tx-x_1\|^2+\lambda\|Tx-x_2\|^2-\lambda(1-\lambda)\|x_1-x_2\|^2\\ &\leq (1-\lambda)\left(-\frac{(\beta+\delta+\varepsilon+\eta)\lambda^2}{\alpha+\gamma+\varepsilon+\eta}\|x_1-x_2\|^2\right.\\ &\qquad\left.-\frac{2(\varepsilon+\eta)\lambda}{\alpha+\gamma+\varepsilon+\eta}\langle Tx-x_1,x_1-x_2\rangle\right)\\ &\quad+\lambda\left(-\frac{(\beta+\delta+\varepsilon+\eta)(1-\lambda)^2}{\alpha+\gamma+\varepsilon+\eta}\|x_1-x_2\|^2\right.\\ &\qquad\left.+\frac{2(\varepsilon+\eta)(1-\lambda)}{\alpha+\gamma+\varepsilon+\eta}\langle Tx-x_2,x_1-x_2\rangle\right)\\ &\quad-\lambda(1-\lambda)\|x_1-x_2\|^2\\ &= -\frac{(\beta+\delta+\varepsilon+\eta)\lambda^2(1-\lambda)}{\alpha+\gamma+\varepsilon+\eta}\|x_1-x_2\|^2\\ &\quad-\frac{(\beta+\delta+\varepsilon+\eta)\lambda(1-\lambda)^2}{\alpha+\gamma+\varepsilon+\eta}\|x_1-x_2\|^2\\ &\quad+\frac{2(\varepsilon+\eta)\lambda(1-\lambda)}{\alpha+\gamma+\varepsilon+\eta}\|x_1-x_2\|^2\\ &\quad-\lambda(1-\lambda)\|x_1-x_2\|^2\\ &= -\frac{(\alpha+\beta+\gamma+\delta)\lambda(1-\lambda)}{\alpha+\gamma+\varepsilon+\eta}\|x_1-x_2\|^2 \leq 0\end{aligned}$$

and hence $x \in F(T)$. Thus $F(T)$ is convex. Similarly, we can obtain the desired result in the case of $\alpha+\beta+\gamma+\delta \geq 0$ and $\alpha+\beta+\zeta+\eta > 0$. □

Lemma 4.3. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma > 0$ *and* $\varepsilon+\eta \geq 0$;

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta > 0$ *and* $\zeta+\eta \geq 0$.

Then T is quasi-nonexpansive.

Proof. From (3.1) we have that for any $x \in C$ and for any $y \in F(T)$,

$$\begin{aligned}\alpha\|Tx-Ty\|^2 + \beta\|x-Ty\|^2 + \gamma\|Tx-y\|^2 + \delta\|x-y\|^2 \\ +\varepsilon\|x-Tx\|^2 + \zeta\|y-Ty\|^2 + \eta\|(x-Tx)-(y-Ty)\|^2 \\ = (\alpha+\gamma)\|Tx-y\|^2 + (\beta+\delta)\|x-y\|^2 + (\varepsilon+\eta)\|x-Tx\|^2 \leq 0.\end{aligned}$$

From $\alpha+\gamma > 0$ we obtain that

$$\|Tx-y\|^2 \leq -\frac{\beta+\delta}{\alpha+\gamma}\|x-y\|^2 - \frac{\varepsilon+\eta}{\alpha+\gamma}\|x-Tx\|^2.$$

Since $-\frac{\beta+\delta}{\alpha+\gamma} \leq 1$ from $\alpha+\beta+\gamma+\delta \geq 0$ and $-\frac{\varepsilon+\eta}{\alpha+\gamma} \leq 0$ from $\varepsilon+\eta \geq 0$, we obtain that

$$\|Tx-y\|^2 \leq \|x-y\|^2$$

and hence

$$\|Tx-y\| \leq \|x-y\|.$$

Thus T is quasi-nonexpansive. Similarly, we can obtain the desired result for the case of $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta > 0$ and $\zeta+\eta \geq 0$. □

Moreover we obtain the following.

Lemma 4.4. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\gamma)\lambda+\varepsilon+\eta < \alpha+\gamma+\varepsilon+\eta$;

(2) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\beta)\lambda+\zeta+\eta < \alpha+\beta+\zeta+\eta$.

Then $(1-\lambda)T+\lambda I$ is quasi-nonexpansive.

Proof. Let $S = (1-\lambda)T+\lambda I$. As in the proof of Theorem 3.1, we have that S is an $\left(\frac{\alpha}{1-\lambda}, \frac{\beta}{1-\lambda}, \frac{\gamma}{1-\lambda}, -\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta, \frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}, \frac{\zeta+\beta\lambda}{(1-\lambda)^2}, \frac{\eta+\alpha\lambda}{(1-\lambda)^2}\right)$-widely more generalized hybrid mapping from C into H and $F(S) = F(T)$. Furthermore, we obtain that

$$\begin{aligned}\frac{\alpha}{1-\lambda}+\frac{\beta}{1-\lambda}+\frac{\gamma}{1-\lambda}-\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta &= \alpha+\beta+\gamma+\delta \geq 0,\\ \frac{\alpha}{1-\lambda}+\frac{\gamma}{1-\lambda} &= \frac{\alpha+\gamma}{1-\lambda} > 0,\\ \frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}+\frac{\eta+\alpha\lambda}{(1-\lambda)^2} &= \frac{(\alpha+\gamma)\lambda+\varepsilon+\eta}{(1-\lambda)^2} \geq 0.\end{aligned}$$

By Lemma 4.3 S is quasi-nonexpansive. Similarly, we can obtain the desired result for the case of the condition (2). □

Now we first obtain the following mean convergence theorem for widely more generalized hybrid mappings in a Hilbert space.

Theorem 4.1. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ *and* $\varepsilon + \eta \geq 0$*;*

(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ *and* $\zeta + \eta \geq 0$.

Then for any $x \in C(T; 0) = \{z \mid T^n z \in C, \ \forall n \in \mathbb{N} \cup \{0\}\}$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

is weakly convergent to a fixed point p of T, where P is the metric projection from H onto $F(T)$ and $p = \lim_{n\to\infty} PT^n x$.

Proof. Let $x \in C(T; 0)$. We first consider the case of the condition (2). Since $F(T)$ is non-empty and by Lemma 4.3 T is quasi-nonexpansive, we obtain that

$$\|T^{n+1}x - y\| \leq \|T^n x - y\|$$

for any $n \in \mathbb{N} \cup \{0\}$ and for any $y \in F(T)$ and hence $\{T^n x\}$ is bounded for any $x \in C(T; 0)$. Since

$$\|S_n x - y\| \leq \frac{1}{n} \sum_{k=0}^{n-1} \|T^k x - y\| \leq \|x - y\|$$

for any $n \in \mathbb{N} \cup \{0\}$ and for any $y \in F(T)$, $\{S_n x \mid n = 0, 1, \ldots\}$ is also bounded. Therefore there exist a strictly increasing sequence $\{n_i\}$ and $p \in H$ such that $\{S_{n_i} x \mid i = 0, 1, \ldots\}$ is weakly convergent to p. Since C is closed and convex, C is weakly closed. Thus $p \in C$. We show that $p \in F(T)$. Since T is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H, we obtain that

$$\begin{aligned} &\alpha\|Tz - T^{k+1}x\|^2 + \beta\|z - T^{k+1}x\|^2 + \gamma\|Tz - T^k x\|^2 + \delta\|z - T^k x\|^2 \\ &\quad +\varepsilon\|z - Tz\|^2 + \zeta\|T^k x - T^{k+1}x\|^2 + \eta\|(z - Tz) - (T^k x - T^{k+1}x)\|^2 \leq 0 \end{aligned}$$

for any $k \in \mathbb{N} \cup \{0\}$ and for any $z \in C$. By (2.2) we obtain that

$$\begin{aligned} &\|(z - Tz) - (T^k x - T^{k+1}x)\|^2 \\ &\quad = \|z - Tz\|^2 + \|T^k x - T^{k+1}x\|^2 - 2\langle z - Tz, T^k x - T^{k+1}x\rangle \\ &\quad = \|z - Tz\|^2 + \|T^k x - T^{k+1}x\|^2 + \|z - T^k x\|^2 + \|Tz - T^{k+1}x\|^2 \\ &\qquad -\|z - T^{k+1}x\|^2 - \|Tz - T^k x\|^2. \end{aligned}$$

Thus we obtain that

$$\begin{aligned} &(\alpha + \eta)\|Tz - T^{k+1}x\|^2 + (\beta - \eta)\|z - T^{k+1}x\|^2 + (\gamma - \eta)\|Tz - T^k x\|^2 \\ &\quad +(\delta + \eta)\|z - T^k x\|^2 + (\varepsilon + \eta)\|z - Tz\|^2 + (\zeta + \eta)\|T^k x - T^{k+1}x\|^2 \leq 0. \end{aligned}$$

From

$$\begin{aligned}&(\gamma-\eta)\|Tz-T^kx\|^2\\&\quad=(\alpha+\gamma)(\|z-Tz\|^2+\|z-T^kx\|^2-2\langle z-Tz,z-T^kx\rangle)\\&\qquad-(\alpha+\eta)\|Tz-T^kx\|^2,\end{aligned}$$

we obtain that

$$\begin{aligned}&(\alpha+\eta)\|Tz-T^{k+1}x\|^2+(\beta-\eta)\|z-T^{k+1}x\|^2\\&\quad+(\alpha+\gamma)(\|z-Tz\|^2+\|z-T^kx\|^2-2\langle z-Tz,z-T^kx\rangle)\\&\quad-(\alpha+\eta)\|Tz-T^kx\|^2+(\delta+\eta)\|z-T^kx\|^2\\&\quad+(\varepsilon+\eta)\|z-Tz\|^2+(\zeta+\eta)\|T^kx-T^{k+1}x\|^2\le 0.\end{aligned}$$

and hence

$$\begin{aligned}&(\alpha+\eta)(\|Tz-T^{k+1}x\|^2-\|Tz-T^kx\|^2)+(\beta-\eta)\|z-T^{k+1}x\|^2\\&\quad+(\alpha+\gamma+\delta+\eta)\|z-T^kx\|^2-2(\alpha+\gamma)\langle z-Tz,z-T^kx\rangle\\&\quad+(\alpha+\gamma+\varepsilon+\eta)\|z-Tz\|^2+(\zeta+\eta)\|T^kx-T^{k+1}x\|^2\le 0.\end{aligned}$$

By $\alpha+\beta+\gamma+\delta\ge 0$, we obtain that

$$-(\beta-\eta)=-(\beta+\delta)+\delta+\eta\le\alpha+\gamma+\delta+\eta.$$

From this inequality and $\zeta+\eta\ge 0$, we obtain that

$$\begin{aligned}&(\alpha+\eta)(\|Tz-T^{k+1}x\|^2-\|Tz-T^kx\|^2)\\&\quad+(\beta-\eta)(\|z-T^{k+1}x\|^2-\|z-T^kx\|^2)\\&\quad-2(\alpha+\gamma)\langle z-Tz,z-T^kx\rangle+(\alpha+\gamma+\varepsilon+\eta)\|z-Tz\|^2\le 0\end{aligned}$$

for any $k\in\mathbb{N}\cup\{0\}$ and for any $z\in C$. Summing up these inequalities with respect to $k=0,1,\dots,n-1$ and dividing by n, we obtain that

$$\begin{aligned}&\frac{\alpha+\eta}{n}(\|Tz-T^nx\|^2-\|Tz-x\|^2)+\frac{\beta-\eta}{n}(\|z-T^nx\|^2-\|z-x\|^2)\\&\quad-2(\alpha+\gamma)\langle z-Tz,z-S_nx\rangle+(\alpha+\gamma+\varepsilon+\eta)\|z-Tz\|^2\le 0.\end{aligned}$$

Replacing n by n_i, we obtain that

$$\begin{aligned}&\frac{\alpha+\eta}{n_i}(\|Tz-T^{n_i}x\|^2-\|Tz-x\|^2)+\frac{\beta-\eta}{n_i}(\|z-T^{n_i}x\|^2-\|z-x\|^2)\\&\quad-2(\alpha+\gamma)\langle z-Tz,z-S_{n_i}x\rangle+(\alpha+\gamma+\varepsilon+\eta)\|z-Tz\|^2\le 0.\end{aligned}$$

Letting $i\to\infty$, we obtain that

$$-2(\alpha+\gamma)\langle z-Tz,z-p\rangle+(\alpha+\gamma+\varepsilon+\eta)\|z-Tz\|^2\le 0.$$

Putting $z=p$, we obtain that

$$(\alpha+\gamma+\varepsilon+\eta)\|p-Tp\|^2\le 0.$$

Since $\alpha+\gamma+\varepsilon+\eta>0$, we obtain that $Tp=p$. Since by Lemmas 4.1 and 4.2 $F(T)$ is closed and convex, the metric projection P from H onto $F(T)$ is well-defined. By Lemma 2.1 there exists $q \in F(T)$ such that $\{PT^n x \mid n = 0, 1, \dots\}$ is convergent to q. To complete the proof, we show that $q = p$. Note that the metric projection P satisfies

$$\langle z - Pz, Pz - u \rangle \geq 0$$

for any $z \in H$ and for any $u \in F(T)$; see [17]. Therefore

$$\langle T^k x - PT^k x, PT^k x - y \rangle \geq 0$$

for any $k \in \mathbb{N} \cup \{0\}$ and for any $y \in F(T)$. Since P is the metric projection and T is quasi-nonexpansive, we obtain that

$$\begin{aligned} \|T^n x - PT^n x\| &\leq \|T^n x - PT^{n-1} x\| \\ &\leq \|T^{n-1} x - PT^{n-1} x\|, \end{aligned}$$

that is, $\{\|T^n x - PT^n x\| \mid n = 0, 1, \dots\}$ is non-increasing. Therefore we obtain

$$\begin{aligned} \langle T^k x - PT^k x, y - q \rangle &\leq \langle T^k x - PT^k x, PT^k x - q \rangle \\ &\leq \|T^k x - PT^k x\| \cdot \|PT^k x - q\| \\ &\leq \|x - Px\| \cdot \|PT^k x - q\|. \end{aligned}$$

Summing up these inequalities with respect to $k = 0, 1, \dots, n-1$ and dividing by n, we obtain

$$\left\langle S_n x - \frac{1}{n} \sum_{k=0}^{n-1} PT^k x, y - q \right\rangle \leq \frac{\|x - Px\|}{n} \sum_{k=0}^{n-1} \|PT^k x - q\|.$$

Since $\{S_{n_i} x \mid i = 0, 1, \dots\}$ is weakly convergent to p and $\{PT^n x \mid n = 0, 1, \dots\}$ is convergent to q, we obtain that

$$\langle p - q, y - q \rangle \leq 0.$$

Putting $y = p$, we obtain

$$\|p - q\|^2 \leq 0$$

and hence $q = p$. This completes the proof.

Similarly, we can obtain the desired result for the case of the condition (1). □

Moreover we obtain the following.

Theorem 4.2. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\gamma)\lambda+\varepsilon+\eta < \alpha+\gamma+\varepsilon+\eta$*;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\beta)\lambda+\zeta+\eta < \alpha+\beta+\zeta+\eta$.

Then for any $x \in C(T; \lambda) = \{z \mid ((1-\lambda)T + \lambda I)^n z \in C, \ \forall n \in \mathbb{N} \cup \{0\}\}$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} ((1-\lambda)T + \lambda I)^k x$$

is weakly convergent to a fixed point p of T, where P is the metric projection from H onto $F(T)$ and $p = \lim_{n\to\infty} P((1-\lambda)T + \lambda I)^n x$.

Proof. Let $S = (1-\lambda)T + \lambda I$. As in the proof of Theorem 3.1, we have that S is an $\left(\frac{\alpha}{1-\lambda}, \frac{\beta}{1-\lambda}, \frac{\gamma}{1-\lambda}, -\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta, \frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}, \frac{\zeta+\beta\lambda}{(1-\lambda)^2}, \frac{\eta+\alpha\lambda}{(1-\lambda)^2}\right)$-widely more generalized hybrid mapping from C into H and

$$\begin{aligned}
\frac{\alpha}{1-\lambda} + \frac{\beta}{1-\lambda} + \frac{\gamma}{1-\lambda} - \frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta &= \alpha+\beta+\gamma+\delta \geq 0, \\
\frac{\alpha}{1-\lambda} + \frac{\gamma}{1-\lambda} &= \frac{\alpha+\gamma}{1-\lambda} > 0, \\
\frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2} + \frac{\eta+\alpha\lambda}{(1-\lambda)^2} &= \frac{(\alpha+\gamma)\lambda+\varepsilon+\eta}{(1-\lambda)^2} \geq 0.
\end{aligned}$$

By Theorem 4.1 $\{S_n x\}$ is weakly convergent to $p \in F(S) = F(T)$. Since by Lemmas 4.1 and 4.2 $F(S)$ is closed and convex, the metric projection P from H onto $F(S)$ is well-defined. Since by Lemma 4.4 S is quasi-nonexpansive, we obtain that

$$\|S^{n+1}x - y\| \leq \|S^n x - y\|$$

for any $n \in \mathbb{N} \cup \{0\}$ and for any $y \in F(S)$. Therefore we can obtain the desired result as in the proof of Theorem 4.1.

Similarly, we can obtain the desired result for the case of the condition (2). $\square$

Theorem 4.3. *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\gamma)\lambda+\varepsilon+\eta < \alpha+\gamma+\varepsilon+\eta$*;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\beta)\lambda+\zeta+\eta < \alpha+\beta+\zeta+\eta$.

Suppose that for any $x \in C$, there exist $m \in \mathbb{R}$ and $y \in C$ such that $0 \leq (1-\lambda)m \leq 1$ and $Tx = x + m(y-x)$. Then for any $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1}((1-\lambda)T + \lambda I)^k x$$

is weakly convergent to a fixed point p of T, where P is the metric projection from H onto $F(T)$ and $p = \lim_{n\to\infty} P((1-\lambda)T + \lambda I)^n x$.

Proof. Let $S = (1-\lambda)T + \lambda I$. Since $S = (1-\lambda)T + \lambda I$ is a mapping from C into itself, we have $C(T;\lambda) = \{z \mid ((1-\lambda)T + \lambda I)^n z \in C,\ \forall n \in \mathbb{N} \cup \{0\}\} = C$. Therefore by Theorem 4.2 we obtain the desired result. $\square$

Example 4.1. Let $H = \mathbb{R}$, let $C = \left[0, \frac{\pi}{2}\right]$, let $Tx = (1+2x)\cos x - 2x^2$ and let $\alpha = 1$, $\beta = \gamma = 11$, $\delta = -22$, $\varepsilon = \zeta = -12$ and $\eta = 1$. Then T is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H, $\alpha+\beta+\gamma+\delta = 1 \geq 0$ and $\alpha+\gamma+\varepsilon+\eta = 1 > 0$. Let $\lambda = \frac{2+3\pi}{3(1+\pi)}$ and $m = 1+\pi$. Then $0 \leq (1-\lambda)m = \frac{1}{3} < 1$ and $0 \leq (\alpha+\gamma)\lambda+\varepsilon+\eta = \frac{\pi-3}{1+\pi} < 1 = \alpha+\gamma+\varepsilon+\eta$. Let $y = x + \frac{(1+2x)(\cos x - x)}{1+\pi}$ for any $x \in C$. Then $Tx = x + m(y-x)$ and $y \in C$. Therefore by Theorem 4.3 for any $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1}((1-\lambda)T + \lambda I)^k x$$

is weakly convergent to a fixed point p of T, where P is the metric projection from H onto $F(T)$ and $p = \lim_{n\to\infty} P((1-\lambda)T + \lambda I)^n x$.

Acknowledgements. The authors are grateful to Professor Wataru Takahashi for his suggestions and comments.

References

[1] J.-B. Baillon, *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, C. R. Acad. Sci. Sér. A–B **280** (1975), 1511–1514.

[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922), 133–181.

[3] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123–145.

[4] F. E. Browder, *Convergence theorems for sequences of nonlinear operators in Banach spaces*, Math. Z. **100** (1967), 201–225.

[5] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl. **20** (1967), 197–228.

[6] P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (2005), 117–136.

[7] K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge University Press, Cambridge, 1990.

[8] M. Hojo, W. Takahashi, and J.-C. Yao, *Weak and strong mean convergence theorems for super hybrid mappings in Hilbert spaces*, Fixed Point Theory **12** (2011), 113–126.

[9] S. Iemoto and W. Takahashi, *Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space*, Nonlinear Anal. **71** (2009), 2082–2089.

[10] S. Itoh and W. Takahashi, *The common fixed point theory of singlevalued mappings and multivalued mappings*, Pacific J. Math. **79** (1978), 493–508.

[11] T. Kawasaki and W. Takahashi, *Fixed point and nonlinear ergodic theorems for new nonlinear mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **13** (2012), 529–540.

[12] ———, *Existence and mean approximation of fixed points of generalized hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **14** (2013), 71–87.

[13] P. Kocourek, W. Takahashi, and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **14** (2010), 2497–2511.

[14] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*, SIAM J. Optim. **19** (2008), 824–835.

[15] ———, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. (Basel) **91** (2008), 166–177.

[16] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **81** (1981), 253–256.

[17] ———, *Nonlinear Functional Analysis. Fixed Points Theory and its Applications*, Yokohama Publishers, Yokohama, 2000.

[18] ———, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2009.

[19] ———, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79–88.

[20] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), 417–428.

[21] W. Takahashi and J.-C. Yao, *Fixed point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces*, Taiwanese J. Math. **15** (2011), 457–472.

Communicated by *Wataru Takahashi*

T. Kawasaki
College of Engineering, Nihon University, Fukushima 963–8642, Japan
E-mail: toshiharu.kawasaki@nifty.ne.jp

T. Kobayashi
College of Engineering, Nihon University, Fukushima 963–8642, Japan
E-mail: tekoba@ge.ce.nihon-u.ac.jp

SOME RESULTS ON DERIVATIONS OF BCI-ALGEBRAS

R. A. Borzooei, O. Zahiri

Received January 24, 2012 ; revised February 26, 2013

Abstract. In this paper, by considering the notions of left-right and right-left derivations of BCI-algebras, we generalize some results on regular derivations and classify this derivation in P-semisimple BCI-algebra and BCK-algebra. Then we make a congruence relation, for any derivation d of X and defined the concept of conjugate derivations. In the sequel, we show that the set of all equivalence classes of X with respect to this relation forms a BCI-algebra and we denote it by X/d. Finally, we get some interesting result about these quotient algebras.

1 Introduction

BCK/BCI-algebras are algebraic structures, introduced by K. Iséki and Y. Imai in 1966, that describe fragments of the propositional calculus involving implication known as BCK/BCI-logics. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. They were soon joined by many other researchers to develop various aspects of the BCK/BCI-algebra theory. These algebras have been extensively studied since their introduction.

In the theory of rings and near-rings, the properties of derivations are an important topic to study and several authors have studied derivations in rings and near-rings, see [4, 5, 13, 18]. In non-commutative rings, the notion of derivations is extended to α-derivations, left derivations and central derivations. The properties of α-derivations and central derivations were discussed in several papers with respect to the ring structures. In [12], Y. B. Jun and X. L. Xin applied the notion of derivation in ring and near-ring theory to BCI-algebras. They also introduced a new concept called a regular derivation in BCI-algebras, investigated some of its properties, defined a d-invariant ideal and gave some conditions for an ideal to be d-invariant. In [21], J. Zhan and Y. L. Liu generalized this concept and introduced the notion of f-derivation on BCI-algebras. Since then several authors [1, 2, 11, 12] have studied the theory of derivation in BCI-algebras. Now, in this paper we verify the (r, l) and (l, r)-derivations of BCI and BCK-algebras and explain some useful properties of these derivations as mentioned in the abstract and attempt to generalize some of the results which proved in [1, 12].

2 Preliminaries

Definition 2.1. [9, 10] A *BCI-algebra* is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following conditions:
($BCI1$) $((x * y) * (x * z)) * (z * y) = 0$;
($BCI2$) $x * 0 = x$;
($BCI3$) $x * y = 0$ and $y * x = 0$ imply $y = x$.

In a BCI-algebra $(X, *, 0)$, we can define a partial order on X by $x \leq y \Leftrightarrow x * y = 0$, for all $x, y \in X$, which is called the partial order induced by BCI-algebra X. In each BCI-algebra X , the following hold: hold (see [20]):

2010 *Mathematics Subject Classification.* 06F35, 03G25.
Key words and phrases. BCI-algebra, derivation, d-invariant, regular, P-semisimple, quotient.

$(BCI4)$ $x * x = 0$;
$(BCI5)$ $0 * (x * y)^n = (0 * x^n) * (0 * y^n)$;
$(BCI6)$ $0 * (0 * (0 * x)) = 0 * x$;
$(BCI7)$ $(x * y) * z = (x * z) * y$;
$(BCI8)$ $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$, for any $z \in X$,

where, $x * y^0 = x$ and $x * y^{n+1} = (x * y^n) * y$, for all $n \in \mathbb{N} \cup \{0\}$. Let $(X, *, 0)$ be a BCI-algebra. We denote $x \wedge y = y * (y * x)$, for all $x, y \in X$. By $(BCI7)$, it is clear that $(x \wedge y) * x = (y * (y * x)) * x = (y * x) * (y * x) = 0$ and so $x \wedge y \leq x$, for all $x, y \in X$. Note that it is not the greatest lower bound of x and y. Moreover, when X is a commutative BCK-algebra, then $x \wedge y$ is the greatest lower bound of x and y. The set $B = \{x \in X | 0 * x = 0\}$ is called BCK-part of X. A BCI-algebra X is called a *BCK-algebra* if $B = X$. Clearly, $x * (0 * (0 * x)) \in B$, for any $x \in X$ and 0 is the least element of X. Moreover, $P = \{x \in X | 0 * (0 * x) = x\}$ is called *P-semisimple* part of a BCI-algebra X. It is the set of all minimal elements of X, with respect the partial order induced by X. The BCI-algebra X is called a *P-semisimple* BCI-algebra if $P = X$. In a P-semisimple BCI-algebra, the following hold (see [20]):

$(P1)$ $(x * z) * (y * z) = x * y$;
$(P2)$ $0 * (0 * x) = x$;
$(P3)$ $x * y = 0$ implies $x = y$;
$(P4)$ $x * a = x * b$ implies $a = b$ (right cancellation law);
$(P5)$ $a * x = b * x$ implies $a = b$ (left cancellation law).

Let X be a BCI-algebra and P be its P-semisimple part. It is clear that $G(X) = \{x \in X | 0 * x = x\} \subseteq P$. Moreover, $x * (x * a) = a$ and $0 * x \in P$, for any $x \in X$ and $a \in P$. Moreover,

$(P6)$ $a * x \in P$, for any $a \in P$ and $x \in X$.

For any $a \in P$, we define $V(a) = \{x \in X | a * x = 0\}$, which is called the branch of X with respect to a. On the other hand, $X = \bigcup\limits_{a \in P} V(a)$ and $V(a) \cap V(b) = \emptyset$, for any distinct elements $a, b \in P$. The set of all branches of X formed a partition for X. If $x \in V(a)$ and $y \in V(b)$, then $x * y \in V(a * b)$, for any $x, y \in X$ and $a, b \in P$. If $x \in V(a)$, for some $a \in P$, then

$(P7)$ $\{y \in X | x * y = 0 \quad \text{or} \quad y * x = 0\} \subseteq V(a)$.

A nonempty subset S of X is called a *subalgebra* of X if $x * y \in S$, for any $x, y \in S$. The P-semisimple part of X is a subalgebra of X. For more details, we refer to [3, 7, 14, 15, 19, 20, 22].

Definition 2.2. [9, 10] A subset I of X is called an *ideal* of X if (i) $0 \in I$; (ii) $x * y \in I$ and $x \in I$ imply $y \in I$, for all $x, y \in X$. This definition follows from the concept of deductive system in basic logic algebras (BL-algebras), MV-algebras and residuated lattices. An ideal I of X is called a *closed ideal* if $x * y \in I$, for any $x, y \in I$.

Let X and Y be two BCI-algebras. Then the map $f : X \to Y$ is called a *homomorphism* if $f(x * y) = f(x) * f(y)$, for any $x, y \in X$. A homomorphism f is called an *isomorphism* if f is one-to-one and onto. We say X is *isomorphic* to Y, symbolically $X \cong Y$, if there exists an isomorphism $f : X \to Y$. Note that, if $f : X \to Y$ is a homomorphism, then $f(0) = f(x * x) = f(x) * f(x) = 0$ and so $f(x) = f(0 * (0 * x)) = 0 * (0 * (f(x))$, for any $x \in P$. Therefore, if P is the P-semisimple part of X, then $f(x)$ belong to the P-semisimple part of Y, for any $x \in P$.

Definition 2.3. [16] The BCI-algebra X is said to be *commutative* if $x = x \wedge y$, whenever $x \leq y$, for all $x, y \in X$.

Definition 2.4. [6] A BCI-algebra X is said to be *branchwise commutative* if $x \wedge y = y \wedge x$, for all $x, y \in V(a)$ and all $a \in P$.

Theorem 2.5. *[12] A BCI-algebra X is commutative if and only if it is branchwise commutative.*

Definition 2.6. [12] Let X be a BCI-algebra. By a left-right derivation (briefly, (l, r)-derivation) of X we mean a self-map d of X satisfying the identity $d(x * y) = (d(x) * y) \wedge (x * d(y))$, for all $x, y \in X$. If d satisfies the identity $d(x * y) = (x * d(y)) \wedge (d(x) * y)$, for all $x, y \in X$, then we say that d is a right-left derivation (briefly, (r, l)-derivation). Moreover, if d is both (l, r) and (r, l)-derivation, we say that d is a derivation. An (r, l)-derivation ((l, r)-derivation) d is called *regular* if $d(0) = 0$.

Note that the map $d : X \to X$ defined by $d(x) = 0 * (0 * x)$, for all $x \in X$, is an (l, r) derivation of X (See [12], Proposition 3.7).

Proposition 2.7. *[1] Every (l, r)-derivation (or (r, l)-derivation) of any BCK-algebra is regular.*

Proposition 2.8. *[12] Let d be an (l, r)-derivation of a BCI-algebra X. Then d satisfies the following properties:*
(i) *$d(x) \in P$, for all $x \in P$;*
(ii) *d is identity on P if and only if $d(0) = 0$;*
(iii) *$d(x) = d(x) \wedge x$, for all $x \in X$.*

Proposition 2.9. *[12] Let d be an (r, l)-derivation of a BCI-algebra X. Then d satisfies the following properties:*
(i) *$d(a) \in G(X)$, for all $a \in G(X)$;*
(ii) *$d(a) \in P$, for all $a \in P$;*
(iii) *d is identity on P if and only if $d(0) = 0$;*
(iv) *$d(x) = x \wedge d(x)$, for all $x \in X$ if and only if $d(0) = 0$.*

Proposition 2.10. *[12] Let d be a regular (r, l)-derivation of a commutative BCI-algebra X. Then*
(i) *x and $d(x)$ belong to the same branch, for all $x \in X$.*
(ii) *d is also an (l, r)-derivation.*

Proposition 2.11. *[12] Let d be a regular derivation of a BCI-algebra X. Then the following hold:*
(i) *$d(x) \leq x$, for all $x \in X$;*
(ii) *$d(x) * y \leq x * d(y)$, for all $x, y \in X$;*
(iii) *$d(x * y) = d(x) * y \leq d(x) * d(y)$, for all $x, y \in X$;*
(iv) *$d^{-1}(0)$ is a subalgebra of X and $d^{-1}(0) \subseteq B$.*

Note that d need not be a derivation in the proof of Proposition 2.11. If d is an (r, l)-derivation of X, then (i), (ii) and (iii) in Proposition 2.11, are also true (see [12], proof of Proposition 3.14). In fact, in the proves of (i), (ii) and (iii) the authors used the properties of (r, l)-derivation.

Definition 2.12. [8] An element x of a BCI-algebra X is called *nilpotent* if $0 * x^n = 0$, for some $n \in \mathbb{N}$ and the least natural number n satisfying $0 * x^n = 0$ is called the *period* of x and is denoted by $o(x)$. Moreover, the set of all nilpotent elements of X is denoted by $N(X)$. It is a closed ideal of X.

Definition 2.13. [12] Let d be a self-map of a BCI-algebra X. An ideal A of X is said to be d-invariant if $d(A) \subseteq A$.

3 (l, r) and (r, l)-derivations of BCI-algebras

From now on, in this part, X is a BCI-algebra, B is a BCK-part and P is a P-semisimple part of X, unless otherwise stated.

In this section, we verify some useful properties of (l, r) and (r, l)-derivations of BCI-algebra X and used this properties to classify (l, r) and (r, l)-derivations in P-semisimple BCI-algebras. We show that any (r, l) or (l, r)-derivation of X is an one-to-one map, if X is a P-semisimple BCI-algebra. Moreover, any (r, l) or (l, r)-derivation which is one-to-one, is an identity map in BCK-algebras. Then we find the relation between d-invariant ideals of X and $d(0)$, for any (l, r) and (r, l)-derivations of X. Then we defined a congruence relation for (r, l) derivation d of BCI-algebra X we denote by X/d the set of all equivalence classes of this relation. Also, we prove that X/d is a BCI-algebra. Finally, we define the concept of conjugate for derivations and show that if d and d' are conjugate derivations of a BCI-algebra X, then $X/d \cong X/d'$.

Lemma 3.1. *Let d be a self map of X and $a = d(0)$.*

(i) *If d is an (r, l)-derivation of X, then $x * d(x)$ and $d(x) * x$ belong to the branch $V(d(0))$.*

(ii) *If d is an (l, r)-derivation of X, then $x * d(x) \in V(0 * a)$ and $d(x) * x \in V(a)$, for any $x \in X$.*

Proof. (i) Let $x \in X$. Since d is an (r, l)-derivation of X and $0 \in G(X)$, then by Proposition 2.9(i), $a = d(0) \in G(X)$ and so $0 * a = a$. Moreover,

$$\begin{aligned} d(0) &= d((a * x) * (a * x)) \\ &= d((a * (a * x)) * x), \quad \text{by } (BCI7) \\ &= ((a * (a * x)) * d(x)) \wedge (d(a * (a * x)) * x) \\ &\leq (a * (a * x)) * d(x). \end{aligned}$$

Since $a \in P$ and $a * d(x), a * x \in X$, then by $(P6)$, $a * (a * d(x)), a * (a * x) \in P$ and so $(a * (a * d(x))) * x, (a * (a * x)) * d(x) \in P$. Then by $(BCI7)$ and $(BCI1)$, $d(0) = (a*(a*x))*d(x) = (a*d(x))*(a*x) \leq x*d(x)$. Hence $x*d(x) \in V(a)$. By Proposition 2.9(i) and $(BCI8)$, we have $0*(x*d(x)) \leq 0*a = a$, and so $(BCI5)$ implies $(0*x)*(0*d(x)) \leq a$. Since $a \in P$ and any element of P is minimal element of X we have $(0 * x) * (0 * d(x)) = a$. Now, $(BCI1)$, implies $a \leq d(x) * x$ and so $d(x) * x, \in V(a)$. In a similar way, we can show that $d(x) * x \in V(a)$.
(ii) Let $x \in X$. Since $0 \in G(X)$, then by Proposition 2.8(i), $a = d(0) \in P$. Moreover, $d(0) = d(x*x) = (d(x)*x) \wedge (x*d(x)) \leq d(x)*x$ and so $d(x)*x \in V(a)$. Hence by $(BCI8)$, $0 * (d(x) * x) \leq 0 * a$ and so $(BCI5)$ implies $0 * a = 0 * (d(x) * x) = (0 * d(x)) * (0 * x)$. Now, by $(BCI1)$, we get $0 * a \leq x * d(x)$ and so $x * d(x) \in V(0 * a)$. □

In [12], Y. B. Jun and X. L. Xin proved that (r, l)-derivation ((l, r)-derivation) d is regular if and only if $d(x) = x$, for any $x \in P$. In Theorem 3.2, we will show that if there exists $x \in X$ such that $d(x) = x$, then d is regular. These theorems help us when we want to define regular and irregular (r, l) and (l, r)-derivations. For example if $f : X \to X$ is a self maps such that $f(x) = x$, for some $x \in X$ and $f(0) \neq 0$, then f is not an (r, l) or (l, r)-derivation.

Theorem 3.2. *Let d be an (l,r)-derivation ((r,l)-derivation of X) of X. Then the following are equivalent:*

(i) *d is regular;*

(ii) *d is the identity on P;*

(iii) *there exists $x \in X$ such that $d(x) = x$;*

(iv) *there exists $x \in X$ such that $d(x)$ and x belong to the same branch of X.*

Proof. If d is a regular (l,r)-derivation of X, then by Proposition 2.8(ii), d is identity on P. The proof of $(ii) \to (iii)$ and $(iii) \to (iv)$ are straightforward. Let $d(x), x \in V(a)$, for some $x \in X$ and $a \in P$. Then by the properties of the branches in BCI-algebras, we have $x * d(x) \in V(a * a) = V(0)$. Moreover, by Lemma 3.1(ii), we have $x * d(x) \in V(0 * d(0))$, hence $x * d(x) \in V(0 * d(0)) \cap V(0)$. It follows that $0 * d(0) = 0$ and so by Proposition 2.8(i), $d(0) = 0 * (0 * d(0)) = 0 * 0 = 0$. Therefore, d is a regular (r,l)-derivation of X. The proof of other case is similar. □

Let $|X| = n$ and d be an (r,l)-derivation ((l,r)-derivation) of X. If there exists $x \in X$ such that $d(x) = x$, then by Theorem 3.2, d is regular. Therefore, the number of (r,l)-derivations ((l,r)-derivations) of X which are not regular, is less than $(n-1)^n$.

In Proposition 3.3, we omit the conditions $d(0) = 0$ and attempt to obtain the results of Proposition 2.11, for (r,l)-derivations instead of derivation.

Proposition 3.3. *Let d be an (r,l)-derivation of X. Then*

(i) *there exists $a \in P$ such that $d(x) \leq x * a$, for all $x \in X$ if and only if $d(0) = a$.*

(ii) *$d(x) * x = d(0)$, for all $x \in X$.*

(iii) *$d(x) * (y * d(0)) \leq (x * d(0)) * d(y)$, for any $x, y \in X$.*

(iv) *$d(x * y) \leq (x * d(0)) * (d(y) * d(0))$, for any $x, y \in X$.*

Proof. (i) Let $x \in X$. Then $d(x) = d(x * 0) = (x * d(0)) \wedge (d(x) * 0) = (x * d(0)) \wedge d(x)$. Hence $d(x) \leq x * d(0)$. Conversely, let $d(x) \leq x * a$, for all $x \in X$. Then $d(0) = d(0 * 0) = (0 * a) \wedge a = 0 * a = a$.
(ii) Let $x \in X$. By (i), we have $d(x) \leq x * d(0)$. Therefore, by ($BCI7$) and ($BCI8$) we get $d(x) * x \leq (x * d(0)) * x = 0 * d(0)$. Now, Proposition 2.9(i), implies $d(x) * x = d(0)$.
(iii) Let $x \in X$. Then $d(x) \leq d(x * 0) \leq x * d(0)$ and so by ($BCI8$), $d(x) * (y * d(0)) \leq (x * d(0)) * (y * d(0))$, for any $y \in X$. By the similar way, we obtain $(x * d(0)) * (y * d(0)) \leq (x * d(0)) * d(y)$. Therefore, $d(x) * (y * d(0)) \leq (x * d(0)) * d(y)$.
(iv) Let $x, y \in X$. By (i) and ($BCI7$), we have $d(x * y) \leq (x * y) * d(0) = (x * d(0)) * y$. Moreover, by (ii) and ($BCI7$), we obtain $d(y) * d(0) \leq y$ and so ($BCI8$) implies $(x * d(0)) * y \leq (x * d(0)) * (d(y) * d(0))$. Hence $d(x * y) \leq (x * d(0)) * (d(y) * d(0))$. □

Let X be an (r,l)-derivation of X. By Proposition 3.3(ii), $d(x) * x = d(y) * y$, for any $x, y \in X$.
Note that Jun and Xin in [12] have shown that if d is a regular (r,l)-derivation of a commutative BCI-algebra X, then x and $d(x)$ both belong to the same branch of X, for all $x \in X$ and d is also an (l,r)-derivation of X. In Theorem 3.5, we omit the regular condition. We prove that if X is a commutative BCI-algebra, then any (r,l)-derivation of X is an (l,r)-derivation of X.

Lemma 3.4. *Let d be an (r,l)-derivation of X. Then $(x * d(x)) * (d(y) * y)$, $(x * d(x)) * (y * d(y))$, $(d(x) * x) * (d(y) * y)$ and $(d(x) * x) * (y * d(y))$ belong to the B.*

Proof. Let $x, y \in X$. By Lemma 3.1(i), $x * d(x) \in V(d(0))$ and $d(x) * x \in V(d(0))$, for any $x \in X$. Therefore, $(x * d(x)) * (d(y) * y) \in V(d(0) * d(0)) = V(0) = B$. By the similar way, we can obtain $(x * d(x)) * (y * d(y))$, $(d(x) * x) * (d(y) * y)$ and $(d(x) * x) * (y * d(y))$ belong to B. □

Theorem 3.5. *Let X be a commutative BCI-algebra.*

(i) *If d is an (r, l)-derivation of X, then d is a regular derivation of X if and only if x and $d(x)$ both belong to the same branch of X, for some $x \in X$.*

(ii) *If d is an (r, l)-derivation of X, then d is also an (l, r)-derivation of X and so d is a derivation of X.*

(iii) *If d is an (l, r)-derivation of X such that $d(0) \in G(X)$, then d is also an (r, l)-derivation of X.*

Proof. (i) Let $x \in X$ such that $x, d(x) \in V(a)$, for some $a \in P$. Then by Theorem 3.2, d is a regular (r, l)-derivation of X. Hence by Proposition 2.10, d is a regular derivation of X. Conversely, if d is an (r, l)-derivation of X, then d is an (r, l) and (l, r) derivation and so by Proposition 2.10, x and $d(x)$ both belong to the same branch of X, for all $x \in X$.
(ii) Let $x, y \in X$. Then there are $a, b \in P$ such that $x * d(y) \in V(a)$ and $d(x) * y \in V(b)$ and so $(x * d(y)) * (d(x) * y) \in V(a * b)$. By Lemma 3.4, we have $(x * d(x)) * (d(y) * y) \in B$ and so

$$
\begin{aligned}
0 &= 0 * [(x * d(x)) * (d(y) * y)] \\
&= [(0 * x) * (0 * d(x))] * [(0 * d(y)) * (0 * y)], \quad \text{by } (BCI5) \\
&= [(0 * [(0 * d(y)) * (0 * y)]) * (0 * d(x))] * x, \quad \text{by } (BCI7) \\
&= [[(0 * (0 * d(y))) * (0 * (0 * y))] * (0 * d(x))] * x, \quad \text{by } (BCI5) \\
&= [((0 * x) * (0 * d(y))) * (0 * (0 * y))] * (0 * d(x)), \quad \text{by } (BCI7) \\
&= [(0 * (x * d(y))) * (0 * (0 * y))] * (0 * d(x)), \quad \text{by } (BCI5) \\
&= [(0 * (0 * d(x))) * (0 * (0 * y))] * (x * d(y)), \quad \text{by } (BCI7) \\
&= [0 * (0 * (d(x) * y))] * (x * d(y)), \quad \text{by } (BCI5) \\
&= (0 * (x * d(y))) * (0 * (d(x) * y)), \quad \text{by } (BCI7) \\
&= 0 * [(x * d(y)) * (d(x) * y)], \quad \text{by } (BCI7)
\end{aligned}
$$

Hence $(x * d(y)) * (d(x) * y) \in V(0) \cap V(a * b)$ and so $a * b = 0$. Since $a, b \in P$, then by $(P3)$, $a = b$. By Theorem 2.5, we obtain $(x * d(y)) \wedge (d(x) * y) = (d(x) * y) \wedge (x * d(y))$. Therefore, d is a derivation of X.
(iii) Since $d(0) \in G(X)$, then by Lemma 3.1(ii), $x * d(x), d(x) * x \in V(d(0))$, for any $x \in X$. Hence $(x * d(x)) * (d(y) * y) \in B$, for any $x, y \in X$. Similar to the proof of (ii), d is an (r, l)-derivation of X, too. □

Lemma 3.6. *Let d be an (r, l)-derivation of X. If there exists $x \in X$ such that $d(x) \in B$, then $d(0) = 0 * x$.*

Proof. Let $x \in X$, such that $d(x) \in B$. Then $0 * d(x) = 0$. Since d is an (r, l)-derivation and $0 * d(x) \in P$, we have $d(0 * x) = 0 * d(x)$ and so $d(0 * x) = 0$. Hence $(0 * x) * d(0 * x) = 0 * x$ and so $(0 * x) * d(0 * x) \in V(0 * x)$. Moreover, by Lemma 3.1, $(0 * x) * d(0 * x) \in V(a)$, where $a = d(0)$. Therefore, $a = 0 * x$. □

Note that, if d is an (r, l)-derivation on X and $d(x) \in B$, for some $x \in X$, then by Lemma 3.6 and Proposition 2.9(i), we have $0 * (0 * x) = 0 * x$. Hence if $y \in X$ such that $d(y) \in B$, then $0 * x = 0 * y$.

By Proposition 2.11, we know that if d is a regular derivation of X, then $d^{-1}(0)$ is a subalgebra of X and $d^{-1}(0) \subseteq B$. In the next proposition, we omit the regularity of d and generalize this result.

Proposition 3.7. *Let d be a derivation of X and $a = d(0)$.*
(i) *If $d(y) = d(0)$, then $0 * y = 0$. Moreover, $d(a) = 0$;*
(ii) *$d^{-1}(\{0, a\})$ is a subalgebra of X;*
(iii) *$d^{-1}(\{0, a\}) \subseteq V(0) \cup V(a)$.*

Proof. (i) Let $y \in X$ such that $d(y) = d(0)$. Then by Proposition 2.9(i), we have $0 * d(y) = 0*a = a$. Now, since d is an (r, l)-derivation of X, then $d(0*y) = (0*d(y))*(d(0)*y) \leq 0*d(y)$ and $0 * d(y) \in P$ and so $d(0 * y) = 0 * d(y)$. On the other hand, d is an (l, r)-derivation and $d(0 * y) = (d(0) * y) \wedge (0 * d(y)) \leq d(0) * y$. By $0 * y \in P$, Proposition 2.8(i) and $(P6)$, $d(0*y), d(0)*y \in P$ and so $d(0*y) = d(0)*y$. Hence $a = 0*d(y) = d(0*y) = d(0)*y = a*y$ and so by $(BCI7)$, $0 * y = (a * y) * a = a * a = 0$. Therefore, $0 * y = 0$.
Using Proposition 2.9(i), we get $d(a) = d(0 * a) = (d(0) * a) \wedge (0 * d(a)) = 0 \wedge (0 * d(a)) = 0$ and so $d(a) = 0$.
(ii) By (i), $0 \in d^{-1}(\{0, a\})$. Let $x, y \in d^{-1}(\{0, a\})$. Then we have $d(x) = 0$ or $d(x) = a$ and $d(y) = 0$ or $d(y) = a$.
(1) If $d(x) = a$ and $d(y) = 0$, then

$$\begin{aligned} d(x*y) &= (d(x)*y) \wedge (x*d(y)), \quad \text{since } d \text{ is an } (l,r)\text{-derivation} \\ &= (d(x)*y), \text{ since } d(x)*y \in P \\ &= a*y, \quad \text{since } d(x) = a \\ &= (0*a)*y, \quad \text{by Proposition 2.9(i)} \\ &= (0*y)*a, \quad \text{by } (BCI7) \\ &= a*a = 0, \text{ by Lemma 3.6.} \end{aligned}$$

(2) If $d(x) = 0$ and $d(y) = a$, then $d(x*y) = (d(x)*y) \wedge (x*d(y)) = (0*y) \wedge (x*d(y))$. Since $0 * y \in P$, we have $d(x * y) = 0 * y$. Now, by (i) we have $0 * y = 0$. Therefore, $d(x * y) = 0$ and so $x * y \in d^{-1}(\{0\})$.
(3) If $d(x) = 0 = d(y)$, then

$$\begin{aligned} d(x*y) &= (d(x)*y) \wedge (x*d(y)), \quad \text{since } d \text{ is an } (r,l)\text{-derivation} \\ &= (0*y) \wedge (x*d(y)) \\ &= 0*y, \quad \text{since } 0*y \in P \\ &= d(0) = a, \quad \text{by Lemma 3.6} \end{aligned}$$

(4) If $d(x) = a = d(y)$, then by (i), $d(d(x)) = d(a) = 0$. Hence $d(x) \in d^{-1}(0)$ and $y \in d^{-1}(a)$ and so by (2), $d(x)*y \in d^{-1}(0)$. Now, since $d(x*y) = (d(x)*y) \wedge (x*d(y))$ and $d(x) * y = a * y \in P$ we have $d(x * y) = d(x) * y \in d^{-1}(0)$. Therefore, (1), (2), (3), and (4) imply $d(x * y) \in d^{-1}(\{0, a\})$ and $0 \in d^{-1}(\{0, a\})$ and so $d^{-1}(\{0, a\})$ is a subalgebra of X.
(iii) Let $x \in d^{-1}(\{0, a\})$. Then $d(x) = 0$ or $d(x) = a$. If $d(x) = 0$, then by Lemma 3.6, $a = d(0) = 0 * x$ and so Proposition 2.9(i) implies $a = 0 * a = 0 * (0 * x)$. Hence $x \in V(a)$. If $d(x) = a$, then (i) implies, $0*x = 0$ and so $x \in V(0)$. Therefore, $d^{-1}(\{0, a\}) \subseteq V(0) \cup V(a)$. □

Theorem 3.8. *Let X be a BCK-algebra.*

(i) *Let d be an (l,r)-derivation of X. If d is an one-to-one map, then d is the identity map.*

(ii) *Let d be an (r,l)-derivation of X. If d is an one-to-one map, then d is the identity map.*

Proof. (i) Let d be an one-to-one (l,r)-derivation of X and $x \in X$. Then by Proposition 2.7, d is regular and

$$d(d(x) * x) = (d(d(x)) * x) \wedge (d(x) * d(x)) = (d(d(x)) * x) \wedge 0 = 0 \tag{3.1}$$

$$d(x * d(x)) = (d(x) * d(x)) \wedge (x * d(d(x))) = 0 \wedge (x * d(d(x))) = 0 \tag{3.2}$$

Since d is one-to-one, then (3.1) and (3.2) imply $x * d(x) = 0 = d(x) * x$. Now, by $(BCI3)$, we obtain $d(x) = x$. Therefore, d is an identity map on X.
(ii) Similar to (i). □

Let d be a regular (r,l)-derivation of X. By Proposition 2.9, $d(P) \subseteq P$, $d(y) = y$, for any $y \in P$ and $0 = d(0) = d(0 * x) = (0 * d(x)) \wedge (d(0) * x) = 0 * d(x)$, for any $x \in B$ and so $d(B) \subseteq B$. Hence $d|_B : B \to B$ is a regular (r,l)-derivation of B. Moreover, if $d|_B$ is one-to-one, then by Theorem 3.8, $d(x) = x$, for any $x \in B$. Now, let d be an one-to-one regular (r,l)-derivation of X. Then d is identity on P and B. Let $x \in X$. Since $x*(0*(0*x)) \in B$ we have $x*(0*(0*x)) = d(x*(0*(0*x))) = (d(x)*(0*(0*x)))\wedge(x*d(0*(0*x))) \leq d(x)*(0*(0*x))$. On the other hand, since d is regular, then by Proposition 2.11(i), $d(x) \leq x$. Hence by $(BCI8)$, we get $d(x) * (0 * (0 * x)) \leq x * (0 * (0 * x))$. Therefore, if d is an (r,l)-regular derivation of X, then $d(x) * (0 * (0 * x)) = x * (0 * (0 * x))$, for any $x \in X$.

Theorem 3.9. (i) *Let d be a regular (r,l)-derivation of X. Then d is one-to-one if and only if $d(x) = 0$, implies $x = 0$, for any $x \in X$.*

(ii) *Let d be a derivation of X. Then $d(x) = 0$ implies $x = 0$, for any $x \in X$ if and only if $d(x) = x$, for any $x \in X$.*

(iii) *Any one-to-one regular derivation of X is an identity map.*

Proof. (i) Let $d(x) = 0$ imply $x = 0$, for any $x \in X$. If $d(x) = d(y)$, for some $x, y \in X$, then $d(x) * d(y) = 0$. Since d is a regular (r,l)-derivation, then by Proposition 2.11 (iii), $d(x * y) = 0$ and so $x * y = 0$. By the similar way, we can obtain $y * x = 0$. Therefore, by $(BCI3)$, we have $x = y$. The proof of the converse is straightforward.
(ii) Let $x \in X$ and $d(y) = 0$, implies $y = 0$, for any $y \in X$. Since d is an (l,r)-derivation of X, we have $d(x*d(x)) = (d(x)*d(x))\wedge(x*d(d(x))) = 0\wedge(x*d(d(x))) = 0$ and so $x*d(x) = 0$. Moreover, since d is an (l,r)-derivation, then $d(d(x) * x) = (d(x) * d(x)) \wedge (d(d(x)) * x) = 0 \wedge (d(d(x)) * x) = 0$ and so $d(x) * x = 0$. Hence by $(BCI3)$ we obtain $x = d(x)$. Therefore, d is an identity map on X. The proof of the converse is straightforward.
(iii) Let d be an one-to-one regular derivation of X. Then by (i), $d^{-1}(\{0\}) = \{0\}$. Hence by (ii), $d(x) = x$, for any $x \in X$. □

Definition 3.10. Let d be a derivation of X. The element $x \in X$ is called *d-nilpotent* if there exists $n \in \mathbb{N}$, such that $0 * d(x)^n = d(0)$. Moreover, d is called *nilpotent* if and only if any element of X is d-nilpotent. For any $x \in X$, the least natural number n satisfies $0 * d(x)^n = d(0)$, is denoted by $od(x)$.

Example 3.11. Let $X = \{0, 1, 2, 3, 4, 5\}$. Define the binary operation "*" on X by the following table:

*	0	1	2	3	4	5
0	0	0	3	2	3	2
1	1	0	5	4	3	2
2	2	2	0	3	0	3
3	3	3	2	0	2	0
4	4	2	1	5	0	3
5	5	3	4	1	2	0

Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 0, & x = 0, 1, \\ 2, & x = 2, 4, \\ 3, & x = 3, 5. \end{cases}$$

Then d is a derivation of X (see [12], Example 3.6). Clearly, $0, 1$ are d-nilpotent and $od(x) = 1$. It can be easily verified that for all $x \in \{2, 3, 4, 5\}$, $od(x) = 3$.

Lemma 3.12. *If d is an (l, r)-derivation of X and $a \in P$, then $d(a * x^k) = d(a) * x^k$, for any $x \in X$ and $k \in \mathbb{N}$.*

Proof. Let $x \in X$. By Proposition 2.8(i) and $(P6)$, $d(a) * x \in P$ and $d(a * x) = (d(a) * x) \wedge (a * d(x)) \leq d(a) * x$ and so $d(a * x) = d(a) * x$. Now, let $k > 1$ and $d(a * x^{k-1}) = d(a) * x^{k-1}$. Then

$$\begin{aligned} d(a * x^k) &= [d(a * x^{k-1}) * x] \wedge [(a * x^{k-1}) * d(x)] \\ &= [(d(a) * x^{k-1}) * x] \wedge [(a * x^{k-1}) * d(x)] \\ &= (d(a) * x^{k-1}) * x, \quad \text{by Proposition 2.8(i) and } P6 \\ &= d(a) * x^k. \end{aligned}$$

Therefore, $d(a * x^k) = d(a) * x^k$, for any $a \in P$, $x \in X$ and $k \in \mathbb{N}$. □

Lemma 3.13. *Let d be a derivation of X. Then*

$$0 * d(x)^n = \begin{cases} 0 * x^n, & n \text{ is even} \\ d(0) * x^n, & \text{otherwise} \end{cases}$$

Proof. Let $x \in X$. We want to show that $0 * d(x)^n = d^n(0) * x^n$, where $d^{i+1}(x) = d(d^i(x))$, for any $i, n \in \mathbb{N}$. Since d is an (r, l)-derivation of X and $0 * d(x) \in P$, we have $0 * d(x) = d(0 * x) = (d(0) * x) \wedge (0 * d(x)) = d(0) * x$. Now, let $n > 1$ and $0 * d(x)^m = d^m(0) * x^m$, for any $m \leq n - 1$. By Proposition 2.9(i) and $(P6)$, $(d^{n-1}(0) * x^{n-1}) \in P$. Then

$$\begin{aligned} 0 * d(x)^n &= (0 * d(x)^{n-1}) * d(x) \\ &= (d^{n-1}(0) * x^{n-1}) * d(x) \\ &= [(d^{n-1}(0) * x^{n-1}) * d(x)] \wedge [d(d^{n-1}(0) * x^{n-1}) * x], \quad \text{since } (d^{n-1}(0) * x^{n-1}) \in P \\ &= d((d^{n-1}(0) * x^{n-1}) * x), \quad \text{since } d \text{ is an } (r, l)\text{-derivation} \\ &= d(d^{n-1}(0) * x^n). \end{aligned}$$

Hence $0 * d(x)^n = d(d^{n-1}(0) * x^n)$. Since $d^{n-1}(0) \in P$, then by Lemma 3.12, $0 * d(x)^n = d(d^{n-1}(0) * x^n) = d^n(0) * x^n$. Using Proposition 3.7(i), we get $0 * d(x)^n = 0 * x^n$, whenever n is an even integer and $0 * d(x)^n = d(0) * x^n$, where n is an odd integer. □

We know that if X is a BCK-algebra, then any derivation of X is regular (see [1], Proposition 3.1). In the next theorem we show that if there exists an odd integer n such that $0 * x^n = 0$, for any $x \in X$, then any derivation of X is regular.

Theorem 3.14. *Let d be a derivation of X and $x \in X$.*
(i) *If $o(x) = 2k+1$, for some $k \in \mathbb{N}$, then x is d-nilpotent.*
(ii) *If d is regular, then x is nilpotent if and only if x is d-nilpotent.*
(iii) *If there exists an odd integer n such that $0 * x^n = 0 = 0 * d(x)^n$, for some $x \in X$, then d is regular.*

Proof. (i) Let $n = 2k+1$ and $o(x) = n$, for some $k \in \mathbb{N}$. Then

$$\begin{aligned} 0 * d(x)^n &= d(0) * x^n, \quad \text{by Lemma 3.13} \\ &= (0 * d(0)) * x^n, \quad \text{by Proposition 2.9(i)} \\ &= (0 * x^n) * d(0), \quad \text{by } (BCI7) \\ &= 0 * d(0), \quad \text{since } o(x) = n \\ &= d(0), \quad \text{by Proposition 2.9(i)} \end{aligned}$$

(ii) Let d be a regular derivation of X. Then $0 * d(x)^n = 0 * x^n$, whenever n is even integer and $0 * d(x)^n = d(0) * x^n = 0 * x^n$, whenever n is an odd integer and so x is nilpotent if and only if x is d-nilpotent.
(iii) Let $n \in \mathbb{N}$ be odd and such that $0 * x^n = 0 = 0 * d(x)^n$, for some $x \in X$. Then by Lemma 3.12, we have $d(0) = d(0 * x^n) = d(0) * x^n$. Since n is an odd integer by Lemma 3.13, we have $d(0) * x^n = 0 * d(x)^n = 0$ and so $d(0) = 0$. Therefore, d is a regular derivation. □

Note that, if X is a BCI-algebra such that $0 * x^n = 0$, for any $x \in X$ and some odd integer $n \in \mathbb{N}$, then by Theorem 3.14(iii), any derivation of X is regular.
In the next theorem, we want to classify the (r,l) and (l,r)-derivations of X, when X is a P-semisimple algebra.

Theorem 3.15. *Let X be a P-semisimple BCI-algebra and $d : X \to X$ be a map.*
(i) *d is an (r,l)-derivation of X if and only if there exists $a \in G(X)$ such that $d(x) = x*a$, for any $x \in X$;*
(ii) *d is an (l,r)-derivation of X if and only if there exists $a \in X$ such that $d(x) = a * (0 * x)$, for any $x \in X$;*
(iii) *If d is an (r,l) or (l,r)-derivation of X, then d must be one-to-one;*
(iv) *If d is an (r,l)-derivation of X, then d is a (l,r)-derivation. The converse is true, whenever $d(0) \in G(X)$.*

Proof. (i) Let d be an (r,l)-derivation of X and $x \in X$. Then by Proposition 2.9(i), $d(0) \in G(x)$ and $d(x) = d(x * 0) = (x * d(0)) \wedge (d(x) * 0) \le x * d(0)$. Since any element of X is minimal we obtain $d(x) = x * d(0)$. Conversely, let there exist $a \in G(X)$ such that $d(x) = x * a$, for any $x \in X$. Then $d(x * y) = (x * y) * a$ and $(x * d(y)) \wedge (d(x) * y) = x * d(y)$. Moreover,

$$\begin{aligned} x * d(y) = x * (y * a) &= (0 * (0 * x)) * (y * a), \quad \text{since } X \text{ is a } P\text{-semisimple } BCI\text{-algebra} \\ &= (0 * (y * a)) * (0 * x) = ((0 * y) * (0 * a)) * (0 * x), \quad \text{by } (BCI5) \\ &= ((0 * y) * (0 * x)) * a, \quad \text{by } (BCI7) \text{ and Proposition 2.9(i)} \\ &= ((0 * (0 * x)) * y) * a, \quad \text{by } (BCI7) \\ &= (x * y) * a, \quad \text{since } X \text{ is a } P\text{-semisimple } BCI\text{-algebra} \\ &= d(x * y). \end{aligned}$$

Hence d is an (r,l)-derivation of X.
(ii) Let d be an (l,r)-derivation of X and $x \in X$. Then $d(x) = d(0*(0*x)) = (d(0)*(0*x)) \wedge (0*(d(0*x))) = d(0)*(0*x)$. Conversely, let there exists $a \in X$, such that $d(x) = a*(0*x)$, for any $x \in X$. Then by properties $(P1)$, $(P2)$ and $(P3)$ of P-semisimple BCI-algebras we have

$$\begin{aligned} d(x*y) = a*(0*(x*y)) &= (0*(0*a))*(0*(x*y)) \\ &= (0*(0*(x*y)))*(0*a) = (x*y)*(0*a) \\ &= (x*(0*a))*y = (((0*(0*x))*(0*a)))*y \\ &= ((0*(0*a))*(0*x))*y = (a*(0*x))*y \\ &= d(x)*y = (d(x)*y) \wedge (x*d(y)). \end{aligned}$$

Therefore, d is an (l,r)-derivation of X.
(iii) Let d be an (r,l)-derivation of X and $d(x) = d(y)$, for some $x, y \in X$. Then by (i), we have $x*d(0) = d(x) = d(y) = y*d(0)$. The left cancellation law in the P-semisimple BCI-algebras implies $x = y$. Hence d is one-to-one. Now, let d be an (l,r)-derivation of X and $d(x) = d(y)$, for some $x, y \in X$. Then by (ii), we get $d(0)*(0*x) = d(x) = d(y) = d(0)*(0*y)$. The right cancellation law in P-semisimple BCI-algebras implies $0*x = 0*y$ and so $x = 0*(0*x) = 0*(0*y) = y$. Therefore, d is one-to-one.
(iv) Let d be an (r,l)-derivation of X, then by (i), there exists $a \in G(X)$ such that $d(x) = x*a$, for any $x \in X$ and so $d(0) = 0*d(0) = 0*(0*a) = a$. Hence $d(x) = (0*(0*x))*a$, for any $x \in X$. Since $a \in G(X)$, then by $(BCI7)$ we get $d(x) = a*(0*x)$, for any $x \in X$. Now, by (ii), we have d is an (l,r)-derivation of X. Moreover, if d is an (l,r)-derivation of X such that $d(0) \in G(X)$. Then by (ii), we have $d(x) = a*(0*x) = (0*a)*(0*x) = (0*(0*x))*a = x*a$, for any $x \in X$. Hence (i), implies d is an (r,l)-derivation of X. □

By Theorem 3.15, we conclude that if X is a P-semisimple BCI-algebra and $|X| = n$, for some $n \in \mathbb{N}$, then X has n (l,r)-derivations and has $|G(X)|$ (r,l)-derivations.

Corollary 3.16. (i) *The BCI-algebra X is P-semisimple if and only if every (l,r)-derivations of X is one-to-one.*

(ii) *If d is an (r,l)-derivation of X, then $d(x) = d(y)$ implies $0*x = 0*y$, for any $x, y \in X$.*

Proof. (i) Let X be a P-semisimple BCI-algebra. Then by Theorem 3.15 (iii), any (l,r)-derivation of X is one-to-one. Conversely, let any (l,r)-derivation of X is one-to-one. Then $d : X \to X$, was defined by $d(x) = 0*(0*x)$ is an (l,r)-derivation and so is one-to-one. Therefore, BCK part of X is equal to $\{0\}$ and so X is a P-semisimple BCI-algebra.
(ii) Let $d(x) = d(y)$, for some $x, y \in X$. Since d is an (r,l)-derivation and $0*d(x) \in P$, then $d(0*x) = 0*d(x) = 0*d(y) = d(0*y)$. By Theorem 3.15(iii), $d|_P$ is an one-to-one map. Hence $0*x = 0*y$. □

Note that, if d is an (r,l)-derivation of X and $d(x) = d(y)$, for some $x, y \in X$, then by Corollary 3.16(ii), we conclude that x, y belong to the same branch of X.

Corollary 3.17. *Let d and d' be two (r,l) or (l,r)-derivations of X. If there exists $x \in X$, such that $d(x) = d'(x)$, then $d|_P = d'|_P$.*

Proof. Let d and d' be two (l,r)-derivations of X and $d(x) = d'(x)$, for some $x \in X$. Then $d(0) = d(x*x) = (d(x)*x) \wedge (x*d(x)) = (d'(x)*x) \wedge (x*d'(x)) = d'(x*x) = d'(0)$. Since $d|_P$ and $d|_{P'}$ are two (r,l)-derivations of P, then by Theorem 3.15(ii), we conclude that $d|_P = d'|_P$. By the similar way, we obtain the following issue for (r,l)-derivations. □

We know that if d is a regular derivation of X, then every ideal of X is d-invariant (see [12], Proposition 3.20). In the next theorem, we indicate the d-invariant ideals of X, for any (r, l) or (l, r)-derivation of X. It is depend on $d(0)$.

Theorem 3.18. *Let A be an ideal of X.*

(i) *If d is an (r, l)-derivation of X, then A is d-invariant if and only if $d(0) \in A$.*

(ii) *If d is an (l, r)-derivation of X, then A is d-invariant if and only if $d(0) \in A$.*

Proof. (i) Let A be an d-invariant ideal of X. Then clearly, $d(0) \in A$. Conversely, let $d(0) \in A$ and $x \in A$. Then by Proposition 3.3(ii), $d(x) * x = d(0) \in A$, $x \in A$ and A is an ideal of X and so $d(x) \in A$. Therefore, $d(A) \subseteq A$ and so A is a d-invariant ideal of X.
(ii) Clearly, if A is a d-invariant ideal of X, then $d(0) \in A$. Conversely, let $d(0) \in A$ and $x \in A$. Then by Proposition 2.8(iii), we have $d(x) = d(x) \wedge x = x * (x * d(x))$ and so $(BCI7)$ implies $d(x) * x = (x * x) * (x * d(x)) = 0 * (x * d(x))$. By Lemma 3.1(ii), we have $x * d(x) \in V(0 * a)$, where $a = d(0)$ and so by $(BCI8)$, $0 * (0 * a) = 0 * (x * d(x))$. Hence by Proposition 2.8(i), $d(x) * x = 0 * (0 * a) = a$, which implies $d(x) * x \in A$ and $x \in A$ and so $d(x) \in A$. Therefore, A is d-invariant. □

Theorem 3.18 implies that if d is an (r, l) or (l, r)-derivation of X and A is a d-invariant ideal of X, then any ideal containing A is a d-invariant ideal of X.

Definition 3.19. [20] For each ideal I of X the relation θ_I defined by $(x, y) \in \theta_I$ if and only if $x * y, y * x \in I$, is a congruence relation on X and the algebra $(X/I, *, I_0)$ is a BCI-algebra, where $I_x = \{y \in X | (x, y) \in \theta_I\}$, $X/I = \{I_x | x \in X\}$ and $I_x * I_y = I_{x*y}$, for all $x, y \in X$.

Proposition 3.20. *([20], Proposition 1.5.1) If θ is a congruence relation of BCI-algebra X, then the class of X, which contains 0 is a closed ideal of X.*

In the next theorem, we defined a relation Φ on X, for a regular (r, l)-derivation d of X. We show that it is a congruence relation and the algebra $(X/\Phi, *, [0])$ is a BCI-algebra, where X/Φ is the set of all equivalence classes of X with respect to Φ.

Theorem 3.21. *Let d be an (r, l)-regular derivation of X.*

(i) *The relation Φ defined by $x\Phi y$ if and only if $x * d(y) \in B$ and $y * d(x) \in B$, for any $x, y \in X$, is a congruence relation of X.*

(ii) *If $[x]$ is an equivalence class of x with respect to Φ and X/Φ be the set of all equivalence classes of X with respect to Φ, then $(X/\Phi, *, [0])$ is a P-semisimple BCI-algebra.*

Proof. (i) Since $d(0) = 0$, then by Lemma 3.1(i), we have $x * d(x) \in B$, for any $x \in X$ and so Φ is reflexive. Clearly, Φ is symmetric. Now, let $x\Phi y$ and $y\Phi z$, for some $x, y, z \in X$. Then $x * d(y), y * d(x) \in B$ and $y * d(z), z * d(y) \in B$. Since B is a subalgebra of X, we have $(x * d(y)) * (z * d(y)) \in B$ and so $(BCI1)$, implies $x * z \in B$. Since d is a regular (r, l)-derivation, then $0 = d(0) = d(0 * (x * z)) = 0 * d(x * z)$ and so $0 = 0 * d(x * z) = 0 * ((x * d(z)) \wedge (d(x) * z)) = 0 * (x * d(z))$. Hence $x * d(z) \in B$. By the similar way, we obtain $z * d(x) \in B$ and so $x\Phi z$. Therefore, Φ is transitive. Moreover, if $x\Phi y$ and $u\Phi v$, for some $x, y, u, v \in X$, then $x * d(y), y * d(x) \in B$ and $u * d(v), v * d(u) \in B$. Also,

$$
\begin{aligned}
&0 * [(x * u) * d(y * v)] \\
&\quad = (0 * (x * u)) * (0 * d(y * v)), \quad \text{by } (BCI5) \\
&\quad = ((0 * x) * (0 * u)) * (0 * (d(y) * d(v))), \quad \text{by Proposition 2.11(iii) and } (BCI5) \\
&\quad = ([0 * (0 * (d(y) * d(v)))] * (0 * u)) * x, \quad \text{by } (BCI7) \\
&\quad = ([(0 * (0 * d(y)))(0 * (0 * d(v)))] * (0 * u)) * x, \quad \text{by } (BCI5) \text{ and } (BCI7) \\
&\quad = ([(0 * (0 * u))(0 * (0 * d(v)))] * (0 * d(y))) * x, \quad \text{by } (BCI7) \\
&\quad = ([0 * (0 * (u * d(v)))] * (0 * d(y))) * x, \quad \text{by } (BCI7) \\
&\quad = (0 * (0 * d(y))) * x, \quad \text{since } u * d(v) \in B \\
&\quad = (0 * x) * (0 * d(y)) = 0 * (x * d(y)), \quad \text{by } (BCI5) \text{ and } (BCI7) \\
&\quad = 0, \quad \text{since } x * d(y) \in B
\end{aligned}
$$

Hence $(x * u) * d(y * v) \in B$. By the similar way, we have $(y * v) * d(x * u) \in B$ and so $(x * u)\Phi(y * v)$. Therefore, Φ is a congruence relation of X.
(ii) By (i), $(X/\Phi, *, [0])$ is an algebra of type (2,0). Clearly, X/Φ satisfies $(BCI1)$ and $(BCI2)$. Let $[x] * [y] = [0] = [y] * [x]$, for some $x, y \in X$. Then $[x * y] = [0] = [y * x]$ and so $(x * y) * d(0) \in B$ and $0 * d(x * y) \in B$. Since d is an (r, l)-derivation of X and $0 * (x * d(y)) \in P$, then $0 * d(x * y) = (0 * (x * d(y))) \wedge (0 * (d(x) * y)) = 0 * (x * d(y))$ and so $0 * (x * d(y)) \in B$. Hence $0 * (x * d(y)) \in B \cap P = \{0\}$ and so $x * d(y) \in B$. By the similar way, we obtain $y * d(x) \in B$. Hence $x\Phi y$ and so $(X/\Phi, *, [0])$ is a BCI-algebra. Now, by Proposition 2.11(i), we have $(0 * (0 * x)) * d(x) = (0 * d(x)) * (0 * x)) = 0 * (d(x) * x) = 0$ and $x * d(0 * (0 * x)) = x * (0 * (0 * x)) \in B$. Therefore, $[0] * ([0] * [x]) = [x]$, for any $x \in X$ and so X/Φ is a P-semisimple BCI-algebra. □

Lemma 3.22. *Let $a \in G(X)$. Then $a * (x * y) = (a * x) * (0 * y)$, for any $x, y \in X$.*

Proof. Let $x, y \in X$.

$$
\begin{aligned}
a * (x * y) &= (0 * a) * (x * y), \quad \text{since } a \in G(X) \\
&= (0 * (x * y)) * a = ((0 * x) * (0 * y)) * a, \quad \text{by } (BCI5) \text{ and } (BCI7) \\
&= ((0 * a) * (0 * y)) * x, \quad \text{by } (BCI7) \\
&= (a * (0 * y)) * x = (a * x) * (0 * y), \quad \text{by } (BCI7) \quad .
\end{aligned}
$$

□

Theorem 3.23. *Let d be a derivation of X and $a = d(0)$. Then the relation θ, was defined by $x\theta y$ if and only if $x * d(y) \in V(a)$ and $y * d(x) \in V(a)$, for any $x, y \in X$ is a congruence relation of X.*

Proof. (i) By Lemma 3.1(i), θ is reflexive. Clearly, θ is symmetric. Now, let $x\theta y$ and $y\theta z$, for some $x, y, z \in X$. Then $x * d(y), y * d(x) \in V(a)$ and $y * d(z), z * d(y) \in V(a)$ and so $(x * d(y)) * (z * d(y)) \in V(a * a) = B$. By $(BCI1)$, we have $x * z \in B$ and so

$$
\begin{aligned}
a = d(0) = d(0 * (x * z)) &= (0 * d(x * z)) \wedge (d(0) * (x * z)) \\
&= 0 * d(x * z) = 0 * ((x * d(z)) \wedge (d(x) * z)) = 0 * (x * d(z)).
\end{aligned}
$$

By Proposition 2.9(i), $a = 0 * a = 0 * (0 * (x * d(z)))$. Hence by $(BCI7)$ and $(BCI4)$, $x * d(z) \in V(a)$. By the similar way, $z * d(x) \in V(a)$. Therefore, θ is transitive. Now, let $x\theta y$ and $u\theta v$, for some $x, y, u, v \in X$. Then $x * d(y), y * d(x) \in V(a)$ and $u * d(v), v * d(u) \in V(a)$. Now, we show that $(x * u) * d(y * v) \in V(a)$. For this,

$$\begin{aligned}
& a*((x*u)*d(y*v)) \\
= \;& (a*(x*u))*(0*d(y*v)), \quad \text{by Lemma 3.22} \\
= \;& ((a*x)*(0*u))*(0*d(y*v)), \quad \text{by Lemma 3.22} \\
= \;& ((a*x)*(0*u))*((0*(y*d(v)))\wedge(0*(d(y)*v))), \quad \text{by } (BCI5) \\
= \;& ((a*x)*(0*u))*(0*(y*d(v))), \quad \text{since } 0*(y*d(v))\in P \\
= \;& ((a*x)*(0*u))*((0*y)*(0*d(v))), \quad \text{by } (BCI5) \\
= \;& ((a*((0*y)*(0*d(v))))*(0*u))*x, \quad \text{by } (BCI7) \\
= \;& (((a*(0*y))*(0*(0*d(v))))*(0*u))*x, \quad \text{by Lemma 3.22} \\
= \;& (((a*(0*u))*(0*(0*d(v))))*(0*y))*x, \quad \text{by } (BCI7) \\
= \;& (((a*(0*(u*d(v)))))*(0*y))*x, \quad \text{by } (BCI5) \text{ and Lemma 3.22} \\
= \;& ((a*x)*(0*y))*(0*(u*d(v))), \quad \text{by } (BCI7) \\
= \;& (d(0*x)*(0*y))*(0*(u*d(v))), \quad \text{by Lemma 3.12} \\
= \;& d((0*x)*(0*y))*(0*(u*d(v))), \quad \text{by Lemma 3.12} \\
= \;& d(0*(x*y))*(0*(u*d(v))), \quad \text{by } (BCI5) \\
= \;& ((0*d(x*y))\wedge(d(0)*(x*y)))*(0*(u*d(v))), \quad \text{since } d \text{ is an } (r,l)\text{-derivation} \\
= \;& (0*d(x*y))*(0*(u*d(v))), \quad \text{since } 0*d(x*y)\in P \\
= \;& ((0*(x*d(y)))\wedge(0*(d(x)*y)))*(0*(u*d(v)), \quad \text{since } d \text{ is an } (r,l)\text{-derivation} \\
= \;& (0*(x*d(y)))*(0*(u*d(v))), \quad \text{since } 0*(x*d(y))\in P \\
= \;& 0*((x*d(y))*(u*d(v))), \quad \text{by } (BCI5) \\
= \;& 0, \quad \text{since } (x*d(y))*(u*d(v)))\in V(a*a)=B
\end{aligned}$$

Hence $a \leq (x*u)*d(y*v)$ and so $(x*u)*d(y*v) \in V(a)$. By the similar way, $(y*v)*d(x*u) \in V(a)$. Therefore, θ is a congruence relation of X. □

By Theorem 3.23, if d is a derivation of X and $a = d(0)$, then the relation θ, was defined by $x\theta y$ if and only if $x * d(y) \in V(a)$ and $y * d(x) \in V(a)$, for any $x, y \in X$ is a congruence relation of X. Let [x] be the congruence class of X containing x, with respect to θ, for any $x \in X$ and $X/\theta = \{[x]|x \in X\}$. Then by Theorem 3.23, $(X/\theta, *, [0])$ is an algebra of type $(2, 0)$, where $[x] * [y] = [x * y]$, for any $x, y \in X$. In Theorem 3.24, we show that it is a BCI-algebra and we denote this BCI-algebra by X/d.

Theorem 3.24. *Let d be a derivation of X and θ be the relation, which is defined in the Theorem 3.23. Then $(X/\theta, *, [0])$ is a P-semisimple BCI-algebra.*

Proof. Let $a = d(0)$. By definition of "$*''$, we conclude that $X/\theta = \{[x]|x \in X\}$ satisfies in $(BCI1)$ and $(BCI2)$. Now, let $[x] * [y] = [0] = [y] * [x]$. Then $[x * y] = [0] = [y * x]$ and so $0 * d(x * y) \in V(a)$. Since d is an (r, l)-derivation of X, then by $(BCI5)$ we get

$$0*d(x*y) = 0*((x*d(y))\wedge(d(x)*y)) = (0*(x*d(y)))\wedge(0*(d(x)*y)) = 0*(x*d(y)).$$

Hence by Proposition 2.9(i), $0 * (0 * (x * d(y))) \in V(0 * a) = V(a)$ and so $x * d(y) \in V(a)$. By the similar way, we obtain $y * d(x) \in V(a)$. Therefore, $[x] = [y]$ and so $(X/\theta, *, [0])$ is a BCI-algebra. Now, let $x \in X$. Then by Lemma 3.1, $(0*(0*x))*d(x) = (0*d(x))*(0*x) = 0*(d(x)*x) = 0*a = a$. Hence $(0*(0*x))*d(x) \in V(a)$. Since d is an (r, l)-derivation of X, then $d(0*(0*x)) = 0*(0*d(x))$. By Lemma 3.22, Lemma 3.1 and $(BCI6)$ we obtain $a*(x*d(0*(0*x))) = a*(x*(0*(0*d(x)))) = (a*x)*(0*(0*(0*d(x)))) = (a*x)*(0*d(x)) = a*(x*d(x)) = 0$. Hence $x*d(0*(0*x)) \in V(a)$. Therefore, $[x] = [0*(0*x)]$ and so X/d is a P-semisimple BCI-algebra. □

Lemma 3.25. *Let d be an (r,l)-derivation ((l,r)-derivation) of X and $f : X \to X$ be an isomorphism. Then the map $f \circ d \circ f^{-1}$ is an (r,l)-derivation ((l,r)-derivation) of X.*

Proof. Let d be an (r,l)-derivation of X and $x, y \in X$. Then

$$\begin{aligned} f \circ d \circ f^{-1}(x * y) &= f(d(f^{-1}(x) * f^{-1}(y))) \\ &= f((f^{-1}(x) * df^{-1}(y)) \wedge (df^{-1}(x) * f^{-1}(y))) \\ &= (f(f^{-1}(x)) * f(df^{-1}(y))) \wedge (f(df^{-1}(x)) * f(f^{-1}(y))) \\ &= (x * f \circ d \circ f^{-1}(y)) \wedge (f \circ d \circ f^{-1}(x) * y). \end{aligned}$$

Therefore, $f \circ d \circ f^{-1}$ is an (r,l)-derivation of X. By the similar way, we can prove that, if d is an (l,r)-derivation of X, then $f \circ d \circ f^{-1}$ is an (l,r)-derivation too. □

Definition 3.26. Let d and d' be two derivations of X. We say d and d' are conjugate if there exists an isomorphism $f : X \to X$, such that $d' = f \circ d \circ f^{-1}$.

Theorem 3.27. *Let d and d' be two derivations of X. If d and d' are conjugate, then $X/d \cong X/d'$.*

Proof. Since d and d' are conjugate, there exists an isomorphism $f : X \to X$, such that $d' = f \circ d \circ f^{-1}$. Let $g : X/d \to X/d'$, was defined by $g([x]) = [f(x)]$, for any $x \in X$. We show that g is an isomorphism. Let $[x] = [y]$, for some $x, y \in X$. Then $x * d(y) \in V(d(0))$ and $y * d(x) \in V(d(0))$. Since f is an isomorphism we have $f(d(0)) \in P$. Moreover, $d(0) * (x * d(y)) = 0$ and f is a homomorphism. Hence $f(d(0)) * (f(x * d(y))) = f(d(0) * (y * d(x))) = 0$ and so $f(x * d(y)) \in V(f(d(0)))$. By the similar way, we get $f(y * d(x)) \in V(f(d(0)))$. Hence $f(x) * f \circ d(y) \in V(f \circ d(0))$ and $f(y) * f \circ d(x) \in V(f \circ d(0))$. Since $f^{-1} : X \to X$ is an isomorphism, then $f^{-1}(0) = 0$ and so $f(x) * f \circ d \circ f^{-1} \circ f(y) \in V(f \circ d(f^{-1}(0)))$ and $f(y) * f \circ d \circ f^{-1} \circ f(x) \in V(f \circ d(f^{-1}(0)))$. Hence $f(x) * d'(f(y)) \in V(d'(0))$ and $f(y) * d'(f(x)) \in V(d'(0))$ and so $f(x)\theta' f(y)$, where θ' is a congruence relation induced by d' in Theorem 3.23. Therefore, $[f(x)] = [f(y)]$ and so g is well defined. Now, let $[f(x)] = [f(y)]$, for some $x, y \in X$. Then $f(x) * d'(f(y)) \in V(d'(0))$ and $f(y) * d'(f(x)) \in V(d'(0))$. Since $d' = f \circ d \circ f^{-1}$ and f is a homomorphism, we have $f(x * (d(f^{-1}(f(y))))) \in V(f(d(0)))$ and $f(y * (d(f^{-1}(f(x))))) \in V(f(d(0)))$. Hence $x * d(y) = f^{-1}(f(x * d(y))) \in V(d(0))$ and $y * d(x) = f^{-1}(f(y * d(x)))) \in V(d(0))$ and so $[x] = [y]$. Therefore, g is one-to-one. Clearly, g is onto. Moreover, if $x, y \in X$, then $g([x] * [y]) = g([x * y]) = [f(x * y)] = [f(x) * f(y)] = [f(x)] * [f(y)] = g([x]) * g([y])$ and so g is an isomorphism. Therefore, $X/d \cong X/d'$. □

Example 3.28. Let $X = \{0, 1, 2, 3\}$. Define the binary operation "$*$" on X by the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
a	3	3	3	0

Then $(X, *, 0)$ is a BCK-algebra (see [20]). Let $d : X \to X$ and $d' : X \to X$ be defined by $d(x) = x$ and $d'(x) = 0$, for any $x \in X$. Clearly, d and d' are two derivations of X. Since X is a BCK-algebra $X/d = \{[0]\} = X/d'$. We show that d and d' are not conjugate. If there exists an isomorphism $f : X \to X$, such that $d = f \circ d' \circ f^{-1}$, then $x = d(x) = f(d'(f^{-1}(x))) = 0$, for any $x \in X$, which is impossible. Moreover, if $d' = f \circ d \circ f^{-1}$, then $0 = d'(x) = f(d(f^{-1}(x))) = x$, for any $x \in X$, which is impossible. Hence d and d' are not conjugate. But $X/d \cong X/d'$. Therefore, the converse of Theorem 3.27 is not true, in general.

Theorem 3.29. *Let d be a derivation of X and $I = [0]$, with respect to equivalence relation induced by d. Then $X/d \cong X/I$.*

Proof. By Theorem 3.23, θ is a congruence relation of X and so by Proposition 3.20, I is a closed ideal of X. Let $f : X/I \to X/d$, was defined by $f(I_x) = [x]$. We show that f is a BCI-isomorphism. Let $[x] = [y]$, for some $x, y \in X$. Then $x * y, y * x \in I$ and so $(x * y) * d(0) \in V(d(0))$ and $0 * d(x * y) \in V(d(0))$. Hence by $0 * d(x * y) \in P$ we get $0 * d(x * y) = d(0)$. Since d is an (l, r)-derivation of X and $0 * d(x) * y \in P$, by $(BCI5)$ we have $d(0) = 0 * d(x * y) = [0 * ((d(x) * y))] \wedge [0 * (x * d(y))] = 0 * (d(x) * y) = (0 * d(x)) * (0 * y)$. Hence by $(BCI1)$ we get $y * d(x) \in V(d(0))$. By the similar way, we have $x * d(y) \in V(d(0))$ and so $[x] = [y]$. Therefore, f is well defined. Clearly, f is an onto map. Moreover, $f(I_x * I_y) = f(I_{x*y}) = [x * y] = f(I_x) * f(I_y)$, for any $x, y \in X$ and so f is an onto homomorphism. Now, let $f(I_x) = f(I_y)$, for some $x, y \in X$. Then $x * d(y) \in V(d(0))$ and $y * d(x) \in V(d(0))$. Since d is an (l, r)-derivation of X, then

$$\begin{aligned}
0 * d(x * y) &= 0 * ((d(x) * y) \wedge (x * d(y))) \\
&= [0 * (d(x) * y)] \wedge [0 * (x * d(y))] \quad \text{by } (BCI5) \\
&= 0 * (d(x) * y) \quad \text{Since } 0 * (d(x) * y) \in P \\
&= (0 * d(x)) * (0 * y) \leq y * d(x) \quad \text{by } (BCI5) \text{ and } (BCI1)
\end{aligned}$$

Hence $y * d(x) \in V(d(0))$ and $(P7)$ imply $0 * d(x * y) \in V(d(0)))$. Moreover, by Lemma 3.22 and Proposition 2.9(i), we obtain $d(0) * ((x * y) * d(0)) = (d(0) * (x * y)) * (0 * d(0)) = (d(0) * (x * y)) * d(0)$. Since d is an (l, r)-derivation of X and $0 \in P$, then by Lemma 3.12, $(d(0) * (x * y)) = d(0 * (x * y)) = (0 * d(x * y)) \wedge (d(0) * (x * y) = 0 * d(x * y)$ and so $d(0) * ((x * y) * d(0)) = (0 * d(x * y)) * d(0) = [0 * ((x * d(y))) \wedge (d(x) * y)] * d(0) = (0 * ((x * d(y))) * d(0) = (0 * d(0)) * d(0)$. By Proposition 2.9(i), we get $d(0) * ((x * y) * d(0)) = 0$ and so $(x * y) * d(0) \in V(d(0))$. Hence $[x * y] = [0] = I$. By the similar way, we get $[y * x] = [0] = I$ and so $I_x = I_y$. Therefore, f is one-to-one map and so f is an isomorphism. □

Theorem 3.30. *Let d be a derivation of a P-semisimple BCI-algebra X. Then $X \cong X/d$.*

Proof. Let $g : X/d \to X$ be defined by $g([x]) = x$, for any $x \in X$. If $[x] = [y]$, for some $x, y \in X$, then $x * d(y) \in V(d(0))$ and so $d(0) \leq x * d(y)$. Hence $d(0) = x * d(y)$. Since X is P-semisimple by Lemma 3.1(i), we have $y * d(y) = d(0)$. Hence the left cancellation law implies $x = y$. Clearly, g is an onto map. Moreover, $g([x] * [y]) = g([x * y]) = x * y = g(x) * g(y)$. Therefore, g is an isomorphism. □

Corollary 3.31. *X is a P-semisimple BCI-algebra if and only if $X/d \cong X$, for some derivation d of X.*

Proof. If there exists a derivation d on X such that $X \cong X/d$, then by Theorem 3.24 X is a P-semisimple BCI-algebra. Conversely, since X is a P-semisimple BCI-algebra, then it can be easily shown that the identity map $Id : X \to X$ is a derivation on X. Thus by Theorem 3.30, $X/d \cong X$. □

4 Conclusions and future works

In this paper, we used the concept of derivation of BCI-algebra, which introduced by Y. B. Jun and X. L. Xin ([12]) and obtained some useful theorem on BCI-algebras. Then we characterized (r, l) and (l, r) derivations on P-semisimple BCI-algebras and attempted to obtain the set of all d invariant ideal, for any (r, l) and (l, r) derivation d. Finally, we constructed a congruence relation from a derivation d and make a quotient BCI-algebra, which was denoted by X/d and verify some properties of it.

References

[1] H. A. S. Abujabal, N. O. Al-shehri, Some results on derivations of BCI-algebras, *Jr. of Natural sciences and Mathematics.*, **46**(2) (2006), 13-19.

[2] H. A. S. Abujabal, N. O. Al-sheri, On left derivations of BCI-algebras, *Soochow Journal of Mathematics.*, **33**(3) (2007), 435-444.

[3] M. Aslam, A. B. Thaheem, A note on P-semisimple BCI-algebras, *Math. Japon.*, **36**(1) (1991), 39-45.

[4] H. E. Bell, L. C. Kappe, Rings in which derivations satisfy certain algebraic conditions, *Acta Math. Hungar.*, **53**(3-4), (1989), 339-346.

[5] H. E. Bell, G. Mason, On derivations in near-rings, Near-Rings and Near-Fields (Tübingen, 1985), *North-HollandMath. Stud.*, **137**, North-Holland, Amsterdam, (1987), 31-35.

[6] M. A. Chaudhry, Branchwise commutative BCI-algebras, *Math. Japon.*, **37**(1) (1992) 163170.

[7] M. Daoji, BCI-algebras and Abelian groups, *Math. Japon.*, **32** (1987) 749-756.

[8] W. P. Huang, Nil-radical in BCI-algebra, *Math. Japon.*, **37**(2) (1992), 363-366.

[9] Y. Imai, K. Iseki, On axiom system of propositional calculi, XIV, *Japan Acad.*, **42** (1966), 19-22.

[10] K. Iseki, An algebra related with a propositional calculus, *Japan Acad.*, **42** (1966), 26-29.

[11] M. A. Javed, M. Aslam, A note on f-derivations of BCI-algebras, *Commun. Korean Math. Soc.*, **24** (2009), 321-331.

[12] Y. B. Jun, X. L. Xin, On derivations of BCI-algebras, *Inform. Sci.*, **159** (2004), 167-176.

[13] K. Kaya, Prime rings with a-derivations Hacettepe, *Bull. Mater. Sci. Eng.*, **16-17**, (1987-1988), 63-71

[14] T. D. Lei, C. C. Xi, P-radical in BCI-algebras, *Math. Japon.*, **30**(4) (1985), 511-517.

[15] J. Meng, Y. B. Jun, E. H. Roh, BCI-algebras of order 6, *Math. Japon.*, **47**(1) (1998), 33-43.

[16] J. Meng, X. L. Xin, Commutative BCI-algebras, *Math. Japon.*, **37**(3)(1992), 569-572.

[17] F. Nisar, On F-derivations of BCI-algebras, *Journal of Prime Research in Mathematics.*, **5** (2009), 176-191.

[18] E. C. Posner, Derivations in prime rings, *Proc. Amer. Math. Soc.*, **8**, (1957), 1093-1100.

[19] X. L. Xin, E. H. Roh, J. C. Li, Some results on the BCI-G-part of BCI-algebras, *Far East J. Math. Sci.* Special Volume (Part III), (1997), 363-370.

[20] H. Yisheng, BCI-algebra, *Science Press,* China, (2006).

[21] J. Zhan, Y. L. Liu, On f-derivations of BCI-algebras, *Internat. J. of Math. Math. Sci.*, **11** (2005), 1675-1684.

[22] Q. Zhang, Some other characterizations of P-semisimple BCI-algebras, *Math. Japon.*, **36**(5) (1991), 815-817.

Communicated by *Milan Vlach*

R. A. Borzooei, Department of Mathematics, Shahid Beheshti University, G. C, Tehran, Iran.
`E-mail address: borzooei@sbu.ac.ir`

O. Zahiri, Department of Mathematics, Shahid Beheshti University, G. C, Tehran, Iran.
`E-mail address:o.zahiri@yahoo.com`

Scientiae Mathematicae Japonicae 77,No.1 (2014) 45-57

TIGHTLY BORDERED CONVEX AND CO-CONVEX SETS

Douglas S. Bridges

Received July 30, 2012 ; revised May 24, 2013

Abstract.We investigate conditions under which a convex or co-convex set in a normed space is tightly bordered, in the sense that a point of the set that is bounded away from its boundary lies in the interior of the set. The investigation lies entirely within a constructive framework.

1 Introduction

We say that a subset S of a metric space is **tightly bordered**, or has a **tight border,** if $x \in S^{\circ}$ for each $x \in S$ with $\rho(x, \partial S) > 0$.[1] Every open set is tightly bordered. With classical logic, the law of excluded middle (**LEM**) leads to every subset of a metric space being tightly bordered. In constructive mathematics,[2] things are not so simple: if $x \in S$ and $\rho(x, \partial S) > 0$, then it is absurd that $x \notin S^{\circ}$; but this information does not, of itself, enable us to compute $r > 0$ such that the ball $B(x, r)$ is contained in S. This is part of a more general difficulty in constructive geometry and analysis: namely, placing a point in a set (a positive conclusion) when all we know is that it cannot fail to belong therein (negative information). This situation really can arise in constructive practice, so it makes sense to try to provide conditions on the set S that ensure, constructively, that it is tightly bordered. We discuss such conditions in this paper, which can be regarded as a continuation of work begun in [9]. That work arose naturally in a constructive study of the Dirichlet problem (for more on which, see [6]); our present study was motivated by an ongoing search for the 'right' definition of a differential manifold in constructive analysis.

Our visual intuition suggests that when we are dealing with a convex set or a co-convex set—that is, the complement of a convex one—in a normed space, we might be able to prove the tightness of the border.[3] In fact, as the Brouwerian examples in the final section of this paper show, even for such relatively special sets, we cannot expect to do that without additional hypotheses. Our main purpose is to discuss, in Sections 2 and 3, conditions under which a convex set C or its complement

$$\sim C \equiv \{x \in X : \forall_{y \in C} (\|x - y\| > 0)\}$$

is tightly bordered. In particular, we show that if, in a Banach space, a convex set C has inhabited interior and $C \cup \sim C$ is dense in X, then both C and $\sim C$ are tightly bordered (Propositions 5 and 13). In the course of our discussion, we also deal with a number of classically trivial, but constructively significant, geometric properties of convex and co-convex sets.

[1] We do not require that ∂S be located: that is, that the distance from any point of X to ∂S exist. Instead, we are using Richman's convention about distance expressions (see [9]), under which, for example, the expression $\rho(x, \partial S) > 0$ means that there exists $r > 0$ such that $\rho(x, y) > r$ for each $y \in \partial S$.

[2] That is, roughly, mathematics with intuitionistic, rather than classical, logic, and with an appropriate foundation such as those presented in [1, 2, 11]. For more on this type of constructive mathematics in practice, see [3, 4, 7, 8].

[3] For convex sets we can often establish results that hold more generally in classical, but not in constructive, analysis. For example, every convex subset C of $\mathbf{R}^n$ with positive Lebesgue measurable is located—that is, $\rho(x, C) \equiv \inf\{\|x - y\| : y \in C\}$ exists for each $x \in \mathbf{R}^n$; see [5].

2 Tightly bordered convex sets Although we assume that the reader has access to one or more of such books on constructive analysis as [3, 4, 7, 8, 13], it is convenient for all if we quote two results.

Lemma 1 *let C be a convex subset of a normed space X, let $\xi \in C^{\circ}$, and let $r > 0$ be such that the ball $B(\xi, r)$ is contained in C. Let $z \neq \xi$, $0 < t < 1$, and $z' = t\xi + (1-t)\,z$. If $B(z, tr)$ intersects C, then $B(z', t^2 r) \subset C$* ([8], Lemma 5.1.1).

Note that for points x, y in a normed space,

$$[x, y] \equiv \{tx + (1-t)\,y : 0 \leqslant t \leqslant 1\}.$$

We adopt other natural notations for 'intervals' joining x and y without further comment.

Proposition 2 *Let C be a subset of a Banach space such that $C \cup {\sim}C$ is dense, let $x \in C$ and $y \in {\sim}C$, and let $\varepsilon > 0$. Then there exists $z \in \partial C$ such that $\rho\,(z, [x, y]) < \varepsilon$* ([9], Proposition 8).

Lemma 1 and Proposition 2 are two of several results in convex geometry that will be found throughout the paper. Here is the next one.

Proposition 3 *Let C be a convex subset of a Banach space X such that C° is inhabited. Then $\overline{C}^{\circ} = C^{\circ}$.*

Proof. Construct $\xi \in C$ and $r > 0$ such that $B(\xi, r) \subset C$. Let $x \in \overline{C}^{\circ}$. In trying to prove that $x \in C^{\circ}$, we may assume that $\|x - \xi\| > r$. Pick s such that $0 < s < r$ and $\overline{B}(x, s) \subset \overline{C}$. Let

$$0 < t < \frac{s}{\|x - \xi\|} \quad \text{and} \quad z = \frac{1}{1-t}x - \frac{t}{1-t}\xi.$$

Then $0 < t < 1$ and $x = t\xi + (1-t)z$. Moreover, $\|x - z\| = t\,\|x - \xi\| < s$, so $z \in \overline{C}$. Hence $B(z, tr)$ intersects C, and therefore, by Lemma 1, $B(x, t^2 r) \subset C$. Thus $x \in C^{\circ}$. ∎

Proposition 4 *Let C be a convex subset of a Banach space X such that $C \cup {\sim}C$ is dense in X and $\overline{C}^{\circ} = C^{\circ}$. Then C is tightly bordered.*

Proof. Let x be a point of C with $\rho\,(x, \partial C) > 0$. Choose r such that $0 < 2r < \rho\,(x, \partial C)$, and consider any $y \in B(x, r)$. Suppose that $y \in {\sim}C$. Applying Proposition 2, we can find $z \in \partial C$ and $t \in [0, 1]$ such that $\|z - (1-t)\,x - ty\| < r$. Then

$$\begin{aligned} \|x - z\| &\leqslant \|x - (1-t)\,x - ty\| + \|z - (1-t)\,x - ty\| \\ &< \|x - y\| + r < 2r, \end{aligned}$$

which contradicts our choice of r. It follows that $y \notin {\sim}C$. Since y is arbitrary, we conclude that $B(x, r) \cap {\sim}C$ is empty, and hence, by the density of $C \cup {\sim}C$, that $B(x, r) \subset \overline{C}$. Thus $x \in \overline{C}^{\circ} = C^{\circ}$. ∎

Proposition 5 *Let C be a convex subset of a Banach space X such that C° is inhabited and $C \cup {\sim}C$ is dense in X. Then C is tightly bordered. Moreover, if ξ is an interior point of C and $\|x - \xi\| < \rho\,(\xi, \partial C)$, then $x \in C^{\circ}$.*

Proof. Propositions 3 and 4 together show that C is tightly bordered. Given $\xi \in C^\circ$ and $x \in X$ with $\|x - \xi\| < \rho(\xi, \partial C)$, pick $r > 0$ such that $\|x - \xi\| + 3r < \rho(\xi, \partial C)$ and $B(\xi, r) \subset C$. In proving that $x \in C^\circ$, we may assume that $\|x - \xi\| > r/2$. Let

$$t = \frac{r}{\|x - \xi\| + r} \quad \text{and} \quad z \equiv \frac{1}{1-t}x - \frac{t}{1-t}\xi.$$

Then $0 < t < 1$, $x = t\xi + (1-t)z$,

$$\|\xi - z\| = \frac{1}{1-t}\|x - \xi\| < \|x - \xi\| + r,$$

and $z \neq \xi$. Now choose $\zeta \in C \cup {\sim}C$ such that $\|z - \zeta\| < tr$. If $\zeta \in {\sim}C$, then by Proposition 2, there exist $\eta \in [\xi, \zeta]$ and $y \in \partial C$ such that $\|\eta - y\| < tr$; in that case,

$$\begin{aligned}
\|\xi - y\| &\leqslant \|\xi - \eta\| + \|\eta - y\| \\
&\leqslant \|\xi - \zeta\| + tr \\
&\leqslant \|\xi - z\| + \|z - \zeta\| + r \\
&\leqslant \|x - \xi\| + r + tr + r < \rho(\xi, \partial C),
\end{aligned}$$

a contradiction. Hence $\zeta \in C \cap B(z, tr)$. Applying Lemma 1, we now see that $B(x, t^2 r) \subset C$, so $x \in C^\circ$. ■

Recall that the **metric complement** of a set S in a metric space X is the set

$$-S \equiv \{x \in X : \rho(x, S) > 0\},$$

and (from [9]) that S is **coherent** if $-{\sim}S \subset S$.

Proposition 6 *Let C be a coherent, convex subset of a normed space X such that $C \cup {\sim}C$ is dense in X. Then C is tightly bordered.*

Proof. Let $x \in C$ with $\rho(x, \partial C) > 0$, and choose $r > 0$ such that $\rho(x, \partial C) \geqslant 3r$. Given $y \in B(x, r)$, suppose there exists $z \in {\sim}C$ such that $\|y - z\| < r$. Since $C \cup {\sim}C$ is dense in X, we can apply Proposition 2 to produce $b \in [x, z]$ such that $\rho(b, \partial C) < r$; whence

$$\begin{aligned}
\rho(x, \partial C) &\leqslant \|x - b\| + \rho(b, \partial C) \\
&\leqslant \|x - z\| + r \\
&\leqslant \|x - y\| + \|y - z\| + r < 3r,
\end{aligned}$$

a contradiction from which we conclude that $\rho(y, {\sim}C) \geqslant r$. It follows from the coherence of C that $y \in C$. Hence $B(x, r) \subset C$ and therefore $x \in C^\circ$. ■

We shall return to coherence towards the end of the next section.

3 Tightly bordered co-convex sets When can we be sure that the *complement* of a convex subset of a normed space is tightly bordered? Our answer depends on some additional results on convex geometry—in particular, one on boundary crossings (Proposition 11), improving Proposition 5.1.5 of [8].

Proposition 7 *The interior of a convex subset C of a normed space X is convex. If also C° is inhabited, then it is dense in C. Moreover, $-C^\circ = -C$.*

Proof. If $x, y \in C^\circ$, then there exists $r > 0$ such that $B(x, r) \subset C$ and $B(y, r) \subset C$. Given t with $0 \leqslant t \leqslant 1$, we see that $z \equiv tx + (1-t)y$ belongs to C. In order to prove that $z \in C^\circ$, we may assume that $\|z - x\| > r/2$ and $\|z - y\| > r/2$, in which case $0 < t < 1$. Since $y \in B(y, tr) \cap C$, we now see from Lemma 1 that $B(z, t^2 r) \subset C$; whence $z \in C^\circ$. Thus C° is convex.

Now assume that there is a point $\xi \in C^\circ$. For each $x \in C$, either $\|x - \xi\|$ is so small that $x \in C^\circ$ or else $x \neq \xi$. In the latter case, Lemma 1 tells us that for each $t \in (0,1)$, the point $t\xi + (1-t)x$ belongs to C°. Letting $t \to 0$, we see that $x \in \overline{C^\circ}$.

Finally, since $-C \subset -C^\circ$ and C° is dense in C, it readily follows that $-C^\circ = -C$. ∎

Proposition 8 *Let X be a normed space, and C a convex subset of X with inhabited interior. Then $-C$ is dense in both $\sim C$ and $\sim C^\circ$.*

Proof. By Proposition 7, C° is convex, and $-C^\circ = -C$. Applying Lemma 5.1.4 of [8] to C°, we find that $-C$ is dense in $\sim C^\circ$. But $-C \subset \sim C \subset \sim C^\circ$, so $-C$ is also dense in $\sim C$. ∎

We digress briefly in order to establish the conclusion of the preceding proposition under different hypotheses. This requires us to state the following **ridiculously useful lemma** (Lemma 5.1.3 of [8]):

> *Let X be a normed space, let x_1, x_2 be distinct points of X, and let $x_3 = \lambda x_1 + (1-\lambda) x_2$ with $\lambda \neq 0, 1$. For all $\alpha, \beta > 0$, if $\|x - x_1\| \leqslant \alpha / |\lambda|$ and $\|y - x_2\| \leqslant \beta / |1 - \lambda|$, then $\|\lambda x + (1-\lambda) y - x_3\| \leqslant \alpha + \beta$.*

Lemma 9 *Let C be an inhabited, convex subset of a finite-dimensional Banach space X, and let $x \in -(-C)$. Then $\neg\neg(x \in C)$.*

Proof. Translating if necessary, we may assume that $0 \in C$. Let n be the dimension of X; if $n = 0$, then the conclusion is trivial; so we may assume that $n \geqslant 1$. Fix $r > 0$ such that $B(x, r) \subset -(-C)$. In order to derive a contradiction, assume that $x \notin C$. Suppose that C contains n linearly independent vectors $x_1, \ldots, x_n$. Then if contains a nondegenerate ball $B(y, t)$ in the interior of the (convex) simplex with vertices $0, x_1, \ldots, x_n$. Since $x \notin C$, we have $\|x - y\| \geqslant t$. Let $z = \lambda x + (1-\lambda) y$, where

$$\lambda = 1 + \frac{r}{2\|x - y\|}$$

Then $\|z - x\| = r/2$ and

$$y = \frac{\lambda}{\lambda - 1} x - \frac{1}{\lambda - 1} z.$$

Pick s such that

$$0 < s < \min\left\{t, \frac{r}{2(\lambda - 1)}\right\},$$

and apply the ridiculously useful lemma with $x_1 = x, x_2 = z, \alpha = 0$, and $\beta = s$. We find that if $\|z - \zeta\| < s(\lambda - 1)$, then

$$\left\| \left(\frac{\lambda}{\lambda - 1} x - \frac{1}{\lambda - 1} \zeta \right) - y \right\| < s < t,$$

so

$$\frac{\lambda}{\lambda - 1} x - \frac{1}{\lambda - 1} \zeta \in C.$$

Also,

$$\begin{aligned}\|\zeta - x\| &\leqslant \|\zeta - z\| + \|z - x\| \\ &< s(\lambda - 1) + \frac{r}{2} < r,\end{aligned}$$

so $\zeta \in -(-C)$. If $\zeta \in C$, then, since

$$x = \frac{1}{\lambda}\zeta + \left(1 - \frac{1}{\lambda}\right)\left(\frac{\lambda}{\lambda - 1}x - \frac{1}{\lambda - 1}\zeta\right),$$

the convexity of C yields $x \in C$, contradicting our assumption that $x \notin C$. We conclude that $\zeta \notin C$ for each $\zeta \in B(z, s(\lambda - 1))$; whence $z \subset -C$, which is also absurd, since $B(z, r/2) \subset B(x, r) \subset -(-C)$. It follows from all this that that C cannot contain n linearly independent vectors in X.

Next, suppose that for some k with $1 \leqslant k \leqslant n$, we have proved that C cannot contain k linearly independent vectors in X. Suppose that C contains $k - 1$ linearly independent vectors, and let V be the finite-dimensional subspace of X spanned by those vectors. If there exists $y \in C \cap -V$, then C contains k linearly independent vectors, a contradiction. Hence $C \subset \overline{V} = V$. Since $k - 1 < n$, there exists a point $z \in B(x, r) \cap -V$; then $z \in -C \cap B(x, r)$, which is impossible. This completes the inductive proof that C cannot contain k linearly independent vectors for any k with $1 \leqslant k \leqslant n$. It now follows that $C = \{0\}$, so we can find an element y of $B(x, r)$ with positive norm. Then $y \in -C \cap B(x, r)$, a final contradiction that ensures that $\neg(x \notin C)$. ■

Proposition 10 *Let C be an inhabited, located, convex subset of a finite-dimensional Banach space, such that $\sim C$ is inhabited. Then $\sim C$ is dense in $-C$.*

Proof. Let $x \in \sim C$. If $\rho(x, -C) > 0$, then $x \in -(-C)$, so, by Lemma 9, $\neg\neg(x \in C)$, a contradiction. Hence $\rho(x, -C) = 0$. ■

We now return to our main path, with the promised improvement on Proposition 5.1.1 of [8].

Proposition 11 *Let X be a Banach space, C a convex subset of X such that $C \cup \sim C$ is dense in X, and ξ an interior point of C. Let $z \in -C$, and for each $t \in [0, 1]$ write*

$$z_t \equiv (1 - t)\xi + tz.$$

Then the following hold:

(i) $\gamma(\xi, z) \equiv \inf\{t \in [0, 1] : z_t \in C\}$ *exists, and* $0 < \gamma(\xi, z) < 1$.

(ii) $z_{\gamma(\xi,z)}$ *is the unique intersection of* $[\xi, z]$ *with* ∂C.

(iii) *If* $\gamma(\xi, z) < t \leqslant 1$, *then* $z_t \in C^\circ$.

(iv) *If* $0 \leqslant t < \gamma(\xi, z)$, *then* $z_t \in -C$.

Moreover, the mapping $(\xi, z) \rightsquigarrow z_{\gamma(\xi,z)}$ is continuous at each point of $C^\circ \times -C$.

Proof. By Proposition 7, C° is convex and dense in C, and $-C^\circ = -C$. On the other hand, Proposition 8 shows that $-C$ is dense in $\sim C$. Hence $C^\circ \cup -C^\circ$ is dense in $C \cup \sim C$ and therefore in X. Moreover, since Proposition 8 also gives $-C^\circ$ dense in $\sim C^\circ$, we have

$$\partial C^\circ = \overline{C^\circ} \cap \overline{\sim C^\circ} = C \cap \overline{-C^\circ} = C \cap \overline{-C} = C \cap \overline{\sim C} = \partial C.$$

Applying Proposition 5.1.5 of [8] to C°, and again using both the density of C° in C and the identity $-C^\circ = -C$, we now see that $\gamma(\xi, z)$ exists and satisfies (i)–(iv), and that $\gamma : C^\circ \times -C \to \partial C$ is pointwise continuous. ■

One more lemma and we are ready to deal with co-convex sets and tight borders.

Lemma 12 *Let X be a Banach space, and C a convex subset of X with inhabited interior such that $C \cup \sim C$ is dense in X. Then $\partial(\sim C) = \partial C$.*

Proof. It is clear that $\partial C \subset \partial(\sim C)$. For the reverse inclusion, first fix ξ in C° and $r > 0$ such that $B(\xi, r) \subset C$. Given $v \in \partial(\sim C)$, we have $v \neq \xi$. Set $z \equiv 2v - \xi$ and note that $v \in (\xi, z)$. Taking $x_1 = v, x_2 = z, x_3 = \xi, \lambda = 2$, and $\alpha = \beta = r/2$ in the ridiculously useful lemma, we see that for each $y \in B(z, r/2)$ and each $u \in B(v, r/4)$,

$$\|(2u - y) - \xi\| < \frac{r}{2} + \frac{r}{2} = r$$

and therefore $2u - y \in C$. It follows that if also $y \in C$, then

$$u = \frac{1}{2}\left((2u - y) + y\right) \in C$$

for each $u \in B(v, r/4)$. But then $\rho(v, \sim C) \geqslant r/4$, so $v \notin \partial(\sim C)$, a contradiction from which we conclude that $y \notin C$ for each $y \in B(z, r/2)$. Hence $\rho(z, C) \geqslant r/2$, and so $z \in -C$. By Proposition 11, there exists a unique $t \in (0, 1)$ such that $w \equiv (1 - t)\xi + tz$ belongs to ∂C, $y \in C^\circ$ for all $y \in [\xi, w)$, and $y \in -C$ for all $y \in (w, z]$.

Given $\varepsilon > 0$, pick a point ζ in the open segment (ξ, v) such that $0 < \|\zeta - v\| < \varepsilon$ and $\zeta \neq w$. Since $\zeta \in (\xi, z)$, either $\zeta \in (\xi, w)$ or $\zeta \in (w, z)$. In the latter case, since $v \in (\zeta, z)$, we have $v \in (w, z)$ and so $v \in -C$; but this is absurd, since it puts v in $(\sim C)^\circ$ and thereby contradicts the choice of v as an element of $\partial(\sim C)$. Hence, in fact, $\zeta \in (\xi, w)$ and therefore $\zeta \in C^\circ$. Since ε is arbitrarily small and, by definition of $\partial(\sim C)$, there are points of $\sim C$ arbitrarily close to v, it follows that $v \in \partial C$. Thus $\partial(\sim C) \subset \partial C$. ■

Proposition 13 *Let X be a Banach space, and C a convex subset of X with inhabited interior such that $C \cup \sim C$ is dense in X. Then $\sim C$ is tightly bordered.*

Proof. Consider any $x \in \sim C$ with $\rho(x, \partial(\sim C)) > 0$. By Lemma 12, $\rho(x, \partial C) > 0$. Let $0 < r < \rho(x, \partial C)$, and apply Proposition 8 to produce $z \in -C \cap B(x, r)$. Given $\xi \in B(x, r)$, suppose that $\xi \in C^\circ$. By Proposition 11, there exists a unique point y in $[\xi, z] \cap \partial C$. But then y belongs to the convex set $B(x, r)$, so $\rho(x, \partial C) < r$—a contradiction. Hence $\xi \notin C^\circ$, so $\xi \in -C^\circ$ (since C° is an open set) and therefore, by Proposition 7, $\xi \in -C$. It now follows that $B(x, r) \subset -C \subset \sim C$ whence $x \in (\sim C)^\circ$. ■

The hypothesis that $C \cup \sim C$ be dense in X appears in most of the preceding results. One situation in which it arises is when C is located; another is given by the next proposition.

Proposition 14 *Let C be a coherent, convex subset of a normed space such that $\sim C$ is located. Then $C \cup \sim C$ is dense in X.*

Proof. Given x in X and $\varepsilon > 0$, we have either $\rho(x, {\sim}C) < \varepsilon$ or $\rho(x, {\sim}C) > 0$. In the latter case, $x \in -{\sim}C$ and therefore, by coherence, $x \in C$. Since $\varepsilon > 0$ is arbitrary, we conclude that $C \cup {\sim}C$ is dense in X. ■

As a partial converse to Proposition 14, we have:

Proposition 15 *Let C be a convex subset of a normed space X such that C° is inhabited and $C \cup {\sim}C$ is dense in X. Then C is coherent; in fact, if $\rho(x, {\sim}C) > 0$, then $x \in C^\circ$.*

Proof. Fix $\xi \in C^\circ$. Given $x \in -{\sim}C$, pick $r > 0$ such that $B(\xi, 2r) \subset C^\circ$ and $\rho(x, {\sim}C) > 2r$. In order to prove that $x \in C$, we may assume that $\|x - \xi\| > r$. Compute t such that $0 < t < 1$ and

$$\frac{t}{1-t}\|x - \xi\| < r,$$

and let

$$z = \frac{1}{1-t}x - \frac{t}{1-t}\xi.$$

Then $x = t\xi + (1-t)z$ and $\|x - z\| < r$. Hence

$$B(z, tr) \subset B(z, r) \subset B(x, 2r) \subset -{\sim}C.$$

Since $C \cup {\sim}C$ is dense in X, it follows that $B(z, tr)$ intersects C. Lemma 1 now shows us that $B(x, t^2 r) \subset C$; whence $x \in C^\circ$. ■

We conclude this section with two more results about borders of convex subsets. The first of these will be used in the Brouwerian examples in Section 4.

Proposition 16 *Let C be an inhabited, located, convex subset of a Hilbert space H. Then for each $x \in {\sim}C$, $\rho(x, \partial C)$ exists and equals $\rho(x, C)$.*

Proof. Replacing C by $\overline{C}$, we may assume that C is closed in H. Let $x \in {\sim}C$. By a well-known extension of Theorem 4.3.1 of [8], there exists a unique $z \in C$ such that $\|x - z\| = \rho(x, C)$. If $\rho(x, \partial C) < \|x - z\|$, then there exists $\zeta \in C$ such that $\|x - \zeta\| < \rho(x, C)$, which is absurd; hence $\rho(x, \partial C) \geqslant \|x - z\|$. It remains to prove that $z \in \partial C$; for that, it will suffice to show that for each $\varepsilon > 0$, there exists $\zeta \in {\sim}C$ with $\|z - \zeta\| < 3\varepsilon$. Either $\|x - z\| < 3\varepsilon$, in which case we can take $\zeta = x$, or else, as we assume, $\|x - z\| > 2\varepsilon$. Letting

$$t = \frac{2\varepsilon}{\|x - z\|} \quad \text{and } y = tx + (1-t)z,$$

we see that $0 < t < 1$ and $\|y - z\| = 2\varepsilon$. Since C is located, $C \cup {\sim}C$ is dense in H; so there exists $\zeta \in C \cup {\sim}C$ such that $\|y - \zeta\| < \varepsilon$. Then

$$\begin{aligned}
\|x - \zeta\| &\leqslant \|x - y\| + \|y - \zeta\| \\
&< (1-t)\|x - z\| + \varepsilon \\
&= \|x - z\| - 2\varepsilon + \varepsilon \\
&< \|x - z\| = \rho(x, C),
\end{aligned}$$

from which it follows that $\zeta \in {\sim}C$. Also,

$$\|z - \zeta\| \leqslant \|y - \zeta\| + \|y - z\| < 3\varepsilon.$$

This completes the proof that $z \in \partial C$. ■

Corollary 17 *Let C be an inhabited, located, convex subset of a Hilbert space with $\sim C$ inhabited. Then $\partial(\sim C) = \partial C$.*

Proof. Clearly, $\partial C \subset \partial(\sim C)$. To prove the reverse inequality, consider $x \in \partial(\sim C)$, and first observe that, by definition of $\partial(\sim C)$, there are points of $\sim C$ arbitrarily close to x. Since C is convex and located in the Hilbert space, there exists $z \in \overline{C}$ such that $\|x - z\| = \rho(x, C)$. Given $\varepsilon > 0$, we have either $x \neq z$ or $\|x - z\| < \varepsilon$. In the first case, $x \in -C \subset (\sim C)^{\circ}$, which is impossible since $x \in \partial(\sim C)$. Thus we have $\|x - z\| < \varepsilon$, and so, since $z \in \overline{C}$, there exists $y \in C$ with $\|x - y\| < \varepsilon$. Hence, $\varepsilon > 0$ being arbitrary, there are points of C arbitrarily close to x, which ensures that $x \in \partial C$. Thus $\partial(\sim C) \subset \partial C$, as required. ■

4 Limiting Brouwerian counterexamples In this section we present two Brouwerian counterexamples.[4] The first shows why we needed some of the hypotheses for the results in Sections 2 and 3.

Brouwerian Example 1 [LEM]. *A convex subset C of $\mathbf{R}$ that has inhabited interior, has tightly bordered complement, but for which none of the following properties can be derived:*

(i) *$C \cup \sim C$ is dense in $\mathbf{R}$.*

(ii) *∂C is located in $\mathbf{R}$.*

(iii) *$\partial(\sim C) = \partial C$.*

(iv) *C is tightly bordered.*

Let P be any proposition such that $\neg\neg P$ holds, and define

$$C \equiv [0,1] \cup \{x \in [0,2] : P\}.$$

This set is convex and contains $1/2$ in its interior. If $C \cup \sim C$ is dense in $\mathbf{R}$, then we can choose $x \in C \cup \sim C$ with $|x - 3/2| < 1/2$. If $x \in \sim C$, then $\neg P$, which is absurd; so $x \in C$ and therefore P holds. If ∂C is located, then either $\rho(3/4, \partial C) > 1/4$ or $\rho(3/4, \partial C) < 1/2$. The first case is ruled out, since it implies that $2 \notin \partial C$ and hence that $\neg P$ holds. Thus there exists $x \in \partial C$ with $x > 1/2$, and therefore P holds. Next we observe that $2 \in \partial(\sim C)$: indeed, $(2, \infty) \subset \sim C$, and, since $\neg\neg P$ holds, $(1,2) \subset \sim\sim C$. However, if $2 \in \partial C$, then $C \cap (1,2]$ is inhabited, so P holds. On the other hand, if $x \in \sim C$ and $\rho(x, \partial(\sim C)) > 0$, then either $x < 0$ or $x > 2$, so $x \in \sim C$. Thus $\sim C$ is tightly bordered.

Finally, if $x \in \partial C$ and $|x - 1| < 1$, then we must have $\neg P$, a contradiction; whence $\rho(1, \partial C) \geqslant 1$; it follows that if C is tightly bordered, then $1 \in C^{\circ}$, so $C \cap (1,2]$ is inhabited and therefore P holds.

For the remaining three Brouwerian examples, each connected with the hypothesis in Proposition 5 that C has inhabited interior, we remind the reader of two essentially nonconstructive classical principles:

> **The limited principle of omniscience, LPO:** For each binary sequence $(a_n)_{n \geqslant 1}$, either $a_n = 0$ for all n or else there exists n such that $a_n = 1$,
>
> **Markov's principle, MP:** For each binary sequence $(a_n)_{n \geqslant 1}$, if it is impossible that $a_n = 0$ for all n, then there exists n such that $a_n = 1$.

[4] "A Brouwerian counterexample is not a counterexample in the usual sense; it is *evidence* that a statement does not admit of a constructive proof" ([7], page 3).

LPO is equivalent to the statement

$$\forall_{x\in\mathbf{R}}\,(x=0\vee|x|>0)\,.$$

MP, which is weaker than **LPO**, is equivalent to

$$\forall_{x\in\mathbf{R}}\,(\neg\,(x=0)\Rightarrow x\neq 0)\,,$$

where '$x\neq 0$' means '$|x|>0$'.

Brouwerian Example 2 [LPO]. *An inhabited, balanced, convex subset C of $\mathbf{R}$ such that C and ∂C are compact, $\sim C$ is open and located, both C and $\sim C$ are tightly bordered, but we cannot determine that C° is either empty or inhabited.*

Take any nonnegative, small real number a and let $C=[-a,a]$. Then C is inhabited by 0 and is compact; ∂C $(=\{-a,a\})$ is located; $\sim C=-C$ is open and are located. However, if C° is inhabited, then $a\neq 0$; and if $C^\circ=\varnothing$, then $a=0$.

We can turn this into a Markovian example—one in which the derivability of the property under examination leads to that of Markov's principle—by choosing $a\geqslant 0$ such that $\neg\neg(a=0)$. Then C° is open if and only if $a\neq 0$. Thus we have:

Brouwerian Example 3 [MP]. *An inhabited, balanced, convex subset C of $\mathbf{R}$ such that C and ∂C are compact, $\sim C$ is open and located, both C and $\sim C$ are tightly bordered, C° cannot be empty, but we cannot determine that C° is inhabited.*

We now give a much more complicated Brouwerian example, showing that if we replace $\mathbf{R}$ by a Hilbert subspace of $l_2(\mathbf{R})$, then we can replace **MP** by **LPO** in Brouwerian Example 3:

Brouwerian Example 4 [LPO]. *A balanced, compact, tightly bordered, convex subset C of a Hilbert space such that ∂C is compact, $\sim C$ is located and open (and hence tightly bordered), C° cannot be empty, but we cannot determine that C° is inhabited.*

Let $(a_n)_{n\geqslant 1}$ be a binary sequence with $a_1=1$ and at most one other term equal to 1. Let $(e_n)_{n\geqslant 1}$ be an orthonormal basis of unit vectors in the Hilbert space $l_2(\mathbf{R})$, and let H be the linear subspace

$$\left\{\sum_{n=1}^{\infty}a_n\langle x,e_n\rangle\,e_n : x\in l_2(\mathbf{R})\right\}.$$

We first prove that H is closed in $l_2(\mathbf{R})$ and is therefore a Hilbert space. Let $\left(x^{(n)}\right)_{n\geqslant 1}$ be a sequence in H that converges to a limit $x^\infty\in l_2(\mathbf{R})$. For each n, there exists $z_n\in l_2(\mathbf{R})$ such that $x^{(n)}=\sum_{k=1}^{\infty}a_k\langle z_n,e_k\rangle\,e_k$. Thus for each $k\geqslant 1$, $a_k\langle z_n,e_k\rangle\to\langle x^\infty,e_k\rangle$ as $n\to\infty$, and so

$$\begin{aligned}a_k\langle x^\infty,e_k\rangle &= a_k\lim_{n\to\infty}a_k\langle z_n,e_k\rangle\\ &=\lim_{n\to\infty}a_k^2\langle z_n,e_k\rangle=\lim_{n\to\infty}a_k\langle z_n,e_k\rangle=\langle x^\infty,e_k\rangle\,.\end{aligned}$$

Hence $x^\infty=\sum_{k=1}^{\infty}a_k\langle x^\infty,e_k\rangle\,e_k\in H$.

Call a pair (λ^+,λ^-) of nonnegative sequences **acceptable** if there exists ν such that

▷ $\lambda_n^+=\lambda_n^-=0$ for all $n\geq\nu$, and

▷ $\sum_{n=1}^{\infty}\left(\lambda_n^+ + \lambda_n^-\right) = \sum_{n=1}^{\nu}\left(\lambda_n^+ + \lambda_n^-\right) = 1$
(where, for example, λ_n^+ is the nth term of λ^+).

Let C be the closure in H of the set S of all points of the form

$$\sum_{n=1}^{\infty} n^{-1}\left(\lambda_n^+ - \lambda_n^-\right) a_n e_n,$$

where the sequence pair (λ^+, λ^-) is admissible. It is straightforward to prove the following facts:

(a) C is a balanced, convex subset of H that contains 0.

(b) If $x \in C$, then $|\langle x, e_1\rangle| \leqslant 1$.

(c) If $a_n = 0$ for all $n \geqslant 2$, then $H = \mathbf{R}e_1$, $C = \{te_1 : -1 \leqslant t \leqslant 1\}$, which is both compact and tightly bordered, $\partial C = \{-e_1, e_1\}$, and $\sim C$ is both located and open in H.

(d) If there exists $N \geqslant 2$ such that $a_N = 1$, then $H = \operatorname{span}\{e_1, e_N\}$, C is the closed convex hull of $\left\{\pm e_1, \pm N^{-1}e_N\right\}$ and is compact; ∂C is the compact closure of the parallelogram with vertices $\pm e_1, \pm N^{-1}e_N$; and $\sim C$ is both located and open in H. Moreover, by elementary Euclidean geometry, $B(0, r_N) \subset C$, where

$$r_N = N^{-1}\cos\left(\tan^{-1}\frac{1}{N}\right),$$

and $\rho(x, \partial C) \leqslant r_N$ for each $x \in C$.

(e) It is impossible that C has empty interior.

To prove that C is totally bounded, fix $\varepsilon > 0$ and let F be a finite ε-approximation to the set $C \cap \mathbf{R}e_1$. Pick N such that $\sum_{n=N+1}^{\infty} n^{-2} < \varepsilon^2$. If $a_n = 1$ for some n with $2 \leqslant n \leqslant N$, then (as noted at (d) above) C is compact. So we may assume that $a_n = 0$ whenever $2 \leqslant n \leqslant N$. Given an acceptable sequence pair (λ^+, λ^-), define

$$x = \sum_{n=1}^{\infty} n^{-1}\left(\lambda_n^+ - \lambda_n^-\right) a_n e_n \in S. \tag{1}$$

Since $\langle x, e_1\rangle e_1 \in C \cap \mathbf{R}e_1$, there exists $y \in F$ with $\|\langle x, e_1\rangle e_1 - y\| < \varepsilon$. Then

$$\begin{aligned} \|x - y\| &\leqslant \|\langle x, e_1\rangle e_1 - y\| + \left\|\sum_{n=N+1}^{\infty} n^{-1}\left(\lambda_n^+ - \lambda_n^-\right) a_n\right\| \\ &< \varepsilon + \left(\sum_{n=N+1}^{\infty} n^{-2}\right)^{1/2} < 2\varepsilon. \end{aligned}$$

Thus F is a finite 2ε-approximation to S. Since $\varepsilon > 0$ is arbitrary, we see that S is totally bounded; whence C is totally bounded and hence, being complete, compact.

To prove that C is tightly bordered, let $x \in C$ and $0 < r < \rho(x, \partial C)$. Since $e_1 \in \partial C$, we have

$$0 \neq e_1 - x = (1 - \langle x, e_1\rangle) e_1 - \sum_{n=2}^{\infty} a_n \langle x, e_n\rangle e_n$$

so either $\langle x, e_1 \rangle \neq 1$ or there exists $n \geqslant 2$ with $a_n \neq 0$. Since in the latter event C is tightly bordered and therefore $x \in C^{\circ}$, we may assume that $\langle x, e_1 \rangle \neq 1$; similarly, since $-e_1 \in \partial C$, we may assume that $\langle x, e_1 \rangle \neq -1$. Thus either $|\langle x, e_1 \rangle| > 1$, in which case $x \notin C$, a contradiction; or else, as must be the case, $-1 < \langle x, e_1 \rangle < 1$. Now choose an integer $N > 1/r$. If $a_n = 1$ for some $n > N$, then $\rho(x, \partial C) \leqslant r_n < r_N < r$, a contradiction; hence $a_n = 0$ for all $n > N$. It follows that

– either there exists n with $2 \leqslant n \leqslant N$ and $a_n = 1$, in which case C is tightly bordered and so $x \in C^{\circ}$;

– or else $a_n = 0$ for all $n \geqslant 2$, when $x = \langle x, e_1 \rangle e_1 \in \{te_1 : -1 < t < 1\} = C^{\circ}$.

Thus C is tightly bordered.

Turning now to $\sim C$, we first observe that since C is located and therefore $C \cup \sim C$ is dense in H, in order to prove that $\sim C$ is located, it will suffice to prove that $\rho(x, \sim C)$ exists for each $x \in C$. Given such x and $\varepsilon > 0$, choose a positive integer N such that $r_N < \varepsilon/2$. If $a_n = 1$ for some n with $2 \leqslant n \leqslant N$, then $\sim C$ is clearly located, being the outside of a parallelogram; so we may assume that $a_n = 0$ for $2 \leqslant n \leqslant N$. It follows that C is a subset of the closed convex hull of $\left\{\pm e_1, \pm N^{-1} e_N\right\}$. Given $x \in X$, and writing

$$T \equiv \{te_1 : -1 < t < 1\},$$

we have either $\rho(x, T) > 0$ or $\rho(x, T) < \varepsilon/2$. In the first case, there exists $m > N$ such that $a_m = 1$, so $\sim C$ is located. In the second case, pick $y \in T$ such that $\|x - y\| < \varepsilon/2$. Then $y + N^{-1} e_m \in \sim C$ and

$$\left\|x - \left(y + N^{-1} e_m\right)\right\| \leqslant \|x - y\| + N^{-1} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Putting all this together, we see that for each $x \in C$ and each $\varepsilon > 0$, either $\rho(x, \sim C)$ exists or there exists $y \in \sim C$ such that $\|x - y\| < \varepsilon$. Hence $\sim C$ is located. It follows from Proposition 11 of [9] that ∂C is located; since ∂C is a closed subset of the compact set C, it is therefore compact.

To prove that $\sim C$ is open, let $x \in \sim C$. Since $C \cap \mathbf{R} e_1$ is closed, located, and convex, there exists $z \in C \cap \mathbf{R} e_1$ such that $\rho(x, C \cap \mathbf{R} e_1) = \|x - z\|$; then $x \neq z$, so

$$0 < d \equiv \rho(x, C \cap \mathbf{R} e_1).$$

Either $\rho(x, \mathbf{R} e_1) > 0$ or $\rho(x, \mathbf{R} e_1) < d/2$. In the first case, choose a positive integer $N > 1/\rho(x, \mathbf{R} e_1)$. We may assume that $a_n = 0$ for $2 \leqslant n \leqslant N$; so C is a subset of the closed convex hull of $\left\{\pm e_1, \pm N^{-1} e_N\right\}$. If $\|x - y\| < \rho(x, \mathbf{R} e_1) - 1/N$, then $\rho(y, \mathbf{R} e_1) > 1/N$, so $y \in \sim C$. Hence $\rho(x, C) \geqslant \rho(x, \mathbf{R} e_1) - 1/N$, and therefore $x \in -C = (\sim C)^{\circ}$. This leaves us with the case $\rho(x, \mathbf{R} e_1) < d/2$, in which, if $|\langle x, e_1 \rangle| < 1$, then

$$\rho(x, \mathbf{R} e_1) = \|x - \langle x, e_1 \rangle e_1\| \geqslant \rho(x, C \cap \mathbf{R} e_1),$$

a contradiction. Hence $|\langle x, e_1 \rangle| \geqslant 1$ and therefore either $\langle x, e_1 \rangle \leqslant -1$ or $\langle x, e_1 \rangle \geqslant 1$. We illustrate with the latter alternative. We have

$$\begin{aligned} |\langle x, e_1 \rangle| - 1 &= \|\langle x, e_1 \rangle e_1 - e_1\| \\ &\geqslant \|x - e_1\| - \|x - \langle x, e_1 \rangle e_1\| \geqslant d - \frac{d}{2} = \frac{d}{2}, \end{aligned}$$

so $|\langle x, e_1\rangle| \geqslant 1 + d/2$. It follows that if $\|x - y\| < d/2$, then

$$\begin{aligned}|\langle y, e_1\rangle| &\geqslant |\langle x, e_1\rangle| - |\langle x, e_1\rangle - \langle y, e_1\rangle| \\ &\geqslant 1 + \frac{d}{2} - \|x - y\| > 1\end{aligned}$$

and therefore $y \notin C$. Hence $B(x, d/2) \subset -C$ and therefore $x \in -C = (\sim C)^{\circ}$. This completes the proof that $\sim C$ is open.

Finally, suppose that C° is inhabited; then, since C is convex and balanced, $0 \in C^{\circ}$. Pick $r > 0$ such that $B(0, r) \subset C$ and compute N such that $r_N < r$. If $a_n = 1$ for some $n \geqslant N$, then $\rho(0, \partial C) = r_N$ and there exist points of $\sim C$ within r of 0, a contradiction. Hence $a_n = 0$ for all $n \geqslant N$. By testing $a_2, \ldots, a_{N-1}$, we can show that either $a_n = 0$ for all n or else there exists $n < N$ such that $a_n = 1$.

5 A Final Remark In several of our results, we have used the hypothesis that $C \cup \sim C$ is located, where C is an inhabited convex subset of the ambient normed space X. Could it be that that hypothesis actually implies that C is located? If $X = \mathbf{R}$, then the answer is 'yes'. To see this, first translate C to ensure that it contains 0, and set $a_1 = 0$. Fixing $x > 0$, pick $b_1 \in C \cup \sim C$ such that $0 < b_1 - x < x/2$. If $b_1 \in C$, then $x \in [a_1, b_1] \subset C$; so we may assume that $b_1 \in \sim C$. Let $c_1 = (b_1 - a_1)/2$, and pick $y_1 \in C \cup \sim C$ such that $y_1 > 0$ and $|c_1 - y_1| < \min\{2^{-1}\varepsilon, c_1/6\}$. If $y_1 \in C$, set $a_2 = y_1$ and $b_2 = b_1$. If $y_1 \in \sim C$, set $a_2 = a_1$ and $b_2 = y_1$. At this stage, we have $a_2 \in C$ and $b_2 \in \sim C$ such that $0 < a_2 < \frac{2}{3}(b_1 - a_1)$. Continuing on in this way, we construct an increasing sequence $(a_n)_{n\geqslant 1}$ in C and a decreasing sequence $(b_n)_{n\geqslant 1}$ in $\sim C$ such that $0 < b_n - a_n \to 0$ as $n \to \infty$. These sequences have a common limit $a_\infty \in \partial C$, and $\rho(x, C)$ exists and equals $x - a_\infty$. The case $x < 0$ is handled similarly. Since $(-\infty, 0) \cup (0, \infty)$ is dense in $\mathbf{R}$, we conclude that C is located.

However, when we move from one to two dimensions, we have a Brouwerian counterexample to the locatedness of C. Given any proposition P, take $X = \mathbf{R}^2$ and

$$C = ([0, 1] \cup \{x \in [0, 2] : P\}) \times \{0\} \subset \mathbf{R}^2.$$

Then $\sim C$ contains the dense subset $\{(x, y) \in \mathbf{R}^2 : y \neq 0\}$ of $\mathbf{R}^2$ and so is itself dense in $\mathbf{R}^2$. But if C is located, then (cf. Brouwerian Example 1 above) we can easily derive $P \vee \neg P$.

References

[1] P. Aczel and M. Rathjen: *Notes on Constructive Set Theory*, Report No. 40, Institut Mittag-Leffler, Royal Swedish Academy of Sciences, 2001.

[2] P. Aczel and M. Rathjen: *Constructive Set Theory*, in preparation.

[3] E. Bishop: *Foundations of Constructive Analysis*, McGraw-Hill, New York, 1967.

[4] E. Bishop and D.S. Bridges: *Constructive Analysis*, Grundlehren der Math. Wiss. **279**, Springer Verlag, Heidelberg, 1985.

[5] D.S. Bridges: 'Locatedness, convexity, and Lebesgue measurability', Quart. J. Math. Oxford (2) **39**, 411–421, 1988.

[6] D.S. Bridges, M. McKubre-Jordens: 'Solving the Dirichlet problem constructively', J. Logic Anal. **5**(3), 1-22, 2013.

[7] D.S. Bridges and F. Richman: *Varieties of Constructive Mathematics*, London Math. Soc. Lecture Notes **97**, Cambridge Univ. Press, Cambridge, U.K., 1987.

[8] D.S. Bridges, L.S. Vîţă: *Techniques of Constructive Analysis*, Universitext, Springer New York, 2006.

[9] D.S. Bridges, F. Richman, Wang Yuchuan: 'Sets, complements and boundaries', Proc. Koninklijke Nederlandse Akad. Wetenschappen (Indag. Math.) **7**(4), 425–445, 1996.

[10] D.S. Bridges, A. Calder, W.H. Julian, R. Mines, F. Richman, 'Locating metric complements in $\mathbb{R}^n$', in *Constructive Mathematics* (F. Richman, ed.), Springer Lecture Notes in Math. **873**, 241–249, 1981.

[11] P. Martin-Löf: 'An intuitionistic theory of types', in *Twenty-five Years of Constructive Type Theory* (G. Sambin, J. Smith, eds), 127–172, Oxford Logic Guides **36**, Clarendon Press, Oxford, 1998.

[12] F. Richman, D.S. Bridges, A. Calder, W.H. Julian, R. Mines: 'Compactly generated Banach spaces', Arch. Math. **36**, 239–243, 1981.

[13] A.S. Troelstra and D. van Dalen: *Constructivism in Mathematics: An Introduction* (two volumes), North Holland, Amsterdam, 1988.

Communicated by *Mariko Yasugi*

Author's address: Department of Mathematics & Statistics, University of Canterbury, Private Bag 4800, Christchurch 8140, New Zealand.
email: d.bridges@math.canterbury.ac.nz

ASYMPTOTIC MOMENTS OF SYMMETRIC SELF-NORMALIZED SUMS

Yan Liu

Received May 27, 2013; revised June 15, 2013

Abstract. We give a general and explicit formula for the moments of the limiting distribution of symmetric self-nomalized sum of i.i.d. random variables, which belong to the domain of attraction of a stable law. The result shows that the finite order moments for symmetric self-normalized sums are always finite. As an application, tail index can be estimated through our result by using moment estimators.

1 Introduction and preliminaries. The self-normalized method has been focused on in these two decades, and many interesting results are obtained. (See [2], [4], [5], [7].) In this paper, we extend the result for the moments of symmetric self-normalized sum in [5] to a more explicit one.

Consider a sequence $\{X_i\}_{i=1,\ldots,n}$ that X_i's are assumed to be independent and identically distributed and belong to the domain of attraction of a stable law G, the parameter of attracting stable law G is denoted by α. More specifically, we assume that the density function g of the stable distribution G satisfies

$$x^{\alpha+1}g(x) \to r, \quad x^{\alpha+1}g(-x) \to l, \tag{1.1}$$

where $0 < \alpha < 2$, $r + l > 0$. Also U_n and V_n^2 are defined as

$$U_n = \frac{X_1 + \cdots + X_n}{n^{1/\alpha}} \tag{1.2}$$

and

$$V_n^2 = \frac{|X_1|^2 + \cdots + |X_n|^2}{n^{2/\alpha}}. \tag{1.3}$$

To have the limiting distribution of $S_n(2)$ $(= U_n/V_n)$ exist, we further assume that

$$EX_i = 0 \quad \text{if } 1 < \alpha < 2. \tag{1.4}$$

The limiting distribution of $S_n(2)$ is denoted by $S(2)$.

It is shown in [5] that if $\alpha \neq 1$, the moments of $S(2)$ can be derived from

$$\frac{1}{\pi}\int_0^\infty \varphi(t)e^{-st}\,dt = \int_0^\infty e^{-s^2t^2/2}\mathcal{D}(t)\,dt, \tag{1.5}$$

where

$$\varphi(t) = Ee^{iS(2)t} = \lim_{n\to\infty} Ee^{iS_n(2)t}, \tag{1.6}$$

2010 *Mathematics Subject Classification.* 60F05, 62G20.
Key words and phrases. self-normalized sums, asymptotic moments, tail index.

the characteristic function of the limiting distribution of $S(2)$, and

$$\mathcal{D}(t) = (1-\alpha)(2\pi^{-3})^{1/2}\frac{rD_{\alpha-2}(-it)+lD_{\alpha-2}(it)}{rD_{\alpha}(-it)+lD_{\alpha}(it)}, \tag{1.7}$$

$D_\nu(z)$ $(z \in \mathbb{C})$ is the parabolic cylinder functions. (See [6].) Here are two important properties of parabolic cylinder functions for calculation.

$$\frac{d}{dz}D_\nu(z) - \frac{z}{2}D_\nu(z) + D_{\nu+1}(z) = 0; \tag{1.8}$$

$$\frac{d}{dz}D_\nu(z) + \frac{z}{2}D_\nu(z) - \nu D_{\nu-1}(z) = 0. \tag{1.9}$$

The calculation of the moments depends on the expansion of (1.5). We first decompose the left hand side into the form of power series.

$$\begin{aligned}
\frac{1}{\pi}\int_0^\infty \varphi(t)e^{-st}\,dt &= \frac{1}{\pi}\int_0^\infty \sum_{k=0}^\infty \frac{\varphi^{(k)}(0)}{k!}t^k e^{-st}\,dt \\
&= \sum_{k=0}^\infty \frac{\varphi^{(k)}(0)}{k!}\frac{1}{\pi}\int_0^\infty t^k e^{-st}\,dt \\
&= \sum_{k=0}^\infty \frac{1}{\pi}\varphi^{(k)}(0)s^{-k-1}.
\end{aligned} \tag{1.10}$$

Secondly, the right hand side of (1.5) can be written as follows.

$$\begin{aligned}
\int_0^\infty e^{-s^2t^2/2}\mathcal{D}(t)dt &= \sum_{k=0}^\infty \frac{\mathcal{D}^{(k)}(0)}{k!}\int_0^\infty e^{-s^2t^2/2}t^k\,dt \\
&= \sum_{k=0}^\infty \frac{\mathcal{D}^{(k)}(0)}{k!}\int_0^\infty e^{-u}\left(\frac{2u}{s^2}\right)^{\frac{k}{2}}\cdot\frac{1}{s\sqrt{2u}}\,du \quad (\text{change the variable } u = \frac{s^2t^2}{2}) \\
&= \sum_{k=0}^\infty \frac{\mathcal{D}^{(k)}(0)}{k!}2^{\frac{k-1}{2}}\Gamma(\frac{k+1}{2})s^{-k-1}.
\end{aligned} \tag{1.11}$$

Equating coefficients of like powers of s^{-1} in (1.10) and (1.11), we can see that

$$E(S(2)^k) = i^k\varphi^{(k)}(0) = \frac{\mathcal{D}^{(k)}(0)}{k!}2^{\frac{k-1}{2}}\Gamma(\frac{k+1}{2})\pi. \tag{1.12}$$

The main purpose of this paper is to derive a general and explicit formula to calculate the moments of the distribution $S(2)$ when $r = l$.

2 The Main result.

We assume X_i's are symmetric, i.e., $r = l$, then $\mathcal{D}(t)$ becomes

$$\mathcal{D}(t) = (1-\alpha)(2\pi^{-3})^{1/2}\frac{D_{\alpha-2}(-it)+D_{\alpha-2}(it)}{D_{\alpha}(-it)+D_{\alpha}(it)} \equiv (1-\alpha)(2\pi^{-3})^{1/2}\mathcal{A}(t). \tag{2.1}$$

For symmetry case, we simplify the notation of the limiting distribution $S(2)$ by S.

Theorem 2.1. *Let S be defined above. Then for any $m = 1, 2, \dots,$ $E(S^{2m-1}) = 0$ and*

$$E(S^{2m}) = \frac{(2m-1)!!}{(2m)!} 2\{(D_{\alpha-2}^{(2m)}(0) - D_{\alpha}^{(2m)}(0)) - (1-\alpha)\sum_{k=0}^{m-1} \frac{(-1)^{m-k}\mathcal{A}^{(2k)}(0)}{(2(m-k))!!(2k)!} D_{\alpha}^{2(m-k)}(0)\}, \tag{2.2}$$

where $\mathcal{A}^{(2k)}(0)$ satisfies $\mathcal{A}(0) = D_{\alpha-2}(0)/D_{\alpha}(0)$ and

$$\mathcal{A}^{(2k)}(0) = \frac{(-1)^k}{1-\alpha} 2(D_{\alpha-2}^{(2k)}(0) - D_{\alpha}^{(2k)}(0)) - 2\sum_{l=0}^{k-1} \frac{\mathcal{A}^{(2l)}(0)}{(2(k-l))!!(2l)!}(-1)^{(k-l)} D_{\alpha}^{2(k-l)}(0)\}. \tag{2.3}$$

Furthermore, suppose

$$D_{\nu}^{(2k)}(0) = \eta^0(k) + \sum_{j=1}^{k} \eta^j(k)\nu^j, \tag{2.4}$$

then $\eta^j(k)$ satisfies

$$\begin{cases} \eta^j(k) = -\sum_{t=0}^{k-j} \binom{k-t}{j} \nu^{k-t}(k-1) + \frac{2k-1}{2}\eta^j(k-1) \quad \text{for } j \geq 0; \\ \nu^1(k) = k!\left((-1)^k + \sum_{l=1}^{k} \frac{(2l-1)!!(-1)^{k-l}}{2^l l!}\right); \\ \nu^j(k) = -\sum_{l=1}^{k-j+2} \frac{(2k-1)!!}{(2k-(2l-1))!!2^{l-1}} \nu^{j-1}(k-l) - k\nu^j(k-1) \quad \text{for } j \geq 2; \\ \nu^1(0) = 1,\ \nu^j(0) = 0 \ \text{ for } j \geq 2; \quad \nu^0(k) = 0 \ \text{ for any } k \geq 0; \\ \eta^j(k) = 0 \ \text{ for } j > k; \quad \eta^k(k) = (-1)^k \ \text{ for any } k \geq 0. \end{cases} \tag{2.5}$$

Corollary 2.2. *The finite order moments for self-normalized sum of i.i.d random variables in the domain of attraction of a stable law are always finite.*

3 Proof of Theorem 2.1. Set

$$A_{\nu}(t) = D_{\nu}(-it) + D_{\nu}(it), \tag{3.1}$$

then

$$\mathcal{A}(t) = \frac{D_{\alpha-2}(-it) + D_{\alpha-2}(it)}{D_{\alpha}(-it) + D_{\alpha}(it)} = \frac{A_{\alpha-2}(t)}{A_{\alpha}(t)}. \tag{3.2}$$

From (1.12), the moment of the limiting distribution can be simply written as

$$E(S^k) = \frac{(k-1)!!}{k!} i^k (1-\alpha)\mathcal{A}^{(k)}(0). \tag{3.3}$$

Note that $\frac{d}{dt}D_{\nu}(-it) = -i\frac{d}{dz}D_{\nu}(z)\Big|_{z=-it}$ and $\frac{d}{dt}D_{\nu}(it) = i\frac{d}{dz}D_{\nu}(z)\Big|_{z=it}$, it is obvious that

$$A_k(\nu) \equiv \frac{d^k}{dt^k}A_{\nu}(t)\Big|_{t=0} = \begin{cases} 0 & \text{if k is odd;} \\ (-1)^{k/2} 2D_{\nu}^{(k)}(0), & \text{if k is even,} \end{cases} \tag{3.4}$$

where $D_{\nu}^{(k)}(0) = \frac{d^k}{dz^k}D_{\nu}(z)\Big|_{z=0}$. To prove the first statement, we use a recursive formula for nth derivative.

Lemma 3.1 ([9])**.**

$$\left(\frac{u(x)}{v(x)}\right)^{(n)} = \frac{1}{v(x)}\left(u^{(n)}(x) - n!\sum_{j=1}^{n}\frac{v(x)^{(n+1-j)}}{(n+1-j)!(j-1)!}\left(\frac{u(x)}{v(x)}\right)^{(j-1)}\right). \tag{3.5}$$

Applying the formula to the case that $u(x) = A_{\alpha-2}(x)$ and $v(x) = A_\alpha(x)$, we have

$$\mathcal{A}^{(2k)}(0) = \frac{1}{1-\alpha}(A_{2k}(\alpha-2) - A_{2k}(\alpha)) - \sum_{l=0}^{k-1}\binom{2k}{2l}A_{2(k-l)}(\alpha)\mathcal{A}^{(2l)}(0). \tag{3.6}$$

The first result is straightforward from (3.3).

Next, we show the second half of Theorem 2.1. Differentiating (1.8) and (1.9) iteratively, we have

$$D_\nu^{(k)}(z) - \frac{z}{2}D_\nu^{(k-1)}(z) - \frac{k-1}{2}D_\nu^{(k-2)}(z) + D_{\nu+1}^{(k-1)}(z) = 0; \tag{3.7}$$

$$D_\nu^{(k)}(z) + \frac{z}{2}D_\nu^{(k-1)}(z) + \frac{k-1}{2}D_\nu^{(k-2)}(z) - \nu D_{\nu-1}^{(k-1)}(z) = 0. \tag{3.8}$$

Thus $D_\nu^{(k)}(0)$ can be derived from

$$D_\nu^{(k)}(0) = \frac{k-1}{2}D_\nu^{(k-2)}(0) - D_{\nu+1}^{(k-1)}(0); \tag{3.9}$$

$$D_\nu^{(k)}(0) = -\frac{k-1}{2}D_\nu^{(k-2)}(0) + \nu D_{\nu-1}^{(k-1)}(0). \tag{3.10}$$

In the case when k is odd, rewrite $2k+1$ as k, then

$$\begin{aligned} D_\nu^{(2k+1)}(0) &= -kD_\nu^{(2k-1)}(0) + \nu(\frac{2k-1}{2}D_{\nu-1}^{(2k-2)}(0) - D_\nu^{(2k-1)}(0)) \\ &= -kD_\nu^{(2k-1)}(0) - \nu\sum_{l=1}^{k}\frac{(2k-1)!!}{(2k-2l+1)!!2^{l-1}}D_\nu^{(2k-2l+1)}(0) + \nu\frac{(2k-1)!!}{2^k}D_{\nu-1}(0). \end{aligned}$$

This is a recurrence formula for $D_\nu^{(2k+1)}(0)$. If we can expand it, then it must be the product of a polynomial of ν and $D_{\nu-1}(0)$. Let $\nu^j(k)$ denote the coefficient of ν^j in the case of $(2k+1)$th derivative.

For the initial values, we can see that $\nu^1(0) = 1$, $\nu^j(0) = 0$ for all $j \geq 2$ and $\nu^0(k) = 0$ for all $k \geq 0$ from (1.9). After some painful calculation, we have

$$\nu^1(k) = k!\left((-1)^k + \sum_{l=1}^{k}\frac{(2l-1)!!(-1)^{k-l}}{2^l l!}\right); \tag{3.11}$$

$$\nu^j(k) = -\sum_{l=1}^{k-j+2}\frac{(2k-1)!!}{(2k-(2l-1))!!2^{l-1}}\nu^{j-1}(k-l) - k\nu^j(k-1) \quad \text{for } j \geq 2. \tag{3.12}$$

From the recurrence formula, one can see that the highest degree of the polynomial is $k+1$, which can be shown by the induction. Using this property reversely, one also can see that $\nu^j(k) = 0$ for any j and k satisfying $j \geq k+2$.

Corollary 3.2.

$$\nu^{k+1}(k) = (-1)^k, \quad \nu^k(k) = (-1)^k \frac{k}{2}. \tag{3.13}$$

Proof. Applying this result to (3.12),

$$\nu^{k+1}(k) = -\nu^k(k-1) \tag{3.14}$$

holds, and since the initial value $\nu^1(0) = 1$, we have

$$\nu^{k+1}(k) = (-1)^k. \tag{3.15}$$

Also applying the result to $\nu^k(k)$,

$$\nu^k(k) = -\nu^{k-1}(k-1) - \frac{1}{2}, \tag{3.16}$$

which implies

$$\nu^k(k) = (-1)^k \frac{k}{2}, \tag{3.17}$$

since $\nu^0(0) = 0$. □

On the other hand, when k is even, rewrite $2k+2$ as k and we have

$$D_\nu^{(2k+2)}(0) \quad = \quad \frac{2k+1}{2} D_\nu^{(2k)}(0) - D_{\nu+1}^{(2k+1)}(0)$$

Here, let $\eta^j(k)$ denote the coefficient of ν^j in the case of $2k$th derivative. Then we have

$$\eta^j(k) = -\sum_{t=0}^{k-j} \binom{k-t}{j} \nu^{k-t}(k-1) + \frac{2k-1}{2} \eta^j(k-1) \quad \text{for } j \geq 0. \tag{3.18}$$

From (3.15), $\eta^j(k) = 0$ if $j > k$ and $\eta^k(k) = -\nu^k(k-1) = (-1)^k$. □

4 Examples.

4.1 Mathematica code. This section provides Mathematica code. The functions $f(j,k)$ and $g(j,k)$ denote the function $\nu^j(k)$ and $\eta^j(k)$ in the previous section, respectively. The function $A(n,a)$ is corresponding to $1/2\ A_{2n}(\alpha)$, while $CA(n,a)$ represents the function $\mathcal{A}^{(2n)}(0)$ above. Lastly, function $M(n,a)$ indicates the $2n$th moment of the limit distribution S.

```
f[1, k_] :=
  k! ((-1)^k + Sum[(2 l - 1)!! (-1)^(k - 1)/(2^l l!), {l, 1, k}]);
f[j_, k_] :=
  If[j == 0, 0,
   If[k <= -1, 0,
    If[k > -1 && j >= 2,
      -(2 k - 1)!! Sum[
```

```
        f[j - 1, k - l]/((2 k - (2 l - 1))!! 2^(l - 1)), {l, 1, k - j + 2}]
          - k f[j, k - 1]]]]];
g[j_, k_] :=
  If[j > k, 0,
   If[j == k, (-1)^k,
    If[k > 0, -Sum[
        Binomial[k - t, j] f[k - t, k - 1], {t, 0, k - j}] + (2 k -
         1) g[j, k - 1]/2, 0]]];
A[n_, a_] := (-1)^n g[0, n] + (-1)^n Sum[g[t, n] a^t, {t, 1, n}];
CA[n_, a_] :=
  If[n > 0, (A[n, a - 2] - A[n, a])/(1 - a) -
    Sum[CA[t, a] Binomial[2 n, 2 t] A[n - t, a], {t, 1, n - 1}], 0];
M[n_, a_] :=
  Simplify[(2 n - 1)!!/(2 n)! (-1)^n (1 - a) CA[n, a], a > 0];
```

4.2 Some results and knowledge. Using the code above, we obtain the general result for the moments of symmetric self-normalized moments and some special cases of $\alpha = 0.5$, $\alpha = 1.5$ and $\alpha = 2$.

Table 1: The $2k$th moments of symmetric self-normalized sum for the case of $\alpha = 0.5, 1.5, 2$

k	$E(S^{2k})$	$\alpha = 0.5$	$\alpha = 1.5$	$\alpha = 2$
1	1	1	1	1
2	$1+\alpha$	1.5	2.5	3
3	$1+3\alpha+2\alpha^2$	3	10	15
4	$1/3(3+20\alpha+34\alpha^2+17\alpha^3)$	7.875	55.625	105
5	$1/3(3+40\alpha+130\alpha^2+155\alpha^3+62\alpha^4)$	26.25	397.5	945
6	$1/15(15+383\alpha+2118\alpha^2+4514\alpha^3+4146\alpha^4+1382\alpha^5)$	106.838	3471.56	10395

When $\alpha = 2$, the limiting distribution S is standard normal distribution. From Table 1, we can see the result is corresponding to the moments we can obtain from other methods.

4.3 Tail index estimation Hill's estimator is proposed to be an estimator for tail index. It is defined as

$$\hat{\alpha}_H = \Big(\frac{1}{k}\sum_{j=1}^{k} \log X_{n,n-j+1} - \log X_{n,n-k}\Big)^{-1}, \tag{4.1}$$

where $X_n, 1 \leq \cdots \leq X_{n,n}$ are the order statistics of $X_1, \ldots, X_n$. (See [3].) As an alternative to it, we can apply the result above to the derivation of the tail index of the random variables in the domain of stable distribution. This is achieved by moment estimators after calculating the asymptotic moments for the self-normalized sums.

As an example, we use the fourth moment estimator of the self-normalized sums, which is denoted by

$$\hat{\alpha} = \frac{1}{K}\sum_{i=1}^{K} S_i^4 - 1, \tag{4.2}$$

where K is the number of blocks, for the self-normalized sums, of the original samples.

We compare the performance of (4.2) with Hill's estimator by means of Monte Carlo experiments. All numerical results in this paper are based on 250 simulations. The sample sizes are 200 and 2000. The latter size is typical for current financial data, and the former is relatively small for the observation of the behaviors of the estimates in small samples. In each case, we evaluate the performance of the estimators on the basis of i.i.d. random variables and dependent ones. The number in the parentheses after the name of the distribution is the tail index of random variables.

For dependent case, we follow the examples in [1]. The MA process $Y_t = X_t + X_{t-1}$, where the X_t are i.i.d. stable with the tail index α is considered. The other stochastic process Y_t, stochastic volatility, is defined as follows:

$$Y_t = U_t X_t H_t, \tag{4.3}$$

$$U_t \quad \text{i.i.d discrete uniform on } -1 \text{ and } 1, \tag{4.4}$$

$$X_t = \sqrt{57/Z_t}, \quad Z_t \sim \text{i.i.d. } \chi_1^2, \tag{4.5}$$

$$H_t = 0.1Q_t + 0.9H_{t-1}, \quad Q_t \sim \text{i.i.d. } \mathcal{N}(0,1). \tag{4.6}$$

This process is denoted by S.V.(1). Its marginal distribution has a Student-t with 1 degree of freedom.

For Hill's estimator, it is known that $k = O(n^{2/3})$ is optimal ([3], [8]). However, k is not specified since it is sensitive to the sample size and the assumed model. For simplicity, we use $k = \lceil n^{2/3} \rceil$ in all simulations, where $\lceil \cdot \rceil$ is the ceiling function. On the other hand, let T be the number of samples for a self-normalized sum. To guarantee K is large enough, we use bootstrap samples for the self-normalized sums. For each distribution, we report the true value of α, the optimal T for the estimation of α, the mean and the root mean squared error (RMSE) of each estimator in the case that $K = 5n$.

Table 2: Monte Carlo experiment with $n = 200$

Distribution	α	T	Mean ($\hat{\alpha}_H$)	Mean ($\hat{\alpha}$)	RMSE ($\hat{\alpha}_H$)	RMSE ($\hat{\alpha}$)
Stable(0.5)	0.5	8	0.468	0.506	0.082	0.092
Stable(1)	1	10	0.988	0.996	0.169	0.186
Stable(1.5)	1.5	12	1.690	1.475	0.366	0.334
Stable(2)	2	16	2.574	1.995	0.710	0.516
Student(0.5)	0.5	5	0.527	0.493	0.093	0.082
Student(1.5)	1.5	19	1.290	1.493	0.296	0.366
MA(0.5)	0.5	6	0.489	0.511	0.014	0.010
MA(1)	1	8	1.019	1.007	0.045	0.053
MA(1.5)	1.5	9	1.690	1.492	0.172	0.200
MA(2)	2	12	2.631	2.016	0.608	0.433
S.V.(1)	1	5	1.617	1.020	0.461	0.021

From Table 2, the moment estimator of the self-normalized sums can be sufficiently accurate even in the small sample cases. The RMSEs of both Hill's estimator and the moment estimator become larger as the tail index increases. The difference between two estimators is that the RMSE of Hill's estimator becomes large more sharply than that of the moment estimator as the tail index grows larger than 1 in the stable case.

Table 3: Monte Carlo experiment with $n = 2000$

Distribution	α	T	Mean ($\hat{\alpha}_H$)	Mean ($\hat{\alpha}$)	RMSE ($\hat{\alpha}_H$)	RMSE ($\hat{\alpha}$)
Stable(0.5)	0.5	23	0.484	0.500	0.040	0.037
Stable(1)	1	34	0.999	1.001	0.079	0.078
Stable(1.5)	1.5	43	1.753	1.505	0.292	0.148
Stable(2)	2	46	3.877	2.005	1.898	0.170
Student(0.5)	0.5	11	0.502	0.502	0.039	0.030
Student(1.5)	1.5	79	1.414	1.505	0.139	0.208
MA(0.5)	0.5	17	0.485	0.502	0.003	0.002
MA(1)	1	23	1.006	1.002	0.010	0.012
MA(1.5)	1.5	28	1.792	1.500	0.129	0.024
MA(2)	2	35	3.919	1.996	3.767	0.055
S.V.(1)	1	5	1.900	0.969	0.836	0.003

As shown in Table 3, we can see that the moment estimators attain to the true tail index if we choose a proper T for each case. The RMSEs of two estimators are lower for the larger sample size, while the comparison between two estimators is almost similar to the sample size $n = 200$. However, we also find that the behavior of the moment estimators in the student case and the S.V case are a little different from the stable case and MA case from both tables. The representation of the tail of t-distribution is more complicate than that of stable, since there is a second term in the representation of the former. Nevertheless, the moment estimator of self-normalized sums performs well in the estimation of the tail index.

Acknowledgements The author would like to thank his Ph.D. supervisor, Prof. Masanobu Taniguchi, for his precious advice which greatly improved the paper. The author is also grateful to an anonymous referee for many helpful comments.

References

[1] Danielsson, J., de Haan, L., Peng, L., de Vries, C.G., 2001. Using a bootstrap method to choose the sample fraction in tail index estimation. Journal of Multivariate analysis 76, 226–248.

[2] Griffin, P.S., Mason, D.M., 1991. On the asymptotic normality of self-normalized sums, in: Mathematical Proceedings of the Cambridge Philosophical Society, Cambridge Univ Press. pp. 597–610.

[3] Hall, P., 1982. On some simple estimates of an exponent of regular variation. Journal of the Royal Statistical Society. Series B (Methodological) , 37–42.

[4] Klüppelberg, C., Mikosch, T., 1996. The integrated periodogram for stable processes. The Annals of Statistics , 1855–1879.

[5] Logan, B., Mallows, C., Rice, S., Shepp, L., 1973. Limit distributions of self-normalized sums. The Annals of Probability 1, 788–809.

[6] Magnus, W., Oberhettinger, F., 1954. Formulas and theorems for the functions of mathematical physics. Chelsea Pub. Co.

[7] Peña, V.H., Lai, T.L., Shao, Q.M., 2009. Self-normalized processes: Limit theory and Statistical Applications. Springer.

[8] Resnick, S., Stărică, C., 1998. Tail index estimation for dependent data. The Annals of Applied Probability 8, 1156–1183.

[9] Xenophontos, C., 2007. A formula for the nth derivative of the quotient of two functions.

Communicated by *Masanobu Taniguchi*

Yan Liu:
Department of Pure and Applied Mathematics
Graduate School of Fundamental Science and Engineering, Waseda University
3-4-1, Okubo, Shinjuku-ku, Tokyo, 169-8555, Japan
Email: great-rainbow@ruri.waseda.jp

Scientiae Mathematicae Japonicae 77,No.1 (2014) 69-81

SOME DOUBLE SEQUENCE SPACES DEFINED BY A SEQUENCE OF ORLICZ FUNCTIONS OVER n-NORMED SPACES

Kuldip Raj, Sunil K. Sharma and Ajay K. Sharma

Received May 7, 2013 ; revised June 24, 2013

Abstract. In the present paper we introduce some double sequence spaces defined by a sequence of Orlicz functions $\mathcal{M} = (M_{k,l})$ over n-normed spaces. We study some topological properties and prove some inclusion relations between these spaces.

1 Introduction and Preliminaries The initial works on double sequences is found in Bromwich [4]. Later on, it was studied by Hardy [11], Moricz [17], Moricz and Rhoades [18], Tripathy ([31], [32]), Başarir and Sonalcan [2] and many others. Hardy [10] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [34] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [21] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly Cesaro summable double sequences. Nextly, Mursaleen [20] and Mursaleen and Edely [22] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{m,n})$ into one whose core is a subset of the M-core of x. More recently, Altay and Başar [1] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively and also examined some properties of these sequence spaces and determined the α-duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(v)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Now, recently Başar and Sever [3] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. By the convergence of a double sequence we mean the convergence in the Pringsheim sense i.e. a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in N$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > n$ see [25]. We shall write more briefly as P-convergent. The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l.

The notion of difference sequence spaces was introduced by Kızmaz [11], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et. and Çolak [5] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let s be a non-negative integer, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta^s) = \{x = (x_k) \in w : (\Delta^s x_k) \in Z\},$$

2010 *Mathematics Subject Classification.* 40A05,46A45.

Key words and phrases. Paranorm space, Difference sequence space, Orlicz function, Musielak-Orlicz function, n-normed space.

where $\Delta^s x = (\Delta^s x_k) = (\Delta^{s-1} x_k - \Delta^{s-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^s(x_k) = \sum_{v=0}^{s} (-1)^v \begin{pmatrix} s \\ v \end{pmatrix} x_{k+v}.$$

Taking $s = 1$, we get the spaces which were introduced and studied by Kızmaz [11].
The concept of 2-normed spaces was initially developed by Gähler[6] in the mid of 1960's, while that of n-normed spaces one can see in Misiak[19]. Since then, many others have studied this concept and obtained various results, see Gunawan ([7],[8]) and Gunawan and Mashadi [9]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

1. $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and
4. $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called a n-norm on X and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$.
For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm, $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$ and $||.||_E$ denotes the Euclidean norm. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.
A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k \to \infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,p \to \infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Let l_∞, c and c_0 be the sequence spaces of bounded, convergent and null sequences $x = (x_k)$, respectively.

A sequence $x \in l_\infty$ is said to be almost convergent if all Banach limits of x coincide. Lorentz [13] proved that

$$\hat{c} = \Big\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} x_{k+s} \quad \text{exists, uniformly in } \ s \Big\}.$$

Maddox ([14], [15]) has defined x to be strongly almost convergent to a number L if

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - L| = 0, \quad \text{uniformly in } \ s.$$

Let $p = (p_k)$ be a sequence of strictly positive real numbers. Nanda [24] has defined the following sequence spaces:

$$[\hat{c}, p] = \Big\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - L|^{p_k} = 0, \quad \text{uniformly in } \ s \Big\},$$

$$[\hat{c}, p]_0 = \Big\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+s}|^{p_k} = 0, \quad \text{uniformly in } \ s \Big\}$$

and

$$[\hat{c}, p]_\infty = \Big\{ x = (x_k) : \sup_{s,n} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s}|^{p_k} < \infty \Big\}.$$

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.
Lindenstrauss and Tzafriri [12] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \Big\{ x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty \Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf \Big\{ \rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1 \Big\}.$$

It is shown in [12] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. An Orlicz function M satisfies Δ_2-condition if and only if for any constant $L > 1$ there exists a constant $K(L)$ such that $M(Lu) \leq K(L)M(u)$ for all values of $u \geq 0$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.
Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$, for all $x \in X$;
2. $p(-x) = p(x)$, for all $x \in X$;

3. $p(x+y) \leq p(x) + p(y)$, for all $x, y \in X$;

4. if (σ_n) is a sequence of scalars with $\sigma_n \to \sigma$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\sigma_n x_n - \sigma x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [33], Theorem 10.4.2, P-183). For more details about sequence spaces see ([16], [23], [26], [27], [28], [29], [30]) and references therein.
Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space and $p = (p_{k,l})$ be bounded sequence of positive real numbers. By $S(n-X)$ we denote the space of all sequences defined over $(X, ||\cdot, \cdots, \cdot||)$. In the present paper we define the following sequence spaces:
$\Big[\hat{c}^2, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big](\Delta^s) =$

$$\Big\{x = (x_{k,l}) \in S(n{-}X) : P{-}\lim_{q,r} \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}} = 0,$$

$$\text{uniformly in } m \text{ and } n, \text{ for some } L \text{ and } \rho > 0 \Big\},$$

$\Big[\hat{c}^2, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta^s) =$

$$\Big\{x = (x_{k,l}) \in S(n{-}X) : P{-}\lim_{q,r} \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}} = 0,$$

$$\text{uniformly in } m \text{ and } n,\ \rho > 0 \Big\},$$

and
$\Big[\hat{c}^2, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta^s) =$

$$\Big\{x = (x_{k,l}) \in S(n{-}X) : \sup_{q,r} \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}} < \infty,\ \rho > 0\Big\}.$$

If we take $\mathcal{M}(x) = x$, we get
$\Big[\hat{c}^2, p, ||\cdot, \cdots, \cdot||\Big](\Delta^s) =$

$$\Big\{x = (x_{k,l}) \in S(n{-}X) : P{-}\lim_{q,r} \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big(||\frac{\Delta^s x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_{k,l}} = 0,$$

$$\text{uniformly in } m \text{ and } n, \text{ for some } L \text{ and } \rho > 0 \Big\},$$

$\Big[\hat{c}^2, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta^s) =$

$$\Big\{x = (x_{k,l}) \in S(n - X) : P - \lim_{q,r} \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big(||\frac{\Delta^s x_{k+l,m+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_{k,l}} = 0,$$

$$\text{uniformly in } m \text{ and } n,\ \rho > 0 \Big\},$$

and
$\Big[\hat{c}^2, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta^s) =$

$$\Big\{x = (x_{k,l}) \in S(n{-}X) : \sup_{q,r} \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_{k,l}} < \infty,\ \rho > 0\Big\}.$$

If we take $p = (p_{k,l}) = 1$ for all $k \in \mathbb{N}$, we get
$\Big[\hat{c}^2, \mathcal{M}, ||\cdot, \cdots, \cdot||\Big](\Delta^s) =$

$$\Big\{x = (x_{k,l}) \in S(n{-}X) : P{-}\lim_{q,r} \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] = 0,$$

$$\text{uniformly in } m \text{ and } n, \text{ for some } L \text{ and } \rho > 0 \Big\},$$

$\Big[\hat{c}^2, \mathcal{M}, ||\cdot, \cdots, \cdot||\Big]_0(\Delta^s) =$

$$\Big\{x = (x_{k,l}) \in S(n - X) : P - \lim_{q,r} \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] = 0,$$

$$\text{uniformly in } m \text{ and } n,\ \rho > 0 \Big\},$$

and $\Big[\hat{c}^2, \mathcal{M}, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta^s) =$

$$\Big\{x = (x_{k,l}) \in S(n{-}X) : \sup_{q,r} \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] < \infty,\ \rho > 0\Big\}.$$

The following inequality will be used throughout the paper. Let $p = (p_{k,l})$ be a double sequence of positive real numbers with $0 < p_{k,l} \leq \sup_{k,l} = H$ and let $K = \max\{1, 2^{H-1}\}$. Then for the factorable sequences $\{a_{k,l}\}$ and $\{b_{k,l}\}$ in the complex plane, we have

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq K(|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}). \tag{1.1}$$

The main aim of this paper is to study some double sequence spaces defined by a sequence of Orlicz functions over n-normed spaces in more general setting. We also make an effort to study some topological properties and some inclusion relations between these spaces.

2 Main Results

Theorem 2.1 *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions and $p = (p_{k,l})$ be bounded sequence of positive real numbers, then the spaces $\Big[\hat{c}^2, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big](\Delta^s)$, $\Big[\hat{c}^2, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta^s)$ and $\Big[\hat{c}^2, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta^s)$ are linear spaces.*

Proof. Let $x = (x_{k,l})$, $y = (y_{k,l}) \in \Big[\hat{c}^2, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta^s)$ and α, β be any scalars. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_{q,r} \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}} = 0$$

and

$$\lim_{q,r} \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big[M_{k,l}\Big(||\frac{\Delta^s y_{k+m,l+n}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}} = 0.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_{k,l})$ is non-decreasing convex function and by using inequality (1.1), we have

$$\begin{aligned}
&\tfrac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big[M_{k,l}\Big(||\frac{\Delta^s(\alpha x_{k+m,l+n} + \beta y_{k+m,l+n})}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}} \\
\leq\ & \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big[M_{k,l}\Big(||\frac{\Delta^s \alpha x_{k+m,l+n}}{\rho_3}, z_1, \cdots, z_{n-1}|| + ||\frac{\Delta^s \beta y_{k+m,l+n}}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}} \\
\leq\ & \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho_1}, z_1, \cdots, z_{n-1}|| + ||\frac{\Delta^s y_{k+m,l+n}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}} \\
\leq\ & K \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}} \\
+\ & K \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big[M_{k,l}\Big(||\frac{\Delta^s y_{k+m,l+n}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}} \\
& \to 0 \ \text{ as } \ n \to \infty, \quad \text{uniformly in } \ m \text{ and } n.
\end{aligned}$$

So that $\alpha x + \beta y \in \Big[\hat{c}^2, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta^s)$. Thus $\Big[\hat{c}^2, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta^s)$ is a linear space. Similarly, we can prove that $\Big[\hat{c}^2, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta^s)$ and $\Big[\hat{c}^2, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big](\Delta^s)$ are linear spaces.

Theorem 2.2 *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions and $p = (p_{k,l})$ be a bounded sequence of positive real numbers, then the space $\Big[\hat{c}^2, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta^s)$ is a paranormed space with respect to the paranorm defined by*

$$g(x) = \inf\Big\{\rho^{\frac{p_{(q,r)}}{H}} : \Big(\frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}} \leq 1\Big\},$$

where $H = \max(1, \sup_{k,l} p_{k,l} < \infty)$.

Proof. Clearly $g(x) \geq 0$ for $x = (x_{k,l}) \in \Big[\hat{c}^2, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta^s)$. Since $M_{k,l}(0) = 0$, we get $g(0) = 0$.
Conversely, suppose that $g(x) = 0$, then

$$\inf\Big\{\rho^{\frac{p_{(q,r)}}{H}} : \Big(\frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}} \leq 1\Big\} = 0.$$

This implies that for a given $\epsilon > 0$, there exists some $\rho_\epsilon (0 < \rho_\epsilon < \epsilon)$ such that

$$\Big(\frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho_\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}} \leq 1.$$

Thus

$$\Big(\tfrac{1}{(q,r)}\sum_{k,l=1,1}^{q,r}\Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\epsilon},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}}$$

$$\begin{aligned}
&\leq \Big(\frac{1}{(q,r)}\sum_{k,l=1,1}^{q,r}\Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho_\epsilon},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}}\\
&\leq 1,
\end{aligned}$$

for each q,r. Suppose that $x_{k,l}\neq 0$ for each $k,l\in\mathbb{N}$. This implies that $\Delta^s x_{k+m,l+n}\neq 0$, for each $k,l\in\mathbb{N}$. Let $\epsilon\to 0$, then $||\frac{\Delta^s x_{k+m,l+n}}{\epsilon},z_1,\cdots,z_{n-1}||\to\infty$. It follows that

$$\Big(\frac{1}{(q,r)}\sum_{k,l=1,1}^{q,r}\Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\epsilon},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}}\to\infty,$$

which is a contradiction. Therefore, $\Delta^s x_{k+m,l+n}=0$ for each k,l and thus $x_{k,l}=0$ for each $k,l\in\mathbb{N}$. Let $\rho_1>0$ and $\rho_2>0$ be such that

$$\Big(\frac{1}{(q,r)}\sum_{k,l=1,1}^{q,r}\Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}}\leq 1$$

and

$$\Big(\frac{1}{(q,r)}\sum_{k,l=1,1}^{q,r}\Big[M_{k,l}\Big(||\frac{\Delta^s y_{k+m,l+n}}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}}\leq 1$$

for each m and n. Let $\rho=\rho_1+\rho_2$. Then by using Minkowski's inequality, we have

$$\Big(\tfrac{1}{(q,r)}\sum_{k,l=1,1}^{q,r}\Big[M_{k,l}\Big(||\frac{\Delta^s(x_{k+m,l+n}+y_{k+m,l+n})}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}}$$

$$\begin{aligned}
&\leq \Big(\frac{1}{(q,r)}\sum_{k,l=1,1}^{q,r}\Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}+\Delta^s y_{k+m,l+n}}{\rho_1+\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}}\\
&\leq \Big(\frac{1}{(q,r)}\sum_{k,l=1,1}^{q,r}\Big[\frac{\rho_1}{\rho_1+\rho_2}M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\\
&+ \frac{\rho_2}{\rho_1+\rho_2}M_{k,l}\Big(||\frac{\Delta^s y_{k+m,l+n}}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}}\\
&\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{1}{(q,r)}\sum_{k,l=1,1}^{q,r}\Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}}\\
&+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{1}{(q,r)}\sum_{k,l=1,1}^{q,r}\Big[M_{k,l}\Big(||\frac{\Delta^s y_{k+m,l+n}}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}}\\
&\leq 1.
\end{aligned}$$

Since ρ's are non-negative, so we have
$g(x+y)$

$$
\begin{aligned}
&= \inf\Big\{\rho^{\frac{p_{(q,r)}}{H}} : \Big(\frac{1}{(q,r)}\sum_{k,l=1,1}^{q,r}\Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}+\Delta^s y_{k+m,l+n}}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}} \le 1\Big\}\\
&\le \inf\Big\{\rho_1^{\frac{p_{(q,r)}}{H}} : \Big(\frac{1}{(q,r)}\sum_{k,l=1,1}^{q,r}\Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho_1}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}} \le 1\Big\}\\
&+ \inf\Big\{\rho_2^{\frac{p_{(q,r)}}{H}} : \Big(\frac{1}{(q,r)}\sum_{k,l=1,1}^{q,r}\Big[M_{k,l}\Big(||\frac{\Delta^s y_{k+m,l+n}}{\rho_2}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}} \le 1\Big\}.
\end{aligned}
$$

Therefore,

$$g(x+y) \le g(x)+g(y).$$

Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x) = \inf\Big\{\rho^{\frac{p_{(q,r)}}{H}} : \Big(\frac{1}{(q,r)}\sum_{k,l=1,1}^{q,r}\Big[M_{k,l}\Big(||\frac{\Delta^s \lambda x_{k+m,l+n}}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}} \le 1\Big\}.$$

Then

$$g(\lambda x) = \inf\Big\{(|\lambda| t)^{\frac{p_{(q,r)}}{H}} : \Big(\frac{1}{(q,r)}\sum_{k,l=1,1}^{q,r}\Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{t}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}} \le 1\Big\}.$$

where $t = \frac{\rho}{|\lambda|}$. Since $|\lambda|^{p_{(q,r)}} \le \max(1, |\lambda|^{\sup p_{k,l}})$, we have

$$g(\lambda x) \le \max(1, |\lambda|^{\sup p_{k,l}}) \inf\Big\{t^{\frac{p_{(q,r)}}{H}} : \Big(\frac{1}{(q,r)}\sum_{k,l=1,1}^{q,r}\Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{t}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}} \le 1\Big\}.$$

So, the fact "scalar multiplication is continuous" follows from the above inequality. This completes the proof of the theorem.

Theorem 2.3 *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions. Then the following statements are equivalent :*
(i) $\Big[\hat{c}^2, p, ||\cdot,\cdots,\cdot||\Big]_\infty(\Delta^s) \subseteq \Big[\hat{c}^2, \mathcal{M}, p, ||\cdot,\cdots,\cdot||\Big]_\infty(\Delta^s)$*;*
(ii) $\Big[\hat{c}^2, p, ||\cdot,\cdots,\cdot||\Big]_0(\Delta^s) \subseteq \Big[\hat{c}^2, \mathcal{M}, p, ||\cdot,\cdots,\cdot||\Big]_\infty(\Delta^s)$*;*
(iii) $\sup_{q,r} \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} [M_{k,l}(t)]^{p_{k,l}} < \infty$*, where* $t = ||\frac{\Delta^s x_{k+m,l+n}}{\rho}, z_1,\cdots,z_{n-1}|| > 0$.

Proof. (i) $\Longrightarrow$ (ii) is obvious. Since $\Big[\hat{c}^2, p, ||\cdot,\cdots,\cdot||\Big]_0(\Delta^s) \subseteq \Big[\hat{c}^2, p, ||\cdot,\cdots,\cdot||\Big]_\infty(\Delta^s)$.

(ii) $\Longrightarrow$ (iii). Suppose $\Big[\hat{c}^2, p, ||\cdot,\cdots,\cdot||\Big]_0(\Delta^s) \subseteq \Big[\hat{c}^2, \mathcal{M}, p, ||\cdot,\cdots,\cdot||\Big]_\infty(\Delta^s)$ and let (iii) does not hold. Then for some $t > 0$

$$\sup_{q,r} \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} [M_{k,l}(t)]^{p_{k,l}} = \infty,$$

and therefore there is a sequence $(q,r)_i$ of positive integers such that

$$\frac{1}{(q,r)_i}\sum_{k,l=1,1}^{(q,r)_i}[M_{k,l}(i^{-1})]^{p_{k,l}} > i, \quad i=1,2,\cdots \tag{2.1}$$

Define $x=(x_{k,l})$ by

$$(x_{k,l})=\begin{cases} i^{-1}, & 1\leq k\leq (q,r)_i, \quad i=1,2,\cdots \\ 0, & k\geq (q,r)_i. \end{cases}$$

Then $x=(x_{k,l})\in \Big[\hat{c}^2,p,||\cdot,\cdots,\cdot||\Big]_0(\Delta^s)$ but $x=(x_{k,l})\notin \Big[\hat{c}^2,\mathcal{M},p,||\cdot,\cdots,\cdot||\Big]_\infty(\Delta^s)$ which contradicts (ii). Hence (iii) must hold.

(iii) $\Longrightarrow$ (i). Suppose $x=(x_{k,l})\in \Big[\hat{c}^2,p,||\cdot,\cdots,\cdot||\Big]_\infty(\Delta^s)$ and $x=(x_{k,l})\notin \Big[\hat{c}^2,\mathcal{M},p,||\cdot,\cdots,\cdot||\Big]_\infty(\Delta^s)$. Then

$$\sup_{q,r}\frac{1}{(q,r)}\sum_{k,l=1,1}^{q,r}\Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}=\infty. \tag{2.2}$$

Let $t=||\frac{\Delta^s x_{k+m,l+n}}{\rho},z_1,\cdots,z_{n-1}||$ for each k, l and fixed m,n, then by eqn.(2.2)

$$\sup_{q,r}\frac{1}{(q,r)}\sum_{k,l=1,1}^{q,r}[M_{k,l}(t)]^{p_{k,l}}=\infty,$$

which contradicts (iii). Hence (i) must hold.

Theorem 2.4 *Let $1\leq p_{k,l}\leq \sup_{k,l} p_{k,l}<\infty$. Then the following statements are equivalent:*
(i) $\Big[\hat{c}^2,\mathcal{M},p,||\cdot,\cdots,\cdot||\Big]_0(\Delta^s)\subseteq \Big[\hat{c}^2,p,||\cdot,\cdots,\cdot||\Big]_0(\Delta^s)$;
(ii) $\Big[\hat{c}^2,\mathcal{M},p,||\cdot,\cdots,\cdot||\Big]_0(\Delta^s)\subseteq \Big[\hat{c}^2,p,||\cdot,\cdots,\cdot||\Big]_\infty(\Delta^s)$;
(iii) $\inf_{q,r}\frac{1}{(q,r)}\sum_{k,l=1,1}^{q,r}[M_{k,l}(t)]^{p_{k,l}}>0, \quad t>0.$

Proof. (i) $\Longrightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (iii) Suppose $\Big[\hat{c}^2,\mathcal{M},p,||\cdot,\cdots,\cdot||\Big]_0(\Delta^s)\subseteq \Big[\hat{c}^2,p,||\cdot,\cdots,\cdot||\Big]_\infty(\Delta^s)$ and let (iii) does not hold. Then

$$\inf_{q,r}\frac{1}{(q,r)}\sum_{k,l=1,1}^{q,r}[M_{k,l}(t)]^{p_{k,l}}=0, \quad t>0. \tag{2.3}$$

We can choose an index sequence $(q,r)_i$ such that

$$\frac{1}{(q,r)_i}\sum_{k,l=1,1}^{(q,r)_i}[M_{k,l}(i)]^{p_{k,l}}<i^{-1}, \quad i=1,2,...$$

Define the sequence $x = (x_{k,l})$ by

$$(x_{k,l}) = \begin{cases} i, & 1 \le k \le (q,r)_i, \quad i = 1, 2, ... \\ 0, & \qquad k \ge (q,r)_i. \end{cases}$$

Thus by eqn.(2.3), $x = (x_{k,l}) \in \left[\hat{c}^2, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\right]_0 (\Delta^s)$ but $x = (x_{k,l}) \notin \left[\hat{c}^2, p, ||\cdot, \cdots, \cdot||\right]_\infty (\Delta^s)$ which contradicts (ii). Hence (iii) must hold.

(iii) $\Longrightarrow$ (i) Let $x = (x_{k,l}) \in \left[\hat{c}^2, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\right]_0 (\Delta^s)$. That is,
(2.4)
$$\lim_{q,r} \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \left[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\right]^{p_{k,l}} = 0, \ \ \text{uniformly in } \ m \text{ and } n.$$

Suppose (iii) hold and $x = (x_{k,l}) \notin \left[\hat{c}^2, p, ||\cdot, \cdots, \cdot||\right]_0 (\Delta^s)$. Then for some number $\epsilon_0 > 0$ and index $(q,r)_0$, we have $||\frac{\Delta^s x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}|| \ge \epsilon_0$, for some $m > m'$ and $1 \le k, l \le (q,r)_0$. Therefore

$$[M_{k,l}(\epsilon_0)]^{p_{k,l}} \le \left[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\right]^{p_{k,l}}$$

and consequently by eqn.(2.3)

$$\lim_{q,r} \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} [M_{k,l}(\epsilon_0)]^{p_{k,l}} = 0,$$

which contradicts (iii). Hence $\left[\hat{c}^2, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\right]_0 (\Delta^s) \subseteq \left[\hat{c}^2, p, ||\cdot, \cdots, \cdot||\right]_0 (\Delta^s)$.

Theorem 2.5 *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions. Let $1 \le p_{k,l} \le \sup_{k,l} p_{k,l} < \infty$. Then*

$$\left[\hat{c}^2, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\right]_\infty (\Delta^s) \subseteq \left[\hat{c}^2, p, ||\cdot, \cdots, \cdot||\right]_0 (\Delta^s)$$

holds if

(2.5)
$$\lim_{q,r} \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} [M_{k,l}(t)]^{p_{k,l}} = \infty, \quad t > 0.$$

Proof. Suppose $\left[\hat{c}^2, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\right]_\infty (\Delta^s) \subseteq \left[\hat{c}^2, p, ||\cdot, \cdots, \cdot||\right]_0 (\Delta^s)$ and let eqn.(2.5) does not hold. Therefore there is a number $t_0 > 0$ and an index sequence $(q,r)_i$ such that

(2.6)
$$\frac{1}{(q,r)_i} \sum_{k,l=1,1}^{(q,r)_i} [M_{k,l}(t_0)]^{p_{k,l}} \le N < \infty, \ \ i = 1, 2, ...$$

Define a sequence $x = (x_{k,l})$ by

$$(x_{k,l}) = \begin{cases} t_0, & 1 \le k, l \le (q,r)_i, \quad i = 1, 2, ... \\ 0, & \qquad k, l \ge (q,r)_i. \end{cases}$$

Clearly, $x=(x_{k,l}) \in \Big[\hat{c}^2, \mathcal{M}, p, ||\cdot,\cdots,\cdot||\Big]_\infty(\Delta^s)$ but $x=(x_{k,l}) \notin \Big[\hat{c}^2, p, ||\cdot,\cdots,\cdot||\Big]_0(\Delta^s)$. Hence eqn.(2.5) must hold.
Conversely, if $x=(x_{k,l}) \in \Big[\hat{c}^2, \mathcal{M}, p, ||\cdot,\cdots,\cdot||\Big]_\infty(\Delta^s)$, then for each m and n

$$(2.7) \qquad \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}} \leq N < \infty.$$

Suppose that $x=(x_{k,l}) \notin \Big[\hat{c}^2, p, ||\cdot,\cdots,\cdot||\Big]_0(\Delta^s)$. Then for some number $\epsilon_0 > 0$ there is a number m_0

$$||\frac{\Delta^s x_{k+m,l+n}}{\rho}, z_1,\cdots,z_{n-1}|| \geq \epsilon_0, \quad \text{for } \ m,n \geq m_0.$$

Therefore

$$[M_{k,l}(\epsilon_0)]^{p_{k,l}} \leq \Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}},$$

and hence for each k,l,m and n we get

$$\frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} [M_{k,l}(\epsilon_0)]^{p_{k,l}} \leq N < \infty,$$

for some $N > 0$, which contradicts eqn. (2.5). Hence

$$\Big[\hat{c}^2, \mathcal{M}, p, ||\cdot,\cdots,\cdot||\Big]_\infty(\Delta^s) \subseteq \Big[\hat{c}^2, p, ||\cdot,\cdots,\cdot||\Big]_0(\Delta^s).$$

This completes the proof.

Theorem 2.6 *Suppose $\mathcal{M}=(M_{k,l})$ be a sequence of Orlicz functions and let $1 \leq p_{k,l} \leq \sup_{k,l} p_{k,l} < \infty$. Then*

$$\Big[\hat{c}^2, p, ||\cdot,\cdots,\cdot||\Big]_\infty(\Delta^s) \subseteq \Big[\hat{c}^2, \mathcal{M}, p, ||\cdot,\cdots,\cdot||\Big]_0(\Delta^s)$$

holds if

$$(2.8) \qquad \lim_{q,r} \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} [M_{k,l}(t)]^{p_{k,l}} = 0, \quad t > 0.$$

Proof. Let $\Big[\hat{c}^2, p, ||\cdot,\cdots,\cdot||\Big]_\infty(\Delta^s) \subseteq \Big[\hat{c}^2, \mathcal{M}, p, ||\cdot,\cdots,\cdot||\Big]_0(\Delta^s)$. Suppose that eqn.(2.8) does not hold. Then for some $t_0 > 0$,

$$(2.9) \qquad \lim_{q,r} \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} [M_{k,l}(t)]^{p_{k,l}} = L \neq 0.$$

Define $x=(x_{k,l})$ by

$$(x_{k,l}) = t \sum_{v=0}^{k-m} (-1)^m \begin{pmatrix} m+k-v-1 \\ k-v \end{pmatrix}$$

for $k = 1, 2, ...$ Then $x = (x_{k,l}) \notin \Big[\hat{c}^2, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta^s)$ but $x = (x_{k,l}) \in \Big[\hat{c}^2, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta^s)$. Hence eqn. (2.8) must hold.
Conversely, let $x = (x_{k,l}) \in \Big[\hat{c}^2, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta^s)$. Then for every k, l, m and n, we have

$$||\frac{\Delta^s x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}|| \leq N < \infty.$$

Therefore

$$\Big[M_{k,l}||\frac{\Delta^s x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big]^{p_{k,l}} \leq [M_{k,l}(N)]^{p_{k,l}}$$

and

$$\lim_{q,r} \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} \Big[M_{k,l}\Big(||\frac{\Delta^s x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}} \leq \lim_{q,r} \frac{1}{(q,r)} \sum_{k,l=1,1}^{q,r} [M_{k,l}(N)]^{p_{k,l}} = 0.$$

Hence $x = (x_{k,l}) \in \Big[\hat{c}^2, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta^s)$. This completes the proof.

Acknowledgement : The authors thank the referee for his valuable suggestions that improved the presentation of the paper.

References

[1] B. Altay and F. Başar, *Some new spaces of double sequences*, J. Math. Anal. Appl., **309** (2005), 70-90.

[2] M. Başarir, and O. Sonalcan, *On some double sequence spaces*, J. Indian Acad. Math., **21** (1999), 193-200.

[3] F. Başar and Y. Sever, *The space $\mathcal{L}_q$ of double sequences*, Math. J. Okayama Univ., **51** (2009), 149-157.

[4] T. J. Bromwich, *An introduction to the theory of infinite series*, Macmillan and Co. Ltd., New York (1965).

[5] M. Et and R. Çolak, *On some generalized difference sequence spaces*, Soochow J. Math., **21** (1995), 377-386.

[6] S. Gähler, *2- metrische Raume und ihre topologishe Struktur*, Math. Nachr., **26** (1963), 115-148.

[7] H. Gunawan, *On n-inner product, n-norms, and the Cauchy-Schwartz inequality*, Sci. Math. Jpn., **5** (2001), 47-54.

[8] H. Gunawan, *The space of p-summable sequences and its natural n-norm*, Bull. Aust. Math. Soc., **64** (2001), 137-147.

[9] H. Gunawan and M. Mashadi, *On n-normed spaces*, Int. J. Math. Math. Sci., **27** (2001), 631-639.

[10] G. H. Hardy, *On the convergence of certain multiple series*, Proc. Camb. Phil., Soc., **19** (1917), 86-95.

[11] H. Kızmaz, *On certain sequence spaces*, Cand. Math. Bull., **24** (1981), 169-176.

[12] J. Lindenstrauss and L. Tzafriri, *On Orlicz seequence spaces*, Israel J. Math., **10** (1971), 379-390.

[13] G. G. Lorentz, *A contribution to the theory of divergent series*, Acta Math., **80** (1948), 167-190.

[14] I. J. Maddox, *Spaces of strongly summable sequences*, Quart. J. Math., **18** (1967), 345-355.

[15] I. J. Maddox, *A new type of convergence*, Math. Proc. Camb. Phil. Soc., **83** (1978), 61-64.

[16] L. Maligranda, *Orlicz spaces and interpolation*, Semin. Math. 5, Polish Academy of Science, (1989).

[17] F. Moricz, *Extension of the spaces c and c_0 from single to double sequences*, Acta Math. Hung., **57** (1991), 129-136.

[18] F. Moricz and B. E. Rhoades, *Almost convergence of double sequences and strong regularity of summability matrices*, Math. Proc. Camb. Phil. Soc., **104** (1988), 283-294.

[19] A. Misiak, *n-inner product spaces*, Math. Nachr., **140** (1989), 299-319.

[20] M. Mursaleen, *Almost strongly regular matrices and a core theorem for double sequences*, J. Math. Anal. Appl., **293** (2004), 523-531.

[21] M. Mursaleen and O. H. H. Edely, *Statistical convergence of double sequences*, J. Math. Anal. Appl., **288** (2003), 223-231.

[22] M. Mursaleen and O. H. H. Edely, *Almost convergence and a core theorem for double sequences*, J. Math. Anal. Appl., **293** (2004), 532-540.

[23] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, **1034** (1983).

[24] S. Nanda, *Strongly almost convergent sequences*, Bull. Calcutta Math. Soc., **76** (1984), 236-240.

[25] A. Pringsheim, *Zur Theori der zweifach unendlichen Zahlenfolgen*, Math. Ann. **53** (1900), 289-321.

[26] K. Raj, A. K. Sharma and S. K. Sharma, *A Sequence space defined by Musielak-Orlicz functions*, Int. J. Pure Appl. Math., **67** (2011), 475-484.

[27] K. Raj, S. K. Sharma and A. K. Sharma, *Some difference sequence spaces in n-normed spaces defined by Musielak-Orlicz function*, Armen. J. Math., **3** (2010), 127-141.

[28] K. Raj and S. K. Sharma, *Some sequence spaces in 2-normed spaces defined by Musielak-Orlicz functions*, Acta Univ. Sapientiae Math., **3** (2011), 97-109.

[29] W. Raymond, Y. Freese and J. Cho, *Geometry of linear 2-normed spaces*, N. Y. Nova Science Publishers, Huntington, (2001).

[30] A. Şahiner, M. Gürdal, S. Saltan and H. Gunawan, *Ideal Convergence in 2-normed spaces*, Taiwanese J. Math., **11** (2007), 1477-1484.

[31] B. C. Tripathy, *Generalized difference paranormed statistically convergent sequences defined by Orlicz function in a locally convex spaces*, Soochow J. Math., **30** (2004), 431-446.

[32] B. C. Tripathy, *Statistically convergent double sequences*, Tamkang J. Math., **34** (2003), 231-237.

[33] A. Wilansky, *Summability through Functional Analysis*, North- Holland Math. Stud. **85** (1984).

[34] M. Zeltser, *Investigation of double sequence spaces by Soft and Hard Analytical Methods*, Diss. Math. Univ. Tartu., **25**, Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu (2001).

Communicated by *Yasunao Hattori*

School of Mathematics Shri Mata Vaishno Devi University Katra-182320, J&K, India.
Email : kuldeepraj68@rediffmail.com
Email : sunilksharma42@yahoo.co.in

NEW SORT OF GENERALIZED CLOSED SETS *

H.Maki, M.Lellis Thivagar, C.Santhini and M.Anbuchelvi

Received May 7, 2013

Abstract.

In this paper we introduce the notion of $\hat{\omega}$-closed sets in topological spaces and obtain some of the basic properties of this class of sets. Further more their relationships with other generalized closed sets are investigated. It turns out that this class lies between the class of a-closed sets and the class of $g\alpha$-closed sets. And some $\hat{\omega}$-closure formulas in subspaces are investigated in the end of the present paper.

1 Introduction

The concept of generalized closed sets play a significant role in General Topology and they are now the research topics of many topologists world wide. Levine [20] introduced generalized closed sets in general topology as a generalization of closed sets. This concept was found to be useful and many results in general topology were improved. Extensive research on generalizing closed ness was done in recent years as the notions of semi-generalized closed[6], generalized semi-closed[5], generalized α-closed [23],α-generalized closed [24] and generalized semi-pre closed [11] were investigated. Recently in [15], Erdal Ekici introduced a-open sets. Lellis Thivagar et al.[2] introduced $\alpha\hat{g}$-closed sets and studied their basic properties. In this present paper we introduce a new class of sets namely $\hat{\omega}$-closed sets in topological spaces (cf. Definition 3.1). In Section 3, we study some of its basic properties and its relationships with other generalized closed sets are investigated. It turns out that this class lies between the class of a-closed sets and the class of $g\alpha$-closed sets (cf. Remark 3.13 below). In Section 4, some characterizations of $\hat{\omega}$-closed sets are discussed and investigated. In Section 5, the concept of the $\hat{\omega}$-closure of each subset of a topological space (X,τ) is introduced and its fundamental properties are investigated (Theorem 5.2). Moreover, a new topology of X, say $\tau_{\hat{\omega}}$, is introduced and investigated by using the $\hat{\omega}$-closures (Definition 5.4, Theorem 5.5). In the last Section 6, we investigate some $\hat{\omega}$-closure formulas in a subspace of a given topological space (Theorem 6.2 and Theorem 6.3). They are proved in the end of the present section (cf. the subsection (V)). To prove them, we need some topological properties in subspaces; i.e., (I) ordinary interior formula in subspaces, (II) δ-interor formula and δ-closure formula in subspaces, (III) a-closure formula in subspaces, (IV) (IV-1) α-closure formula in subspaces, (IV-2) $\alpha\hat{g}$-closedness in subspaces, (IV-3) $\hat{\omega}$-closedness in subspaces.

2 Preliminaries

Throughout the present paper, (X,τ) (or simply X) represents a topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset A of $X, cl(A), int(A)$ and A^c denote the closure of A, the interior of A and the complement of A respectively. Sometimes, A^c is denoted by $X \setminus A$.

Let us recall the following definitions, which are useful in the sequel.

*2010 Math. Subject classification–:Primary: 54A05.

Key words and phrases —$\hat{\omega}$-closed sets,$\hat{\omega}$-open sets,$\alpha\hat{g}$-closed sets,g-closed sets, ω-closed sets, a-open sets, α-open sets, preopen sets.

Definition 2.1 A subset A of a space (X,τ) is called *δ-closed* [38] (e.g., [15]) if $A = cl_\delta(A)$, where $cl_\delta(A) = \{x \in X | int(cl(U)) \cap A \neq \emptyset$ for every open set U containing $x\}$. The complement of δ-closed set is called δ-open. The δ-interior of A is defined by $int_\delta(A) := \{x \in X | int(cl(U)) \subseteq A$ for some open set U containing $x\}$. Sometimes, $cl_\delta(A)$ (resp. $int_\delta(A)$) is denoted by $\delta cl(A)$ (resp. $\delta int(A)$) in Section 6.

Remark 2.2 For a subset A of $(X,\tau), A$ is δ-open in (X,τ) if and only if $A = int_\delta(A)$ holds.

Definition 2.3 A subset A of a space (X,τ) is called θ-closed [38] (e.g., [7]) if $A = cl_\theta(A)$, where $cl_\theta(A) := \{x \in X | cl(U) \cap A \neq \emptyset$ for every open set U containing $x\}$. The complement of θ-closed set is called θ-open.

Definition 2.4 A subset A of a topological space (X,τ) is called a
(i) *semi-open set* [19] (e.g., [16]) if $A \subseteq cl(int(A))$;
(ii) *preopen set* [26] (e.g., [16]) if $A \subseteq int(cl(A))$;
(iii) *α-open set* [29] (e.g., [2]) if $A \subseteq int(cl(int(A))$;
(iv) *β-open set* [1] (or *semi-preopen set* [4]; e.g., [2]) if $A \subseteq cl(int(cl(A))$;
(v) *regular open set* [2] if $A = int(cl(A))$;
(vi)*a-open set* [15] (e.g., [14, Definition 1(v)]) if $A \subset int(cl(int_\delta(A))$.

The complement of the above mentioned sets are called *semi-closed, preclosed, α-closed, β-closed, regular closed* and *a-closed* respectively. The *semi-closure* (resp. *preclosure, α-closure, β-closure, a-closure*) of a subset A of X is the intersection of all semi-closed (resp. preclosed, α-closed, β-closed, a-closed) sets containing A and is denoted by $scl(A)$ (resp. $pcl(A)$, $\alpha cl(A), \beta cl(A), acl(A)$).

Definition 2.5 A subset A of a space (X,τ) is called a
(i) *generalized closed* (briefly *g-closed*) *set* [20] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X;
(ii) *α-generalized closed* (briefly *αg-closed*) *set* [24] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X;
(iii) *generalized α-closed* (briefly *$g\alpha$-closed*) *set* [23] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in X;
(iv) *generalized semi-closed* (briefly *gs-closed*) *set* [2] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X;
(v) *generalized semi-pre closed* (briefly *gsp-closed*) *set* [11] if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X (note: $spcl(A) = \beta cl(A)$);
(vi) *generalized preclosed* (briefly *gp-closed*) *set* [33] (e.g., [22]) if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X;
(vii) *ω-closed set*[36] (=*$\hat{g}$-closed set* [37]) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X;
(viii) *$\alpha\hat{g}$-closed set* [2] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\hat{g}$-open in X.

The complement of a g-closed (resp. gs-closed, αg-closed, $g\alpha$-closed, gs-closed, gsp-closed, gp-closed, ω-closed, $\alpha\hat{g}$-closed) set is called *g-open* (resp. *gs-open, $\alpha\hat{g}$-open, $g\alpha$-open, gs-open, gsp-open, gp-open, ω-open, $\alpha\hat{g}$-open*).

Theorem 2.6 *Let A and B be subsets of a topological space (X,τ). Then*
(i) $acl(A) \subseteq cl_\delta(A)$.
(ii) $acl(A) \subseteq cl_\theta(A)$.

(iii) $scl(A) \subseteq acl(A)$.
(iv) ([14]) *If* $A \subseteq B$,*then* $acl(A) \subseteq acl(B)$.
(v) ([14]) $acl(acl(A)) = acl(A)$.
(vi) ([14]) $acl(A \cup B) = acl(A) \cup acl(B)$.

Proof. (i) It is well known that $int(cl_\delta(A)) \subseteq cl_\delta(A)$ for any subset A of X. Since $cl_\delta(A)$ is τ-closed (e.g., [7]), we have $cl(int(cl_\delta(A))) \subseteq cl(cl_\delta(A)) = cl_\delta(A)$ and so $A \cup cl(int(cl_\delta(A))) \subseteq A \cup cl_\delta(A) = cl_\delta(A)$. By [15, Theorem 4], $acl(A) = A \cup cl(int(cl_\delta(A))) \subseteq cl_\delta(A)$.

(ii) It is well known that $cl_\delta(A) \subseteq cl_\theta(A)$ for any subset A of X. Then, by (i), (ii) is shown.

(iii) It is well known that $cl(A) \subseteq cl_\delta(A)$ for any subset A of X. Hence $int(cl(A) \subseteq int(cl_\delta(A)) \subseteq cl(int(cl_\delta(A)))$ which implies $scl(A) = A \cup int(cl(A)) \subseteq A \cup cl(int(cl_\delta(A)))$; and so, by [15, Theorem 4], $scl(A) \subseteq acl(A)$.

(iv) (v) (vi) By [14, Theorem 5],[15], it is well known that the family of all a-open sets in (X, τ), say $aO(X, \tau)$, forms a topology of X; then $acl(A) = aO(X, \tau)$-$cl(A)$ holds for any set A of (X, τ). Therefore, (iv), (v) and (vi) are shown. □

Notation 2.7 The family of all semi-open sets (resp. preopen sets, α-open sets, $\hat{g}$-open sets (or ω-open sets), $\alpha\hat{g}$-open sets, δ-open sets, a-open sets, $\hat{\omega}$-open sets) in a topological space (X, τ) is denoted by $SO(X, \tau)$ (resp. $PO(X, \tau)$, τ_α or $\alpha O(X, \tau)$, $\hat{g}O(X, \tau)$ (or $\omega O(X, \tau)$), $\alpha\hat{g}O(X, \tau)$, $\delta O(X, \tau)$, $aO(X, \tau)$, $\hat{\omega}O(X, \tau)$). By $SC(X, \tau)$ (resp. $PC(X, \tau)$, $\alpha C(X, \tau)$, $\hat{g}C(X, \tau)$ (or $\omega C(X, \tau)$), $\alpha\hat{g}C(X, \tau)$, $\delta C(X, \tau)$, $aC(X, \tau)$, $\hat{\omega}C(X, \tau)$), we denote the family of all semi-closed sets (resp. preclosed sets, α-closed sets, $\hat{g}$-closed sets (or ω-closed sets), $\alpha\hat{g}$-closed sets, δ-closed sets, a-closed sets, $\hat{\omega}$-closed sets) in a topological space (X, τ).

3 $\hat{\omega}$-closed sets

In this section we introduce a new class of sets called $\hat{\omega}$-closed sets in topological spaces and investigate some of their properties.

Definition 3.1 A subset A of a topological space (X, τ) is called $\hat{\omega}$-closed subset of X if $acl(A) \subseteq U$ whenever $A \subseteq U$ and U is a $\alpha\hat{g}$-open subset of X. The complement of $\hat{\omega}$-closed subset is called $\hat{\omega}$-open.

Proposition 3.2 *In a topological space* (X, τ), *the following statements hold.*
(i) *Every* a*-closed subset is* $\hat{\omega}$*-closed.*
(ii) *Every* θ*-closed subset* $\hat{\omega}$*-closed .*
(iii) *Every* δ*-closed subset is* $\hat{\omega}$*-closed.*
(iv) *Every regular closed subset is* $\hat{\omega}$*-closed.*

Proof. (i) Let A be an a-closed subset and U be any $\alpha\hat{g}$-open subset containing A. Since A is a-closed, $acl(A) = A$. Therefore, $acl(A) \subseteq U$ and hence A is $\hat{\omega}$-closed in X.

(ii) Let A be a θ-closed subset of X and U be any $\alpha\hat{g}$-open subset containing A. Since A is θ-closed, $cl_\theta(A) = A$ which implies that $cl_\theta(A) \subseteq U$ and by Theorem 2.6 (ii), $acl(A) \subseteq cl_\theta(A) \subseteq U$ and hence A is $\hat{\omega}$-closed set.

(iii) Let A be a δ-closed subset of X and U is any $\alpha\hat{g}$-open set containing A. Since A is δ-closed, $cl_\delta(A) = A$ and so $cl_\delta(A) \subseteq U$. By Theorem 2.6(i), $acl(A) \subseteq cl_\delta(A) \subseteq U$ and hence A is $\hat{\omega}$-closed.

(iv) Since every regular closed subset is a-closed [15, Remark 5], the proof follows from (i). □

Remark 3.3 The reversible implications of Proposition 3.2 are not true from the following example.

Example 3.4 Let $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. Here, $\{c, d\}$ is $\hat{\omega}$-closed in X but not a-closed, θ-closed, δ-closed or regular closed. Indeed, we see that $SO(X,\tau) = \{\emptyset,\ \{a\},\ \{a,d\},\ \{b,c\},\ \{b,c,d\},\ \{a,b,c\},\ X\}$, $PO(X,\tau) = \{\emptyset,\ \{a\},\ \{b\},\ \{c\},$ $\{a,b\},\ \{a,c\},\{b,c\},\ \{a,c,d\},\ \{a,b,c\},\ \{a,b,d\},\ X\}$, $\tau_\alpha = \{\emptyset,\ \{a\},\ \{b,c\},\ \{a,b,c\},\ X\}$, $\hat{g}O(X,\tau)$ (i.e., $=\omega O(X,\tau)$)$= \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, X\}=\alpha\hat{g}O(X,\tau)$, $\delta O(X,\tau) = \tau$, $aO(X,\tau) = \tau_\alpha$; and so we have that $\hat{\omega}C(X,\tau) = \{\emptyset,\ \{d\},\ \{a,d\},\ \{b,d\},\ \{c,d\},$ $\{a,c,d\},\ \{a,b,d\},\ \{b,c,d\},\ X\}$.

Proposition 3.5 *The following statements are true in a topological space* (X,τ).
(i) *Every $\hat{\omega}$-closed subset is $g\alpha$-closed.*
(ii) *Every $\hat{\omega}$-closed subset is αg-closed.*
(iii) *Every $\hat{\omega}$-closed subset is gs-closed.*
(iv) *Every $\hat{\omega}$-closed subset is preclosed.*

Proof. (i) Let A be a $\hat{\omega}$-closed subset of X and U be any α-open subset of X containing A. Since every α-open set is $\alpha\hat{g}$-open (cf. Definition 2.5(viii)), U is an $\alpha\hat{g}$-open set containing A. By hypothesis, $acl(A) \subseteq U$. Since every a-open set is α-open (cf. [15, Remark 5]), $\alpha cl(A) \subseteq acl(A) \subseteq U$ and hence A is $g\alpha$-closed.

(ii) It follows from the fact that every $g\alpha$-closed set is αg-closed (cf. Definition 2.5(ii)(iii)).

(iii) It follows from the fact that every αg-closed is gs-closed (cf. Definition 2.5(ii)(iv)).

(iv) It follows from the fact that every $g\alpha$-closed is preclosed (cf. [10, Theorem 2.3]). □

Remark 3.6 Since every preclosed set is β-closed (resp.*gp*-closed, *gsp*-closed) , every $\hat{\omega}$-closed set is β-closed (resp. *gp*-closed, *gsp*-closed).

Remark 3.7 The reversible implications of Proposition 3.5 are not always true from the following examples.

Example 3.8 Let $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$.

Then, the set $\{b, c\}$ is αg-closed, $g\alpha$-closed, preclosed, *gs*-closed, *gp*-closed,*gsp*-closed but not $\hat{\omega}$-closed in (X,τ). Indeed, we have the following list of the families: $SO(X,\tau) = PO(X,\tau)=\tau_\alpha= \{\emptyset,\ \{a\},\ \{a,b\},\ \{a,c\},\ \{a,d\},\ \{a,c,d\},\ \{a,b,d\},\ \{a,b,c\},\ X\}$, $\hat{g}O(X,\tau)$(i.e., $=\omega O(X,\tau)$)$=\{\emptyset,\ \{a\},\ \{a,b\},\ X\}$, $\alpha\hat{g}O(X,\tau)=P(X) \setminus \{\{c,d\},\ \{b,c,d\}\}$, $\delta O(X,\tau) = \{\emptyset,$ $X\}=aO(X,\tau)$ and so we have that $\hat{\omega}C(X,\tau)=\{\emptyset, \{b,c,d\}, X\}$. By $gC(X,\tau)$ (resp. $g\alpha C(X,\tau)$, $\alpha gC(X,\tau)$, $gsC(X,\tau)$, $gpC(X,\tau)$, $gspC(X,\tau)$ (or $g\beta C(X,\tau)$)), we denote the family of all g-closed (resp. $g\alpha$-closed, αg-closed, *gs*-closed, *gp*-closed, *gsp*-closed (or $g\beta$-closed)) sets of (X,τ). Then, $g\alpha C(X,\tau)= \{\emptyset,\ \{b\},\ \{c\},\ \{d\},\ \{b,c\},\ \{b,d\},\ \{c,d\},\ \{b,c,d\},\ X\}$, $gC(X,\tau) = P(X) \setminus \{\{a\},\ \{b\},\ \{a,b\}\}$ and $\alpha gC(X,\tau) = gsC(X,\tau)= gpC(X,\tau)=gspC(X,\tau)= P(X) \setminus$ $\{\{a\},\ \{a,b\}\}$.

Remark 3.9 The following examples show that $\hat{\omega}$-closedness is independent from closedness, g-closedness and α-closedness.

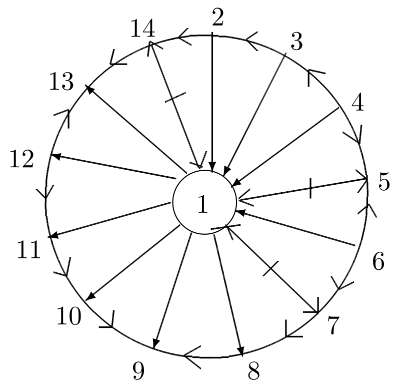
Example 3.10 Let $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ (cf. Example 3.8 above). Then the set $\{c, d\}$ is closed, α-closed, preclose, g-closed, but not $\hat{\omega}$-closed in (X,τ).

Example 3.11 Let $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then the set $\{c\}$ is $\hat{\omega}$-closed, but not g-closed in (X,τ) (and so it is not closed in (X,τ)). Indeed, we see that $SO(X,\tau) = P(X) \setminus \{\{c\},\ \{d\},\ \{c,d\}\}$, $PO(X,\tau) = \{\emptyset,\ \{a\},\ \{b\},\ \{a,b\},\ \{a,b,c\},$ $\{a,b,d\},\ X\}$, $\tau_\alpha = PO(X,\tau)$, $\hat{g}O(X,\tau)$(i.e., $=\omega O(X,\tau)$) $= \{\emptyset,\ \{a\},\ \{a,b\},\ \{a,b,c\},\ X\}$, $\alpha\hat{g}O(X,\tau) = \{\emptyset,\ \{a\},\ \{b\},\ \{c\},\ \{a,b\},\ \{a,c\},\ \{b,c\},\ \{a,b,d\},\ \{a,b,c\},\ X\}$, $\delta O(X,\tau) = \{\emptyset,$ $\{a\},\ \{b\},\ \{a,b\},\ X\}$, $aO(X,\tau) = \{\emptyset,\ \{a\},\ \{b\},\ \{a,b\},\ \{a,b,d\},\ \{a,b,c\},\ X\}$; and so we have that $\hat{\omega}C(X,\tau) = \{\emptyset,\ \{c\},\ \{d\},\ \{c,d\},\ \{a,c,d\},\ \{b,c,d\},\ X\}$ and $gC(X,\tau) = \{\emptyset,\ \{d\},\ \{a,d\},$ $\{b,d\},\ \{c,d\},\ \{a,c,d\},\ \{a,b,d\},\ \{b,c,d\},\ X\}$.

Example 3.12 Let $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. Then the set $\{b, d\}$ is $\hat{\omega}$-closed, but not α-closed in (X, τ) (cf. Example 3.4 above).

Thus the class of $\hat{\omega}$-closed sets properly contains the class of a-closed sets and is properly contained in the class of $g\alpha$-closed sets.

Remark 3.13 The following diagram shows the relationships of $\hat{\omega}$-closed sets with some other sets: $A \to B$ represents A implies B but not conversely and $A \nleftrightarrow$B represents A and B are independent.



1.$\hat{\omega}$-closed 2.a-closed 3.δ-closed 4.θ-closed 5.closed 6.regular closed 7.g-closed 8.gs-closed 9.gsp-closed 10.gp-closed 11.αg-closed 12.$g\alpha$-closed 13.preclosed 14.α-closed.

4 Properties of $\hat{\omega}$-closed sets

In the present section, the concept of $\hat{\omega}$-closed sets is characterized by using some kernels, closures of sets etc (cf. Theorem 4.10, Definition 4.9, Notation 4.6) and some fundamental properties of the family of all $\hat{\omega}$-closed sets are investigated (cf. Theorems 4.11,4.12).

Theorem 4.1 *If A is $\hat{\omega}$-closed in a topological space(X, τ), then the following statements hold.*

(i) *$acl(A) \setminus A$ does not contain any non-empty $\alpha\hat{g}$-closed set.*

(ii) *$acl(A) \setminus A$ does not contain any non-empty α-closed set.*

(iii) *$acl(A) \setminus A$ does not contain any non-empty closed set.*

Proof. Let A be $\hat{\omega}$-closed in a topological space (X, τ).

(i) Assume that $acl(A) \setminus A$ contains an $\alpha\hat{g}$-closed subset F of (X, τ). Then $A \subseteq X \setminus F$, where $X \setminus F$ is an $\alpha\hat{g}$-open subset of (X, τ). Since A is $\hat{\omega}$-closed in (X, τ), $acl(A) \subseteq X \setminus F$ and hence $F \subseteq X \setminus acl(A)$. Then $F \subseteq (acl(A) \setminus A) \cap (X \setminus acl(A)) = \emptyset$. Thus, $acl(A) \setminus A$ does not contain any non-empty $\alpha\hat{g}$-closed subset of (X, τ).

(ii) (resp. (iii)) Assume that $acl(A) \setminus A$ contains an α-closed (resp. closed) subset F of (X, τ). Then F is $\alpha\hat{g}$-closed. By (i), it is obtained $F = \emptyset$; and hence $acl(A) \setminus A$ does not contain any nonempty α-closed set (resp. closed set). $\square$

Remark 4.2 The converse of Theorem 4.1 (iii) is not always possible from the following example. Moreover, the converse of Theorem 4.1 (i) shall be investigated in the end of Section 6 beow (cf. Remark 6.28, Theorem 6.29) using some properties on subspaces stated in Section 6.

Example 4.3 Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Let $A = \{c\}$. Then $acl(A) \setminus A = X \setminus \{c\} = \{a, b, d\}$ does not contain any non-empty closed set. However A is not a $\hat{\omega}$-closed subset of (X, τ) (cf. Example 3.8 above).

Theorem 4.4 *Let A be a $\hat{\omega}$-closed set in (X, τ). Then A is a-closed if and only if $acl(A) \setminus A$ is $\alpha\hat{g}$-closed in (X, τ).*

Proof. (Necessity) Since A is a-closed set in X, $acl(A) = A$ and so $acl(A) \setminus A = \emptyset$ which is $\alpha\hat{g}$-closed.

(Sufficiency) Since A is $\hat{\omega}$-closed set, by Theorem 4.1(i), $acl(A) \setminus A$ does not contain any nonempty $\alpha\hat{g}$-closed set which implies $acl(A) \setminus A = \emptyset$. Hence $acl(A) = A$ and so A is a-closed.

□

Theorem 4.5 *Let A be a $\hat{\omega}$-closed set in (X, τ). If $A \subseteq B \subseteq acl(A)$, then B is also a $\hat{\omega}$-closed set in (X, τ).*

Proof. Let $B \subseteq U$ where U is $\alpha\hat{g}$-open in X. Then $A \subseteq U$. Since A is $\hat{\omega}$-closed, $acl(A) \subseteq U$. Also $B \subseteq acl(A)$ implies $acl(B) \subseteq acl(A) \subseteq U$. Hence B is also $\hat{\omega}$-closed in X. □

Sometimes, we need the following notation.

Notation 4.6 In a topological space (X, τ), we introduce the following four subsets of X (cf. e.g., [12, p.260] for (i) below).

(i) • $X_{\mathcal{ND}} := \{x \in X |\ \{x\}$ is nowhere dense in $(X, \tau)\}$ (i.e., $X_{\mathcal{ND}} = \{x \in X |\ int(cl(\{x\})) = \emptyset\}$); and

• $X_{\mathcal{PO}} := \{x \in X |\ \{x\}$ is preopen in $(X, \tau)\}$ (i.e., $X_{\mathcal{PO}} = \{x \in X |\ \{x\} \subseteq int(cl(\{x\}))\}$).

(Note: in [12, p.260], $X_{\mathcal{ND}}$ (resp. $X_{\mathcal{PO}}$) above is denoted by X_1 (resp. X_2).)

(ii) • $X_{\hat{\omega}\mathcal{O}} := \{x \in X |\ \{x\}$ is $\hat{\omega}$-open in $(X, \tau)\}$; and

• $X_{\alpha\hat{g}\mathcal{C}} := \{x \in X |\ \{x\}$ is $\alpha\hat{g}$-closed in $(X, \tau)\}$.

Theorem 4.7 (i) ([17, Lemma 2]; e.g., [9], [12, p.260]) *Let x be a point of (X, τ). Every singleton $\{x\}$ is either nowhere dense or preopen; and $X = X_{\mathcal{ND}} \cup X_{\mathcal{PO}}$ (disjoint union;) holds, where $X_{\mathcal{ND}} \cap X_{\mathcal{PO}} = \emptyset$.*

(ii) *For each point x of (X, τ), the singleton $\{x\}$ is $\alpha\hat{g}$-closed or $\{x\}^c$ is $\hat{\omega}$-closed; and so $X = X_{\alpha\hat{g}\mathcal{C}} \cup X_{\hat{\omega}\mathcal{O}}$ holds.*

(iii) *For any topological space (X, τ), $X_{\hat{\omega}\mathcal{O}} \subseteq X_{\mathcal{PO}}$ holds.*

Proof. (i) The first statement is the well known fact ([17, Lemma 2]). The proof of the disjointness, i.e., $X_{\mathcal{ND}} \cap X_{\mathcal{PO}} = \emptyset$, is shown as follows: let $x \in X_{\mathcal{ND}} \cap X_{\mathcal{PO}}$; then $int(cl(\{x\})) = \emptyset$ and $int(cl(\{x\})) \supseteq \{x\}$; and so $\{x\} = \emptyset$. This contradicts the first setting of the point x. Thus we confirm the disjoint union of X.

(ii) For a point $x \in X$, suppose $\{x\}$ is not $\alpha\hat{g}$-closed in X, i.e., $x \notin X_{\alpha\hat{g}\mathcal{C}}$. Then $\{x\}^c$ is not $\alpha\hat{g}$-open and the only $\alpha\hat{g}$-open containing $\{x\}^c$ is the space X itself; and so $\{x\}^c \subseteq X$. Therefore, $acl(\{x\}^c) \subseteq X$ and so $\{x\}^c$ is $\hat{\omega}$-closed, i.e., $x \in X_{\hat{\omega}\mathcal{C}}$.

(iii) We recall that every $\hat{\omega}$-open set is preopen (Theorem 3.5(iv)). Let $x \in X_{\hat{\omega}\mathcal{O}}$; and so $\{x\}$ is $\hat{\omega}$-open; and hence $\{x\}$ is preopen (i.e., $x \in X_{\mathcal{PO}}$). □

Remark 4.8 (cf. Theorem 4.7 (ii)) (i) For the following topological space (X, τ), it is shown that $X_{\alpha\hat{g}\mathcal{C}} \cap X_{\hat{\omega}\mathcal{O}} = \emptyset$; and so X has a disjoint union: $X = X_{\alpha\hat{g}\mathcal{C}} \cup X_{\hat{\omega}\mathcal{O}}$ (disjoint union). Let $X := \{a, b, c, d\}$ and $\tau := \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then, we have $X_{\alpha\hat{g}\mathcal{C}} = \{c, d\}$ and $X_{\hat{\omega}\mathcal{O}} = \{a, b\}$. Indeed, it is shown that $SO(X, \tau) = P(X) \setminus \{\{c\}, \{d\}, \{c, d\}\}$,

$PO(X,\tau)$=$\tau_\alpha = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X\}$, $\hat{g}O(X,\tau)$ (i.e., $=\omega O(X,\tau)$)=$\{\emptyset,$ $\{a\}, \{b\}, \{d\}, \{a,b\}, X\}$ and so $\alpha\hat{g}C(X,\tau)$=$P(X)\setminus\{\{a\}, \{b\}, \{a,b\}\}$ and so $X_{\alpha\hat{g}\mathcal{C}} = \{a,b\}$; moreover, we show $\delta O(X,\tau) = \tau, aC(X,\tau)$=$\{\emptyset, \{c\}, \{d\}, \{c,d\}, \{b,c,d\}, \{a,c,d\}, X\}$ and hence $\hat{\omega}O(X,\tau)$=$\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X\}$ and $X_{\hat{\omega}\mathcal{O}} = \{a,b\}$.

(ii) We have an example of a topological space (X,τ) such that $X_{\alpha\hat{g}\mathcal{C}} \cap X_{\hat{\omega}\mathcal{O}} \neq \emptyset$. Let $X := \{a,b,c\}$ and $\tau := \{\emptyset, \{a\}, \{b,c\}, X\}$. Then, it is shown that $X_{\alpha\hat{g}\mathcal{C}} = \{a\}$ and $X_{\hat{\omega}\mathcal{O}} = \{a,b,c\}$. Indeed, $SO(X,\tau) = \tau, PO(X) = P(X), \tau_\alpha = \tau, \hat{g}O(X,\tau)$(i.e., $=\omega O(X,\tau)) = P(X)$ and $\alpha\hat{g}O(X,\tau) = \tau$; and so $X_{\alpha\hat{g}C} = \{a\}$. Moreover we see that $\delta O(X,\tau) = aO(X,\tau) = \tau$; and hence we have that $\hat{\omega}C(X,\tau) = P(X)$ and $X_{\hat{\omega}\mathcal{O}} = \{a,b,c\}$.

We characterlize the $\hat{\omega}$-closedness in (X,τ) using Theorem 4.7 and the following definition.

Definition 4.9 The intersection of all $\alpha\hat{g}$-open subsets of (X,τ) containing A is called the *$\alpha\hat{g}$-kernel of A* and is denoted by $\alpha\hat{g}ker(A)$. Namely, • $\alpha\hat{g}ker(A) := \bigcap\{U | U$ is $\alpha\hat{g}$-open in X and $A \subseteq U\}$.

Theorem 4.10 *For a subset A of (X,τ), the following properties are equivalent:*

(1) *A is $\hat{\omega}$-closed;*

(2) *$acl(A) \subseteq \alpha\hat{g}ker(A)$ holds;*

(3) (i) *$acl(A) \cap X_{\mathcal{ND}} \subseteq A$ and* (ii) *$acl(A) \cap X_{\mathcal{PO}} \subseteq \alpha\hat{g}ker(A)$ hold;*

(4) (i) *$acl(A) \cap X_{\alpha\hat{g}\mathcal{C}} \subseteq A$ and* (ii) *$acl(A) \cap X_{\hat{\omega}\mathcal{O}} \subseteq \alpha\hat{g}ker(A)$ hold.*

Proof. (1)⇒(2) Suppose that A is $\hat{\omega}$-closed in X. Let $x \in acl(A)$ and $x \notin \alpha\hat{g}ker(A)$. Then there exists an $\alpha\hat{g}$-open set U in X such that $A \subseteq U$ and $x \notin U$. Since A is $\hat{\omega}$-closed, $acl(A) \subseteq U$. But $x \notin U$ and so this implies $x \notin acl(A)$, a contradiction.

(2)⇒(1) Suppose that $acl(A) \subseteq \alpha\hat{g}ker(A)$. Let U be any $\alpha\hat{g}$-open set containing A. Then $\alpha\hat{g}ker(A) \subseteq U$ which implies $acl(A) \subseteq U$. Hence A is $\hat{\omega}$-closed.

(2)⇒(3) (i) To show $X_{\mathcal{ND}} \cap acl(A) \subseteq A$, let $x \in X_{\mathcal{ND}} \cap acl(A)$; and we suppose that $x \notin A$. Since $x \in X_{\mathcal{ND}}$, $int(cl(\{x\})) = \emptyset$ and hence $cl(int(cl(\{x\}))) \subseteq \{x\}$. Then,$\{x\}$ is an α-closed subset of X. By [2], $\{x\}$ is an $\alpha\hat{g}$-closed subset of X and then $X \setminus \{x\}$ is $\alpha\hat{g}$-open subset of X. By the equivalence: (1)⇔ (2) above, it is obtained that A is $\hat{\omega}$-closed in X. Since $x \notin A$, $A \subseteq X\setminus\{x\}$ holds and so $acl(A) \subseteq X\setminus\{x\}$. Thus, $x \notin acl(A)$, this contradicts the first of the point x. Therefore, $X_{\mathcal{ND}} \cap acl(A) \subseteq A$.

(ii) Suppose that $acl(A) \subseteq \alpha\hat{g}ker(A)$. Then, we have $X_{\mathcal{PO}} \cap acl(A) \subseteq acl(A) \subseteq \alpha\hat{g}ker(A)$.

(3)⇒(2) Suppose that A is a subset of X such that $X_{\mathcal{ND}} \cap acl(A) \subseteq A$ and $X_{\mathcal{PO}} \cap acl(A) \subseteq \alpha\hat{g}ker(A)$. Then, $acl(A) = X \cap acl(A) = (\{X_{\mathcal{ND}} \cup X_{\mathcal{PO}}) \cap acl(A)$= $(\{X_{\mathcal{ND}} \cap acl(A)) \cup (X_{\mathcal{PO}} \cap acl(A)) \subseteq A \cup \alpha\hat{g}ker(A) = \alpha\hat{g}ker(A)$.

(2)⇒(4) (i) Let $x \in acl(A) \cap X_{\alpha\hat{g}\mathcal{C}}$ and assume that $x \notin A$. Since $x \in X_{\alpha\hat{g}\mathcal{C}}$ and $x \notin A$, $X\setminus\{x\}$ is $\alpha\hat{g}$-open in X and $A \subseteq X\setminus\{x\}$. Using Definition 4.9, it is shown that $\alpha\hat{g}ker(A) \subset X \setminus \{x\}$. And so, by the assumption (2) and the first setting of $x, x \in acl(A) \subset X \setminus \{x\}$; the obtained property $x \in X \setminus \{x\}$ contradicts the first setting of x.

(ii) By (2), it is obtained that $acl(A) \cap X_{\hat{\omega}\mathcal{O}} \subseteq acl(A) \subseteq \alpha\hat{g}ker(A)$.

(4)⇒(2) By Theorem 4.7(ii) and (3), it is shown that $acl(A) = acl(A) \cap X$= $(acl(A) \cap X_{\alpha\hat{g}\mathcal{C}}) \cup (acl(A) \cap X_{\hat{\omega}\mathcal{O}}) \subseteq A \cup \alpha\hat{g}ker(A) = \alpha\hat{g}ker(A)$. That is, $acl(A) \subseteq \alpha\hat{g}ker(A)$ holds.
□ In the end of the present section, we show more fundamental properties, Theorem 4.11 and Theorem 4.12 with some additional assumptions.

Theorem 4.11 *If A and B are $\hat{\omega}$-closed in (X,τ), then $A \cup B$ is $\hat{\omega}$-closed in (X,τ).*

Proof. Suppose $A \cup B \subseteq U$ where U is $\alpha\hat{g}$-open in X. Then $A \subseteq U$ and $B \subseteq U$. Since A and B are $\hat{\omega}$-closed, $acl(A) \subseteq U$ and $acl(B) \subseteq U$. Thus $acl(A \cup B) = acl(A) \cup acl(B) \subseteq U$ (cf. Theorem 2.6 (vi)). Hence $A \cup B$ is $\hat{\omega}$-closed in (X, τ). $\square$

Theorem 4.12 *Suppose that $\{A_i : i \in J\}$ is a family of $\hat{\omega}$-closed subsets of (X, τ).*

(i) *If $(*1)$ $\alpha\hat{g}ker(A_i) \setminus A_i$ does not contain any $\hat{\omega}$-open singleton* (i.e., $(\alpha\hat{g}ker(A_i) \setminus A_i) \cap X_{\hat{\omega}\mathcal{O}} = \emptyset$) *for each $i \in J$, then $\bigcap\{A_i : i \in J\}$ is $\hat{\omega}$-closed in (X, τ).*

(ii) *If $(*2)$ $\alpha\hat{g}ker(A_i) \setminus A_i$ does not contain any preopen singleton* (i.e., $(\alpha\hat{g}ker(A_i) \setminus A_i) \cap X_{\mathcal{PO}} = \emptyset$) *for each $i \in J$, then $\bigcap\{A_i : i \in J\}$ is $\hat{\omega}$-closed in (X, τ).*

(iii) *Especially, if $X_{\mathcal{PO}} = \emptyset$ or $X_{\hat{\omega}\mathcal{O}} = \emptyset$, then $\bigcap\{A_i : i \in J\}$ is $\hat{\omega}$-closed in (X, τ).*

Proof. Throghout the present proof, we put $B := \bigcap\{A_i : i \in J\}$.

(i) In order to prove that B is $\hat{\omega}$-closed, it is enough to show that (cf. (4) in Theorem 4.10):
$(*3)$ $X_{\alpha\hat{g}\mathcal{C}} \cap acl(B) \subseteq B$ and $(*4)$ $X_{\hat{\omega}\mathcal{O}} \cap acl(B) \subseteq \alpha\hat{g}ker(B)$ hold.

Since each A_i is $\hat{\omega}$-closed in X, it is shown that $X_{\alpha\hat{g}\mathcal{C}} \cap acl(B) \subseteq X_{\alpha\hat{g}\mathcal{C}} \cap acl(A_i) \subseteq A_i$ for each $i \in J$. Hence we see $X_{\alpha\hat{g}\mathcal{C}} \cap acl(B) \subseteq \bigcap\{A_i : i \in J\} = B$; and so $(*3)$ is proved. For the proof of $(*4)$, we assume the contrary that $x \notin \alpha\hat{g}ker(B)$ holds for any point $x \in X_{\hat{\omega}\mathcal{O}} \cap acl(B)$. Then, there exists an $\alpha\hat{g}$-open set U in X such that $x \notin U$ and $B \subseteq U$; and so $x \notin B$. There exists $i_0 \in J$ such that $x \notin A_{i_0}$. Since $x \in X_{\hat{\omega}\mathcal{O}} \cap acl(B) \subseteq X_{\hat{\omega}\mathcal{O}} \cap acl(A_{i_0})$, we have $x \in acl(A_{i_0})$; an so Theorem 4.10 (2) for A_{i_0} shows that $x \in \alpha\hat{g}ker(A_{i_0})$. Since $x \notin A_{i_0}$, we have that $x \in X_{\hat{\omega}\mathcal{O}} \cap (\alpha\hat{g}ker(A_{i_0}) \setminus A_{i_0}) \subseteq \alpha\hat{g}ker(A_{i_0}) \setminus A_{i_0}$. Namely, the subset $\alpha\hat{g}ker(A_{i_0}) \setminus A_{i_0}$ contains a $\hat{\omega}$-open singleton $\{x\}$. This contradicts the hypothesis $(*1)$ of (i); and hence $(*4)$ is proved. Therefore, $B := \bigcap\{A_i | i \in J\}$ is $\hat{\omega}$-closed subsets of (X, τ) (cf. Theorem 4.10 (4)$\Leftrightarrow$(1)).

(ii) We recall that $X_{\hat{\omega}\mathcal{O}} \subseteq X_{\mathcal{PO}}$ holds (Theorem 4.7 (iii)). Then, it follows from assumption $(*2)$ of (ii) that $(\alpha\hat{g}ker(A_i) \setminus A_i) \cap X_{\hat{\omega}\mathcal{O}} = \emptyset$ holds (i.e., $(*1)$ of (i) above). By (i), the present property (ii) is obtained.

(iii) The assumption $X_{\mathcal{PO}} = \emptyset$ (resp. $X_{\hat{\omega}\mathcal{O}} = \emptyset$) implies the assumption of (i) (resp. (ii)) above; and so (iii) is proved. $\square$

Theorem 4.13 (i) *If $\{A_i | i \in J\}$ is a family of $\hat{\omega}$-closed sets of (X, τ) such that $acl(A_i) \cap X_{\hat{\omega}\mathcal{O}} \subset A_i$ for each $i \in J$, then $\bigcap\{A_i | i \in J\}$ is $\hat{\omega}$-close in (X, τ).*

(ii) *If $\{A_i | i \in J\}$ is a family of $\hat{\omega}$-closed sets of (X, τ) such that $acl(A_i) \cap X_{\mathcal{PO}} \subset A_i$ for each $i \in J$, then $\bigcap\{A_i | i \in J\}$ is $\hat{\omega}$-closed in (X, τ).*

Proof. We put $B := \bigcap\{A_i | i \in J\}$.

(i) It follows from assumption that $acl(B) \cap X_{\hat{\omega}\mathcal{O}} \subseteq acl(A_i) \cap X_{\hat{\omega}\mathcal{O}} \subseteq A_i$ for each $i \in J$; and so $acl(B) \cap X_{\hat{\omega}\mathcal{O}} \subseteq \bigcap\{A_j | j \in J\} = B$ holds. Since $E \subset \alpha\hat{g}ker(E)$ holds in general (cf. Definition 4.9), where E is a subset of X, we see that $acl(B) \cap X_{\hat{\omega}\mathcal{O}} \subseteq \alpha\hat{g}ker(B)$ holds. Namely, the condition (ii) of Theorem 4.10 (4) for the set B above is satisfied. On the other hands, we prove the condition (i) of Theorem 4.10 (4) for the set B. Indeed, since each A_j is $\hat{\omega}$-closed, we see that $acl(B) \cap X_{\alpha\hat{g}\mathcal{C}} \subseteq acl(A_j) \cap X_{\alpha\hat{g}\mathcal{C}} \subseteq A_j$ holds for each $j \in J$ (cf. (i) in Theorem 4.12 (4) for A_j); and so $acl(B) \cap X_{\alpha\hat{g}\mathcal{C}} \subseteq \bigcap\{A_j | j \in J\} = B$; hence this shows that the condition (i) of Theorem 4.10 (4) is satisfied for the set B. Therefore, by Theorem 4.10 (4)$\Leftrightarrow$ (1), B is $\hat{\omega}$-closed in X.

(ii) We recall that $X_{\hat{\omega}\mathcal{O}} \subseteq X_{\mathcal{PO}}$ (cf. Theorem 4.7 (iii)). Then, it follows from assumption of (ii) that $acl(A_i) \cap X_{\hat{\omega}\mathcal{O}} \subset A_i$ for each $i \in J$. Therefore, by (i) above, it is obtained that B is $\hat{\omega}$-closed in X. $\square$

5 $\hat{\omega}$-closure

In the present section, we define the concept of a $\hat{\omega}$-closure of each subset of a topological space (X,τ), say $\hat{\omega}cl(A)$ for a subset A of X, (cf. Definition 5.1 below) and investigate some fundamental properties of $\hat{\omega}$-closures (cf. Theorem 5.2 below). Moreover, a class of subsets of (X,τ), say $\tau_{\hat{\omega}}$, is introduced (cf. Definition 5.4(ii) below). We introduce the following definition and notation.

Definition 5.1 (i) By $\hat{\omega}C(X,\tau)$ (resp. $\hat{\omega}O(X,\tau)$), we denote the family of all $\hat{\omega}$-closed sets (resp. $\hat{\omega}$-open sets) in (X,τ).

(ii) The intersection of all $\hat{\omega}$-closed subsets of (X,τ) containing A is called the *$\hat{\omega}$-closure of* A and it is denoted by $\hat{\omega}cl(A)$. Namely, for a subset A of (X,τ), • $\hat{\omega}cl(A) := \bigcap\{F|A \subseteq F, F \in \hat{\omega}C(X,\tau)\}$.

Theorem 5.2 *Let A and B be subsets of (X,τ). Then the following statements hold.*

(i) $A \subseteq \hat{\omega}cl(A)$;

(ii) *If A is $\hat{\omega}$-closed in (X,τ), then* $\hat{\omega}cl(A) = A$;

(iii) $\hat{\omega}cl(\emptyset) = \emptyset$ *and* $\hat{\omega}cl(X) = X$;

(iv) *If $A \subseteq B$,then* $\hat{\omega}cl(A) \subseteq \hat{\omega}cl(B)$;

(v) $\hat{\omega}cl(A\cap B) \subseteq \hat{\omega}cl(A) \cap \hat{\omega}cl(B)$;

(vi) $\hat{\omega}cl(A\cup B) = \hat{\omega}cl(A) \cup \hat{\omega}cl(B)$;

(vii) $\hat{\omega}cl(\hat{\omega}cl(A)) = \hat{\omega}cl(A)$.

Proof. (i) By Definition 5.1(ii), $\hat{\omega}cl(A) := \bigcap\{F : A \subseteq F, F \in \hat{\omega}C(X,\tau)\}$ for a subset A of X; and it is clear that $A \subseteq \hat{\omega}cl(A)$.

(ii) For a given $\hat{\omega}$-closed subset A of (X,τ) i.e., $A \in \hat{\omega}C(X,\tau)$, it is shown that $\hat{\omega}cl(A) \subseteq A$ (cf. Definition 5.1(ii)); and so, using (i), $A = \hat{\omega}cl(A)$.

(iii) Using (ii), (iii) is obviously obtained, because $\emptyset \in \hat{\omega}C(X,\tau)$ and $X \in \hat{\omega}C(X,\tau)$.

(iv) Let $x \notin \hat{\omega}cl(B)$. Then, there exists a $\hat{\omega}$-closed set F in (X,τ) such that $x \notin F, B \subseteq F$ and so $A \subseteq B \subseteq F$. It is shown that there exist a $\hat{\omega}$-closed set F in (X,τ) such that $x \notin F$ and $A \subseteq F$; and, by Definition 5.1(ii), $x \notin \hat{\omega}cl(A)$. Therefore, $\hat{\omega}cl(A) \subseteq \hat{\omega}cl(B)$.

(v) Since $A\cap B \subseteq A$ and $A \cap B \subseteq B$, by (i), it is shown that $\hat{\omega}cl(A\cap B) \subseteq \hat{\omega}cl(A)$ and $\hat{\omega}cl(A\cap B) \subseteq \hat{\omega}cl(B)$;and hence $\hat{\omega}cl(A\cap B) \subseteq \hat{\omega}cl(A) \cap \hat{\omega}cl(B)$.

(vi) Since $A\cup B \supseteq A$ and $A\cup B \supseteq B$, by (iv), it is shown that $\hat{\omega}cl(A\cup B) \supseteq \hat{\omega}cl(A)$ and $\hat{\omega}cl(A\cup B) \supseteq \hat{\omega}cl(B)$;and so $\hat{\omega}cl(A\cup B) \supseteq \hat{\omega}cl(A) \cup \hat{\omega}cl(B)$. On the other hand, if $x \notin \hat{\omega}cl(A) \cup \hat{\omega}cl(B)$, then $x \notin \hat{\omega}cl(A)$ and $x \notin \hat{\omega}cl(B)$. Therefore, there exists ω-closed sets F_A and F_B of (X,τ) such that $x \notin F_A, x \notin F_B$ and $B \subseteq F_B$ and $A \subseteq F_A$. By Theorem 4.11, $F_A \cup F_B$ is a $\hat{\omega}$-closed set in X and $A\cup B \subseteq F_A \cup F_B$. It is shown that there exist a $\hat{\omega}$-closed set $F_A \cup F_B$ such that $x \notin F_A \cup F_B$ and $A \cup B \subseteq F_A \cup F_B$. Therefore, $x \notin \hat{\omega}cl(A\cup B)$ and so $\hat{\omega}cl(A\cup B) \subseteq \hat{\omega}cl(A) \cup \hat{\omega}cl(B)$. Thus, we have $\hat{\omega}cl(A\cup B) = \hat{\omega}cl(A) \cup \hat{\omega}cl(B)$.

(vii) By (i), it is shown that $\hat{\omega}cl(A) \subseteq \hat{\omega}cl(\hat{\omega}cl(A))$. On the other hand, if $x \notin \hat{\omega}cl(A)$, then there exists a $\hat{\omega}$-closed set F in X such that $x \notin F$ and $\hat{\omega}cl(A) \subseteq F$ (cf. (i),(iv)). Using Definition 5.1 (ii), we see $x \notin \hat{\omega}cl(\hat{\omega}cl(A))$. Thus, $\hat{\omega}cl(\hat{\omega}cl(A)) \subseteq \hat{\omega}cl(A)$. Therefore, $\hat{\omega}cl(\hat{\omega}cl(A)) = \hat{\omega}cl(A)$. □

Remark 5.3 It is not claimed that, in Theorem 5.2, if $\hat{\omega}cl(A) = A$ then $A \in \hat{\omega}C(X,\tau)$, where $A \subseteq X$. The $\hat{\omega}cl(A)$ is the intersection of all $\hat{\omega}$-closed sets containing A (cf. Definition 5.1(ii)); by the present paper, it is not proved that the intersection of $\hat{\omega}$-closed sets is $\hat{\omega}$-closed. Under some assumptions, the intersection of $\hat{\omega}$-closed sets is $\hat{\omega}$-closed (cf. Theorems 4.12, 4.13). Thus, for example, by Theorem 4.13 (ii) and Definition 5.1, it is known that $\hat{\omega}cl(A)$ is $\hat{\omega}$-closed if $(\alpha\hat{g}ker(F) \setminus F) \cap X_{\mathcal{PO}} = \emptyset$ for every $F \in \hat{\omega}C(X,\tau)$ such that $A \subseteq F$.

For a topological space (X,τ), we define the following operation, say $\hat{\omega}cl(\circ) : P(X) \to P(X)$, and two collections, say $\tau_{\hat{\omega}}$ and $\tau_{\hat{\omega}}C$, of subsets of X using the $\hat{\omega}$-closures.

Definition 5.4 Let (X,τ) be a topological space. Then,

(i) $\hat{\omega}cl(\circ) : P(X) \to P(X)$ is defined by $\hat{\omega}cl(\circ)(A) := \hat{\omega}cl(A)$ for every $A \in P(X)$;

(ii) $\tau_{\hat{\omega}} := \{U \subseteq X \mid \hat{\omega}cl(X \setminus U) = X \setminus U\}$; and the element of $\tau_{\hat{\omega}}$ is called a $\tau_{\hat{\omega}}$-open set of (X,τ) (cf. Theorem 5.5(ii) below);

(iii) $\tau_{\hat{\omega}}C := \{V \subseteq X \mid \ X \setminus V \in \tau_{\hat{\omega}}\}$ (i.e., $\tau_{\hat{\omega}}C = \{V \subseteq X \mid \hat{\omega}cl(V) = V\}$);

(iv) $\tau_{\hat{\omega}}\text{-}cl(B) := \bigcap\{F \mid B \subseteq F, F \in \tau_{\hat{\omega}}C$ (i.e., $X \setminus F \in \tau_{\hat{\omega}})\}$ and $\tau_{\hat{\omega}}\text{-}int(B) := \bigcup\{V \mid V \subseteq B, V \in \tau_{\hat{\omega}}\}$ for a subset B of (X,τ).

Theorem 5.5 (i) *The operation of* Definition 5.4 (i), $\hat{\omega}cl(\circ) : P(X) \to P(X)$, *satisfies the Kuratowski closure axioms.*

(ii) *The collection* $\tau_{\hat{\omega}} := \{U \subseteq X \mid \hat{\omega}cl(X \setminus U) = X \setminus U\}$ *forms a topology on* X (cf. Definition 5.4(ii)).

(iii) $\hat{\omega}O(X,\tau) \subseteq \tau_{\hat{\omega}}$ *holds for any topological space* (X,τ).

(iv) *Suppose that if* $A = \hat{\omega}cl(A)$ *then* $A \in \hat{\omega}C(X,\tau)$ (i.e., the converse of Theorem 5.2(ii)). *Then,* $\tau_{\hat{\omega}} = \hat{\omega}O(X,\tau)$ *holds.*

(v) $A \subseteq \tau_{\hat{\omega}}\text{-}cl(A) = \hat{\omega}cl(A)$ *hold and* $\hat{\omega}cl(A)$ *is* τ_{ω}*-closed in* (X,τ) *for a subset* A *of* (X,τ).

(vi) *The following properties for a subset* A *of* (X,τ) *are equivalent:*

(1) A *is* $\tau_{\hat{\omega}}$*-open in* (X,τ) (i.e., $A \in \tau_{\hat{\omega}}$);

(2) A *is open in* $(X,\tau_{\hat{\omega}})$;

(3) $\hat{\omega}cl(X \setminus A) = X \setminus A$;

(4) $\tau_{\hat{\omega}}\text{-}int(A) = A$;

(5) $\tau_{\hat{\omega}}\text{-}cl(X \setminus A) = X \setminus A$.

(vi)' *The following properties for a subset* B *of* (X,τ) *are equivalent:*

(1)' B *is* $\tau_{\hat{\omega}}$*-closed in* (X,τ) (i.e., $X \setminus B \in \tau_{\hat{\omega}}$);

(2)' B *is closed in* $(X,\tau_{\hat{\omega}})$;

(3)' $\hat{\omega}cl(B) = B$;

(4)' $\tau_{\hat{\omega}}\text{-}int(X \setminus B) = X \setminus B$;

(5)' $\tau_{\hat{\omega}}\text{-}cl(B) = B$.

Proof. (i) It follows from Theorem 5.2 (i),(iii),(vi) and (vii) that the operation $\hat{\omega}cl(\circ) : P(X) \to P(X)$ satisfies the Kuratowski closure axioms.

(ii) By (i), it is shown that $\tau_{\hat{\omega}}$ forms a topology on X. Namely, the family $\tau_{\hat{\omega}}$ satisfies the axioms of topology on X. Indeed, we see directly the first one of axioms: if $V_i \in \tau_{\hat{\omega}}$ (for every $i \in \Lambda$) then $\bigcup\{V_i | i \in \Lambda\} \in \tau_{\hat{\omega}}$. Put $V := \bigcup\{V_i | i \in \Lambda\}$. First, we have $X \setminus V \subseteq \hat{\omega}cl(X \setminus V)$ (cf. Theorem 5.2(i)). In order to prove the converse implication $X \setminus V \supseteq \hat{\omega}cl(X \setminus V)$, let $x \notin X \setminus V$. Since $x \in V$, there exists a subset $V_{i_0} (i_0 \in \Lambda)$ such that $x \in V_{i_0}$ and $x \notin X \setminus V_{i_0} = \hat{\omega}cl(X \setminus V_{i_0})$. Then, there exists a $\hat{\omega}$-closed set F such that $x \notin F$ and $X \setminus V_{i_0} \subseteq F$; and so $X \setminus V \subseteq X \setminus V_{i_0} \subseteq F$. These show that $x \notin \hat{\omega}cl(X \setminus V)$; and hence we see $X \setminus V \supseteq \hat{\omega}cl(X \setminus V)$. We see secondly that: if $V_i \in \tau_{\hat{\omega}} (i{=}1, 2)$ then $V_1 \cap V_2 \in \tau_{\hat{\omega}}$. By using Defintion 5.4(ii) and Theorem 5.2 (vi), it is shown that $\hat{\omega}cl(X \setminus (V_1 \cap V_2)){=}\hat{\omega}cl((V_1)^c \cup (V_2)^c){=}\hat{\omega}cl((V_1)^c) \cup \hat{\omega}cl((V_2)^c){=}(V_1)^c \cup (V_2)^c{=}X \setminus (V_1 \cap V_2)$; and hence $V_1 \cap V_2 \in \tau_{\hat{\omega}}$ (cf. Definition 5.4 (ii)). Finnally, by using Theorem 5.2 (iii) and Definition 5.4 (ii), it is shown that $\emptyset \in \tau_{\hat{\omega}}$ (i.e., $\hat{\omega}cl(X \setminus \emptyset) = X \setminus \emptyset$) and $X \in \tau_{\hat{\omega}}$ (i.e., $\hat{\omega}cl(X \setminus X) = X \setminus X$).

(iii) Let $U \in \hat{\omega}O(X,\tau)$. Since $X \setminus U$ is $\hat{\omega}$-closed in (X,τ), by Theorem 5.2(ii), it is shown that $X \setminus U = \hat{\omega}cl(X \setminus U)$; and so, by Definition 5.4(ii), $U \in \tau_{\hat{\omega}}$.

(iv) Let $U \in \tau_{\hat{\omega}}$. Then, $X \setminus U = \hat{\omega}cl(X \setminus U)$ holds; and so, by assumption, $X \setminus U$ is $\hat{\omega}$-closed in (X, τ). Therefore, U is $\hat{\omega}$-open in (X, τ), i.e., $U \in \hat{\omega}O(X, \tau)$.

(v) First we show $\tau_{\hat{\omega}}\text{-}cl(A) \subseteq \hat{\omega}cl(A)$. By Definition 5.4 and (iii) above, it is shown that $\tau_{\hat{\omega}}\text{-}cl(A) \subseteq \bigcap\{F | A \subseteq F,\ X \setminus F \in \hat{\omega}O(X, \tau)\} = \bigcap\{F | A \subseteq F,\ F \in \hat{\omega}C(X, \tau)\} = \hat{\omega}cl(A)$. Secondly, we show $\hat{\omega}cl(A) \subseteq \tau_{\hat{\omega}}\text{-}cl(A)$. Let $x \notin \tau_{\hat{\omega}}\text{-}cl(A)$. There exists a subset $F \in \tau_{\hat{\omega}}C$ such that $x \notin F$ and $A \subseteq F$; and so $X \setminus F \in \tau_{\hat{\omega}}$ and $\hat{\omega}cl(F) = F$. Then, since $A \subseteq \hat{\omega}cl(F) = F$ and $x \notin F$, we have $\hat{\omega}cl(A) \subseteq \hat{\omega}cl(F) = F$ (cf. Theorem 5.2 (vii)) and so $x \notin \hat{\omega}cl(A)$. Thus, we have the required equality: $\tau_{\hat{\omega}}\text{-}cl(A) = \hat{\omega}cl(A)$. Finally, the set $\hat{\omega}cl(A)$ is $\tau_{\hat{\omega}}$-closed in (X, τ), because $\tau_{\hat{\omega}}\text{-}cl(A)$ is $\tau_{\hat{\omega}}$-closed in (X, τ).

(vi) (1)$\Leftrightarrow$(2) Since $\tau_{\hat{\omega}}$ is a topology on X (cf. (ii) above), A is open in the topological space $(X, \tau_{\hat{\omega}})$ if and only if $A \in \tau_{\hat{\omega}}$.

(1)$\Leftrightarrow$(3) It is obvious from Definition 5.4(ii).

(1)$\Rightarrow$(4) It is obvious from Definition 5.4(iv).

(4)$\Leftrightarrow$(5) Since $\tau_{\hat{\omega}}$ is a topology on X, the proofs are obvious.

(5)$\Rightarrow$(1) The $\tau_{\hat{\omega}}$-closure $\tau_{\hat{\omega}}\text{-}cl(X \setminus A)$ is the intersection of $\tau_{\hat{\omega}}$-closed sets containing $X \setminus A$; and so $\tau_{\hat{\omega}}\text{-}cl(X \setminus A)$ is $\tau_{\hat{\omega}}$-closed in (X, τ), because $\tau_{\hat{\omega}}$ is a topology on X (cf. (ii) above). Thus, using the assumption of (5), $X \setminus A$ is $\tau_{\hat{\omega}}$-closed in (X, τ) (i.e., $X \setminus A \in \tau_{\hat{\omega}}C$). Namely, we have $A \in \tau_{\hat{\omega}}$.

(vi)' The proof of (vi)' is clear from (vi), because $B \in \tau_{\hat{\omega}}C$ if and only if $X \setminus B \in \tau_{\hat{\omega}}$, where $B \subseteq X$. $\square$

In the end of the present section, we introduce the $\hat{\omega}$-interior, $\hat{\omega}$-kernel and $\hat{\omega}$-cokernel of subsets of (X, τ).

Definition 5.6 Let A be a subset of (X, τ). The following set is called:

(i) a *$\hat{\omega}$-interior of A*: $\hat{\omega}int(A) := \bigcup\{U \subseteq X |\ U \subseteq A \text{ and } U \in \hat{\omega}O(X, \tau)\}$;

(ii) a *$\hat{\omega}$-kernel of A*: $\hat{\omega}ker(A) := \bigcap\{U \subseteq X |\ A \subseteq U \text{ and } U \in \hat{\omega}O(X, \tau)\}$;

(iii) a *$\hat{\omega}$-cokernel of A*: $\hat{\omega}coker(A) := \bigcup\{F \subseteq X |\ F \subseteq A \text{ and } F \in \hat{\omega}C(X, \tau)\}$.

Remark 5.7 (i) It is not assured that, in the present paper, the union of $\hat{\omega}$-open sets is $\hat{\omega}$-open in (X, τ). The $\hat{\omega}int(A)$ is the union of all $\hat{\omega}$-open sets contained in A (cf. Definition 5.6(i)). Thus, it is not confirmed that $\hat{\omega}int(A)$ is $\hat{\omega}$-open in (X, τ). Under some assumptions, the union of $\hat{\omega}$-open sets is $\hat{\omega}$-open in (X, τ) (cf. Remark 5.3).

(ii) The kernel $\hat{\omega}ker(A)$ of a subset A is the intersecion of all $\hat{\omega}$-open sets containing A (cf. Definition 5.6(ii)). By Theorem 4.11, it is shown that if the cardinality of the family $\hat{\omega}O(X, \tau)$ is finite then $\hat{\omega}ker(A)$ is $\hat{\omega}$-open in (X, τ) for every subset A of (X, τ).

(iii) The cokernel $\hat{\omega}coker(A)$ of a subset A is the union of all $\hat{\omega}$-closed sets contained in A (cf. Definition 5.6(iii)). By Theorem 4.11, it is shown that if the cardinality of the family $\hat{\omega}O(X, \tau)$ is finite then $\hat{\omega}coker(A)$ is $\hat{\omega}$-closed in (X, τ) for every subset A of (X, τ).

The following theorem states the relation between $\hat{\omega}$-interiors and $\hat{\omega}$-closures and the relation between $\hat{\omega}$-kernels and $\hat{\omega}$-cokernels.

Theorem 5.8 *For subsets A and B of (X, τ), the following statements hold.*

(i) (i-1) $\hat{\omega}ker(X \setminus A) = X \setminus \hat{\omega}coker(A)$ (i-2) $\hat{\omega}coker(X \setminus B) = X \setminus \hat{\omega}ker(B)$.

(ii) (ii-1) $\hat{\omega}cl(X \setminus A) = X \setminus \hat{\omega}int(A)$ (ii-2) $\hat{\omega}int(X \setminus B) = X \setminus \hat{\omega}cl(B)$.

Proof. (i) (i-1) Let $x \notin \hat{\omega}ker(X \setminus A)$. There exists a subset $U \in \hat{\omega}O(X, \tau)$ such that $x \notin U$ and $X \setminus A \subseteq U$. Then, $X \setminus U \in \hat{\omega}C(X, \tau), x \in X \setminus U$ and $X \setminus U \subseteq A$. By Definition 5.6(iii), it is shown that $x \in \hat{\omega}coker(A)$ and hence $x \notin X \setminus \hat{\omega}coker(A)$. Therefore, we prove $X \setminus \hat{\omega}coker(A) \subseteq \hat{\omega}ker(X \setminus A)$. In order to prove the converse implication, let $x \notin X \setminus \hat{\omega}coker(A)$. Then $x \in \hat{\omega}coker(A)$. By Definition 5.6(iii), there exists a set $F \in \hat{\omega}C(X,\tau)$ such that $x \in F \subseteq A$; and hence $X \setminus A \subseteq X \setminus F, x \notin X \setminus F$. Then, $X \setminus F \in \hat{\omega}O(X,\tau), X \setminus A \subseteq X \setminus F$ and $x \notin X \setminus F$. By Definition 5.1(ii), it is shown that $x \notin \hat{\omega}ker(X \setminus A)$. Thus we see that $\hat{\omega}ker(X \setminus A) \subseteq X \setminus \hat{\omega}coker(A)$. Therefore, we show that $\hat{\omega}ker(X \setminus A) = X \setminus \hat{\omega}coker(A)$ holds.

(i-2) We applies (i-1) above for the set $A := X \setminus B$, the following equality is obtained: $\hat{\omega}ker(B) = X \setminus \hat{\omega}coker(X \setminus B)$; and so $X \setminus \hat{\omega}ker(B) = \hat{\omega}coker(X \setminus B)$ holds.

(ii) This is proved by an argument similar to that in (i). $\square$

6 Closures in subspaces

In the last section, we investigate some "*$\hat{\omega}$-closure formula in a subspace* $(H, \tau|H)$ " of a given topological space (X,τ) and related properties. Namely, the purpose of this section is to investigate some relations beteween $\hat{\omega}cl_H(B_1 \cap H)$ and $H \cap \hat{\omega}cl(B_1)$, where $B_1 \subseteq X$ and $\hat{\omega}cl_H(B_1 \cap H)$ denotes the $\hat{\omega}$-closure of $B_1 \cap H$ in the subspace $(H, \tau|H)$, i.e., Theorem 6.2 and Theorem 6.3 below; in the end of this section (i.e., (V) below), we prove the theorems above. We define first explicitely the concept of $\hat{\omega}$-closures in subspaces.

Definition 6.1 (cf. Definition 3.1) Let $B \subseteq H \subseteq X$ and $(H, \tau|H)$ a subspace of (X,τ).

(i) By $\hat{\omega}C(H, \tau|H)$ (resp. $\hat{\omega}O(H, \tau|H)$), we denote the family of all $\hat{\omega}$-closed sets (resp. $\hat{\omega}$-open sets) in $(H, \tau|H)$.

(ii) The intersection of all $\hat{\omega}$-closed sets in $(H, \tau|H)$ containing B is called the *$\hat{\omega}$-closure in $(H, \tau|H)$ of B* and it is denoted by $\hat{\omega}cl_H(B)$. Namely, $\hat{\omega}cl_H(B) := \bigcap\{F|\ B \subseteq F$ and F is $\hat{\omega}$-closed in $(H, \tau|H)$ (i.e., $F \in \hat{\omega}C(H, \tau|H))\}$.

Theorem 6.2 *Suppose that H is preopen and closed in (X,τ) and $B_1 \subseteq X$. Then, $\hat{\omega}cl_H(B_1 \cap H) \subseteq H \cap \hat{\omega}cl(B_1)$ holds.*

Theorem 6.3 *Suppose that H is preopen and closed in (X,τ) and $B_1 \subseteq X$. Then, $H \cap \hat{\omega}cl(B_1) \subseteq \hat{\omega}cl_H(B_1 \cap H)$ holds.*

The proofs are stated in the end of the present section (cf. (V) below); we need the following some properties in the subsections (I), (II), (III) and (IV): (I) Odrinary interior formula in subspaces; (II) δ-interior formula and δ-closure formula in subspaces; (III) a-closure formula in subspaces; (IV) $\hat{\omega}$-closedness in subspaces.

In Definition 3.1, the definition of $\hat{\omega}$-closed sets is in constitutive and so complicated; it is defined by two concepts of "generalized closed sets" (=a-closed sets and $\alpha\hat{g}$-closed sets) and their "generalized closed sets" is used by other ""generalized closed sets"" etc. Therefore, in order to prove Theorems 6.2, 6.3 above, we investigate related definitions and properties on subspace of such "generalized closed sets"etc. in below; some properties below are well known.

We recall some implications between the family of the δ-open sets and the family of the preopen sets (cf. Notation 2.7, Remark 3.13 above and Theorem 6.24 below): for a topologival space (X,τ),

$(*1)$ $\delta O(X,\tau) \subseteq \tau \subseteq \omega O(X,\tau) \subseteq PO(X,\tau)$ (cf. Theorem 6.24);

$(*2)$ $\delta O(X,\tau) \subseteq aO(X,\tau) \subseteq \alpha O(X,\tau) \subseteq PO(X,\tau)$;

$(*3)$ $aO(X,\tau) \subseteq \hat{\omega}O(X,\tau) \subseteq g\alpha O(X,\tau) \subseteq PO(X,\tau)$,

where $g\alpha O(X,\tau)$ denotes the family of all $g\alpha$-open sets of (X,τ).

(I) Ordinary interior formula in subspaces. We use the following notation: for a subset U of $(H, \tau|H)$,

$\cdot cl_H(U)$ (or $(\tau|H)$-$cl(U)$) $:= \bigcap\{F \subseteq H|U \subseteq F, F$ is closed in $(H, \tau|H)\}$;

$\cdot int_H(U)$ (or $(\tau|H)$-$int(U)$) $:= \bigcup\{G \subseteq H|G \subseteq U, G$ is open in $(H, \tau|H)\}$.

It is well known that:
($\bullet$1) if H is a subset of (X,τ), then $cl_H(B) = H \cap cl(B)$ holds for every subset B of $(H, \tau|H)$;
($\bullet$2) (i) if H is an open subset of (X,τ), then $cl_H(B_1 \cap H) \supseteq H \cap cl(B_1)$ holds for every subset B_1 of (X,τ);

(ii) if H is a subset of (X,τ), then $cl_H(B_1 \cap H) \subseteq H \cap cl(B_1)$ holds for every subset B_1 of (X,τ).

We recall the following proof of ($\bullet$2) :

Proof of ($\bullet$2). (i) Let $x \notin cl_H(B_1 \cap H)$. There exists an open set V_H of the subspace $(H,\tau|H)$ such that $x \in V_H$ and $V_H \cap (B_1 \cap H) = V_H \cap B_1 = \emptyset$. Since H is open in $(X,\tau), V_H$ is an open set of (X,τ) containing x such that $V_H \cap B_1 = \emptyset$; and so $x \notin cl(B_1) \cap H$.

(ii) Let $x \notin cl(B_1) \cap H$.
Case 1. $x \notin H$: for this case, $x \notin cl_H(B_1 \cap H)$, because $cl_H(B_1 \cap H) \subseteq H$.
Case 2. $x \in H$: for this case, we have $x \notin cl(B_1)$; and so there exists an open set V in (X,τ) such that $x \in V$ and $V \cap B_1 = \emptyset$. Since $x \in V \cap H$, $V \cap H \in \tau|H$ and $(V \cap H) \cap (B_1 \cap H) = \emptyset$, we see $x \notin cl_H(B_1 \cap H)$. □

The following properties below are some interior formulas and related properties in subspaces.

Proposition 6.4 ([13, Lemma 4.2]; e.g., [28, Theorem 5.3 (ii)(ii-2)])
If U_1 is α-open and H is preopen in (X,τ), then $U_1 \cap H$ is preopen in (X,τ).

Theorem 6.5 (i) ([27, Lemma 1.1]) *In (X,τ),*
$int(B_1) \cap H \subseteq int_H(B_1 \cap H)$ holds for any subsets H and B_1 of (X,τ).

(ii) *Let $A \subseteq H \subseteq X$. If H is preopen in (X,τ), then $int_H(A) \subseteq H \cap pint(A) \subseteq H \cap int(cl(A))$ hold.*

(iii) *If H is open in (X,τ), then $int_H(E \cap H) \subseteq int(E) \cap H$ for any subset E of (X,τ).*

Proof. (ii) Let $x \in int_H(A)$. There exists an open subset U_1 of (X,τ) such that $x \in U_1 \cap H \subseteq A$. By Proposition 6.4,$U_1 \cap H$ is preopen in (X,τ). Therefore, we have $x \in pint(A)$ and hence $x \in pint(A) \cap H = (A \cap int(cl(A)) \cap H \subseteq H \cap int(cl(A))$.

(iii) Let $x \in int_H(E \cap H)$. Then there exists an open set V in $(H,\tau|H)$

such that $x \in V \subseteq E \cap H$. Since H is open in (X,τ), V is open in (X,τ) such that $x \in V \subseteq E$. Thus, we see $x \in int(E)$ and hence $x \in int(E) \cap H$. □

Remark 6.6 (i) The following example shows that Theorem 6.5 (iii) is not true under a weak assumption that H is preopen in (X,τ). Namely, let $H := \{a,b\}$ be a preopen subset of a topological space (X,τ), where $X := \{a,b,c\}$ and $\tau := \{\emptyset, \{a\}, \{b,c\}, X\}$; then $int_H(\{b\}) = \{b\} \not\subseteq H \cap int(\{b\}) = \emptyset$.

(ii) Suppose that H is preopen in (X,τ). Then, for any subset E of X, we have the following properties:
(ii-1) $int(E) \cap int(cl(H))) = int_{int(cl(H))}(E \cap int(cl(H)))$ and
$(int_{int(cl(H))}(E \cap int(cl(H)))) \cap H = int(E) \cap H$ hold.
(ii-2) $int_{int(cl(H))}(E \cap H) \subseteq int(E) \cap int(cl(H))$ holds. Indeed, by Theorem 6.5 (i) and (iii), it is shown that:
($*$) if G is open in a topological space (X,τ), then $int_G(B_1 \cap H) = int(B_1) \cap H$ for every subset B_1 of (X,τ).
By using ($*$) above for $G := int(cl(H))$ and $B_1 := E$ and the preopenness of H, the equalities in (ii-1) are shown. The property (ii-2) is shown by using Theorem 6.5 (iii) for the open set $int(cl(H))$ and the subset E of (X,τ).

(II) δ-interior formula and δ-closure formula in subspaces; (from Definition 6.7 to Theorem 6.11). In 1968, Veličko [38] introduced the concept of δ-open sets and δ-closed sets. We recall some definitions, notations and some properties on δ-closed sets in subspaces etc as follows.

Definition 6.7 ([38]) A subset B of H is said to be *δ-open in* $(H, \tau|H)$ if for each point $x \in B$ there exists an open set V in $(H, \tau|H)$ such that $x \in V$ and $int_H(cl_H(V)) \subseteq B$ (cf. Definition 2.1, Remark 2.2). A subset F of H is said to be *δ-closed in* $(H, \tau|H)$ if $H \setminus F$ is δ-open in $(H, \tau|H)$.

It is well known that: ([38], e.g., Definition 2.1, Remark 2.2 above) a subset A of X is δ-open in (X, τ) if and only if $A = \delta int(A)$; and so, A is δ-open in (X, τ) if and only if, for each point $x \in A$, there exists a regular open set U_x in (X, τ) (i.e., $int(cl(U_x)) = U_x$) such that $x \in U_x$ and $U_x \subseteq A$.

Sometimes, we use the following notation:

Notation 6.8 Let $(H, \tau|H)$ be a subspace of (X, τ).
(i) $\delta O(H, \tau|H)$ (or $(\tau|H)_\delta$) $:= \{G|\ G$ is δ-open in $(H, \tau|H)\}$;
(ii) $\delta O(X, \tau)|H$ (or $\tau_\delta|H$) $:= \{G_1 \cap H | G_1 \in \delta O(X, \tau)\}$;
(i)' $\delta C(H, \tau|H) := \{F|F$ is δ-closed in $(H, \tau|H)\}$;
(ii)' $\delta C(X, \tau)|H := \{F_1 \cap H | F_1 \in \delta C(X, \tau)\}$.

Theorem 6.9 *Let H be a subset of (X, τ).*

(i) ([3, Lemma 2.8], e.g., [39, Lemma 2]) *If H is preopen in (X, τ) and B_1 is regular open in (X, τ), then $B_1 \cap H$ is regular open in $(H, \tau|H)$.*

(ii) ([30, Corollary 3], e.g.,[25, Lemma 2.3]) *Let H be a regular open set of (X, τ) and A be a subset of H. Then, A is regular open in (X, τ) if and only if A is regular open in $(H, \tau|H)$.*

(iii) (cf. [21],[35, Lemma 2] (a)) *If H is preopen in (X, τ), then $\delta O(X, \tau)|H \subseteq \delta O(H, \tau|H)$ (or $\tau_\delta|H \subseteq (\tau|H)_\delta$).*

(iii)' *If H is preopen in (X, τ), then $\delta C(X, \tau)|H \subseteq \delta C(H, \tau|H)$.*

(iv) ([21], e.g., [35, Lemma 2] (b)) *If H is an open subset of (X, τ), then $\delta O(H, \tau|H) \subseteq \delta O(X, \tau)|H$.*

(v) ([35, Corollary 1]) *For any open set H of (X, τ), $(\tau|H)_\delta = \tau_\delta|H$,* (i,e., *$\delta O(H, \tau|H) = \delta O(X, \tau)|H$*).

Proof. They are well known; but we recall shortly the proofs of (iii),(iii)' only. **(iii)** Let $B_1 \cap H \in \delta O(X, \tau)|H$, where $B_1 \in \delta O(X, \tau)$. For each point $x \in B_1 \cap H$, there exists an open set U of (X, τ) such that $x \in U$ and $int(cl(U)) \subseteq B_1$ (cf. Definition 2.1, Remark 2.2). Put $V := int(cl(U)) \cap H$. Then, since $int(cl(U))$ is regular open in (X, τ), by (i) it is shown that V is regular open in $(H, \tau|H)$, i.e., $int_H(cl_H(V)) = V$. Therefore, for each point $x \in B_1 \cap H$, there eists an open set V of $(H, \tau|H)$ such that $x \in V$ and $int_H(cl_H(V)) \subseteq B_1 \cap H$; and so $B_1 \cap H$ is δ-open in $(H, \tau|H)$ (cf. Definition 6.7). **(iii)'** It is obtained by (iii) and the following property: **(vi)*** Let F_1 and H_1 be subsets of X. Then $H_1 \setminus (F_1 \cap H_1) = (X \setminus F_1) \cap H_1$ holds. $\square$

Definition 6.10 (cf. Definition 2.1) For a subset A of $(H, \tau|H)$, the δ-closure and δ-interior of A in a subspae $(H, \tau|H)$ are defined respectively:

(i) $\cdot$ $\delta cl_H(A)$:=$\{x \in H | int_H(cl_H(U)) \cap A \neq \emptyset$ for every set $U \in \tau|H$ such that $x \in U\}$;

(ii) $\cdot$ $\delta int_H(A)$:=$\{y \in H | int_H(cl_H(U)) \subseteq A$ for some set $U \in \tau|H$ such that $y \in U\}$.

Theorem 6.11 (i) (i-1) *If H is δ-open in (X,τ), then $\delta int_H(B_1 \cap H) \subseteq \delta int(B_1) \cap H$ holds for every subset B_1 of (X,τ).*

(i-2) *If H is preopen in (X,τ), then $\delta int_H(B_1 \cap H) \supseteq \delta int(B_1) \cap H$ holds for every subset B_1 of (X,τ).*

(ii) (ii-1) ([31, Lemma 4.4]) *If H is open in (X,τ), then $\delta cl_H(B) = \delta cl(B) \cap H$ holds for every subset B of $(H, \tau|H)$.*

(ii-2) *If H is preopen in (X,τ), then $\delta cl_H(B_1 \cap H) \subseteq \delta cl(B_1) \cap H$ holds for every subset B_1 of (X,τ).*

Proof. (i) (i-1) Let $x \in \delta int_H(B_1 \cap H)$. Then, there exists a subset $U \in \tau|H$ such that $x \in U$ and $int_H(cl_H(U)) \subseteq B_1 \cap H$ (cf. Definition 6.10(ii)). Put $V := int_H(cl_H(U))$. Since the subset V is regular open in $(H,\tau|H)$, V is δ-open in $(H,\tau|H)$ (i.e., $V \in \delta O(H,\tau|H)$). By Theorem 6.9 (v) (cf. $H \in \tau$), it is shown that $V \in \delta O(X,\tau)|H$ (i.e., $V = V_1 \cap H$ for some set $V_1 \in \delta O(X,\tau)$). Since $H \in \delta O(X,\tau)$ (by the assumption on H) and $\delta O(X,\tau)$ is a topology of X, we have $V_1 \cap H \in \delta O(X,\tau)$. Since $x \in U \subset int_H(cl_H(U)) = V$ and $V = V_1 \cap H$, we see that the δ-open set V of (X,τ) containes the point x. By the definition of δ-open sets (cf. Defnition 2.1, Remark 2.2), for the point x and the set V, there exists a subset $W \in \tau$ such that $x \in W$ and $int(cl(W)) \subseteq V$. Since $V = int_H(cl_H(U)) \subseteq B_1 \cap H \subseteq B_1$, we have that $W \in \tau$, $x \in W$ and $int(cl(W)) \subseteq B_1$; and hence $x \in \delta int(B_1)$. Thus, we show $x \in \delta int(B_1) \cap H$.

(i-2) Let $x \in \delta int(B_1) \cap H$. There exists a subset $U_1 \in \tau$ such that $x \in U_1$ and $int(cl(U_1)) \subseteq B_1$. Put $V_1 := int(cl(U_1))$. Then, V_1 is regular open in (X,τ) and so $V_1 \in \delta O(X,\tau)$. By Theorem 6.9 (iii) (cf. H is preopen), it is shown that $V_1 \cap H \in \delta O(H,\tau|H)$; and so, since $x \in V_1 \cap H$, there exists a subset $W \in \tau|H$ such that $x \in W$, $int_H(cl_H(W)) \subseteq V_1 \cap H \subseteq B_1 \cap H$. Thus, we have $x \in \delta int_H(B_1 \cap H)$ (cf. Definition 6.10 (ii)).

(ii) (ii-2) We prove: $\delta cl_H(B_1 \cap H) \subseteq \delta cl(B_1) \cap H$ under the assumtion that H is preopen in (X,τ). Let $x \in \delta cl_H(B_1 \cap H)$. Because of $\delta cl_H(B_1 \cap H) \subseteq H$, we have $x \in H$. In order to show $x \in \delta cl(B_1)$, let $V_1 \in \tau$ such that $x \in V_1$. We claim that $int(cl(V_1)) \cap B_1 \neq \emptyset$. Indeed, $V_1 \cap H$ is an open set of $(H,\tau|H)$ containing the point x; since $x \in \delta cl_H(B_1 \cap H)$, we have $(B_1 \cap H) \cap int_H(cl_H(V_1 \cap H)) \neq \emptyset$. Since $int_H(cl_H(V_1 \cap H)) \subseteq int_H[\{cl(V_1 \cap H)\} \cap H]$ (cf. ($\bullet$2) (ii) in (I))
$\subseteq int[cl\{cl(V_1 \cap H)\}] \cap H$ (cf. Theorem 6.5(ii) for a preopen set H)
$\subseteq int(cl(V_1))$, we see that $\emptyset \neq (B_1 \cap H) \cap int_H(cl_H(V_1 \cap H)) \subseteq B_1 \cap int(cl(V_1))$ hold and so we conclude that $B_1 \cap int(cl(V_1)) \neq \emptyset$. ($\circ$) The claim shows that $x \in \delta cl(B_1)$; and hence $x \in \delta cl(B_1) \cap H$. $\square$

(III) a-closure formula in subspaces (from Definition 6.13 to Theorem 6.17). The purpose of the present (III) is to prove Theorem 6.17 below. First we recall and define explicitely their concepts in subspaces. Let $(H,\tau|H)$ be a subspace of (X,τ). A subset B of H is said to be *a-open in* $(H,\tau|H)$, if $B \subseteq int_H(cl_H(\delta int_H(B)))$ holds. A subset F of H is said to be *a-closed in* $(H,\tau|H)$, if $H \setminus F$ is a-open in $(H,\tau|H)$.

Notation 6.12 (i) $aO(H,\tau|H) := \{U|\ U \text{ is } a\text{-open in } (H,\tau|H)\}$;
(ii) $aO(X,\tau)|H := \{U_1 \cap H|\ U_1 \in aO(X,\tau)\}$;
(i)' $aC(H,\tau|H) := \{F|\ F \text{ is } a\text{-closed in } (H,\tau|H)\}$;
(ii)' $aC(X,\tau)|H := \{F_1 \cap H|\ F_1 \in aC(X,\tau)\}$, where $aO(X,\tau)$ (resp. $aC(X,\tau)$) be an the family of all a-open (resp. a-closed) sets of (X,τ) (cf. Notation 2.7).

Definition 6.13 Let B be a subset of $(H,\tau|H)$.
$\cdot$ $acl_H(B) := \bigcap\{F \subseteq H | B \subseteq F, F \text{ is } a\text{-closed in } (H,\tau|H)\}$;
$\cdot$ $aint_H(B) := \bigcup\{G \subseteq H | G \subseteq B, G \text{ is } a\text{-open in } (H,\tau|H)\}$.

In order to prove the following lemma, we need Theorem 6.11(i)(i-2), Theorem 6.5(i) and the well known properties that $(*1)$ $\delta int(E)$ is open in (X,τ), where E is any subset of (X,τ); $(*2)$ $cl(A\cap G)\supseteq cl(A)\cap G$ holds for any set $A\subseteq X$ if G is open in (X,τ); $(*3)$ $cl_H(B)=cl(B)\cap H$ for any subspace $(H,\tau|H)$ and $B\subseteq H$.

Lemma 6.14 *If H is preopen in (X,τ) and U_1 is a subset of (X,τ), then $int_H(cl_H(\delta int_H(U_1\cap H)))\supseteq int(cl(\delta int(U_1)))\cap H$ holds.*

Proof. Put $L:=int_H(cl_H(\delta int_H(U_1\cap H)))$. It follows from Theorem 6.11(i)(i-2) and $(*1)$, $(*2)$ and $(*3)$ above that $L\supseteq int_H(cl_H(H\cap\delta int(U_1)))=int_H(H\cap[cl(H\cap\delta int(U_1))])=int_H(H\cap[cl\{cl(H\cap\delta int(U_1))\}])\supseteq int_H(H\cap[cl\{cl(H)\cap\delta int(U_1)\}])\supseteq int_H(H\cap cl[int\{cl(H)\cap(\delta int(U_1)\}])\supseteq int_H\{H\cap[int(cl(H))\cap cl(\delta int(U_1))]\}\supseteq int[int(cl(H)\cap cl(\delta int(U_1))]\cap H=int(int(cl(H)))\cap int(cl(\delta int(U_1)))\cap H=[int(cl(H))\cap int(cl(\delta int(U_1)))]\cap H\supseteq H\cap int(cl(\delta int(U_1)))$ (cf. $H\subseteq int(cl(H))$ holds). Thus, it is shown that $L\supseteq H\cap int(cl(\delta int(U_1)))$. □

Theorem 6.15 *If H is preopen in (X,τ), then the following statements hold.*
(i) $aO(X,\tau)|H\subseteq aO(H,\tau|H)$*; and*
(ii) $aC(X,\tau)|H\subseteq aC(H,\tau|H)$.

Proof. (i) Let $U_1\cap H\in aO(X,\tau)|H$, where $U_1\in aO(X,\tau)$. By using Lemma 6.14, it is shown that $int_H(cl_H(\delta int_H(U_1\cap H)))\supseteq int(cl(\delta int(U_1)))\cap H\supseteq U_1\cap H$ (cf. $U_1\subseteq int(cl(\delta int_H(U_1))))$; and so $U_1\cap H$ is a-open in $(H,\tau|H)$.

(ii) Let $F_1\cap H\in aC(X,\tau)|H$, where $F_1\in aC(X,\tau)$. Then, $X\setminus F_1\in aO(X,\tau)$; and by (i), $(X\setminus F_1)\cap H\in aO(H,\tau|H)$. It is shown that $H\setminus(F_1\cap H)\in aO(H,\tau|H)$ (cf. (vi)* in Proof of Theorem 6.9); and hence $F_1\cap H$ is an a-closed subset of $(H,\tau|H)$. □

Theorem 6.16 (i) *If H is δ-open in (X,τ), then $aO(H,\tau|H)\subseteq aO(X,\tau)$ holds.*
(ii) *If H is both δ-open and a-closed in (X,τ), then $aC(H,\tau|H)\subseteq aC(X,\tau)$ holds.*

Proof. Let $U\in aO(H,\tau|H)$. Then, $U\subseteq int_H(cl_H(\delta int_H(U)))$ holds. By Theorem 6.5(i)(iii), it is shown that $int_H(cl_H(\delta int_H(U)))=H\cap\{int(cl_H(\delta int_H(U)))\}\subseteq int(cl_H(\delta int_H(U)))=int(H\cap cl(\delta int_H(U)))$
$\subseteq int(cl(\delta int_H(U)))$. Since $\delta int_H(U)\subseteq\delta int(U)\cap H$ holds for the case where H is δ-open (cf. Theorem 6.11(i)(i-1)), we see that $U\subseteq int(cl(\delta int(U)))$ holds, i.e., U is a-open in (X,τ).

(ii) Let $F\in aC(H,\tau|H)$. Then, $H\setminus F\in aO(H,\tau|H)$; and by (i), $H\setminus F\in aO(X,\tau)$. It follows from assumption that $X\setminus H\in aO(X,\tau)$. Therefore, $(H\setminus F)\cup(X\setminus H)=X\setminus F\in aO(X,\tau)$, because the family $aO(X,\tau)$ of the all a-open sets forms a topology of X (cf. [15]); and hence F is a-closed in (X,τ). □

Theorem 6.17 (i) *If H is δ-open in (X,τ), then $acl_H(B_1\cap H)\subseteq acl(B_1)\cap H$ holds for a subset B_1 of (X,τ).*
(ii) *If H is δ-open and a-closed in (X,τ), then $acl(B_1)\cap H\subseteq acl_H(B_1\cap H)$ for a subset B_1 of (X,τ).*

Proof. (i) Let $x\in acl_H(B_1\cap H)$ and F_1 be any a-closed subset of (X,τ) such that $B_1\subseteq F_1$. By Theorem 6.15(ii), $F_1\cap H$ is a-closed in $(H,\tau|H)$. Since $x\in acl_H(B_1\cap H)$, we see that $x\in F_1\cap H\subseteq F_1$; and so $x\in acl(B)\cap H$.

(ii) Let $x\in acl(B_1)\cap H$ and F be any a-closed subset of $(H,\tau|H)$ such that $B_1\cap H\subseteq F$. By Theorem 6.16(ii), F is a-closed in (X,τ). By one of the assumptions on H (i.e., H is δ-open), it is shown that H is a-open and so $X\setminus H$ is a-closed (cf. Remark 3.13; or $(*2)$ before (I) in the top of Section 6). Thus, $F\cup(X\setminus H)$ is an a-closed set in (X,τ) such that

$B_1 \subseteq F \cup (X \setminus H)$. Since $x \in acl(B_1)$, we see that $x \in F \cup (X \setminus H)$ (i.e., $x \in F$); and so $x \in acl_H(B_1 \cap H)$. □

(IV) $\hat{\omega}$-closedness in subspaces (from Notation 6.18 to Theorem 6.30). In the end of the present section (IV), i.e., (IV-3), we investigate some $\hat{\omega}$-closedness in subspaces. We need more properties in the following (IV-1), (IV-2) as follows: (IV-1) (α-closure formula in subspaces); (IV-2) ($\alpha\hat{g}$-closedness in subspaces); (IV-3) ($\hat{\omega}$-closedness in subspaces).

• **(IV-1) (α-closure formula in subspaces**; from Notation 6.18 to Theorem 6.20**)**. The purpose of (IV-1) is to prove Theorem 6.20 below. Some properties below are well known. We need the following notation:

Notation 6.18 (i) $\alpha O(X,\tau)$ (resp. $\alpha C(X,\tau)$) be the family of all α-open sets (resp. α-closed sets) in (X,τ) (cf. Definition 2.4(iii));
· $\alpha O(H,\tau|H)$ (resp. $\alpha C(H,\tau|H)$) be the family of all α-open sets (resp. α-closed sets) in a subspace $(H,\tau|H)$;
• $\alpha O(X,\tau)|H := \{U_1 \cap H |\ U_1 \in \alpha O(X,\tau)\}$;
• $\alpha C(X,\tau)|H := \{V_1 \cap H |\ V_1 \in \alpha C(X,\tau)\}$.
(ii) $\alpha cl_H(B) := \bigcap\{F | B \subseteq F, F \in \alpha C(H,\tau|H)\}$ for a subset $B \subseteq H$.

Theorem 6.19 (i) *If H is any subset of (X,τ), then*
(a) ([18, Proposition 2.1]) *$\alpha O(H,\tau|H) \subseteq \alpha O(X,\tau)|H$ holds;*
(a)' *$\alpha C(H,\tau|H) \subseteq \alpha C(X,\tau)|H$ holds.*
(ii) *If H is preopen in (X,τ), then*
(b) ([27, Lemma 1.1]) *$\alpha O(X,\tau)|H \subseteq \alpha O(H,\tau|H)$ holds;*
(b)' *$\alpha C(X,\tau)|H \subseteq \alpha C(H,\tau|H)$ holds.*

Proof. (i) (a)' (resp. (ii)(b)') is proved by using (i)(a) (resp. (ii)(a)) and (vl)* in Proof of Theorem 6.9. □

Theorem 6.20 *Let B, H and B_1 be subsets of (X,τ) such that $B \subseteq H$ and $B_1 \subseteq X$.*
(i) *If H is preopen in (X,τ), then $\alpha cl_H(B_1 \cap H) \subseteq \alpha cl(B_1) \cap H$ holds.*
(ii) (ii-1) *If H is α-open in (X,τ), then $\alpha cl(B_1) \cap H \subseteq \alpha cl_H(B_1 \cap H)$ holds.*
(ii-2) *If H is any subset of (X,τ) and $B \subseteq H$, then $\alpha cl(B) \cap H \subseteq \alpha cl_H(B)$ holds.*

Proof. (i) Let $x \in \alpha cl_H(B_1 \cap H)$. Let $F_1 \in \alpha C(X,\tau)$ such that $B_1 \subseteq F_1$. Thus, $F_1 \cap H \in \alpha C(X,\tau)|H$. By Theorem 6.19(ii)(b)', it is shown that $F_1 \cap H \in \alpha C(H,\tau|H)$ and $B_1 \cap H \subseteq F_1 \cap H$. Thus we have $x \in F_1 \cap H$ and so $x \in F_1$ which shows that $x \in \alpha cl(B_1)$. Therefore, we see that $x \in \alpha cl(B_1) \cap H$ for the point $x \in \alpha cl_H(B_1 \cap H)$.

(ii) (ii-1) Let $x \in \alpha cl(B_1) \cap H$ and $F \in \alpha C(H,\tau|H)$ such that $B_1 \cap H \subseteq F$. Then, by Theorem 6.19(i)(a)', it is shown that $F \in \alpha C(X,\tau)|H$; and so there exists an α-closed set F_1 of (X,τ) such that $F = F_1 \cap H$. Since H is α-open in (X,τ), $(X \setminus H) \cup F_1$ is an α-closed set of (X,τ) such that $B_1 \subseteq (X \setminus H) \cup F_1$. Since $x \in \alpha cl(B_1) \cap H$, we have $x \in (X \setminus H) \cup F_1$ and so $x \in F_1 \cap H = F$. Therefore, we have that $x \in \alpha cl_H(B_1 \cap H)$ for the point $x \in \alpha cl(B_1) \cap H$.

(ii-2) Let $x \in \alpha cl(B) \cap H$ and $F \in \alpha C(H,\tau|H)$ such that $B \subseteq F$. Then, by Theorem 6.19(i)(a)', it is shown that $F \in \alpha C(X,\tau)|H$; and so there exists an α-closed set F_1 of (X,τ) such that $F = F_1 \cap H$. Then, we have $x \in F_1$ and so $x \in F$, because $x \in \alpha cl(B) \cap H$, $F_1 \in \alpha C(X,\tau)$ and $B \subseteq F_1$. Therefore, we show that $x \in \alpha cl_H(B)$ for the point $x \in \alpha cl(B) \cap H$. □

• **(IV-2) ($\alpha\hat{g}$-closedness in subspaces**; from Notation 6.21 to Theorem 6.29**)**. The purpose of the present (IV-2) is to prove Theorem 6.27 below. We need the following notation:

Notation 6.21 (i) Let $SO(X,\tau)$ (resp. $SC(X,\tau)$) be the family of all semi-open sets (resp. semi-closed sets) in (X,τ) (cf. Definition 2.4(i)); and
let $SO(H,\tau|H)$ (resp. $SC(H,\tau|H)$) be the family of all semi-open sets (resp. semi-closed sets) in a subspace $(H,\tau|H)$. We need the following notations:
• $SO(X,\tau)|H := \{U_1 \cap H|\ U_1 \in SO(X,\tau)\}$ and ;
• $SC(X,\tau)|H := \{F_1 \cap H|\ F_1 \in SC(X,\tau)\}$.

(ii) For a subset $B_1 \subseteq X$ and a subset $B \subseteq H$;
· $sker(B_1) := \bigcap\{U_1 | B_1 \subseteq U_1, U_1 \in SO(X,\tau)\}$; and
• $sker_H(B) := \bigcap\{U | B \subseteq U, U \in SO(H,\tau|H)\}$.

(iii) We recall that $\omega O(X,\tau)$ (resp. $\omega C(X,\tau)$) denotes the family of all ω-open sets (resp. ω-closed sets) in (X,τ) (cf. Definition 2.5(vii), Notation 2.7). Let $\omega O(H,\tau|H)$ (resp. $\omega C(H,\tau|H)$) be the family of all ω-open sets (resp. ω-closed sets) in a subspace $(H,\tau|H)$. We need the following notation:
• $\omega O(X,\tau)|H := \{U_1 \cap H|\ U_1 \in \omega O(X,\tau)\}$; and
• $\omega C(X,\tau)|H := \{F_1 \cap H|\ F_1 \in \omega C(X,\tau)\}$.

(iv) $\alpha\hat{g}O(X,\tau)$ (resp. $\alpha\hat{g}C(X,\tau)$) be the family of all $\alpha\hat{g}$-open sets (resp. $\alpha\hat{g}$-closed sets) in (X,τ) (cf. Definition 2.5(vii), Notation 2.7);
$\alpha\hat{g}O(H,\tau|H)$ (resp. $\alpha\hat{g}C(H,\tau|H)$) be the family of all $\alpha\hat{g}$-open sets (resp. $\alpha\hat{g}$-closed sets) in a subspace $(H,\tau|H)$. We need the following notation:
• $\alpha\hat{g}O(X,\tau)|H := \{U_1 \cap H|\ U_1 \in \alpha\hat{g}O(X,\tau)\}$; and
• $\alpha\hat{g}C(X,\tau)|H := \{F_1 \cap H|\ F_1 \in \alpha\hat{g}C(X,\tau)\}$.

Theorem 6.22 (i) ([34, Theorem 2.3]) *For any subset H of (X,τ),*
$SO(H,\tau|H) \subseteq SO(X,\tau)|H$ holds (cf. Notation 6.21(i) *above*).

(ii) ([32]) *For any subset H of $(X,\tau), SC(H,\tau|H) \subseteq SC(X,\tau)|H$ holds* (cf. Notation 6.21(i) *above*).

(iii) ([32]) *If H is preopen in (X,τ), then $SO(X,\tau)|H \subseteq SO(H,\tau|H)$ holds.* □

Theorem 6.23 *Let $E \subseteq H \subseteq X$. Then, $H \cap sker(E) \subseteq sker_H(E)$ holds.*

Proof. Let $x \in H \cap sker(E)$. Let $U \in SO(H,\tau|H)$ with $E \subseteq U$. By Theorem 6.22(i), there exists a subset $U_1 \in SO(X,\tau)$ such that $U = U_1 \cap H$. Thus $E \subseteq U = U_1 \cap H \subseteq U_1$ and so $sker(E) \subseteq U_1$ because $U_1 \in SO(X,\tau)$ and $E \subseteq U_1$. Since $x \in sker(E)$, we have $x \in U_1$; this implies $x \in U_1 \cap H = U$. Thus, we have that $x \in sker_H(E)$. □

Theorem 6.24 *For every topological space (X,τ), we have the following properties:*

(i) *$\omega C(X,\tau) \subseteq PC(X,\tau)$ holds* (cf. Notation 2.7)*;*

(ii) *for a subset A of $(X,\tau), A$ is δ-open and δ-closed if and onlyif A is preopen and closed.*

Proof. (i) First we claim that every ω-closed set is $g\alpha$-closed in (X,τ). Indeed, let $A \in \omega C(X,\tau)$. And, let U be an α-open set of (X,τ) such that $A \subseteq U$. Since U is semi-open in (X,τ), it is shown that $\alpha cl(A) \subseteq cl(A) \subseteq U$; and so the ω-closed set A is gα-closed in (X,τ) (cf. Definition 2.5(iii)). Finally, it is well known that every gα-closed set is preclosed ([8, the diagram in Section 1], e.g.,[10, Theorems 2.2, 2.3(ii)]); and hence the ω-closed set A is preclosed in (X,τ).

(ii) (Necessity) It is obvious from definitions.
(Sufficiency) Supposed that A is preopen and closed in (X,τ). We first show that $\delta cl(A) \subseteq A$. Indeed, let $x \notin A$. Since $A = cl(A)$, there exists a subset $U \in \tau$ such that $x \in U$ and $U \cap A = \emptyset$; and so $cl(U) \cap A = \emptyset$, because $A \in \tau$. Thus, we have that $int(cl(U)) \cap A = \emptyset$. This shows that $x \notin \delta cl(A)$. ∘ Finally, we prove that $X \setminus A$, say A^c, is δ-closed in (X,τ),

i.e., $\delta cl(A^c) \subseteq A^c$. Indeed, let $x \notin A^c$. Since A is an open set containing x and $A \cap A^c = \emptyset$ holds, we have that $cl(A) \cap A^c = \emptyset$; and so we see that $int(cl(A)) \cap A^c = \emptyset$; and hence $x \notin \delta cl(A^c)$. Thus we see that $\delta cl(A^c) \subseteq A^c$; and hence A is δ-open in (X, τ). $\square$

Theorem 6.25 (i) *If H is preopen and closed in (X, τ), then $\omega C(H, \tau|H) \subseteq \omega C(X, \tau)$ holds.*

(ii) *If H is closed and ω-open in (X, τ), then $\omega O(H, \tau|H) \subseteq \omega O(X, \tau)$ holds.*

Proof. (i) Let $B \in \omega C(H, \tau|H)$. In order to prove $B \in \omega C(X, \tau)$, let $U_1 \in SO(X, \tau)$ such that $B \subseteq U_1$. By Theorem 6.22(iii), it is shown that $U_1 \cap H \in SO(H, \tau|H)$ (cf. the assumption that H is preopen). Since $B \subseteq U_1 \cap H$ and $B \in \omega C(H, \tau|H)$, we have $cl_H(B) \subseteq U_1 \cap H$. We see that $B \subseteq H$ and $cl(B) \subseteq cl(H) = H$; and hence $cl(B) = cl(B) \cap H = cl_H(B) \subseteq U_1 \cap H \subseteq U_1$. Therefore, we have $B \in \omega C(X, \tau)$.

(ii) Let $U \in \omega O(H, \tau|H)$. Then, H is preopen (cf. Theorem 6.24(i)) and $H \setminus U \in \omega C(H, \tau|H)$; and so, by (i), $H \setminus U \in \omega C(X, \tau)$. Hence, we have $H \setminus U = H \cap (X \setminus U) \in \omega C(X, \tau)$. Put $W := X \setminus (H \cap (X \setminus U))$; then $W \in \omega O(X, \tau)$ and $W \cap H = U$; and hence we have $U \in \omega O(X, \tau)$. $\square$

Theorem 6.26 *If H is ω-closed in (X, τ), then*

(i) *$\omega C(X, \tau)|H \subseteq \omega C(H, \tau|H)$ holds; and*

(ii) *$\omega O(X, \tau)|H \subseteq \omega O(H, \tau|H)$ holds.*

Proof. (i) Let $F_1 \cap H \in \omega C(X, \tau)|H$, where $F_1 \in \omega C(X, \tau)$. Since $\omega O(X, \tau)$ forms a topology of X ([36]), $F_1 \cap H \in \omega C(X, \tau)$. By characterization of ω-closed sets ([36]), $F_1 \cap H \in \omega C(X, \tau)$ if and only if $cl(F_1 \cap H) \subseteq sker(F_1 \cap H)$. Hence we have $cl_H(F_1 \cap H) = H \cap [cl(F_1 \cap H)] \subseteq H \cap [sker(F_1 \cap H)] \subseteq sker_H(F_1 \cap H)$ (cf. Theorem 6.23); and so $F_1 \cap H \in \omega C(H, \tau|H)$.

(ii) By definitions, (i) above and the general property (vi)* which is used in Proof of Theorem 6.9(iii)', (ii) is proved. $\square$

Theorem 6.27 (i) (i-1) *If H is ω-open and closed in (X, τ), then $\alpha\hat{g}C(X, \tau)|H \subseteq \alpha\hat{g}C(H, \tau|H)$ holds.*

(i-2) *If H is ω-closed and α-closed in (X, τ), then $\alpha\hat{g}C(H, \tau|H) \subseteq \alpha\hat{g}C(X, \tau)$ holds.*

(ii)(ii-1) *If H is ω-open and closed in (X, τ), then $\alpha\hat{g}O(X, \tau)|H \subseteq \alpha\hat{g}O(H, \tau|H)$ holds.*

(ii-2) *If H is ω-closed and α-closed in (X, τ), then $\alpha\hat{g}O(H, \tau|H) \subseteq \alpha\hat{g}O(X, \tau)|H$ holds.*

(iii) *If H is both α-closed and ω-closed in (X, τ), then $\alpha\hat{g}C(X, \tau)|H \subseteq \alpha\hat{g}C(X, \tau)$ holds.*

(iv) *If V is $\alpha\hat{g}$-open subset of (X, τ) and E is α-open and ω-open in (X, τ), then $V \cup E$ is $\alpha\hat{g}$-open in (X, τ).*

Proof. (i) (i-1) Let $F_1 \cap H \in \alpha\hat{g}C(X, \tau)|H$, where $F_1 \in \alpha\hat{g}C(X, \tau)$. And, let $U \in \omega O(H, \tau|H)$ such that $F_1 \cap H \subseteq U$. We claim that $\alpha cl_H(F_1 \cap H) \subseteq U$ holds (cf. Definition 2.5(viii)). Indeed, since H is ω-open and closed in (X, τ), we have $U \in \omega O(X, \tau)$ (cf. Theorem 6.25(ii)). Since $F_1 \cap H \subseteq U$, we have $F_1 \subseteq U \cup (X \setminus H)$. Now $U \cup (X \setminus H) \in \omega O(X, \tau)$, because $U \in \omega O(X, \tau)$ and $X \setminus H \in \omega O(X, \tau)$ and the union of two ω-open sets is ω-open. Since F_1 is $\alpha\hat{g}$-closed in (X, τ), we have $\alpha cl(F_1) \subseteq U \cup (X \setminus H)$; and so $H \cap \alpha cl(F_1) \subseteq U$. Since H is preopen (cf. Theorem 6.24), using Theorem 6.20(i), we conclude that $\alpha cl_H(F_1 \cap H) \subseteq H \cap \alpha cl(F_1) \subseteq U$; and hence $F_1 \cap H$ is $\alpha\hat{g}$-closed in $(H, \tau|H)$, i.e., $F_1 \cap H \in \alpha\hat{g}C(H, \tau|H)$.

(i-2) Let $B \in \alpha\hat{g}C(H, \tau|H)$. And, let $U_1 \in \omega O(X, \tau)$ such that $B \subseteq U_1$. We claim that $\alpha cl(B) \subseteq U_1$ holds. Indeed, since $B \subseteq H, B = B \cap H \subseteq U_1 \cap H$. Since H is ω-closed in (X, τ), by Theorem 6.26(ii), $U_1 \cap H \in \omega O(H, \tau|H)$. Since B is $\alpha\hat{g}$-closed in $(H, \tau|H)$, we have $\alpha cl_H(B) \subseteq U_1 \cap H$. Since $B \subseteq H$ and H is α-closed in (X, τ), we have $\alpha cl(B) \subseteq \alpha cl(H) = H$ and so $\alpha cl(B) = H \cap \alpha cl(B)$. By Theorem 6.20(ii)(ii-2), it is shown that $\alpha cl(B) = H \cap \alpha cl(B) \subseteq \alpha cl_H(B) \subseteq U_1 \cap H \subseteq U_1$. Hence we have that $B \in \alpha\hat{g}C(X, \tau)$.

(ii)(ii-1) Let $U_1 \cap H \in \alpha\hat{g}O(X, \tau)|H$, where $U_1 \in \alpha\hat{g}O(X, \tau)$. By (i)(i-1), it is shown that $(X \setminus U_1) \cap H \in \alpha\hat{g}C(H, \tau|H)$. Using the general property (vi)* which is used in Proof of Theorem 6.9, we have that $H \setminus (U_1 \cap H) \in \alpha\hat{g}C(H, \tau|H)$ and so $U_1 \cap H \in \alpha\hat{g}O(H, \tau|H)$.

(ii-2) Let $U \in \alpha\hat{g}O(H, \tau|H)$. Using (i)(i-2), we see that $H \setminus U \in \alpha\hat{g}C(X, \tau)$. Using the general property (vi)* in Proof of Theorem 6.9(iii)', we have $(X \setminus U) \cap H \in \alpha\hat{g}C(X, \tau)$; and hence $U_1 := X \setminus [(X \setminus U) \cap H] \in \alpha\hat{g}O(X, \tau)$; then we see that $U = U_1 \cap H$ and so $U \in \alpha\hat{g}O(X, \tau)|H$.

(iii) Let $F_1 \cap H \in \alpha\hat{g}C(X, \tau)|H$, where $F_1 \in \alpha\hat{g}C(X, \tau)$. Let $U \in \omega O(X, \tau)$ such that $F_1 \cap H \subseteq U$. We claim that $\alpha cl(F_1 \cap H) \subseteq U$ holds (cf. Definition 2.5(viii)). Indeed, $F_1 \subseteq U \cup (X \setminus H)$ and $U \cup (X \setminus H)$ is ω-open (i.e., $\hat{g}$-open) in (X, τ); and since F_1 is $\alpha\hat{g}$-closed in (X, τ), we have $\alpha cl(F_1) \subseteq U \cup (X \setminus H)$. And, we have $\alpha cl(F_1 \cap H) \subseteq \alpha cl(F_1) \cap \alpha cl(H) = \alpha cl(F_1) \cap H \subseteq (U \cup (X \setminus H)) \cap H \subseteq U$. Thus, we claimed that $\alpha cl(F_1 \cap H) \subseteq U$ holds. Therefore, $F_1 \cap H \in \alpha\hat{g}C(X, \tau)$.

(iv) It follows from the assumptions on V and E that $X \setminus V$ is $\alpha\hat{g}$-closed and $X \setminus E$ is α-closed and ω-closed in (X, τ). By using (iii) for $H := X \setminus E$, it is shown that $(X \setminus V) \cap (X \setminus E) = X \setminus (V \cup E)$ is $\alpha\hat{g}$-closed in (X, τ) and hence $V \cup E$ is $\alpha\hat{g}$-open in (X, τ). □

Remark 6.28 As a short application of Theorem 6.27(i) above, we have a kind of the converse of Theorem 4.1 with added assumptions as follows.

Theorem 6.29 *Let A be a subset of (X, τ) such that $acl(A)$ is ω-open and closed in (X, τ). If $acl(A) \setminus A$ does not contain any nonempty $\alpha\hat{g}$-closed set of (X, τ), then A is $\hat{\omega}$-closed in (X, τ).*

Proof. Let U be an $\alpha\hat{g}$-open set U of (X, τ) such that $A \subseteq U$. Suppose that $acl(A) \not\subseteq U$ holds. Put $H := acl(A)$. Then $H \cap (X \setminus U) \neq \emptyset$ and, by Theorem 6.27(i)(i-1), it is shown that $H \cap (X \setminus U) \in \alpha\hat{g}C(H, \tau|H)$, because $X \setminus U \in \alpha\hat{g}C(X, \tau)$. Using Theorem 6.27(i)(i-2), we have $H \cap (X \setminus U) \in \alpha\hat{g}C(X, \tau)$ and $H \cap (X \setminus U) \subseteq H \cap (X \setminus A) = acl(A) \setminus A$; and so the subset $acl(A) \setminus A$ contains a nonempty $\alpha\hat{g}$-closed set $H \cap (X \setminus U)$ of (X, τ). This contradicts the assumption of this theorem. Hence we have $acl(A) \subseteq U$; and so A is $\hat{\omega}$-closed in (X, τ). □

• **(IV-3) ($\hat{\omega}$-closedness in subspaces)**. Using some properties in the subsections (I), (II), (III), (IV-1) and (IV-2), we investigate some properties on $\hat{\omega}$-closedness in subspaces as follows:

Theorem 6.30 (i) *If H is preopen and closed in (X, τ), then we have the following properties:*

(a) *$\hat{\omega}C(H, \tau|H) \subseteq \hat{\omega}C(X, \tau)$ and so $\hat{\omega}C(H, \tau|H) \subseteq \hat{\omega}C(X, \tau)|H$;*

(b) *$\hat{\omega}C(X, \tau)|H \subseteq \hat{\omega}C(H, \tau|H)$;*

(c) *$\hat{\omega}O(H, \tau|H) \subseteq \hat{\omega}O(X, \tau)|H$.*

(ii) *Suppose that H is δ-open, ω-closed and α-closed in (X, τ) and $F \subseteq H$. If F is $\hat{\omega}$-closed in (X, τ), then F is $\hat{\omega}$-closed in $(H, \tau|H)$.*

Proof. (i) (a) Let $F \in \hat{\omega}C(H, \tau|H)$. And, let U_1 be any $\alpha\hat{g}$-open subset of (X, τ) such that $F \subseteq U_1$. We claim that $acl(F) \subseteq U_1$ holds (cf. Definition 3.1). We first note that, by the assumption of H, it is shown that H is δ-open and δ-closed (cf. Theorem 6.24(ii)) and so H is ω-open, closed and a-closed in (X,τ) (cf. Remark 3.13). Then, using Theorem 6.27(ii)(ii-1), we have that $U_1 \cap H \in \alpha\hat{g}O(H,\tau|H)$ such that $F \subseteq U_1 \cap H$. Since F is $\hat{\omega}$-closed in $(H,\tau|H)$, we see $acl_H(F) \subseteq U_1 \cap H \subseteq U_1$. Since $F \subseteq H$, we have $acl(F) \subseteq acl(H)$ and hence $acl(F) = acl(F) \cap acl(H) = acl(F) \cap H$. By Theorem 6.17(ii), it is shown that $acl(F) = acl(F) \cap H \subseteq acl_H(F) \subseteq U_1$. Therefore, we have $F \in \hat{\omega}C(X,\tau)$; and $F = F \cap H \in \hat{\omega}C(X,\tau)|H$.

(b) Let $F_1 \cap H \in \hat{\omega}C(X,\tau)|H$, where $F_1 \in \hat{\omega}C(X,\tau)$. Let $U \in \alpha\hat{g}O(H,\tau|H)$ such that $F_1 \cap H \subseteq U$. We claim that $acl_H(F_1 \cap H) \subseteq U$ (cf. Definition 3.1). By Theorem 6.27(ii)(ii-2), there exists a set $U_1 \in \alpha\hat{g}O(X,\tau)$ such that $U = U_1 \cap H \in \alpha\hat{g}O(X,\tau)|H$ and hence $F_1 \cap H \subseteq U_1$. Then, we have that $F_1 \subseteq U_1 \cup (X \setminus H)$ and, by Theorem 6.27(iv), $U_1 \cup (X \setminus H)$ is $\alpha\hat{g}$-open in (X,τ). Since F_1 is $\hat{\omega}$-closed in (X,τ), we have $acl(F_1) \subseteq U_1 \cup (X \setminus H)$. Using Theorem 6.24(ii) and Theorem 6.17(i), we see that $acl_H(F_1 \cap H) \subseteq acl(F_1 \cap H) \cap H \subseteq acl(F_1) \cap H \subseteq (U_1 \cup (X \setminus H)) \cap H \subseteq U_1 \cap H$. Therefore, we claimed that $acl_H(F_1 \cap H) \subseteq U$ holds and hence $F_1 \cap H$ is $\hat{\omega}$-closed in $(H,\tau|H)$.

(c) Let $V \in \hat{\omega}O(H,\tau|H)$. We see that $H \setminus V \in \hat{\omega}C(H,\tau|H) \subseteq \hat{\omega}C(X,\tau)$ (cf. (i)(a)). By the general property (iv)* which is used in Proof of 6.9(iii)', it is shown that $(X \setminus V) \cap H \in \hat{\omega}C(X,\tau)$ and hence $X \setminus [(X \setminus V) \cap H] \in \hat{\omega}O(X,\tau)$. Put $V_1 := X \setminus [(X \setminus V) \cap H]$. Then, we have $V = V_1 \cap H$ holds, where V_1 is $\hat{\omega}$-open in (X,τ); and so $V \in \hat{\omega}O(X,\tau)|H$.

(ii) Let $U \in \alpha\hat{g}O(H,\tau|H)$ such that $F \subseteq U$. We claim that $acl(F) \subseteq U$ holds (cf. Definition 3.1). Indeed, by Theorem 6.27(ii)(ii-2) for the set U, there exists a set $U_1 \in \alpha\hat{g}O(X,\tau)$ such that $U = U_1 \cap H$. Since F is a $\hat{\omega}$-closed subset of (X,τ) such that $F \subseteq U_1$, we have $acl(F) \subseteq U_1$ and hence $acl(F) \cap H \subseteq U_1 \cap H = U$. By Theorem 6.17(i) (cf. Theorem 6.24), it is shown that $acl_H(F) \subseteq U$. Therefore, F is a $\hat{\omega}$-closed subset of $(H,\tau|H)$. $\square$

(V) Proof of Theorem 6.2 and Theorem 6.3.

Proof of Theorem 6.2. Suppose that H is preopen and closed and $B_1 \subset X$ in (X,τ). Let $x \in \hat{\omega}cl_H(B_1 \cap H)$ and F_1 be any $\hat{\omega}$-closed set of (X,τ) such that $B_1 \subseteq F_1$. We claim first that $x \in F_1$. By Theorem 6.30(i)(a) for the set $F_1 \subseteq X$, it is shown that $F_1 \cap H$ is $\hat{\omega}$-closed in $(H,\tau|H)$. Moreover, since $B_1 \cap H \subseteq F_1 \cap H$ and $x \in \hat{\omega}cl_H(B_1 \cap H)$, we can show that $x \in F_1 \cap H$ holds (cf. Definition 6.1(ii)). Namely, we have that $x \in F_1$ for any $\hat{\omega}$-closed set F_1 of (X,τ) such that $B_1 \subseteq F_1$, i.e., $x \in \hat{\omega}cl(B_1)$ (cf. Definition 5.1(ii)). We see secondly that $x \in H$, because $x \in \hat{\omega}cl_H(B_1 \cap H) \subseteq H$. Thus, finally we have that $x \in \hat{\omega}cl(B_1) \cap H$ for any point $x \in \hat{\omega}cl_H(B_1 \cap H)$. Therefore, we see that $\hat{\omega}cl_H(B_1 \cap H) \subseteq H \cap \hat{\omega}cl(B_1)$. $\square$

Proof of Theorem 6.3. Suppose that H is preopen and closed in (X,τ) and $B_1 \subseteq X$. Let $x \in \hat{\omega}cl(B_1) \cap H$ and F be any $\hat{\omega}$-closed set of $(H,\tau|H)$ such that $B_1 \cap H \subseteq F$. It is noted that $F \subseteq H$. We claim that $x \in F$. Indeed, by Theorem 6.30(i)(a) for the set $F \subset H$, it is shown that $F \in \hat{\omega}C(X,\tau)$. Moreover, we have that $F \cup (X \setminus H) \in \hat{\omega}C(X,\tau)$ (cf. Theorem 4.11), because it is shown that $X \setminus H \in \hat{\omega}C(X,\tau)$ (cf. Theorem 6.24(ii), Remark 3.13), and $B_1 \subseteq F \cup (X \setminus H)$. Since $x \in \hat{\omega}cl(B_1)$, we have $x \in F \cup (X \setminus H)$ and so $x \in F$ (cf. Definition 5.1(ii)). Therefore, we have that $x \in \hat{\omega}cl_H(B_1 \cap H)$ (cf. Definition 6.1(ii)). $\square$

References

[1] M.E.Abd El-Monsef, S.N.El-Deeb and R.A.Mahmoud, *β-open sets and β-continuous mappings*, Bull. Fac. Sci., 12(1983), 77-90.

[2] M.E.Abd El-Monsef, S.Rose Mary and M.Lellis Thivagar, *On $\alpha\hat{G}$-closed sets in topological spaces*, Assiut University Journal of Mathematics and Computer Science, 36(2007), 43-51.

[3] A.A.Allam, A.M.Zahran and I.A.Hasanein,*On almost continuous, δ-continuous and set connected mappings*, Ind. J. Pure Appl. Math., 18(11) (1987), 991-996.

[4] D.Andrijević,*Semi-preopen sets*, Mat. Vesnik, 38 (1986),24-32.

[5] S.P.Arya and T.M.Nour,*Chracterizations of s-normal spaces*, Indian J. pure appl.Math. 21(8) (1990),717-719.

[6] P.Bhattacharyya and B.K.Lahiri,*Semi-generalized closed sets in topological spaces*, Indian Math. 29(1987),376-382.

[7] M.Caldas, S.Jafari and M.Kovar,*Some properties of θ-open sets*, Divulgaciones Mathematicas, 12(2)(2004),161-169.

[8] J.Cao, M.Ganster, C.Konstadilaki and I.L.Reilly,*On preclosed sets and their generalizations*, Houston J.Math., 28(2004),771-719.

[9] C.Chattopadhyay,*On strongly pre-open sets and a decomposition of continuity*, Matematnykn Bechnk,57(2005), 121-125.

[10] R.Devi, K.Bhuvaneswari and H.Maki,*Weak form on $g\rho$-closed sets, where $\rho \in \{\alpha, \alpha^*, \rho^{**}\}$, and digital planes*, Mem. Fac. Sci. Kochi Univ. Ser.A.(Math.),25(2004),37-54.

[11] J.Dontchev,*On generalising semi-pre open sets*, Mem. Fac. Sci. Kochi Univ. Ser.A.(Math.),16(1995),35-48.

[12] J.Dontchev, H.Maki,*On sg-closed sets and semi-λ-closed sets*, Questions Answers Gen. Topology, 15(1997), 269-266.

[13] J.N.El-Deeb, I.A.Hasanein, A.S.Mashhour and T.Noiri,*On p-regular spaces*, Bull. Math. de la Soc. Sci. Math. de la R.S. de Roumanie, 27(75)(4) (1983), 311-315.

[14] E.Ekici, *A note on a-open sets and e*-open sets*, Filomat, 22(1)(2008), 89-96.

[15] E.Ekici, *On a-open sets,A*-sets and decompositions of continuity and super-continuity*, Annales Univ. Sci. Budapest,Rolando Eotvos,Sect.Math.,51(1)(2008), 39-51.

[16] S.Jafari, M.Lellis Thivagar and N.Rebecca Paul , *Remarks on $\widetilde{g}\alpha$-Closed Sets in Topological spaces*, International Mathematical Forum,5(2010),no.24,1167-1178.

[17] D.Jankovic and I.L. Reilly, *On semiseparation properties*, Indian J. Pure Appl. Math., 16(9)(1985), 957-964.

[18] G.Lo Faro, *On some properties of α-open sets*,Atti Sem. Mat. fis.Univ.Modena (Italian English summary),29(2)(1980),242-252.

[19] N.Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer.Math.Monthly, 70(1963), 36-41.

[20] N.Levine,*Generalized closed sets in topology*, Rend. Circ. Mat. Palermo, 19(2)(1970),89-96.

[21] P.E.Long and L.L.Herrington,*Basic properties of regular-closed functions*, Rend. Circ. Mat. Palermo, 27(1978),20-28.

[22] H.Maki, J.Umehara and T.Noiri, *Every topological space is pre-$T_{\frac{1}{2}}$*, Mem.Fac.Sci.Kochi Univ Ser. A,Math 17(1996),33-42.

[23] H.Maki, R.Devi and K.Balachandran,*Generalized α-closed sets in Topology*, Bull. Fukuoka Univ.Ed.Part-III,42(1993),13-21.

[24] H.Maki, R.Devi and K.Balachandran,*Associated topologies of generalized α-closed sets and α-generalized closed sets*, Mem. Fac. Sci.Kochi Univ.Ser.A.(Math.),15(1994),51-63.

[25] S.R.Malghan and G.B.Navalagi,*Almost p-regular, p-completely regular and almost-p-completely regular spaces*, Bull.Math. de la Soc. Sci. Math. de Roumanie, 34(82) (1990), 47-53.

[26] A.S.Mashhour, M.E.Abd El-Monsef and S.N.El-Deeb,*On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt,53(1982), 47-53.

[27] A.S.Mashhour, I.A.Hasanein and S.N.El-Deeb,*α-continuous and α-open mappings*,Acta Math.Hung.,41(3-4)(1983),213-218.

[28] A.A.Nasef and H.Maki,*On some maps concerning gp-closed sets and related groups*, Sci. Math. Jpn, 71(2010),55-81;(Online e-2009, 649-675).

[29] O.Njåstad,*On some classes of nearly open sets*, Pacific J. Math., 15(1965), 961-970.

[30] T.Noiri,*On almost open mappings*, Mem. Miyakonojo Tech. Coll., 7(1970), 167-171.

[31] T.Noiri, *A generalization of perfect functions*, J. London Math. Soc. (2), 17(1978),540-544.

[32] T.Noiri, A.S.Mashhour, I.A.Hasanein and S.N.El-Deeb,*A note on S-closed subspaces*, Math.Seminar Notes,10(1982),431-435.

[33] T.Noiri, H.Maki and J.Umehara,*Generalized preclosed functions*, Mem. Fac. Sci. Kochi Univ. (Math.), 19(1998), 13-20.

[34] V.Pipitone and G.Russo,*Spazi semiconness e spazi semiaperti*, Rend.Circ.Matem.Palermo, (2)24(1975),273-285.

[35] S.Raychudhuri and M.N.Mukherjee,*On δ-almost continuity and δ-preopen sets*, Bull.Inst.Math.Acad.Sinica, 21(1993),357-366.

[36] P.Sundaram and M.Sheik John,*Weakly closed sets and weak continuous maps in topological spaces*, Proc.82^{nd} Indian Sci. Cong.(1995),49.

[37] M.K.R.S.Veera Kumar, $\hat{g}$-closed sets in topological spaces, Bull.Allah. Math. Soc, 18(2003),99-112.

[38] N.V.Veličko, *H-closed topological spaces*, Amer.Math.Soc.Transl.,78(1968), 103-118.

[39] I.Zorlutuna,*On Strong forms of Completly irresolute functions*, Chaos,Solitons and Fractals 38, (2008),970-979.

Communicated by *Kohzo Yamada*

H. MAKI:
Wakagidai 2-10-13, Fukutsu-shi, Fukuoka-ken 811-3221, JAPAN.
e-mail: `makih@pop12.odn.ne.jp`

M. LELLIS THIVAGAR:
School of Mathematics, Madurai Kamaraj University
Madurai-625021, Tamil Nadu, INDIA
e-mail: `mlthivagar@yahoo.co.in`

C. SANTHINI:
Department of Mathematics, V.V.Vanniaperumal College For Women
Virudhunagar-626001, Tamil Nadu, INDIA
e-mail: `santhinimdu@gmail.com`

M. ANBUCHELVI:
Department of Mathematics, V.V.Vanniaperumal College For Women
Virudhunagar-626001, Tamil Nadu, INDIA
e-mail: `rsanbuchelvi@gmail.com`

Scientiae Mathematicae Japonicae 77,No.1 (2014) 107-117

A MEMETIC ALGORITHM BASED ON TABU SEARCH FOR K-CARDINALITY TREE PROBLEMS

Qingqiang Guo, Hideki Katagiri

Received May 28, 2013

Abstract. A combinatorial optimization problem, namely k-Cardinality Tree Problem, is to find a subtree with exactly k edges in an undirected graph G, such that the sum of edges' weights is minimal. Since this problem is NP-hard, though many heuristic and metaheuristic methods are widely adopted, the precision of these methods is not well enough. In this paper we shall give a Memetic Algorithm based on Tabu Search for solving this problem. The crossover in Memetic Algorithm acts as a powerful diversitification strategy, which enlarges search area effectively. The experimental results show that the proposed algorithm is superior to existing algorithms both in precision and computing time. We arrive at a conclusion that a well designed hybrid metaheuristic algorithm is efficient for solving the k-Cardinality Tree Problem.

1 Introduction The k-cardinality tree problem (kCTP), also referred to as the k-minimum spanning tree problem, is a combinatorial optimization problem. Let $G = (V, E)$ be an undirected graph, which is made up by connecting a set of vertices V and edges E. Each edge $e \in E$ is attached with a nonnegative value w_e, called a weight. The goal of this problem is to find an acyclic and connected subset with exactly k ($k \leq |V| - 1$) edges of which the total weight is minimized. The subset matching these conditions must form a tree, which we call a k-cardinality tree, denoted by T_k. The problem is mathematically formulated as follows:

$$\begin{aligned} \text{minimize} \;\; & w(T_k) = \sum_{(u,v)\in T_k} w_{(u,v)} \\ \text{subject to} \;\; & T_k \in \mathcal{T}_k \end{aligned}$$

where $u \in V$, $v \in V$, $(u, v) \in E$, T_k is the edges set of a tree, and $\mathcal{T}_k$ is a set containing all feasible solutions in graph G. Figure 1 shows an example of a k-cardinality tree in a connected graph.

Owing to its outstanding combinatorial optimal properties for solving real-world decision making problems, kCTP has been applied in many fields, such as facility layout [1], matrix decomposition [2], telecommunication [3], and image processing [4].

Previous researches have suggested that kCTP is an *NP*-hard problem [5]. Incidentally, this problem can be polynomially solved in two cases. One is that there are only two distinct weights in a graph [5], and the other is that a graph is given as a tree [9].

In the past few years, many algorithms have been proposed to solve kCTP. Quintao *et al.* [6] proposed two integer programming formulations, Multiflow Formulation and a formulation based on the Miller-Tucker-Zemlin constraints, for solving kCTP. Since kCTP is NP-hard, a lot of approximation algorithms were proposed to find near-optimal solutions in polynomial time. At first, an $O(\sqrt{k})$-approximation algorithm for the vertex-weighted

2010 *Mathematics Subject Classification.* 90C27, 90C35, 90C59.
Key words and phrases. k-cardinality tree problem, Tabu search, Memetic algorithm, Combinatorial optimization, Hybrid algorithm.

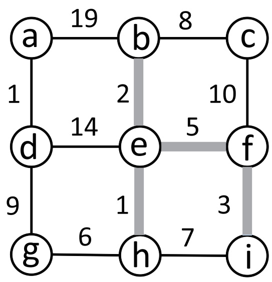


Figure 1: A 4-cardinality tree in a connected graph. The weights on edges are shown and the edges in the 4-cardinality tree are shaded. It's total weight is 11.

problem on grid graphs was proposed by Woeginger [1]. Recently, a polynomial time 2-approximation algorithm for finding the minimum tour that visits k vertices was proposed by Garg [7]. Besides methods mentioned above, various types of heuristic and metaheuristic methods have also been proposed. Heuristics based on greedy strategy and Dynamic Programming (DP) were introduced by Ehrgott *et al.* [8]. A heuristic based on memetic search was proposed in [16], but no computational results were given. In 2007, Blum [9] proposed an improved dynamic programming approach: after a minimum spanning tree for a given graph is obtained, DP is applied in order to obtain an optimal subtree with k edges from the spanning tree. This algorithm has been proved to be efficient even for problems of large size.

Concerning metaheuristics, Tabu Search (TS), Evolutionary Computation and Ant Colony Optimization (ACO) for solving kCTP were studied in [10]. It shows that the performances of these metaheuristics depend on the characteristics of the tackled instances, such as the graph size, degree, and cardinality (the value of k). Recently, a hybrid algorithm based on TS and ACO was constructed by Katagiri *et al.* [11]. Their experimental results using benchmark instances have demonstrated that the hybrid algorithm provides a better performance with solution accuracy over existing metaheuristics.

However, those approaches mentioned above may not be effective in some cases, especially for problems with large size graphs. That is because complexity of the problem increases significantly with size of the graph. In this paper, we present a new MA based on TS for solving the kCTP. The crossover acts as a powerful diversitification strategy, which enlarges search area effectively. The simplified TS brings us an optimal solution in each search area. Nothing else matches its balance of diversity and centralization of solutions. Numerical results show that the proposed algorithm is competitive to classical algorithms from the viewpoint of solution accuracy and computing time.

The remainder of this paper is organized as follows. In Section 2, we present in detail how to generate an initial solution, i.e., k-cardinality tree. In Section 3, we describe the structure of the proposed MA. Section 4 gives a simplified TS. Our experimental results and analysis are described in Section 5. Finally, we conclude the paper in Section 6.

2 Growing a k-cardinality tree The algorithm we consider here uses a semi-greedy approach to solve problems. It grows a k-cardinality tree one edge at a time. Let T be a subset of a k-cardinality tree T_k. We call an edge a $safe\ edge$ if T is still a tree after being added with it. More specifically, it is an edge, one of its vertices belongs to tree T and the other does not. Firstly, a vertex is selected randomly to be the first component of tree T. Then in each step, one safe edge should be added to T until there are k edges in the tree T.

To obtain various k-cardinality trees, a real number $p \in (0,1)$ is generated randomly at each step. If $p \leq p'$, the edge with smallest weight in safe edges will be selected and added to T, else one edge would be selected randomly from safe edges. The value p' determines the range of the heuristic bias. In an extreme case of $p' = 1$, at each step the edge added to T is the best edge in safe edges, thus the construction would be equivalent to Prim's algorithm. It tends to reach k-cardinality trees with a smaller objective function value, but these trees tend to be less diversified. On the contrary, in case $p' = 0$, a k-cardinality tree would be constructed randomly. In this case, it would not be a good initial solution for further searching. It is expected to attain a good balance between the goodness of initial solutions and their diversity by a proper value p'. In this research, we determine the value $p' = 0.85$ due to the results of preliminary numerical experiments.

The pseudo-code is shown in the following:

Growing a k-cardinality tree
T⇐ select one vertex randomly
while k-cardinality tree is not completed **do**
  List ← generate list of safe edges
  p ← generate a value randomly in (0,1)
  if $p \leq p'$
    (u, v) ← an edge with minimal weight in List of safe edges
    T← T ∪ (u, v)
  else
    (u, v) ← an edge randomly selected form List of safe edges
    T← T ∪ (u, v)
  Update the List of safe edges
end while

3 Memetic Algorithm Traditional Evolutionary Computation (eg. Genetic Algorithm) has been applied widely to solve optimization problems because of their good search abilities. However, they may not be efficient to some problems that contain many local optima. For example, it seems difficult to reach the best solution x^* by Evolutionary Computation directly in Figure 2. However, it is easy to find the best solution by local search if the search starts from B. In our study, MA is used as a diversification strategy to reach B easily. Then TS is applied to find optimal solution efficiently.

MA was firstly introduced by Moscato in 1989 [15]. It has both merits of Evolutionary Computation and local search. In this paper, we present a new MA based on TS for solving the kCTP. The core idea is that vertices in a feasible solution (k-cardinality tree) with a good objective function value are usually good components for constructing an optimal solution. Note that a configuration is a list of vertices of a feasible solution. Firstly, we pay attention to the diversity of configurations in each generation. When generating a new generation, repeated configurations should be gotten rid of from the population and the space would be filled up with new configurations. Secondly, to enlarge the search area, crossover is applied to combine all vertices of two configurations and returns a feasible solution with a good objective function value. Finally, to find the optimal solution, TS is applied to each feasible solution generated by crossover. Moreover, to enhance the quality of initial population, one configuration of initial population is generated by Dynamic Programming [9].

The pseudocode of proposed MA is shown in the following:

Memetic Algorithm
P:= Generating Initial Population

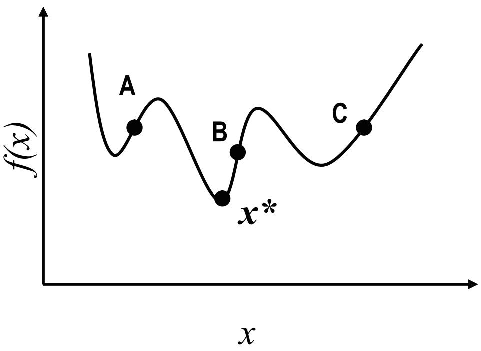


Figure 2: A Minimization Problem

while stop criterion not satisfied **do**
$P' :=$ Crossover (P)
$P' :=$ Renew (P')
$P'' :=$ Tabu Search (P')
$P :=$ Renew (P'')
end while
return the best k-cardinality tree in P,

where P means the population of configurations. Renew $(P) := P/\ P_{repeated} \cup P_{new}$. $P_{repeated}$ and P_{new} are repeated configurations and configurations generated in the way of *section* 2, respectively.

Generating Initial Population The first configuration is obtained by DP, originally introduced in [9]. It has a good objective function value and is considered to be able to improve the quality of the population P. Other configurations in P are generated under the procedure described in *section* 2. Additionally, in order to make sure that the structures of all the individuals are not the same, we compared the newly generated configuration (a list of vertices) with the existing ones. If they are reduplicate, new configuration should be regenerated with $p' := 1 - p'$ momentarily.

Stopping criteria of Memetic Algorithm We define a generation as an *idle generation* if the best objective function value is not improved in that generation. MA stops if idle generation occurs continuously for several times.

Crossover In our study, *crossover*, usually adopted in Evolutionary Computation, is applied for generating new configuration. It enlarges the explored domain, so that the search can escape from local optima easily. The crossover operator is completed by the following two procedures:

Generate spanning tree based on two configurations. Two individuals in P are considered to be parents T_k^C and T_k. If T_k and its cross partner T_k^C have at least one common vertex, a vertex set $V(G^C)$ is defined as: $V(G^C) = V(T_k) \bigcup V(T_k^C)$. Otherwise, edges and vertices should be added to T_k until at least one common vertex is found, by the procedure we described in *section* 2 with $p' := 1 - p'$ momentarily. A spanning tree T^{SP}, which contains all vertices of $V(G^C)$, is constructed under the procedure we introduced in *section* 2.

Generate k-cardinality tree from T^{SP}. DP [9] is applied to the T^{SP} for finding out the best k-cardinality tree. Since DP is very efficient, the crossover operator will help us get a feasible solution with a good objective function value in a very short computing time.

4 Tabu Search After crossover, each offspring should be further improved by Tabu Search (TS). TS, firstly proposed by Glover *et al.* [13] [14], is one of the mostly used metaheuristics for solving combinatorial optimization problems. The most important characteristic of TS is that it uses a concept of *memory* to control movements via a dynamic list of forbidden movements. To be more specific, the solutions which have been searched will be "tabu" (forbidden) to visit for a while. This mechanism allows TS to intensify or diversify its search procedure in order to escape from local optima. Incidentally, TS has also been proved to be effective in solving kCTPs [10].

4.1 Length of Tabu List The core procedure of TS is to forbid some moves based on memory in order to enlarge the search area. In the proposed algorithm, the "tabu" (forbiddance) is applied to edges that have been added or deleted to the k-cardinality tree recently. Tabu lists are used as a memory to record edges that should be forbidden to be added or deleted. $InList$ and $OutList$ are adopted to keep the records of removed edges and added edges, respectively. Tabu tenure, which generally depends on the length of tabu lists, is a period during which it forbids edges in the tabu lists from being added or deleted. The lengths of tabu lists are not dynamically changed in the proposed algorithm, since the computing time explodes as the length increases. The length of tabu list (tl) is defined as follows:

$$tl := \min\left\{ \left\lfloor \frac{|V|}{20} \right\rfloor, \frac{|V|-k}{4}, \frac{k}{4} \right\},$$

where $|V|$ is the number of vertices in G, k is the value of cardinality.

4.2 Aspiration Criterion The "tabu" mechanism, which forbids some of the moves to be employed, helps the algorithm avoid falling into local optima. However, this mechanism may also forbid a move that may reach the best solution. In order to avoid such a situation, a procedure called *aspiration criterion* is used in the proposed algorithm. That is, if $f(T_k^{new}) < \gamma_e$ is satisfied, the movement will be acceptable even if edge e is included in $InList$ or $OutList$. Parameters γ_e called *aspiration level criterion* are given to all of edges and are initially set to be:

$$\gamma_e = \begin{cases} f(T_k^{cur}) & e \in E(T_k^{cur}) \\ \infty & e \notin E(T_k^{cur}). \end{cases}$$

For each explored solution T_k, γ_e is updated as $\gamma_e := f(T_k)$ for each $e \in E(T_k)$.

4.3 Local Search The basic ingredient of TS is local search. Local search is often conducted via some move operators. A move from the current solution to the candidate solution is only performed if the objective function value could be improved. In this study, we propose an efficient local search for solving the k-cardinality tree problem. The basic idea is to translate the current solution T_k to a new one, by exchanging one vertex in T_k with a vertex not in it. Correspondingly, the edges which connect those vertices in T_k should also be updated.

In detail, we define that $V(T_k)$ denotes the vertex set of tree T_k. Neighbourhood vertex set of T_k was defined as: $V_{ADD}(T_k) := \{v|(v,u) \in E, v \notin V(T_k), u \in V(T_k)\}$. Edge (v,u) is called a connecting edge. $V_{RMV}(T_k)$ denotes the the vertex set of tree T_k, which would be removed from T_k after constructing a T_{k+1}.

The procedure of the local search is shown as follows:

while $V_{ADD}(T_k) \neq \emptyset$ **do**
  Growing T_{k+1}
    while $V_{RMV}(T_k) \neq \emptyset$ **do**
      Reconstructing T_k^{new}
      Updating $T_k^{localbest}$
    end while
end while
Updating T_k,
where T_k^{new} is the newly constructed k-cardinality tree.

Growing T_{k+1} **:** A tree T_{k+1} is constructed by adding vertex $v_{add} \in V_{ADD}(T_k)$ as well as its least weight edge to tree T_k. At the same time, v_{add} is deleted from the neighbourhood vertex set $V_{ADD}(T_k)$.

Reconstructing T_k^{new} **:** To reconstruct a k-cardinality tree T_k^{new}, one vertex, called v_{rmv}, should be removed from T_{k+1}. Correspondingly, the edge (or edges) connecting to v_{rmv} is (are) also removed from T_{k+1}. If the v_{rmv} is a leaf vertex (it connects to the tree by only one edge), we obtain a new T_k without further procedure. If a set of the remaining vertices and edges becomes a forest, Kruskal's Aglorithm would be applied to connect the forest into a tree T_k^{new}. Then v_{rmv} will be deleted from the neighbourhood vertex set $V_{RMV}(T_k)$.

Updating $T_k^{localbest}$ **:** If T_k^{new} could be constructed successfully and is better than the local best k-cardinality tree $T_k^{localbest}$, the latter should be updated by T_k^{new}.

Updating T_k **:** If the objective function value of local best tree $T_k^{localbest}$ is better than current tree T_k, the latter should be updated by $T_k^{localbest}$.

5 Experimental study To evaluate the efficiency of MA, we compared the proposed method (MA) with three state-of-the-art existing algorithms. One algorithm is a Hybrid algorithm (HybridK) based on the TS and ACO, originally proposed by Katageri *et.al* [11]. The other two algorithms are tabu search algorithm (TSB) and ant colony optimization (ACOB), both of which are introduced by Blum *et al.* in [10].

We use C as the programming language and compile the program with C-Compiler: Microsoft Visual C++ 2010 Express. MA is tested 10 runs on a PC with Intel Core i7 2.8 GHz CPU (the multi-processor did not process in parallel) and 8 GB RAM under Microsoft Windows 7. The best, mean and worst objective function values and computing time are obtained. Accordingly, the results of existing algorithms are referred to [11]. They executed each method for 30 runs under the condition that TimeLimit = 300 (s). All the metaheuristic approaches were tested on a PC with Celeron 3.06 GHz CPU and RAM 1 GB under Microsoft Windows XP.

The experiments were applied to several famous instances [12] and instances proposed in [11], respectively. Tables 1-6 show the results of these experiments. $|V|$, $|E|$, and $\bar{d}(v)$ indicate the number of vertices, the number of edges and the average number of edges a vertex connecting in a graph, respectively. k denotes the cardinality of kCTP. BKS means the best known solutions which have been obtained by Blum and Blesa through their tremendous experiments [12]. The rows headed "Best", "Mean" and "Worst" provide the best, average, and the worst objective function values, respectively. "–" indicates that the algorithm do not derive solutions within the given time limit. Results highlighted in bold mean that this algorithm beats others. The values marked by $*$ denote that the best known solutions were updated by that algorithm. In addition, columns headed "time" provide the average computing time to reach the best solution. From Table 1 we can see that the precision of MA is not so good, even worse than ACOB in cases of the best objective function values. However, comparing Tables 1 with 2 and 3, we find that as the size of graph becomes large the performance of MA establishes total supremacy to rivals considering "Best value", "Mean value" and "Worst value". We believe that the diversification strategy based on MA enlarges the search area and leads to a better solution.

The $\bar{d}(v)$s of instances in Tables 4, 5, and 6 are larger than those of instances in Tables 1, 2 and 3. In these instances with large $\bar{d}(v)$s, the performance of MA is also outstanding, especially considering the mean objective function values. Furthermore, the MA' deviation between "Best", "Mean" and "Worst" is relatively smaller than those of other algorithms. It can be thought that the MA has a strong robustness.

In consideration of terminate condition is 300 (s) for other algorithms, the computing time of the proposed algorithm is relatively short (most of them are less then 1 second).

6 Conclusion In this paper we proposed a MA based on TS for kCTP. The proposed algorithm enhances the diversity of the configurations in each generation, which helps search escape from local optima even the size of the graph is large. The experiments applied in existing benchmark instances show that MA is able to find optimal (near-optimal) solutions for larger instances within short running time. It can be also observed that a proper combination of metaheuristics is efficient for solving kCTPs. We will do some experiments with much larger size of benchmark problems to show the effectiveness of the proposed algorithm in future.

References

[1] G. J. Woeginger, Computing maximum valued regions, *Acta Cybernet.* 10 (4), pp. 303–315, 1992.

[2] R. Borndorfer, C. Ferreira, A. Martin, Decomposing matrices into blocks, *SIAM Journal on Optimization*, Vol. 9, No. 1, pp. 236-269, 1998.

[3] N. Garg, D. Hochbaum, An $O(\log k)$ approximation algorithm for the k minimum spanning tree problem in the plane, *Algorithmica*, Vol. 18, No.1, pp. 111–121, 1997.

[4] B. Ma, A. Hero, J. Gorman, O. Michel, Image registration with minimum spanning tree algorithm, *IEEE International Conference on Image Processing*, Vancouver, CA, October 2000.

[5] R. Ravi, R. Sundaram, M. V. Marathe, D. J. Rosenkrantz, and S. S. Ravi, Spanning trees short or small, *Proc. 5th Annual ACM-SIAM symposium on Discrete Algorithms.* pp. 546–555, 1994.

[6] F. P. Quintao, A.S. da Cunha, G.R. Mateus, and A. Lucena, The k-Cardinality Tree Problem: Reformulations and Lagrangian Relaxation, *Discrete Applied Mathematics* 158, pp. 1305–1314, 2010.

[7] N. Garg, Saving an epsilon: a 2-approximation for the k-MST problem in graphs, *STOC '05 Proceedings of the thirty-seventh annual ACM symposium on Theory of computing*, pp. 396–402, 2005.

[8] M. Ehrgott, J. Freitag, H.W. Hamacher, F. MaLoli, Heuristics for the k-cardinality tree and subgraph problem. *Asia-Pacifc Journal of Operational Research* 14, pp. 87–114, 1997.

[9] C. Blum. Revisiting dynamic programming for finding optimal subtrees in trees. *European Journal of Operational Research*, 177, pp. 102–115, 2007.

[10] C. Blum, M. Blesa, New metaheuristic approaches for the edge-weighted k-cardinality tree problem, *Computers & Operations Research* 32, pp. 1355-1377, 2005.

[11] H. Katagiri, T. Hayashida, I. Nishizaki and Q. Guo, A hybrid algorithm based on tabu search and ant colony optimization for k-minimum spanning tree problems *Expert Systems with Applications*, 39, pp. 5681–5686, 2012.

[12] KCTLIB, http: //iridia.ulb.ac.be/blum/kctlib/, 2003. (access: 2010/ 6/ 31)

[13] F. Glover, Future paths for integer programming and links to artificial intelligence. *Computers and Operations Research* 5, pp. 533–549, 1986.

[14] F. Glover, M. Laguna, Tabu search. Dordrecht: Kluwer Academic Publishers; 1997.

[15] P. Moscato. On Evolution, Search, Optimization, Genetic Algorithms and Martial Arts: Towards Memetic Algorithms. *Technical Report Caltech Concurrent Computation Program*, Report. 826, California Institute of Technology, Pasadena, California, USA, 1989.

[16] M. J. Blesa, P. Moscato, F. Xhafa. A memetic algorithm for the minimum weighted k-cardinality tree subgraph problem. *Fourth Metaheuristics International Conference*, Porto, pp. 85–90, 2001.

Communicated by *Hiroaki Ishii*

Hideki Katagiri
Graduate School of Engineering, Hiroshima University
1-4-1, Kagamiyama, Higashi-Hiroshima, 739-8527 Japan
katagiri-h@hiroshima-u.ac.jp

Table 1: Results on grid graph.

Graph	k	BKS		MA	time (s)	HybridK	TSB	ACO
$\lvert V\rvert = 225$	40	695	Best	**695**	0.008	**695**	696	**695**
$\lvert E\rvert = 400$			Mean	699.6	0.029	**695.0**	696.0	695.4
$\bar{d}(v) = 3.55$			Worst	728	0.052	**695**	696.0	696.0
(bb45x5_1.gg)	80	*1552	Best	1618	0.013	**1552**	1579	1572
		(1568)	Mean	1636.9	0.029	**1565.1**	1592.7	1581.2
			Worst	1639	0.104	**1572**	1615	1593
	120	*2444	Best	2456	0.038	**2444**	2546	2457
		(2450)	Mean	2468.7	0.092	**2457.9**	2558.5	2520.3
			Worst	2477	0.154	**2465**	2575	2601
	160	*3688	Best	3701	0.027	**3688**	3724	3700
		(3702)	Mean	3714.1	0.085	**3688.0**	3724.9	3704.7
			Worst	3724	0.260	**3688**	3729	3720
	200	5461	Best	**5461**	0.032	**5461**	5462	**5461**
			Mean	**5461.0**	0.055	**5461.0**	5462.4	5469.0
			Worst	**5461**	0.154	**5461**	5463	5485

Table 2: Results on regular graph

Graph	k	BKS		MA	time (s)	HybridK	TSB	ACO
$\|V\| = 1000$	200	3308	Best	3421	0.186	3393	3438	**3312**
$\|E\| = 2000$			Mean	3423.7	0.202	3453.1	3461.4	**3344.1**
$\bar{d}(v) = 4$			Worst	3424	0.226	3517	3517	**3379**
(g1000-4-01.g)	400	7581	Best	**7600**	1.021	7659	7712	7661
			Mean	**7621.8**	1.926	7764.0	7780.2	7703.0
			Worst	**7636**	3.220	7819	7825	7751
	600	12708	Best	**12733**	0.982	12785	12801	12989
			Mean	**12746**	2.035	12836.6	12821.8	13115.6
			Worst	**12759**	4.001	13048	12869	13199
	800	19023	Best	**19033**	1.496	19099	19093	19581
			Mean	**19047.1**	3.682	19101.1	19112.6	19718.7
			Worst	**19060**	12.872	19128	19135	19846
	900	22827	Best	**22827**	0.072	**22827**	22843	23487
			Mean	22829.7	0.176	**22827.0**	22859.2	23643
			Worst	22830	1.006	**22827**	22886	23739

Table 3: Results on regular graph

Graph	k	BKS		MA	time (s)	HybridK	TSB	ACO
$\|V\| = 1000$	200	1018	Best	**1018**	0.166	1034	1036	1036
$\|E\| = 5000$			Mean	**1024.2**	0.451	1048.6	1047.3	1045.9
$\bar{d}(v) = 10.0$			Worst	**1036**	0.849	1063	1056	1056
(steind15.g)	400	2446	Best	**2448**	0.883	2469	2493	2665
			Mean	**2452.4**	1.331	2480.7	2502.5	2806.6
			Worst	**2458**	2.206	2492	2524	2928
	600	4420	Best	**4420**	0.553	4426	4442	5028
			Mean	**4420.7**	0.934	4433.0	4454.6	5398.4
			Worst	**4423**	1.772	4451	4490	5602
	800	7236	Best	**7236**	1.736	**7236**	7252	8457
			Mean	7237.8	2.356	**7237.0**	7272.8	8839.6
			Worst	7239	3.741	**7237**	7308	9006
	900	9248	Best	**9248**	0.068	9256	9283	10873
			Mean	**9248**	0.080	9256.0	9294.2	11166.3
			Worst	**9248**	0.097	9256	9304	11423

Table 4: Results on instances constructed from graph coloring problems

Graph	k	BKS		MA	time (s)	HybridK	TSB	ACO
$\|V\| = 450$	90	135	Best	**135**	0.006	**135**	**135**	**135**
$\|E\| = 8168$			Mean	**135**	0.010	135.1	135.3	135.7
$\bar{d}(v) = 36.30$			Worst	**135**	0.034	137	136	137
(le450_15a.g)	180	336	Best	**336**	0.008	**336**	337	352
			Mean	**336.5**	0.071	337	337.1	374.4
			Worst	**337**	0.268	**337**	338	419
	270	630	Best	**630**	0.163	**630**	**630**	696
			Mean	**630**	0.175	630.1	630.3	839.0
			Worst	**630**	0.196	631	633	913
	360	1060	Best	**1060**	0.014	**1060**	**1060**	1267
			Mean	**1060**	0.020	**1060.0**	1064.1	1461.2
			Worst	**1060**	0.060	**1060**	1118	1566
	405	1388	Best	**1388**	0.014	**1388**	**1388**	1767
			Mean	**1388**	0.018	**1388**	1391.1	1888.7
			Worst	**1388**	0.030	**1388**	1392	2015

Table 5: Results on new instances (1)

Graph	k		MA	time (s)	HybridK	TSB	ACO
$\|V\| = 500$	100	best	**1943**	0	**1943**	1954	**1943**
$\|E\| = 15000$		mean	**1943**	0.008	1950.5	1990.9	2022.3
$\bar{d}(v) = 60$		worst	**1943**	0.016	1966	2023	2241
	200	best	5062	0	**5037**	5063	5517
		mean	5062.8	0.066	**5047.3**	5080.4	7444.4
		worst	**5063**	0.312	5066	5221	9859
	300	best	9760	0.577	**9758**	9821	-
		mean	**9761.7**	0.716	9769.6	9922.6	-
		worst	**9763**	0.998	9795	11696	-
	400	best	**16351**	0.015	**16351**	16373	-
		mean	**16351.0**	0.017	16363.8	16488	-
		worst	**16351**	0.031	16368	17953	-
	450	best	**20929**	0.015	**20929**	20934	-
		mean	**20929**	0.019	**20929**	20945.2	-
		worst	**20929**	0.032	**20929**	20992	-

Table 6: Results on new instances (2)

Graph	k		MA	time (s)	HybridK	TSB	ACO
$\lvert V\rvert = 500$	100	best	1306	0.171	**1294**	1319	1398
$\lvert E\rvert = 30000$		mean	1319	0.278	**1303.7**	1352.8	1743.5
$\bar{d}(v) = 120$		worst	1322	0.780	**1321**	1385	2479
	200	best	**3007**	0.250	3064	3150	4013
		mean	**3012.4**	0.700	3097.1	3934.4	6861.4
		worst	**3020**	1.154	3127	6032	9623
	300	best	**5304**	0.452	5312	5380	-
		mean	**5304.0**	0.479	5312.5	6471.9	-
		worst	**5304**	0.515	5318	8308	-
	400	best	**8582**	0.015	**8582**	8586	-
		mean	**8582**	0.017	**8582**	9540	-
		worst	**8582**	0.031	**8582**	11485	-
	450	best	**10881**	0.015	**10881**	10882	-
		mean	**10881**	0.020	**10881**	11300.4	-
		worst	**10881**	0.032	**10881**	13570	-

THE EULER CHARACTERISTIC AND THE EULER-POINCARÉ FORMULA FOR C^*-ALGEBRAS

Takahiro Sudo

Received April 11, 2013

Abstract. We revisit and study the Euler characteristic for C^*-algebras and obtain the Euler-Poincaré formula for C^*-algebras, as a noncommutative version of the classical Euler-Poincaré formula for spaces.

1 Introduction We first recall a famous fact in homology theory for spaces as follows. Let X be a topological space. The **Euler characteristic** $\chi(X)$ for X (in homology theory) is defined to be the alternative sum:

$$\chi(X) \equiv \chi_*(X) = \sum_{q \geq 0} (-1)^q b_q(X),$$

where $b_q(X)$ is the q-dimensional Betti number of X, that is, the $\mathbb{Z}$-rank of the free abelian direct summand of the q-dimensional (finitely generated) (integral singular) homology group $H_q(X)$ of X (see [3]). The **Euler-Poincaré formula** (in homology theory) is the following equality: for X a finite cell complex,

$$\chi(X) = \sum_{q \geq 0} (-1)^q \alpha_q(X),$$

where $\alpha_q(X)$ is the cardinal number of the set of all q-dimensional cells of X, which may be called the q-dimensional **cell number** of X.

In this paper, especially, we consider several noncommutative algebraic versions of the Euler-Poincaré formula for spaces, namely, the Euler-Poincaré formula(e) for C^*-algebras. For this, we revisit and study the basic and key properties of the Euler characteristic for C^*-algebras, some of which have been already considered by the author [5], but added with some full proofs, extended considerations, and some new attempts. There are five sections after this introduction as follows: 2 The commutative C^*-algebra case; 3 The C^*-algebra extensions; 4 The C^*-algebra crossed products; 5 The C^*-algebra pull-backs and push-outs; 6 KK-theory case. As a note, the equivariant K-theory case and the equivariant KK-theory case, viewed as the K-theory and KK-theory cases for crossed products (in some sense), are also considered in [5], but these cases are not considered in this paper.

Refer to [1] or [7] about the K-theory for C^*-algebras and to [1] about the KK-theory for C^*-algebras

2010 *Mathematics Subject Classification.* Primary 46L05, 46L80, 19K14.
Key words and phrases. C*-algebra, K-theory, Euler characteristic, Euler-Poincaré formula, KK-theory.

2 The commutative C^*-algebra case Let X be a compact Hausdorff space. Denote by $C(X)$ the C^*-algebra of all continuous, complex-valued functions on X with supremum norm and point-wise multiplication. Let $K^0(X)$, $K^1(X)$ be the topological K-theory groups for X. Let $K_0(C(X))$, $K_1(C(X))$ be the K-theory groups for the C^*-algebra $C(X)$, both of which are abelian.

The **Euler characteristic** $\chi(C(X))$ of $C(X)$ is defined by the following difference:

$$\begin{aligned}\chi(C(X)) \equiv \chi_*(C(X)) &= \mathrm{rank}_{\mathbb{Z}} K_0(C(X)) - \mathrm{rank}_{\mathbb{Z}} K_1(C(X)) \\ &\equiv b_0(C(X)) - b_1(C(X)) \in \mathbb{Z} \cup \{\pm\infty\},\end{aligned}$$

where $\mathrm{rank}_{\mathbb{Z}} K_j(C(X)) = b_j(C(X))$ denotes the $\mathbb{Z}$-rank of the free abelian direct summand (i.e. the free abelian part) of $K_j(C(X))$, and we may call it the j-th **Betti number** of $C(X)$

We define the **Euler characteristic** of X in cohomology theory by the alternative sum:

$$\chi^*(X) = \sum_{q\geq 0} (-1)^q b^q(X),$$

where $b^q(X) = \mathrm{rank}_{\mathbb{Z}} H^q(X,\mathbb{Q})$ with $H^q(X,\mathbb{Q})$ the q-th (Alexandar or Čech) cohomology group of X with coefficients in $\mathbb{Q}$ of rational numbers.

The following should be certainly known ([5]):

Proposition 2.1. *Let X be a compact Hausdorff space. Then*

$$\chi^*(X) = \chi(C(X))$$

provided that both sides exist.

Proof. The Chern character is the isomorphism from topological K-theory groups tensored wtih $\mathbb{Q}$ to direct sums of cohomology theory groups with coefficients in $\mathbb{Q}$ given by

$$\begin{aligned}\mathrm{Ch}^0 &: K^0(X)\otimes\mathbb{Q} \to \bigoplus_{n:\text{ even}} H^n(X,\mathbb{Q}) \equiv H^{ev}(X,\mathbb{Q}), \\ \mathrm{Ch}^1 &: K^1(X)\otimes\mathbb{Q} \to \bigoplus_{n:\text{ odd}} H^n(X,\mathbb{Q}) \equiv H^{od}(X,\mathbb{Q})\end{aligned}$$

(see [1]). Thus we have

$$\begin{aligned}\chi^*(X) &= \sum_{q\geq 0}(-1)^q b^q(X) \\ &= \sum_{q\geq 0}(-1)^q \dim_{\mathbb{Q}} H^q(X,\mathbb{Q}) \\ &= \dim_{\mathbb{Q}} H^{ev}(X,\mathbb{Q}) - \dim_{\mathbb{Q}} H^{od}(X,\mathbb{Q}) \\ &= \dim_{\mathbb{Q}} K^0(X)\otimes\mathbb{Q} - \dim_{\mathbb{Q}} K^1(X)\otimes\mathbb{Q} \\ &= \dim_{\mathbb{Q}} K_0(C(X))\otimes\mathbb{Q} - \dim_{\mathbb{Q}} K_1(C(X))\otimes\mathbb{Q} = \chi(C(X))\end{aligned}$$

since $K^j(X) \cong K_j(C(X))$ $(j = 0,1)$. Note that $k\otimes r = 1\otimes kr$ in $\mathbb{Z}\otimes\mathbb{Q}$ and $k\otimes r = kn\otimes\frac{r}{n} = 0\otimes\frac{r}{n} = 0$ in $\mathbb{Z}_n\otimes\mathbb{Q}$ with $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ a cyclic group, so that $\mathbb{Z}\otimes\mathbb{Q}\cong\mathbb{Q}$ and $\mathbb{Z}_n\otimes\mathbb{Q}\cong 0$, i.e., taking tensor product of K-theory groups with $\mathbb{Q}$ changes their $\mathbb{Z}$-rank to $\mathbb{Q}$-rank and kills their torsion part. □

Let X be an n-dimensional, finite cell complex with a cellular decomposition $X = \cup_{k=0}^n D_k$, where D_k is the disjoint union of k-dimensional cells $D_{k,l}$ in X with $1 \leq l \leq n_k \equiv \alpha_k(X)$. Then $C(X)$ has a corresponding composition series $\{\mathfrak{I}_j\}_{j=0}^n$ of closed ideals $\mathfrak{I}_j$ such that $\mathfrak{I}_n = C(X)$ and subquotients isomorphic to direct sums:

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong C_0(D_{n-j}) \cong \oplus_{l=1}^{n_{n-j}} C_0(D_{n-j,l})$$

for $1 \leq j \leq n$, with $\mathfrak{I}_0 = C_0(D_n)$, where $C_0(Y)$ means the C^*-algebra of all continuous complex-valued functions on a locally compact Hausdorff space Y vanishing at infinity. Note that the spectrum of $\mathfrak{I}_j$ is homeomorphic to the union $\cup_{n-j\leq k\leq n} D_k$ and that D_{n-j} is closed in the union. We define a **cellular decomposition** for $C(X)$ to be such a composition series $\{\mathfrak{I}_j\}_{j=0}^n$ of closed ideals corresponding to a cellular decomposition $\cup_{k=0}^n D_k$ of X.

3 The C^*-algebra extensions Let $\mathfrak{A}$ be a C^*-algebra. The **Euler characteristic** $\chi(\mathfrak{A})$ of $\mathfrak{A}$ (in K-theory) is defined by the following difference:

$$\begin{aligned}\chi(\mathfrak{A}) \equiv \chi_*(\mathfrak{A}) &= \mathrm{rank}_{\mathbb{Z}} K_0(\mathfrak{A}) - \mathrm{rank}_{\mathbb{Z}} K_1(\mathfrak{A}) \\ &\equiv b_0(\mathfrak{A}) - b_1(\mathfrak{A}) \in \mathbb{Z} \cup \{\pm\infty\},\end{aligned}$$

where $\mathrm{rank}_{\mathbb{Z}} K_j(\mathfrak{A}) = b_j(\mathfrak{A})$ denotes the $\mathbb{Z}$-rank of the free abelian direct summand (i.e. the free abelian part) of $K_j(\mathfrak{A})$, and we may call it the j-th **Betti number** of $\mathfrak{A}$ (cf. [5]).

The following is certainly known ([5]):

Proposition 3.1. *Let*

$$0 \longrightarrow \mathfrak{I} \xrightarrow{i} \mathfrak{A} \xrightarrow{q} \mathfrak{A}/\mathfrak{I} \longrightarrow 0$$

be a short exact sequence of C^*-algebras, where i is the inclusion map and q is the quotient map. Then

$$\chi(\mathfrak{A}) = \chi(\mathfrak{I}) + \chi(\mathfrak{A}/\mathfrak{I})$$

provided that each term is finite.

Proof. We have the following six-term exact sequence of K-theory groups:

$$\begin{tikzcd} K_0(\mathfrak{I}) \arrow[r, "i_*"] & K_0(\mathfrak{A}) \arrow[r, "q_*"] & K_0(\mathfrak{A}/\mathfrak{I}) \arrow[d, "\partial"] \\ K_1(\mathfrak{A}/\mathfrak{I}) \arrow[u, "\partial"] & K_1(\mathfrak{A}) \arrow[l, "q_*"] & K_1(\mathfrak{I}) \arrow[l, "i_*"] \end{tikzcd}$$

with i_*, q_* the induced maps from the maps i, q respectively, and ∂ the index maps. Let $l_j = b_j(\mathfrak{I})$, $m_j = b_j(\mathfrak{A})$, and $n_j = b_j(\mathfrak{A}/\mathfrak{I})$ ($j = 0, 1$). By exactness of the diagram, we see that

$$\begin{aligned} l_j &= \alpha(\partial(K_{j+1}(\mathfrak{A}/\mathfrak{I}))) + \alpha(i_*(K_j(\mathfrak{I}))) \equiv l_j' + l_j'', \\ m_j &= \alpha(i_*(K_j(\mathfrak{I}))) + \alpha(q_*(K_j(\mathfrak{A}))) \equiv m_j' + m_j'', \\ n_j &= \alpha(q_*(K_j(\mathfrak{A}))) + \alpha(\partial(K_j(\mathfrak{A}/\mathfrak{I}))) \equiv n_j' + n_j'', \end{aligned}$$

and that

$$l_j'' = m_j', \quad m_j'' = n_j', \quad n_j'' = l_{j+1}',$$

where $\alpha(G)$ means the $\mathbb{Z}$-rank of the free abelian direct summand of an abelian group G, which may be called the **Betti number** of G. Therefore, we obtain

$$\begin{aligned}
\chi(\mathfrak{A}) &= m_0 - m_1 = (m_0' + m_0'') - (m_1' + m_1'') \\
&= (l_0'' + n_0') - (l_1'' + n_1') = (l_0'' - l_1'') + (n_0' - n_1') \\
&= (l_0'' - l_1'') + (n_0' - n_1') + (l_0' - n_1'') + (n_0'' - l_1') \\
&= [(l_0' + l_0'') - (l_1' + l_1'')] + [(n_0' + n_0'') - (n_1' + n_1'')] \\
&= \chi(\mathfrak{I}) + \chi(\mathfrak{A}/\mathfrak{I}).
\end{aligned}$$

□

Remark. In fact, finiteness of two terms in the formula above implies that of the other term. Also, (plus or minus) infiniteness of one term implies that of one or two of the other terms, so that the equation is still valid even in such a case. For instance, let $\mathfrak{A} = C_0(\mathbb{Z}) = \oplus_{\mathbb{Z}}\mathbb{C}$ the infinite direct sum of $\mathbb{C}$ over $\mathbb{Z}$ and $\mathfrak{I} = C_0(2\mathbb{Z})$. Then $\mathfrak{A}/\mathfrak{I} \cong C_0(\mathbb{Z} \setminus 2\mathbb{Z})$ with $\mathbb{Z} \setminus 2\mathbb{Z}$ the complement of $2\mathbb{Z}$ in $\mathbb{Z}$ and

$$\chi(\mathfrak{A}) = +\infty = \chi(\mathfrak{I}) + \chi(\mathfrak{A}/\mathfrak{I}) = \infty + \infty.$$

Also, let $\mathfrak{A} = C_0(\mathbb{Z}) \oplus C_0(\mathbb{R} \times \mathbb{Z})$ the direct sum with $C_0(\mathbb{R} \times \mathbb{Z}) \cong C_0(\mathbb{R}) \otimes C_0(\mathbb{Z}) \equiv SC_0(\mathbb{Z})$ the suspension of $C_0(\mathbb{Z})$ and $\mathfrak{I} = C_0(\mathbb{Z})$. Then

$$\chi(\mathfrak{A}) = \infty - \infty = \chi(\mathfrak{I}) + \chi(\mathfrak{A}/\mathfrak{I})$$

if we allow the left hand side to be defined as $\infty - \infty$ which is usually undefined.

Corollary 3.2. *Let $\mathfrak{A}$ be a C^*-algebra. If $\mathfrak{A}$ has a finite composition series $\{\mathfrak{I}_j\}_{j=0}^n$ of closed ideals $\mathfrak{I}_j$, then*

$$\chi(\mathfrak{A}) = \sum_{j=0}^{n} \chi(\mathfrak{I}_j/\mathfrak{I}_{j-1})$$

with $\mathfrak{I}_{-1} = \{0\}$, provided that each Euler characteristic is finite.

Proof. Inductively, by Proposition 3.1 we obtain

$$\begin{aligned}
\chi(\mathfrak{A}) &= \chi(\mathfrak{I}_{n-1}) + \chi(\mathfrak{I}_n/\mathfrak{I}_{n-1}) \\
&= \chi(\mathfrak{I}_{n-2}) + \chi(\mathfrak{I}_{n-1}/\mathfrak{I}_{n-2}) + \chi(\mathfrak{I}_n/\mathfrak{I}_{n-1}) \\
&= \cdots\cdots = \chi(\mathfrak{I}_0) + \chi(\mathfrak{I}_1/\mathfrak{I}_0) + \cdots + \chi(\mathfrak{I}_n/\mathfrak{I}_{n-1}).
\end{aligned}$$

□

Theorem 3.3. *Let X be a finite cell complex with a cellular decomposition $X = \cup_{k=0}^n D_k$ with $D_k = \cup_{l=1}^{n_k} D_{k,l}$ as mentioned in the section 2. Let $\{\mathfrak{I}_j\}_{j=0}^n$ be the cellular decomposition for $C(X)$ corresponding to that of X. Then*

$$\chi(C(X)) = \sum_{j=0}^{n} \chi(\mathfrak{I}_j/\mathfrak{I}_{j-1}) = \sum_{k=0}^{n} (-1)^k n_k = \chi_*(X),$$

where $\mathfrak{I}_{-1} = \{0\}$.

Proof. Since $\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \oplus_{l=1}^{n_{n-j}} C_0(D_{n-j,l})$ and each $D_{n-j,l}$ is the open $(n-j)$-dimensional disk D_{n-j}, which is homeomorphic to $\mathbb{R}^{n-j}$, we see that

$$\begin{aligned} K_k(\mathfrak{I}_j/\mathfrak{I}_{j-1}) &\cong K_k(\oplus_{l=1}^{n_{n-j}} C_0(\mathbb{R}^{n-j})) \\ &\cong \oplus_{l=1}^{n_{n-j}} K_k(C_0(\mathbb{R}^{n-j})) \quad (k=0,1). \end{aligned}$$

Bott periodicity implies that if $n-j$ is even, then

$$K_k(\mathfrak{I}_j/\mathfrak{I}_{j-1}) \cong \oplus_{l=1}^{n_{n-j}} K_k(\mathbb{C}) \cong \begin{cases} \oplus_{l=1}^{n_{n-j}} \mathbb{Z} & \text{if } k=0, \\ 0 & \text{if } k=1, \end{cases}$$

and if $n-j$ is odd, then

$$K_k(\mathfrak{I}_j/\mathfrak{I}_{j-1}) \cong \oplus_{l=1}^{n_{n-j}} K_{k+1}(\mathbb{C}) \cong \begin{cases} 0 & \text{if } k=0, \\ \oplus_{l=1}^{n_{n-j}} \mathbb{Z} & \text{if } k=1. \end{cases}$$

Therefore, we obtain that if n is even, then

$$\begin{aligned} \chi(C(X)) &= \chi(\mathfrak{I}_0) + \chi(\mathfrak{I}_1/\mathfrak{I}_0) + \cdots + \chi(\mathfrak{I}_n/\mathfrak{I}_{n-1}) \\ &= n_n - n_{n-1} + \cdots + n_0 \\ &= \alpha_0(X) - \alpha_1(X) + \cdots - \alpha_{n-1}(X) + \alpha_n(X) = \chi_*(X), \end{aligned}$$

and if n is odd, then

$$\begin{aligned} \chi(C(X)) &= \chi(\mathfrak{I}_0) + \chi(\mathfrak{I}_1/\mathfrak{I}_0) + \cdots + \chi(\mathfrak{I}_n/\mathfrak{I}_{n-1}) \\ &= -n_n + n_{n-1} - \cdots - n_1 + n_0 = \chi_*(X), \end{aligned}$$

where we use the Euler-Poincaré formula for X. □

Remark. This is the Euler-Poincaré formula for commutative C^*-algebras with the cellular decomposition.

Our results Proposition 2.1 and Theorem 3.3 imply that

Corollary 3.4. *Let X be a finite cell complex. Then*

$$\chi^*(X) = \chi_*(X).$$

Remark. This implies a sort of the Euler-Poincaré formula in cohomology theory for spaces. Note that we deal with different singular homology and Ćech cohomology, not dual to each other, but their Euler characteristics are the same. Anyhow, it seems to be well known in algebraic topology.

Elementary C^*-algebras are either the $n \times n$ matrix algebras $M_n(\mathbb{C})$ over $\mathbb{C}$ $(n \geq 1)$ or the C^*-algebra $\mathbb{K}$ of all compact operators on an infinite dimensional, separable Hilbert space, either of which are simple C^*-algebras.

We say that a C^*-algebra $\mathfrak{A}$ has an n-dimensional, **cellular decomposition** with elementary C^*-algebras as constant fibers if $\mathfrak{A}$ has a finite composition series $\{\mathfrak{I}_j\}_{j=0}^n$ of closed ideals $\mathfrak{I}_j$ such that subquotients are isomorphic to direct sums of tensor products:

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \oplus_{l=1}^{n_{n-j}} (C_0(D_{n-j,l}) \otimes E_{n-j,l})$$

for $0 \leq j \leq n$, with $\mathfrak{I}_{-1} = \{0\}$ and each $E_{n-j,l}$ an elementary C^*-algebra and each $D_{n-j,l}$ the $(n-j)$-dimensional open ball, homeomorphic to $\mathbb{R}^{n-j}$. Define by

$$\alpha_p(\mathfrak{A}) = n_p$$

the cardinal number of the direct summands of the corresponding subquotient $\mathfrak{I}_{n-p}/\mathfrak{I}_{n-p-1}$, which may be called the p-dimensional **cell number** of $\mathfrak{A}$.

Theorem 3.5. *Suppose that a C^*-algebra $\mathfrak{A}$ has an n-dimensional, cellular decomposition with elementary C^*-algebras as constant fibers. Then*

$$\begin{aligned}\chi(\mathfrak{A}) &= \alpha_0(\mathfrak{A}) - \alpha_1(\mathfrak{A}) + \cdots + (-1)^{n-1}\alpha_{n-1}(\mathfrak{A}) + (-1)^n\alpha_n(\mathfrak{A}) \\ &= \sum_{p=0}^{n}(-1)^p\alpha_p(\mathfrak{A}).\end{aligned}$$

Proof. Note that

$$K_k(C_0(D_{n-j,l}) \otimes E_{n-j,l}) \cong K_k(C_0(D_{n-j,l})) \quad (k = 0, 1)$$

since $E_{n-j,l}$ are elementary C^*-algebras. Therefore, the proof is the same as that of Theorem 3.3 above. □

Remark. This is the Euler-Poincaré formula for a subclass of the class of type I C^*-algebras.

We say that a C^*-algebra $\mathfrak{A}$ has an n-dimensional, **cellular decomposition** with constant fibers with Euler characteristic finite if $\mathfrak{A}$ has a finite composition series $\{\mathfrak{I}_j\}_{j=0}^n$ of closed ideals $\mathfrak{I}_j$ such that subquotients are isomorphic to direct sums of tensor products:

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \oplus_{l=1}^{n_{n-j}}(C_0(D_{n-j,l}) \otimes \mathfrak{A}_{n-j,l})$$

for $0 \leq j \leq n$, with $\mathfrak{I}_{-1} = \{0\}$ and each $\mathfrak{A}_{n-j,l}$ a C^*-algebra with Euler characteristic finite and each $D_{n-j,l}$ the $(n-j)$-dimensional open ball, homeomorphic to $\mathbb{R}^{n-j}$. Define by

$$\alpha_p(\mathfrak{A}) = n_p$$

the cardinal number of the direct summands of the corresponding subquotient $\mathfrak{I}_{n-p}/\mathfrak{I}_{n-p-1}$, which may be called the p-dimensional **cell number** of $\mathfrak{A}$.

Theorem 3.6. *Suppose that a C^*-algebra $\mathfrak{A}$ has an n-dimensional, cellular decomposition with constant fibers $\mathfrak{A}_{n-j,l}$ with Euler characteristic finite. Then*

$$\chi(\mathfrak{A}) = \sum_{p=0}^{n}(-1)^p \sum_{l=0}^{\alpha_p(\mathfrak{A})} \chi(\mathfrak{A}_{p,l}).$$

Proof. The Künneth formula for K-theory groups of C^*-algebras (see [4] or [1]) implies that

$$\begin{aligned}&K_0(C_0(D_{n-j,l}) \otimes \mathfrak{A}_{n-j,l}) \cong \\ &[K_0(C_0(D_{n-j,l})) \otimes K_0(\mathfrak{A}_{n-j,l})] \oplus [K_1(C_0(D_{n-j,l})) \otimes K_1(\mathfrak{A}_{n-j,l})], \\ &K_1(C_0(D_{n-j,l}) \otimes \mathfrak{A}_{n-j,l}) \cong \\ &[K_0(C_0(D_{n-j,l})) \otimes K_1(\mathfrak{A}_{n-j,l})] \oplus [K_1(C_0(D_{n-j,l})) \otimes K_0(\mathfrak{A}_{n-j,l})]\end{aligned}$$

since $K_k(C_0(D_{n-j,l}))$ are torsion free. Since $D_{n-j,l}$ are homeomorphic to $\mathbb{R}^{n-j}$, if $n-j$ is even, then

$$\begin{aligned}K_0(C_0(D_{n-j,l}) \otimes \mathfrak{A}_{n-j,l}) &\cong \mathbb{Z} \otimes K_0(\mathfrak{A}_{n-j,l}) \cong K_0(\mathfrak{A}_{n-j,l}), \\ K_1(C_0(D_{n-j,l}) \otimes \mathfrak{A}_{n-j,l}) &\cong \mathbb{Z} \otimes K_1(\mathfrak{A}_{n-j,l}) \cong K_1(\mathfrak{A}_{n-j,l}),\end{aligned}$$

and if $n-j$ is odd, then

$$\begin{aligned}K_0(C_0(D_{n-j,l}) \otimes \mathfrak{A}_{n-j,l}) &\cong \mathbb{Z} \otimes K_1(\mathfrak{A}_{n-j,l}) \cong K_1(\mathfrak{A}_{n-j,l}), \\ K_1(C_0(D_{n-j,l}) \otimes \mathfrak{A}_{n-j,l}) &\cong \mathbb{Z} \otimes K_0(\mathfrak{A}_{n-j,l}) \cong K_0(\mathfrak{A}_{n-j,l}).\end{aligned}$$

Therefore, we obtain

$$\chi(\mathfrak{A}) = \sum_{p=0}^{n}(-1)^p \sum_{l=0}^{\alpha_p(\mathfrak{A})} \chi(\mathfrak{A}_{p,l}).$$

□

Recall that a C^*-algebra is said to be nuclear if its maximal and minimal tensor products with any C^*-algebra are the same, i.e., it has the unique C^*-algebra tensor product with any C^*-algebra, but there are other equivalent conditions known (see [1]).

Recall also from [1] that the bootstrap category is defined as the smallest class N of separable nuclear C^*-algebras with the following four properties: 1 The class N contains $\mathbb{C}$; 2 The class N is closed under countable inductive limits; 3 If nonzero two terms of a short exact sequence of C^*-algebras are in N, then so is the third left; 4 The class N is closed under KK-equivalence. Especially, the bootstrap category contains commutative C^*-algebras, and type I C^*-algebras, which have composition series of closed ideals with subquotients of continuous trace, such as tensor products of commutative C^*-algebras with elementary C^*-algebras, and is closed under taking crossed products by $\mathbb{R}$ or $\mathbb{Z}$.

The following is certainly known ([5]):

Proposition 3.7. *Let $\mathfrak{A}$, $\mathfrak{B}$ be C^*-algebras with $\mathfrak{A}$ nuclear and in the bootstrap category. If the K-theory groups of $\mathfrak{A}$ or $\mathfrak{B}$ are torsion free, and if $\chi(\mathfrak{A})$, $\chi(\mathfrak{B})$ are finite, then*

$$\chi(\mathfrak{A} \otimes \mathfrak{B}) = \chi(\mathfrak{A}) \cdot \chi(\mathfrak{B}),$$

where $\cdot$ means the usual multiplication.

Proof. The Künneth formula for K-theory groups of C^*-algebras implies that

$$K_0(\mathfrak{A} \otimes \mathfrak{B}) \cong [K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B})],$$
$$K_1(\mathfrak{A} \otimes \mathfrak{B}) \cong [K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B})].$$

Therefore,

$$\alpha(K_0(\mathfrak{A} \otimes \mathfrak{B})) = \alpha(K_0(\mathfrak{A}))\alpha(K_0(\mathfrak{B})) + \alpha(K_1(\mathfrak{A}))\alpha(K_1(\mathfrak{B})),$$
$$\alpha(K_1(\mathfrak{A} \otimes \mathfrak{B})) = \alpha(K_0(\mathfrak{A}))\alpha(K_1(\mathfrak{B})) + \alpha(K_1(\mathfrak{A}))\alpha(K_0(\mathfrak{B})),$$

where $\alpha(G)$ means the $\mathbb{Z}$-rank of the free abelian direct summand of an abelian group G. Therefore,

$$\begin{aligned}\chi(\mathfrak{A} \otimes \mathfrak{B}) &= \alpha(K_0(\mathfrak{A} \otimes \mathfrak{B})) - \alpha(K_1(\mathfrak{A} \otimes \mathfrak{B})) \\ &= [\alpha(K_0(\mathfrak{A})) - \alpha(K_1(\mathfrak{A}))] \cdot [\alpha(K_0(\mathfrak{B})) - \alpha(K_1(\mathfrak{B}))] \\ &= \chi(\mathfrak{A}) \cdot \chi(\mathfrak{B}).\end{aligned}$$

Note that $\mathbb{Z}^n \otimes \mathbb{Z}^m \cong \oplus^{nm}(\mathbb{Z} \otimes \mathbb{Z})$ with $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$, and $\mathbb{Z} \otimes \mathbb{Z}_n \cong \mathbb{Z}_n$. □

Remark. The finiteness condition in the statement is necessary. For instance, let $\mathbb{T}$ be the 1-dimensional torus. Let $\mathfrak{A} = C_0(\mathbb{Z})$ and $\mathfrak{B} = C(\mathbb{T})$. Then $\mathfrak{A} \otimes \mathfrak{B} \cong \oplus^\infty C(\mathbb{T})$ and $\chi(\mathfrak{A} \otimes \mathfrak{B}) = \infty - \infty$ is undefined usually, and also $\chi(\mathfrak{A}) \cdot \chi(\mathfrak{B}) = \infty \cdot 0$.

Theorem 3.8. *If a C^*-algebra $\mathfrak{A}$ has a finite composition series $\{\mathfrak{I}_j\}_{j=0}^n$ of closed ideals $\mathfrak{I}_j$ such that*

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \oplus_{l=1}^{n_{n-j}}(\mathfrak{A}_{n-j,l} \otimes \mathfrak{B}_{n-j,l})$$

with $\mathfrak{I}_{-1} = \{0\}$ and with either each $\mathfrak{A}_{n-j,l}$ or $\mathfrak{B}_{n-j,l}$ in the bootstrap category and with the K-theory groups of either $\mathfrak{A}_{n-j,l}$ or $\mathfrak{B}_{n-j,l}$ torsion free and with $\chi(\mathfrak{A}_{n-j,l})$ and $\chi(\mathfrak{B}_{n-j,l})$ finite and with $n_{n-j} = \alpha_{n-j}(\mathfrak{A})$. Then

$$\chi(\mathfrak{A}) = \sum_{p=0}^{n} \sum_{l=1}^{\alpha_p(\mathfrak{A})} \chi(\mathfrak{A}_{p,l}) \cdot \chi(\mathfrak{B}_{p,l}).$$

Proof. Combine Theorem 3.6 with Proposition 3.7. □

More generally, in fact we have

Proposition 3.9. *Let $\mathfrak{A}$, $\mathfrak{B}$ be C^*-algebras with $\mathfrak{A}$ in the bootstrap category. If $\chi(\mathfrak{A})$, $\chi(\mathfrak{B})$ are finite, then*

$$\chi(\mathfrak{A} \otimes \mathfrak{B}) = \chi(\mathfrak{A}) \cdot \chi(\mathfrak{B}).$$

Proof. The Künneth formula for K-theory groups of C^*-algebras is that the following short exact sequence holds (see [1]):

$$0 \to K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \xrightarrow{\ i\ } K_*(\mathfrak{A} \otimes \mathfrak{B}) \xrightarrow{\ q\ } \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \to 0$$

where $K_*(\cdot) = K_0(\cdot) \oplus K_1(\cdot)$ and the map i has degree zero and the map q has degree one (plus), and the sequence splits (unnaturally), i.e. it does split but without naturality for maps from the pair $(\mathfrak{A}, \mathfrak{B})$ to $(\mathfrak{A}, \mathfrak{B}')$ with $\mathfrak{B}'$ another C^*-algebra (in general), where $K_p(\mathfrak{A}) \otimes K_q(\mathfrak{B})$ and $\mathrm{Tor}_1^{\mathbb{Z}}(K_p(\mathfrak{A}), K_q(\mathfrak{B}))$ have degree $p + q \pmod 2$. Since the torsion product in the quotient has no free abelian direct summand, we have

$$\alpha(K_*(\mathfrak{A} \otimes \mathfrak{B})) = \alpha(K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B})).$$

The rest of the proof is the same as that in the proof of Proposition 3.7. □

By removing the torsion free condition of Theorem 3.8, one gets

Theorem 3.10. *If a C^*-algebra $\mathfrak{A}$ has a finite composition series $\{\mathfrak{I}_j\}_{j=0}^n$ of closed ideals $\mathfrak{I}_j$ such that*

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \oplus_{l=1}^{n_{n-j}} (\mathfrak{A}_{n-j,l} \otimes \mathfrak{B}_{n-j,l})$$

with $\mathfrak{I}_{-1} = \{0\}$ and with either each $\mathfrak{A}_{n-j,l}$ or $\mathfrak{B}_{n-j,l}$ in the bootstrap category and with $\chi(\mathfrak{A}_{n-j,l})$ and $\chi(\mathfrak{B}_{n-j,l})$ finite and with $n_{n-j} = \alpha_{n-j}(\mathfrak{A})$. Then

$$\chi(\mathfrak{A}) = \sum_{p=0}^{n} \sum_{l=1}^{\alpha_p(\mathfrak{A})} \chi(\mathfrak{A}_{p,l}) \cdot \chi(\mathfrak{B}_{p,l}).$$

Remark. This is the general Euler-Poincaré formula in K-theory of C^*-algebras (extended from the Euler-Poincaré formula for spaces) in the sense that it contains the other formulae obtained above as special cases.

Example 3.11. Let $\mathbb{T}$ be the 1-dimensional torus. We have the following short exact sequence of C^*-algebras:

$$0 \to C_0(\mathbb{R}) \to C(\mathbb{T}) \to \mathbb{C} \to 0$$

as a cellular decomposition for $C(\mathbb{T})$, where the quotient map is the evaluation map at a point of $\mathbb{T}$. Then

$$\chi(C(\mathbb{T})) = \chi(C_0(\mathbb{R})) + \chi(\mathbb{C}) = -1 + 1 = -\alpha_1(C(\mathbb{T})) + \alpha_0(C(\mathbb{T})) = 0.$$

Moreover, let $\mathbb{T}^n$ be the n-dimensional torus. Then

$$\chi(C(\mathbb{T}^n)) = \chi(\otimes^n C(\mathbb{T})) = \Pi^n \chi(C(\mathbb{T})) = 0.$$

Note that $C(\mathbb{T}^n)$ is the universal C^*-algebra generated by n commuting unitaries.

Let S^n be the n-dimensional sphere. We have the following short exact sequence of C^*-algebras:

$$0 \to C_0(\mathbb{R}^n) \to C(S^n) \to \mathbb{C} \to 0$$

as a cellular decomposition for $C(S^n)$. Then

$$\begin{aligned}\chi(C(S^n)) &= \chi(C_0(\mathbb{R}^n)) + \chi(\mathbb{C}) \\ &= (-1)^n + 1 = (-1)^n \alpha_n(C(S^n)) + \alpha_0(C(S^n)) \\ &= \begin{cases} 2 & \text{if } n \text{ even,} \\ 0 & \text{if } n \text{ odd.} \end{cases}\end{aligned}$$

Let X be a contractible compact space, such as products of closed intervals. Then $K_*(C(X))$ is trivial, so that $\chi(C(X)) = 0$. Moreover, the cone $C_0((0,1]) \otimes \mathfrak{A}$ over a C^*-algebra $\mathfrak{A}$ is a contractible C^*-algebra, so that $K_*(C_0((0,1]) \otimes \mathfrak{A})$ is trivial and $\chi(C_0((0,1]) \otimes \mathfrak{A}) = 0$. Refer to [7].

Let $\mathfrak{T}$ be the Toeplitz algebra, which is also the universal C^*-algebra generated by a proper isometry like the unilateral shift. We have the following short exact sequence:

$$0 \to \mathbb{K} \to \mathfrak{T} \to C(\mathbb{T}) \to 0.$$

Thus, since $K_0(\mathbb{K}) \cong \mathbb{Z}$ and $K_1(\mathbb{K}) \cong 0$ we obtain

$$\chi(\mathfrak{T}) = \chi(\mathbb{K}) + \chi(C(\mathbb{T})) = 1 + 0 = 1.$$

Furthermore,

$$\chi(\otimes^n \mathfrak{T}) = \Pi^n \chi(\mathfrak{T}) = 1.$$

Note that $\otimes^n \mathfrak{T}$ is the universal C^*-algebra generated by n $*$-commuting isometries.

Let $\mathcal{O}_n$ be the Cuntz algebra generated by n orthogonal isometries with the sum of their range projections equal to the identity ($2 \leq n < \infty$). Then $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z} = \mathbb{Z}_{n-1}$ and $K_1(\mathcal{O}_n) \cong \mathbb{Z}$, so that $\chi(\mathcal{O}_n) = 0$. Refer to [7].

4 The C^*-algebra crossed products By using several known facts, we consider several cases of crossed products of C^*-algebras by actions of groups, some of which are already known ([5]). Those crossed products are viewed as a sort of twisted tensor products. In the cellular decompositions defined above, one may replace their tensor products (as in Theorem 3.10) with crossed products.

Proposition 4.1. *Let $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ be the crossed product of a C^*-algebra $\mathfrak{A}$ by an action α of $\mathbb{R}$ of reals. Then*

$$\chi(\mathfrak{A} \rtimes_\alpha \mathbb{R}) = -\chi(\mathfrak{A}).$$

Proof. Use the Connes' Thom isomorphism:

$$K_j(\mathfrak{A} \rtimes_\alpha \mathbb{R}) \cong K_{j+1}(\mathfrak{A}) \quad (j = 0, 1).$$

□

Remark. If the action α is trivial, then $\mathfrak{A} \rtimes_\alpha \mathbb{R} \cong \mathfrak{A} \otimes C^*(\mathbb{R}) \cong S\mathfrak{A}$ the suspension of $\mathfrak{A}$ with $C^*(\mathbb{R})$ the group C^*-algebra of $\mathbb{R}$ of reals, isomorphic to $\mathbb{C} \rtimes \mathbb{R}$ the trivial crossed product and to $C_0(\mathbb{R})$ by the Fourier transform.

Corollary 4.2. *Let G be a simply connected, connected solvable Lie group and $C^*(G)$ be the group C^*-algebra of G. Then*

$$\chi(C^*(G)) = (-1)^{\dim G}.$$

Proof. Since G can be viewed as a successive semi-direct product by $\mathbb{R}$, we have

$$C^*(G) \cong C^*(\mathbb{R}) \rtimes \mathbb{R} \cdots \rtimes \mathbb{R}$$

a successive C^*-algebra crossed product by $\mathbb{R}$. □

Proposition 4.3. *Let $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ be the crossed product of a C^*-algebra $\mathfrak{A}$ by an action α of $\mathbb{Z}$ of integers. Then*

$$\chi(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) = 0$$

provided that both $\chi(\mathfrak{A})$ and $\chi(\mathfrak{A} \rtimes_\alpha \mathbb{Z})$ exist.

Proof. The following Pimsner-Voiculescu six-term exact sequence is known ([1]):

$$\begin{array}{ccccc}
K_0(\mathfrak{A}) & \xrightarrow{(\mathrm{id}-\alpha)_*} & K_0(\mathfrak{A}) & \xrightarrow{i_*} & K_0(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \\
\partial \uparrow & & & & \downarrow \partial \\
K_1(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) & \xleftarrow{i_*} & K_1(\mathfrak{A}) & \xleftarrow{(\mathrm{id}-\alpha)_*} & K_1(\mathfrak{A}),
\end{array}$$

where i_* and $(\mathrm{id}-\alpha)_*$ are induced maps from the inclusion map i and the map $\mathrm{id}-\alpha$ with id the identity map, respectively, and ∂ are the index maps. The same reasoning as for the six-term exact diagram for C^*-algebra extensions mentioned above (Proposition 3.1) implies that

$$\chi(\mathfrak{A}) = \chi(\mathfrak{A}) + \chi(\mathfrak{A} \times_\alpha \mathbb{Z})$$

and hence, $\chi(\mathfrak{A} \times_\alpha \mathbb{Z}) = 0$. □

Remark. If the action α is trivial, then $\mathfrak{A} \rtimes_\alpha \mathbb{Z} \cong \mathfrak{A} \otimes C^*(\mathbb{Z})$ with $C^*(\mathbb{Z})$ the group C^*-algebra of $\mathbb{Z}$, isomorphic to $\mathbb{C} \rtimes \mathbb{Z}$ the trivial crossed prdocut and to $C(\mathbb{T})$ by the Fourier transform.

Corollary 4.4. *Let G be a solvable discrete group written as a successive semi-direct product by $\mathbb{Z}$ and $C^*(G)$ be the group C^*-algebra of G. Then*

$$\chi(C^*(G)) = 0.$$

Let $\mathbb{T}_\Theta^n$ be the n-dimensional noncommutative torus generated by n unitaries U_j with the commutation relation: $U_iU_j = e^{2\pi i\theta_{ij}}U_jU_i$ for $1 \leq i,j \leq n$, where $\Theta = (\theta_{ij})_{i,j=1}^n$ is a skew-adjoint $n \times n$ matrix over $\mathbb{R}$. Then

$$\chi(\mathbb{T}_\Theta^n) = 0.$$

Proof. We have

$$C^*(G) \cong C^*(\mathbb{Z}) \rtimes \mathbb{Z} \cdots \rtimes \mathbb{Z}$$

a successive C^*-algebra crossed product by $\mathbb{Z}$. Note also that

$$\mathbb{T}_\theta^n \cong C(\mathbb{T}) \rtimes \mathbb{Z} \cdots \rtimes \mathbb{Z}$$

a successive C^*-algebra crossed product by $\mathbb{Z}$. □

Remark. Note that the C^*-algebra $\mathbb{K}$ of compact operators is written as the crossed product $C_0(\mathbb{Z}) \rtimes_\gamma \mathbb{Z}$ with the action γ the shift. Then $\chi(\mathbb{K}) = 1$ and $\chi(C_0(\mathbb{Z})) = +\infty$. Note also that the corresponding Pimsner-Voiculescu six-term exact sequence becomes

$$\begin{array}{ccccc} \oplus^\infty \mathbb{Z} & \xrightarrow{(\mathrm{id}-\alpha)_*} & \oplus^\infty \mathbb{Z} & \xrightarrow{i_*} & \mathbb{Z} \\ \partial \big\uparrow & & & & \big\downarrow \partial \\ 0 & \xleftarrow{i_*} & 0 & \xleftarrow{(\mathrm{id}-\alpha)_*} & 0 \end{array}$$

It implies that

$$\chi(C_0(\mathbb{Z})) = +\infty = \chi(C_0(\mathbb{Z})) + \chi(\mathbb{K}) = +\infty + 1.$$

In general, it holds that

Proposition 4.5. *For any crossed product $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ of a C^*-algebra $\mathfrak{A}$,*

$$\chi(\mathfrak{A}) = \chi(\mathfrak{A}) + \chi(\mathfrak{A} \rtimes_\alpha \mathbb{Z}).$$

Proof. The proof is the same as that for Proposition 4.3. Note that the equation is still valid even if both sides are not finite. □

5 The C^*-algebra pull-backs and push-outs By using several known facts, we consider the cases of pull-backs and (several) push-outs of C^*-algebras, some of which are already known ([5]). In the cellular decompositions defined above, one may replace their tensor products (as in Theorem 3.10) with the C^*-algebra pull-backs and push-outs.

Consider the following commutative pull-back diagram for C^*-algebras:

$$\begin{array}{ccc} P = \mathfrak{A}_1 \oplus_{\mathfrak{B}} \mathfrak{A}_2 & \xrightarrow{p_2} & \mathfrak{A}_2 \\ p_1 \big\downarrow & & \big\downarrow q_2 \\ \mathfrak{A}_1 & \xrightarrow{q_1} & \mathfrak{B} \end{array}$$

with

$$P = \{(a_1, a_2) \in \mathfrak{A}_1 \oplus \mathfrak{A}_2 \,|\, q_1(a_1) = q_2(a_2) \in \mathfrak{B}\}$$

a pull-back C^*-algebra, where p_1, p_2 are the canonical projections and q_1 or q_2 are surjective $*$-homomorphisms.

Proposition 5.1. *Let $P = \mathfrak{A}_1 \oplus_{\mathfrak{B}} \mathfrak{A}_2$ be a pull-back C^*-algebra. Then*

$$\chi(\mathfrak{A}_1) + \chi(\mathfrak{A}_2) = \chi(P) + \chi(\mathfrak{B}).$$

If each term is finite, then

$$\chi(P) = \chi(\mathfrak{A}_1) + \chi(\mathfrak{A}_2) - \chi(\mathfrak{B}).$$

Proof. Use the Mayer-Vietoris six-term K-theory exact diagram:

$$\begin{array}{ccccc} K_0(P) & \xrightarrow{((p_1)_*, (p_2)_*)} & K_0(\mathfrak{A}_1) \oplus K_0(\mathfrak{A}_2) & \xrightarrow{(q_1)_* - (q_2)_*} & K_0(\mathfrak{B}) \\ \big\uparrow & & & & \big\downarrow \\ K_1(\mathfrak{B}) & \xleftarrow{(q_1)_* - (q_2)_*} & K_1(\mathfrak{A}_1) \oplus K_1(\mathfrak{A}_2) & \xleftarrow{((p_1)_*, (p_2)_*)} & K_1(P) \end{array}$$

(see [1]). □

Corollary 5.2. *Let*

$$P = (\cdots((\mathfrak{A}_1 \oplus_{\mathfrak{B}_1} \mathfrak{A}_2) \oplus_{\mathfrak{B}_2} \mathfrak{A}_3)\cdots) \oplus_{\mathfrak{B}_{n-1}} \mathfrak{A}_n$$

be a successive pull-back C^*-algebra. Then

$$\chi(P) = \sum_{i=1}^{n} \chi(\mathfrak{A}_i) - \sum_{j=1}^{n-1} \chi(\mathfrak{B}_j)$$

where each term is finite.

Example 5.3. In particular, let $\mathfrak{A}_1 = C(X_1)$, $\mathfrak{A}_2 = C(X_2)$, and $\mathfrak{B} = C(X_1 \cap X_2)$ for X_1, X_2 compact Hausdorff spaces with the intersection $X_1 \cap X_2$ non-empty. Then $P \cong C(X_1 \cup X_2)$ with $X_1 \cup X_2$ the union and hence

$$\chi(C(X_1 \cup X_2)) + \chi(C(X_1 \cap X_2)) = \chi(C(X_1)) + \chi(C(X_2)).$$

It follows that

$$\chi^*(X_1 \cup X_2) + \chi^*(X_1 \cap X_2) = \chi^*(X_1) + \chi^*(X_2).$$

Note that there are the following short exact sequences:

$$0 \to C_0(X_i \setminus (X_1 \cap X_2)) \to C(X_i) \to C(X_1 \cap X_2) \to 0,$$
$$0 \to \oplus_{i=1}^{2} C_0(X_i \setminus (X_1 \cap X_2)) \to C(X_1 \cup X_2) \to C(X_1 \cap X_2) \to 0$$

Similarly and inductively, it is obtained that

$$\chi^*(X_1 \cup X_2 \cup X_3) = \sum_{j=1}^{3} \chi^*(X_j) - \sum_{1 \le i < j \le 3} \chi^*(X_i \cap X_j) + \chi^*(X_1 \cap X_2 \cap X_3)$$

and in general,

$$\begin{aligned}\chi^*(\cup_{j=1}^{n} X_n) &= \sum_{j=1}^{n} \chi^*(X_j) - \sum_{1 \le j_1 < j_2 \le n} \chi^*(X_{j_1} \cap X_{j_2}) \\ &\quad + \sum_{1 \le j_1 < j_2 < j_3 \le n} \chi^*(X_{j_1} \cap X_{j_2} \cap X_{j_3}) - \cdots + (-1)^n \chi^*(X_1 \cap X_2 \cap \cdots \cap X_n).\end{aligned}$$

where each term is assumed to be finite.

Next consider the following push-out diagram of C^*-algebras:

$$\begin{tikzcd} \mathfrak{B} \arrow[r, "i_2"] \arrow[d, "i_1"'] & \mathfrak{A}_2 \arrow[d, "j_2"] \\ \mathfrak{A}_1 \arrow[r, "j_1"] & \mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2 = O \end{tikzcd}$$

with O the amalgamated full free product C^*-algebra of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ by a common C^*-subalgebra $\mathfrak{B}$ and with the maps i_k, j_k $(k = 1, 2)$ the inclusion maps. The C^*-algebra O is defined to be the quotient C^*-algebra of the full free product C^*-algebra $\mathfrak{A}_1 * \mathfrak{A}_2$ by the closed ideal generated by the set $\{i_1(b) - i_2(b) \,|\, b \in \mathfrak{B}\}$, where $\mathfrak{A}_1 * \mathfrak{A}_2$ is the universal C^*-algebra generated by elements of $\mathfrak{A}_1$ and those of $\mathfrak{A}_2$, both of which have no relations.

Proposition 5.4. *Let $O = \mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2$ be an amalgamated full free product C^*-algebra. Suppose that there are retractions (i.e., surjective $*$-homomorphisms) τ_j from $\mathfrak{A}_j$ to $\mathfrak{B}$ $(j = 1, 2)$. Then*

$$\chi(\mathfrak{A}_1) + \chi(\mathfrak{A}_2) = \chi(O) + \chi(\mathfrak{B}).$$

If each term is finite, then

$$\chi(O) = \chi(\mathfrak{A}_1) + \chi(\mathfrak{A}_2) - \chi(\mathfrak{B}).$$

Proof. It is known under the assumption [2] that there is an isomorphism:

$$K_j(O) \cong K_j(\mathfrak{A}_1 \oplus_{\mathfrak{B}} \mathfrak{A}_2) \quad (j = 0, 1),$$

where $\mathfrak{A}_1 \oplus_{\mathfrak{B}} \mathfrak{A}_2$ is the pull back C^*-algebra associated with the retractions τ_j. See also [1] and [6]. $\square$

Remark. The assumption on the existence of retractions is somewhat strong and restrictive, so that applications are also limited to such as the cases where $\mathfrak{B}$ is zero or $\mathbb{C}$, i.e., O is the full free product or the unital full free product.

Corollary 5.5. *Let*

$$O = (\cdots((\mathfrak{A}_1 *_{\mathfrak{B}_1} \mathfrak{A}_2) *_{\mathfrak{B}_2} \mathfrak{A}_3)\cdots) *_{\mathfrak{B}_{n-1}} \mathfrak{A}_n$$

be a successive amalgamated full free product C^*-algebra. Suppose that there are retractions from both $(\cdots(\mathfrak{A}_1 *_{\mathfrak{B}_1} \mathfrak{A}_2)\cdots) *_{\mathfrak{B}_{j-1}} \mathfrak{A}_j$ and $\mathfrak{A}_{j+1}$ to the common C^*-subalgera $\mathfrak{B}_j$ $(1 \leq j \leq n-1)$. Then

$$\chi(O) = \sum_{i=1}^{n} \chi(\mathfrak{A}_i) - \sum_{j=1}^{n-1} \chi(\mathfrak{B}_j)$$

where each term is finite.

Example 5.6. Let $C^*(F_n)$ be the full group C^*-algebra of the free group F_n with n generators $(n \geq 1)$, where F_n can be viewed as the n-fold free product group $\mathbb{Z} * \cdots * \mathbb{Z}$ of $\mathbb{Z}$ Then $C^*(F_n)$ can be viewed as the unital n-successive free product C^*-algebra $(\cdots(C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T}))\cdots) *_{\mathbb{C}} C(\mathbb{T})$ with $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ by the Fourier transform. The trivial one-dimensional representations of the free product components are retractions to $\mathbb{C}$. It then follows that

$$\chi(C^*(F_n)) = n \cdot \chi(C(\mathbb{T})) - (n-1) \cdot \chi(\mathbb{C}) = 1 - n.$$

Next consider the following push-out diagram of unital C^*-algebras:

$$\begin{tikzcd}
\mathfrak{B} \arrow[r, "i_2"] \arrow[d, "i_1"'] & \mathfrak{A}_2 \arrow[d, "j_2"] \\
\mathfrak{A}_1 \arrow[r, "j_1"] & \mathfrak{A}_1 \otimes_{\mathfrak{B}} \mathfrak{A}_2 = T
\end{tikzcd}$$

with T the balanced tensor product C^*-algebra of unital C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ by a common C^*-subalgebra $\mathfrak{B}$ and with the maps i_k the inclusion maps and j_k the identity maps $(k = 1, 2)$. The C^*-algebra T is defined as the quotient C^*-algebra of the (maximal) tensor product C^*-algebra $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ by the closed ideal generated by the set $\{j_1(i_1(b)) - j_2(i_2(b)) \mid b \in \mathfrak{B}\}$, where $j_1(i_1(b)) = i_1(b) \otimes 1$ and $j_2(i_2(b)) = 1 \otimes i_2(b)$ in $\mathfrak{A}_1 \otimes \mathfrak{A}_2$.

Proposition 5.7. *Let $T = \mathfrak{A}_1 \otimes_{\mathfrak{B}} \mathfrak{A}_2$ be a balanced tensor product C^*-algebra of unital C^*-algebras with $\mathfrak{B}$ nonzero. Suppose that there are retractions (i.e., surjective $*$-homomorphisms) τ_j from $\mathfrak{A}_j$ to $\mathfrak{B}$ $(j = 1, 2)$ and that $\mathfrak{B}$ commutes with $\mathfrak{A}_1$ and $\mathfrak{A}_2$ and has the same unit with them. Then*

$$\chi(\mathfrak{A}_1) + \chi(\mathfrak{A}_2) = \chi(T) + \chi(\mathfrak{B}).$$

If each term is finite, then

$$\chi(T) = \chi(\mathfrak{A}_1) + \chi(\mathfrak{A}_2) - \chi(\mathfrak{B}).$$

Proof. It is known under the assumption [6] that there is an isomorphism:

$$K_j(T) \cong K_j(\mathfrak{A}_1 \oplus_{\mathfrak{B}} \mathfrak{A}_2) \quad (j = 0, 1),$$

where $\mathfrak{A}_1 \oplus_{\mathfrak{B}} \mathfrak{A}_2$ is the pull-back C^*-algebra associated with the retractions τ_j. □

Remark. The assumption on the existence of retractions as well as the commutativity is somewhat strong and restrictive, so that applications are also limited.

Corollary 5.8. *Let*

$$T = (\cdots((\mathfrak{A}_1 \otimes_{\mathfrak{B}_1} \mathfrak{A}_2) \otimes_{\mathfrak{B}_2} \mathfrak{A}_3) \cdots) \otimes_{\mathfrak{B}_{n-1}} \mathfrak{A}_n$$

be a successive balanced tensor product C^*-algebra. Suppose that there are retractions from both $(\cdots(\mathfrak{A}_1 \otimes_{\mathfrak{B}_1} \mathfrak{A}_2) \cdots) \otimes_{\mathfrak{B}_{j-1}} \mathfrak{A}_j$ and $\mathfrak{A}_{j+1}$ to the common nonzero C^*-subalgera $\mathfrak{B}_j$ $(1 \leq j \leq n-1)$ and that each $\mathfrak{B}_j$ commutes with both $(\cdots(\mathfrak{A}_1 \otimes_{\mathfrak{B}_1} \mathfrak{A}_2) \cdots) \otimes_{\mathfrak{B}_{j-1}} \mathfrak{A}_j$ and $\mathfrak{A}_{j+1}$ and has the same unit with them. Then

$$\chi(T) = \sum_{i=1}^{n} \chi(\mathfrak{A}_i) - \sum_{j=1}^{n-1} \chi(\mathfrak{B}_j),$$

where each term is finite.

Example 5.9. In particular, let $\mathfrak{A}_1 = C(X_1)$, $\mathfrak{A}_2 = C(X_2)$, and $\mathfrak{B} = C(Y)$ for X_1, X_2, Y compact Hausdorff spaces with surjective continuous maps $f_j : X_j \to Y$ $(j = 1, 2)$. Define

$$X_1 \times_Y X_2 = \{(x_1, x_2) \in X_1 \times X_2 \mid f_1(x_1) = f_2(x_2) \in Y\}$$

the balanced product space of X_1 and X_2 over Y. Then

$$C(X_1 \times_Y X_2) \cong C(X_1) \otimes_{C(Y)} C(X_2)$$

with the maps $g_j : C(Y) \to C(X_j)$ $(j = 1, 2)$ defined as $g_j(h) = h \circ f_j$ for $h \in C(Y)$, and note that the maps g_j becomes injective by the surjectivity of the maps f_j. Hence

$$\chi(C(X_1 \times_Y X_2)) + \chi(C(Y)) = \chi(C(X_1)) + \chi(C(X_2)).$$

It follows that

$$\chi^*(X_1 \times_Y X_2) + \chi^*(Y) = \chi^*(X_1) + \chi^*(X_2).$$

Note that there are the following short exact sequences:

$$0 \to C_0((X_1 \times X_2) \setminus (X_1 \times_Y X_2)) \to C(X_1 \times X_2) \to C(X_1 \times_Y X_2) \to 0,$$
$$0 \to C^*(C(Y)) \to C(X_1) \otimes C(X_2) \to C(X_1) \otimes_{C(Y)} C(X_2) \to 0$$

since $X_1 \times_Y X_2$ is closed in $X_1 \times X_2$ by definition, where $C^*(C(Y))$ means the closed ideal of $C(X_1) \otimes C(X_2)$ generated by the set $\{g_1(h) \otimes 1 - 1 \otimes g_2(h) \,|\, h \in C(Y)\}$. Indeed, the elements $g_1(h) \otimes 1 - 1 \otimes g_2(h)$ are zero on $X_1 \times_Y X_2$.

Similarly and inductively define the successive balanced product space

$$(\cdots(X_1 \times_{Y_1} X_2)\cdots) \times_{Y_{n-1}} X_n.$$

It is obtained that

$$\chi(C((\cdots(X_1 \times_{Y_1} X_2)\cdots) \times_{Y_{n-1}} X_n)) + \sum_{j=1}^{n-1} \chi(C(Y_j)) = \sum_{j=1}^{n} \chi(C(X_j)).$$

6 KK-theory case By using several known results, we consider the Euler characteristic of KK-theory groups for C^*-algebras, some of which are already known ([5]).

Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras. Denote by $KK^0(\mathfrak{A}, \mathfrak{B})$ and $KK^1(\mathfrak{A}, \mathfrak{B})$ the KK-theory groups of $\mathfrak{A}$ and $\mathfrak{B}$.

We define the **Euler characteristic** for a pair $(\mathfrak{A}, \mathfrak{B})$ of C^*-algebras in KK-theory to be the following difference:

$$\chi(\mathfrak{A}, \mathfrak{B}) \equiv \mathrm{rank}_{\mathbb{Z}} KK^0(\mathfrak{A}, \mathfrak{B}) - \mathrm{rank}_{\mathbb{Z}} KK^1(\mathfrak{A}, \mathfrak{B}) \in \mathbb{Z} \cup \{\pm\infty\}$$

(cf. [5]).

Proposition 6.1. *Let*

$$0 \longrightarrow \mathfrak{I} \xrightarrow{\ i\ } \mathfrak{A} \xrightarrow{\ q\ } \mathfrak{A}/\mathfrak{I} \longrightarrow 0$$

be a semi-split short exact sequence of σ-unital C^*-algebras, i.e., there is a completely positive, norm decreasing section for q. For any separable C^*-algebra $\mathfrak{D}$,

$$\chi(\mathfrak{D}, \mathfrak{A}) = \chi(\mathfrak{D}, \mathfrak{I}) + \chi(\mathfrak{D}, \mathfrak{A}/\mathfrak{I})$$

and if $\mathfrak{A}$ is separable, then for any σ-unital C^*-algebra $\mathfrak{D}$,

$$\chi(\mathfrak{A}, \mathfrak{D}) = \chi(\mathfrak{I}, \mathfrak{D}) + \chi(\mathfrak{A}/\mathfrak{I}, \mathfrak{D}).$$

Remark. If $\mathfrak{A}$ is nuclear, any short exact sequence of $\mathfrak{A}$ in the middle as above is always semi-split. This fact is the lifting theorem due to Choi and Effros (see [1]). For any C^*-algebra homomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$, the mapping cone sequence: $0 \to S\mathfrak{B} \to C_\varphi \to \mathfrak{A} \to 0$ is semi-split, where C_φ is defined to be the pull-back:

$$\begin{array}{ccc} C_\varphi = \mathfrak{A} \oplus_{\mathfrak{B}} [C_0([0,1)) \otimes \mathfrak{B}] & \xrightarrow{\ p_2\ } & C_0([0,1)) \otimes \mathfrak{B} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle q_0} \\ \mathfrak{A} & \xrightarrow{\ \varphi\ } & \mathfrak{B} \end{array}$$

with q_0 the evaluation map at 0.

Proof. The following six-term exact sequences for KK-theory groups under the respective assumptions are known ([1]):

$$\begin{array}{ccccc} KK^0(\mathfrak{D}, \mathfrak{I}) & \xrightarrow{\ i_*\ } & KK^0(\mathfrak{D}, \mathfrak{A}) & \xrightarrow{\ q_*\ } & KK^0(\mathfrak{D}, \mathfrak{A}/\mathfrak{I}) \\ \uparrow & & & & \downarrow \\ KK^1(\mathfrak{D}, \mathfrak{A}/\mathfrak{I}) & \xleftarrow{\ q_*\ } & KK^1(\mathfrak{D}, \mathfrak{A}) & \xleftarrow{\ i_*\ } & KK^1(\mathfrak{D}, \mathfrak{I}) \end{array}$$

and

$$\begin{array}{ccccc} KK^0(\mathfrak{I},\mathfrak{D}) & \xleftarrow{i_*} & KK^0(\mathfrak{A},\mathfrak{D}) & \xleftarrow{q_*} & KK^0(\mathfrak{A}/\mathfrak{I},\mathfrak{D}) \\ \downarrow & & & & \uparrow \\ KK^1(\mathfrak{A}/\mathfrak{I},\mathfrak{D}) & \xrightarrow{q_*} & KK^1(\mathfrak{A},\mathfrak{D}) & \xrightarrow{i_*} & KK^1(\mathfrak{I},\mathfrak{D}). \end{array}$$

The rest of the proof is the same as that of the K-theory case (Proposition 3.1). □

Let $\mathfrak{A}$ be a C^*-algebra. Denote by $\mathrm{Ext}_1(\mathfrak{A})^{-1} \equiv \mathrm{Ext}(\mathfrak{A})^{-1} \equiv \mathrm{Ext}(\mathfrak{A},\mathbb{K})^{-1}$ the extension group for $(\mathfrak{A},\mathbb{K})$, which is defined as the quotient of strong equivalence classes of invertible extensions E of $\mathfrak{A}$ by $\mathbb{K}$: $0 \to \mathbb{K} \to E \to \mathfrak{A} \to 0$ by the classes of trivial extensions. Set $\mathrm{Ext}_0(\mathfrak{A})^{-1} \equiv \mathrm{Ext}_0(\mathfrak{A},\mathbb{K})^{-1} = \mathrm{Ext}(S\mathfrak{A},\mathbb{K})^{-1} \cong \mathrm{Ext}(\mathfrak{A},S\mathbb{K})^{-1}$ with $S\mathfrak{A} \cong C_0(\mathbb{R}) \otimes \mathfrak{A}$ and $S\mathbb{K}$ the suspensions and set $K^j(\mathfrak{A}) = \mathrm{Ext}_j(\mathfrak{A})^{-1}$ ($j = 0,1$), called the K-homology groups of $\mathfrak{A}$. Refer to [1]. Define

$$\chi^*(\mathfrak{A}) = \mathrm{rank}_{\mathbb{Z}} K^0(\mathfrak{A}) - \mathrm{rank}_{\mathbb{Z}} K^1(\mathfrak{A}) \in \mathbb{Z} \cup \{\pm\infty\},$$

which may be called the **Euler characteristic** of a C^*-algebra $\mathfrak{A}$ in K-homology.

Set $K^*(\cdot) = K^0(\cdot) \oplus K^1(\cdot)$ and $KK^*(\cdot,\cdot) = KK^0(\cdot,\cdot) \oplus KK^1(\cdot,\cdot)$.

Proposition 6.2. *Let $\mathfrak{A},\mathfrak{B}$ be separable C^*-algebras, with $\mathfrak{A}$ in the bootstrap category, and suppose that either $K^*(\mathfrak{A})$ or $K_*(\mathfrak{B})$ is finitely generated. Then*

$$\chi(\mathfrak{A},\mathfrak{B}) = \chi^*(\mathfrak{A}) \cdot \chi_*(\mathfrak{B}),$$

provided that each term is finite.

Proof. Under the assumptions, the Rosenberg-Schochet's Künneth formula for KK-theory groups of C^*-algebras holds (see [1]):

$$0 \to K^*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \xrightarrow{\beta} KK^*(\mathfrak{A},\mathfrak{B}) \xrightarrow{\rho} \mathrm{Tor}_1^{\mathbb{Z}}(K^*(\mathfrak{A}), K_*(\mathfrak{B})) \to 0$$

where the map β has degree zero and the map ρ has degree one. The sequence splits (unnaturally). Since the torsion product in the quotient has no free abelian direct summand, we have

$$\begin{aligned} \chi(\mathfrak{A},\mathfrak{B}) &= \alpha(KK^0(\mathfrak{A},\mathfrak{B})) - \alpha(KK^1(\mathfrak{A},\mathfrak{B})) \\ &= \alpha([K^0(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \oplus [K^1(\mathfrak{A}) \otimes K_1(\mathfrak{B})]) \\ &\quad - \alpha([K^0(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \oplus [K^1(\mathfrak{A}) \otimes K_0(\mathfrak{B})]) \\ &= \alpha(K^0(\mathfrak{A}))\alpha(K_0(\mathfrak{B})) + \alpha(K^1(\mathfrak{A}))\alpha(K_1(\mathfrak{B})) \\ &\quad - \alpha(K^0(\mathfrak{A}))\alpha(K_1(\mathfrak{B})) - \alpha(K^1(\mathfrak{A}))\alpha(K_0(\mathfrak{B})) \\ &= [\alpha(K^0(\mathfrak{A})) - \alpha(K^1(\mathfrak{A}))] \cdot [\alpha(K_0(\mathfrak{B})) - \alpha(K_1(\mathfrak{B}))] \\ &= \chi^*(\mathfrak{A}) \cdot \chi_*(\mathfrak{B}). \end{aligned}$$

□

Example 6.3. Let $\mathfrak{A},\mathfrak{B}$ be C^*-algebras. We have

$$\chi(\mathfrak{A},\mathbb{C}) = \chi^*(\mathfrak{A}), \quad \chi(\mathbb{C},\mathfrak{B}) = \chi_*(\mathfrak{B})$$

with $\chi_*(\mathbb{C}) = 1$ and $\chi^*(\mathbb{C}) = 1$. Indeed, we have

$$KK^j(\mathfrak{A},\mathbb{C}) \cong K^j(\mathfrak{A}), \quad KK^j(\mathbb{C},\mathfrak{B}) \cong K_j(\mathfrak{B})$$

($j = 0,1$). Refer to [1].

Theorem 6.4. *Let $\mathfrak{A}$ be a σ-unital C^*-algebra. Suppose that $\mathfrak{A}$ has a finite composition series $\{\mathfrak{I}_j\}_{j=0}^n$ of closed ideals such that each short exact sequence:*

$$0 \to \mathfrak{I}_{j-1} \to \mathfrak{I}_j \to \mathfrak{I}_j/\mathfrak{I}_{j-1} \to 0$$

is semi-split $(0 \leq j \leq n)$*, where $\mathfrak{I}_{-1} = \{0\}$. For any separable C^*-algebra $\mathfrak{D}$ in the bootstrap category,*

$$\chi(\mathfrak{D}, \mathfrak{A}) = \sum_{j=0}^{n} \chi^*(\mathfrak{D}) \cdot \chi_*(\mathfrak{I}_j/\mathfrak{I}_{j-1}),$$

and if $\mathfrak{A}$ is separable in the bootstrap category, then for any σ-unital C^*-algebra $\mathfrak{D}$,

$$\chi(\mathfrak{A}, \mathfrak{D}) = \sum_{j=0}^{n} \chi^*(\mathfrak{I}_j/\mathfrak{I}_{j-1}) \cdot \chi_*(\mathfrak{D}),$$

provided that each term in both equations is finite.

Proof. Inductively combine Proposition 6.1 with Proposition 6.2. □

Remark. This is viewed as the Euler-Poincaré formula in KK-theory of C^*-algebras. Indeed, the first equation with $\mathfrak{D} = \mathbb{C}$ and the subquotients $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ replaced with finite direct sums of tensor product C^*-algebras corresponds to the Euler-Poincaré formula in K-theory of C^*-algebras (Theorem 3.10), under the semi-split condition.

Proposition 6.5. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ the crossed product of $\mathfrak{A}$ by an action α of $\mathbb{R}$. For any C^*-algebra $\mathfrak{B}$,*

$$\chi(\mathfrak{A} \rtimes_\alpha \mathbb{R}, \mathfrak{B}) = -\chi(\mathfrak{A}, \mathfrak{B}),$$
$$\chi(\mathfrak{B}, \mathfrak{A} \rtimes_\alpha \mathbb{R}) = -\chi(\mathfrak{B}, \mathfrak{A}).$$

Proof. Use the Fack-Skandalis' Thom isomorphism for KK-theory groups:

$$KK^j(\mathfrak{A} \rtimes_\alpha \mathbb{R}, \mathfrak{B}) \cong KK^{j+1}(\mathfrak{A}, \mathfrak{B}),$$
$$KK^j(\mathfrak{B}, \mathfrak{A} \rtimes_\alpha \mathbb{R}) \cong KK^{j+1}(\mathfrak{B}, \mathfrak{A})$$

$(j \in \mathbb{Z}_2)$ (see [1]). □

Proposition 6.6. *Let $\mathfrak{A}$ be a σ-unital C^*-algebra and $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ the crossed product of $\mathfrak{A}$ by an action α of $\mathbb{Z}$. Then for any separable C^*-algebra $\mathfrak{D}$,*

$$\chi(\mathfrak{D}, \mathfrak{A}) = \chi(\mathfrak{D}, \mathfrak{A}) + \chi(\mathfrak{D}, \mathfrak{A} \rtimes_\alpha \mathbb{Z}),$$

and if $\mathfrak{A}$ is separable, then for any σ-unital C^*-algebra $\mathfrak{D}$,

$$\chi(\mathfrak{A}, \mathfrak{D}) = \chi(\mathfrak{A}, \mathfrak{D}) + \chi(\mathfrak{A} \rtimes_\alpha \mathbb{Z}, \mathfrak{D}).$$

When each term in both equations is finite, it follows that

$$\chi(\mathfrak{D}, \mathfrak{A} \rtimes_\alpha \mathbb{Z}) = 0, \quad \chi(\mathfrak{A} \rtimes_\alpha \mathbb{Z}, \mathfrak{D}) = 0.$$

Proof. The following Pimsner-Voiculescu six-term exact sequences for KK-theory groups under the respective assumptions are known ([1]):

$$\begin{tikzcd} KK^0(\mathfrak{D},\mathfrak{A}) \arrow[r, "(\mathrm{id}-\alpha)_*"] & KK^0(\mathfrak{D},\mathfrak{A}) \arrow[r, "i_*"] & KK^0(\mathfrak{D},\mathfrak{A}\rtimes_\alpha \mathbb{Z}) \arrow[d] \\ KK^1(\mathfrak{D},\mathfrak{A}\rtimes_\alpha \mathbb{Z}) \arrow[u] & KK^1(\mathfrak{D},\mathfrak{A}) \arrow[l, "i_*"'] & KK^1(\mathfrak{D},\mathfrak{A}) \arrow[l, "(\mathrm{id}-\alpha)_*"'] \end{tikzcd}$$

where id is the identity map on $\mathfrak{A}$ and i is the inclusion map from $\mathfrak{A}$ to $\mathfrak{A}\rtimes_\alpha \mathbb{Z}$, and

$$\begin{tikzcd} KK^0(\mathfrak{A},\mathfrak{D}) \arrow[d] & KK^0(\mathfrak{A},\mathfrak{D}) \arrow[l, "(\mathrm{id}-\alpha)_*"'] & KK^0(\mathfrak{A}\rtimes_\alpha \mathbb{Z},\mathfrak{D}) \arrow[l, "i_*"'] \\ KK^1(\mathfrak{A}\rtimes_\alpha \mathbb{Z},\mathfrak{D}) \arrow[r, "i_*"] & KK^1(\mathfrak{A},\mathfrak{D}) \arrow[r, "(\mathrm{id}-\alpha)_*"] & KK^1(\mathfrak{A},\mathfrak{D}). \arrow[u] \end{tikzcd}$$

The rest of the proof is the same as that of the K-theory case (Proposition 3.1). □

Proposition 6.7. *Let*

$$\begin{tikzcd} P = \mathfrak{A}_1 \oplus_{\mathfrak{B}} \mathfrak{A}_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathfrak{A}_2 \arrow[d, "q_2"] \\ \mathfrak{A}_1 \arrow[r, "q_1"] & \mathfrak{B} \end{tikzcd}$$

be a pull-back diagram of separable nuclear C^*-algebras. Then for any σ-unital C^*-algebra $\mathfrak{D}$,

$$\chi(\mathfrak{A}_1,\mathfrak{D}) + \chi(\mathfrak{A}_2,\mathfrak{D}) = \chi(\mathfrak{B},\mathfrak{D}) + \chi(P,\mathfrak{D}).$$

Let

$$\begin{tikzcd} P = \mathfrak{A}_1 \oplus_{\mathfrak{B}} \mathfrak{A}_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathfrak{A}_2 \arrow[d, "q_2"] \\ \mathfrak{A}_1 \arrow[r, "q_1"] & \mathfrak{B} \end{tikzcd}$$

be a pull-back diagram of σ-unital nuclear C^*-algebras. Then for any separable C^*-algebra $\mathfrak{D}$,

$$\chi(\mathfrak{D},\mathfrak{A}_1) + \chi(\mathfrak{D},\mathfrak{A}_2) = \chi(\mathfrak{D},P) + \chi(\mathfrak{D},\mathfrak{B}).$$

Proof. Under the respective assumptions, the following six-term exact sequences are known ([1]):

$$\begin{tikzcd} KK^0(\mathfrak{B},\mathfrak{D}) \arrow[r, "(-q_1^*,q_2^*)"] & KK^0(\mathfrak{A}_1,\mathfrak{D}) \oplus KK^0(\mathfrak{A}_2,\mathfrak{D}) \arrow[r, "p_1^*+p_2^*"] & KK^0(P,\mathfrak{D}) \arrow[d] \\ KK^1(P,\mathfrak{D}) \arrow[u] & KK^1(\mathfrak{A}_1,\mathfrak{D}) \oplus KK^1(\mathfrak{A}_2,\mathfrak{D}) \arrow[l, "p_1^*+p_2^*"'] & KK^1(\mathfrak{B},\mathfrak{D}), \arrow[l, "(-q_1^*,q_2^*)"'] \end{tikzcd}$$

and

$$\begin{tikzcd} KK^0(\mathfrak{D},P) \arrow[r, "((p_1)_*,(p_2)_*)"] & KK^0(\mathfrak{D},\mathfrak{A}_1) \oplus KK^0(\mathfrak{D},\mathfrak{A}_2) \arrow[r, "(q_2)_*-(q_1)_*"] & KK^0(\mathfrak{D},\mathfrak{B}) \arrow[d] \\ KK^1(\mathfrak{D},\mathfrak{B}) \arrow[u] & KK^1(\mathfrak{D},\mathfrak{A}_1) \oplus KK^1(\mathfrak{D},\mathfrak{A}_2) \arrow[l, "(q_2)_*-(q_1)_*"'] & KK^1(\mathfrak{D},P). \arrow[l, "((p_1)_*,(p_2)*)"'] \end{tikzcd}$$

The rest of the proof is the same as that of the K-theory case (Proposition 3.1). □

Proposition 6.8. *Let $\mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2 = O$ be the amalgamated full free product of separable nuclear C^*-algebras. Suppose that there are retractions τ_j from $\mathfrak{A}_j$ to $\mathfrak{B}$ $(j = 1, 2)$ and that O is in the bootstrap category. Then for any separable C^*-algebra $\mathfrak{D}$,*

$$\chi(\mathfrak{A}_1, \mathfrak{D}) + \chi(\mathfrak{A}_2, \mathfrak{D}) = \chi(\mathfrak{B}, \mathfrak{D}) + \chi(O, \mathfrak{D}).$$

Let $\mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2 = O$ be the amalgamated full free product of σ-unital nuclear C^*-algebras. Suppose that there are retractions τ_j from $\mathfrak{A}_j$ to $\mathfrak{B}$ $(j = 1, 2)$. Then for any separable C^*-algebra $\mathfrak{D}$ in the bootstrap category,

$$\chi(\mathfrak{D}, \mathfrak{A}_1) + \chi(\mathfrak{D}, \mathfrak{A}_2) = \chi(\mathfrak{D}, O) + \chi(\mathfrak{D}, \mathfrak{B}).$$

Proof. Under the assumption on retractions, we have $K_j(O) \cong K_j(\mathfrak{A}_1 \oplus_{\mathfrak{B}} \mathfrak{A}_2)$ $(j = 0, 1)$ with $\mathfrak{A}_1 \oplus_{\mathfrak{B}} \mathfrak{A}_2$ the pull-back C^*-algebra associated with the retractions τ_j ([2]). It then follows that

$$KK^j(O, \mathfrak{D}) \cong K^j(\mathfrak{A}_1 \oplus_{\mathfrak{B}} \mathfrak{A}_2, \mathfrak{D})$$

$(j = 0, 1)$. Indeed, the K-theory group isomorphisms above, the naturality of the Rosenberg-Schocet's UCT below, and the Five Lemma imply the KK-theory group isomorphisms desired. And then use Proposition 6.7.

Note that the Rosenberg-Schocet's universal coefficient theorem (UCT) is: for separable C^*-algebras A and B, with A in the bootstrap category, the following short exact sequence holds:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\ \delta\ } KK^*(A, B) \xrightarrow{\ \gamma\ } \mathrm{Hom}(K_*(A), K_*(B)) \to 0$$

where the map γ has degree 0 and the map δ has degree 1. The sequence is natural in each variable, and splits (unnaturally). □

Proposition 6.9. *Let $\mathfrak{A}_1 \otimes_{\mathfrak{B}} \mathfrak{A}_2 = T$ be the balanced tensor product of separable nuclear C^*-algebras. Suppose that there are retractions τ_j from $\mathfrak{A}_j$ to $\mathfrak{B}$ $(j = 1, 2)$ and that T is in the bootstrap category and that $\mathfrak{B}$ commutes with $\mathfrak{A}_1$ and $\mathfrak{A}_2$ and has the same unit with them. Then for any separable C^*-algebra $\mathfrak{D}$,*

$$\chi(\mathfrak{A}_1, \mathfrak{D}) + \chi(\mathfrak{A}_2, \mathfrak{D}) = \chi(\mathfrak{B}, \mathfrak{D}) + \chi(T, \mathfrak{D}).$$

Let $\mathfrak{A}_1 \otimes_{\mathfrak{B}} \mathfrak{A}_2 = T$ be the balanced tensor product of σ-unital nuclear C^*-algebras. Suppose that there are retractions τ_j from $\mathfrak{A}_j$ to $\mathfrak{B}$ $(j = 1, 2)$ and that $\mathfrak{B}$ commutes with $\mathfrak{A}_1$ and $\mathfrak{A}_2$ and has the same unit with them.

Then for any separable C^*-algebra $\mathfrak{D}$ in the bootstrap category,

$$\chi(\mathfrak{D}, \mathfrak{A}_1) + \chi(\mathfrak{D}, \mathfrak{A}_2) = \chi(\mathfrak{D}, T) + \chi(\mathfrak{D}, \mathfrak{B}).$$

Proof. Under the assumption on retractions, commutativty, and unity, we have $K_j(O) \cong K_j(\mathfrak{A}_1 \oplus_{\mathfrak{B}} \mathfrak{A}_2)$ $(j = 0, 1)$ with $\mathfrak{A}_1 \oplus_{\mathfrak{B}} \mathfrak{A}_2$ the pull-back C^*-algebra associated with the retractions τ_j ([6]). The rest of the proof is the same as that of Proposition 6.8. □

Corollary 6.10. *Under the same assumptions and notations as in Corollaries* 5.2, 5.5, *and* 5.8 *and as in Propositions* 6.7, 6.8 *and* 6.9, *combined, we have*

$$\chi(P, \mathfrak{D}) = \sum_{i=1}^{n} \chi(\mathfrak{A}_i, \mathfrak{D}) - \sum_{j=1}^{n-1} \chi(\mathfrak{B}_j, \mathfrak{D}),$$

$$\chi(\mathfrak{D}, P) = \sum_{i=1}^{n} \chi(\mathfrak{D}, \mathfrak{A}_i) - \sum_{j=1}^{n-1} \chi(\mathfrak{D}, \mathfrak{B}_j),$$

and

$$\chi(O,\mathfrak{D}) = \sum_{i=1}^{n} \chi(\mathfrak{A}_i,\mathfrak{D}) - \sum_{j=1}^{n-1} \chi(\mathfrak{B}_j,\mathfrak{D}),$$

$$\chi(\mathfrak{D},O) = \sum_{i=1}^{n} \chi(\mathfrak{D},\mathfrak{A}_i) - \sum_{j=1}^{n-1} \chi(\mathfrak{D},\mathfrak{B}_j),$$

and

$$\chi(T,\mathfrak{D}) = \sum_{i=1}^{n} \chi(\mathfrak{A}_i,\mathfrak{D}) - \sum_{j=1}^{n-1} \chi(\mathfrak{B}_j,\mathfrak{D}),$$

$$\chi(\mathfrak{D},T) = \sum_{i=1}^{n} \chi(\mathfrak{D},\mathfrak{A}_i) - \sum_{j=1}^{n-1} \chi(\mathfrak{D},\mathfrak{B}_j),$$

where each term in all the equations above is finite.

Acknowledgement. The author would like to thank the referee for reading the manuscript carefully and giving some comments for revision.

References

[1] B. Blackadar, *K-theory for Operator Algebras*, Second Edition, Cambridge, (1998).

[2] J. Cuntz, *K-groups for free products of C^*-algebras*, Proc. Sympos. Pure Math., **38** Part I, AMS (1982), 81-84.

[3] M. S. J., *Mathematics Dictionary*, (Sugaku Jiten, in Japanese), Math. Soc. Japan, 4th edition, Iwanami (2007).

[4] C. Schochet, *Topological methods for C^*-algebras II: geometric resolutions and the Künneth formula*, Pacific J. Math. Vol. **98**, No. 2, (1982), 443-458.

[5] T. Sudo, *K-theory ranks and index for C^*-algebras*, Ryukyu Math. J., **20** (2007), 43-123.

[6] T. Sudo, *K-theory of the pullback and pushout C^*-algebras*, Scientiae Mathematicae Japonicae, **65**, No. 1 (2007), 53-60, :e2006, 1061-1068.

[7] N.E. Wegge-Olsen, *K-theory and C^*-algebras*, Oxford Univ. Press (1993).

Communicated by *Moto O'uchi*

Department of Mathematical Sciences, Faculty of Science, University of the Ryukyus, Senbaru 1, Nishihara, Okinawa 903-0213, Japan.
Email: sudo@math.u-ryukyu.ac.jp

A SIMPLE PROOF FOR JB*-TRIPLE STRUCTURES IN HILBERT C^*-MODULES

Masaharu Kusuda

Received April 30, 2013; revised July 18, 2013

Abstract. Let A be a C^*-algebra and let X be a Hilbert C^*-module over A. Isidro[**2**] showed the theorem that X becomes a JB*-triple in a canonical way. We give a simple and alternative proof of Isidro's theorem.

1 Introduction. Let X be a JB*-triple, that is, X is a complex Banach space with norm $\|\cdot\|$ which has a continuous triple product $\{\cdot,\cdot,\cdot\}$ from $X \times X \times X$ into X satisfying the following conditions (1) -(5) for all $a, b, x, y, z \in X$:

(1) $\{x, y, z\}$ is symmetric complex linear in x, z and conjugate-linear in y,

(2) every box operator $x \Box x$ on X defined by $(x \Box x)(a) = \{x, x, a\}$ is a hermitian operator on X, i.e., $\|\exp it(x \Box x)\| = 1$ for all $t \in \mathbb{R}$,

(3) $x \Box x$ has non-negative spectrum,

(4) $\|\{x, x, x\}\| = \|x\|^3$.

(5) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$.

Recall that an open connected subset D of a (complex) Banach space X is called a *symmetric domain* if each point x in D is an isolated fixed point of an involutive biholomorphic map s_x on D. Such a map s_x is called a *symmetry* at x. A JB*-triple structure in a Banach space X determines that the open unit disk of X becomes a symmetric domain. In fact, it is well-known that X is a JB*-triple if and only if the open unit disk of X is a symmetric domain (see [**1**, Theorem 2.5.27]). Thus it would be interesting to know what kind of important Banach space has a JB*-triple structure. Isidro[**2, Theorem 1.4**] showed that every Hilbert C^*-module becomes a JB*-triple in a canonical way. The proof is direct in the sense that the conditions (1) - (5) above can be checked. In this paper, we give an alternative proof of Isidro's theorem that every Hilbert C^*-module becomes a JB*-triple in a canonical way, which is simple in the sense that our proof to be given below depends on only well-known fundamental facts.

2 Notation. Recall the definition of a Hilbert C^*-module. Let A be a C^*-algebra. By a *right A-Hilbert module* (or a *right Hilbert C^*-module*), we mean a right A-module X equipped with an A-valued pairing $\langle \cdot\ ,\ \cdot \rangle$, called an A-valued inner product, satisfying the following conditions:

(H1) $\langle \cdot\ ,\ \cdot \rangle$ is sesquilinear. (We make the convention that $\langle \cdot\ ,\ \cdot \rangle$ is conjugate-linear in the first variable and is linear in the second variable.)

(H2) $\langle x\ ,\ y \rangle = \langle y\ ,\ x \rangle^*$ for all $x, y \in X$.

(H3) $\langle x\ ,\ ya \rangle = \langle x\ ,\ y \rangle a$ for all $x, y \in X$ and $a \in A$.

(H4) $\langle x\ ,\ x \rangle \geq 0$ for all $x \in X$, and $\langle x\ ,\ x \rangle = 0$ implies that $x = 0$.

(H5) X is a Banach space with respect to the norm $\|x\| = \|\langle x\ ,\ x \rangle\|^{\frac{1}{2}}$.

2010 *Mathematics Subject Classification.* 46G20, 46L08.
Key words and phrases. Hilbert C^*-module, JB*-triple, C^*-algebra.

Let B be a C^*-algebra. Left B-Hilbert modules are defined similarly except that we require that B should act on the left of X, that the B-valued inner product $\langle \cdot\ ,\ \cdot \rangle$ should be linear in the first variable and is conjugate-linear in the second variable, and that $\langle bx\ ,\ y\rangle = b\langle x\ ,\ y\rangle$ for all $x, y \in X$ and $b \in B$.

Let A and B be C^*-algebras. We denote by ${}_A\langle \cdot\ ,\ \cdot\rangle$ the A-valued inner product on the left A-Hilbert module and by $\langle \cdot\ ,\ \cdot\rangle_B$ the B-valued inner product on the right B-Hilbert module, respectively. By an $A - B$ *Hilbert bimodule*, we mean a left A-Hilbert module and a right B-Hilbert module X satisfying the following condition:

(H6) ${}_A\langle x\ ,\ y\rangle{\cdot}z = x{\cdot}\langle y\ ,\ z\rangle_B$ for all $x, y, z \in X$.

Note that an $A - B$ Hilbert bimodule automatically satisfies the following condition:

(H7) ${}_A\langle xb\ ,\ y\rangle = {}_A\langle x\ ,\ yb^*\rangle$ and $\langle ax,\ y\rangle_B = \langle x\ ,\ a^*y\rangle_B$ for all $x, y \in X, a \in A$ and $b \in B$.

Now we suppose that X is a right A-Hilbert module with the A-inner product $\langle \cdot\ ,\ \cdot\rangle$. We define a linear operator $\Theta_{x,y}$ on X by

$$\Theta_{x,y}(z) = x{\cdot}\langle y\ ,\ z\rangle$$

for all $x, y, z \in X$. We denote by $\mathcal{K}(X)$ the C^*-algebra generated by the set $\{\Theta_{x,y} \mid x, y \in X\ \}$ (see [**3**, Proposition 2.21 and Lemma 2.25]). Then X is a full left $\mathcal{K}(X)$-Hilbert module with respect to the natural left action defined by $t \cdot x = t(x)$ for $t \in \mathcal{K}(X)$, $x \in X$, and the inner product ${}_{\mathcal{K}(X)}\langle x\ ,\ y\rangle \equiv \Theta_{x,y}$. Thus X is a $\mathcal{K}(X) - A$ Hilbert bimodule ([**3**, Lemma 2.30]).

3 Result. Now we briefly review the definition of the linking algebra for a $\mathcal{K}(X) - A$ Hilbert bimodule (see [**3**, p.50] for more details of linking algebras). Let A be a C^*-algebra and let X be a $\mathcal{K}(X) - A$ Hilbert bimodule. We denote by $\widetilde{X}$ the dual Hilbert module of X, which is the set X with the left A-action and the right $\mathcal{K}(X)$-action defined by

$$a \cdot \widetilde{x} = \widetilde{(x \cdot a^*)}, \quad \widetilde{x} \cdot t = \widetilde{(t^* \cdot x)} \quad \text{for } t \in \mathcal{K}(X) \text{ and } a \in A,$$

where we write $\widetilde{x}$ if we view $x \in X$ as an element of $\widetilde{X}$. In addition, $\widetilde{X}$ is an $A - \mathcal{K}(X)$ Hilbert bimodule equipped with the A- and $\mathcal{K}(X)$-valued inner products given by

$${}_A\langle \widetilde{x}\ ,\ \widetilde{y}\rangle = \langle x\ ,\ y\rangle_A, \quad \langle \widetilde{x}\ ,\ \widetilde{y}\rangle_{\mathcal{K}(X)} = {}_{\mathcal{K}(X)}\langle x\ ,\ y\rangle$$

for $x, y \in X$ (see [**3**, p.49] for the details of dual Hilbert C^*-modules). Here we consider the right A-Hilbert module $X \oplus A$. Put

$$\mathcal{L}(X) = \left\{ \begin{pmatrix} t & x \\ \tilde{y} & a \end{pmatrix} \mid t \in \mathcal{K}(X),\ a \in A,\ x, y \in X \right\}.$$

Then each $L = \begin{pmatrix} t & x \\ \tilde{y} & a \end{pmatrix} \in \mathcal{L}(X)$ acts on $X \oplus A$ by

$$\begin{pmatrix} t & x \\ \tilde{y} & a \end{pmatrix} \begin{pmatrix} z \\ b \end{pmatrix} = \begin{pmatrix} tz + xb \\ \langle y, z\rangle_A + ab \end{pmatrix}.$$

The adjoint L^* of L is given by

$$L^* = \begin{pmatrix} t^* & y \\ \widetilde{x} & a^* \end{pmatrix}.$$

Addition and scalar multiplication on $\mathcal{L}(X)$ are defined by the usual formulas for matrices, and in addition, product in $\mathcal{L}(X)$ is given by

$$\begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix} \begin{pmatrix} t' & x' \\ \widetilde{y'} & a' \end{pmatrix} = \begin{pmatrix} tt' + {}_{\mathcal{K}(X)}\langle x, y' \rangle & t \cdot x' + x \cdot a' \\ \widetilde{y} \cdot t' + a \cdot \widetilde{y'} & \langle y, x' \rangle_A + aa' \end{pmatrix}.$$

Then $\mathcal{L}(X)$ is a C^*-subalgebra of the C^*-algebra $\mathcal{A}(X)$ of all adjointable operators on $X \oplus A$. Note that in order that $\mathcal{L}(X)$ should be closed in $\mathcal{A}(X)$ under the operator norm topology, it suffices that X is a *full* left $\mathcal{K}(X)$-Hilbert module (see the proof of [**3**, Lemma 3.20]). Remark that $\mathcal{L}(X)$ does not need to be an imprimitivity bimodule. In fact, it is well-known that more generally, if X is a Hilbert bimodule, the linking algebra $\mathcal{L}(X)$ is a C^*-algebra. We call the C^*-algebra $\mathcal{L}(X)$ the *linking algebra* for X. For ease of notation, we write

$$\mathcal{L}(X) = \begin{pmatrix} \mathcal{K}(X) & X \\ \widetilde{X} & A \end{pmatrix}$$

for the linking algebra for X (see [**3**, p.50] for more details of linking algebras). Now we are in a position to give a proof of the theorem.

Theorem. *Let A be a C^*-algebra and let X be a Hilbert C^*-module over A. Then X becomes a JB*-triple in a canonical way.*

Proof. Without loss of generality, we may assume that X is a right A-Hilbert module. Let $\mathcal{L}(X)$ be the linking algebra for X. Since $\mathcal{L}(X)$ is a C^*-algebra, it is a JB*-triple equipped with the triple product defined by $\{R, S, T\} = \frac{1}{2}(RS^*T + TS^*R)$ for all $R, S, T \in \mathcal{L}(X)$ (cf. [**1**, 1.2.6 and 2.5.33]).

Since we have

$$\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & {}_{\mathcal{K}(X)}\langle x \ , \ y \rangle z \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x\langle y \ , \ z \rangle_A \\ 0 & 0 \end{pmatrix},$$

we see that

$$(*) \qquad \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \right\} = \begin{pmatrix} 0 & \frac{1}{2}(x\langle y \ , \ z \rangle_A + z\langle y \ , \ x \rangle_A) \\ 0 & 0 \end{pmatrix}.$$

Since $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ is a closed vector subspace of $\mathcal{L}(X)$, $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ is a subtriple of $\mathcal{L}(X)$. Thus $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ is a JB*-triple. Equivalently, then the open unit disk D of $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ is a symmetric domain. Define a surjective linear mapping π by $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \to x \in X$. It then follows that π is isometric (cf. [**3**, Lemma 3.20]). Furthermore, clearly π is a biholomorphic map. Take any x from the open unit disk of X and put $R = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in D$. Let s_R be a symmetry at R. Then x is an isolated fixed point of the composition $\pi \circ s_R \circ \pi^{-1}$ which is a symmetry at x. Thus we see that the open unit disk of X is a symmetric domain, which shows that X a JB*-triple. Furthermore, the JB*-triple product $\{\cdot, \cdot, \cdot\}$ on X is given by $\{x, y, z\} = \frac{1}{2}(z\langle y \ , \ x \rangle_A + x\langle y \ , \ z \rangle_A)$, which follows from the equality $(*)$. $\square$

References

[1] C.-H. Chu, *Jordan structures in geometry and analysis*, Cambridge University Press (2012).

[2] J. Isidro, *Holomorphic automorphisms of the unit balls of Hilbert C^*-modules*, Glasgow Math. J., **45** (2003), 249–262.

[3] I. Raeburn and D. P. Williams, *Morita equivalence and continuous trace C^*-algebras,* Math. Surveys Monographs Vol. 60, Amer. Math. Soc. (1998).

Communicated by *Masaru Nagisa*

Department of Mathematics
Faculty of Engineering Science,
Kansai University,
Yamate-cho 3-3-35, Suita, Osaka 564-8680, Japan.
e-mail address: kusuda@kansai-u.ac.jp