# ON CONVERGENCE IN THE ORLICZ SPACE AS A RANKED SPACE

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Abstract. Kita and Yoneda showed that every Cauchy sequence with Orlicz norm r-converges in the Orlicz space considered as a ranked space, and however there exists an r-convergent sequence which is not a Cauchy sequence. We give an another proof and show that every sequence of points in every fundamental sequence r-converges.

**1 Introduction.** The ranked space which was proposed by Kunugi [2] and studied by Nakanishi [3-5], and others, has convergence concepts by using the fundamental sequences. In [1], Kita and Yoneda treated the Orlicz space as a ranked space. Moreover they showed that if a sequence converges with Orlicz norm, then it r-converges to the same point, and there exists a sequence that r-converges and does not converge with Orlicz norm, using a fundamental sequence. In this paper we give an another proof and show that for every fundamental sequence $\{U_n(f_n; \alpha_n, \varepsilon_n)\}$, every sequence $\{g_n : g_n \in U_n(f_n; \alpha_n, \varepsilon_n)\}$ r-converges. Throughout this paper, all the functions are real-valued and measurable on $\mathbf{T} = [-\pi, \pi]$.

**2 Notations and definitions.** Let $p$ be a function on $[0, \infty)$ satisfying the following properties: $p(0) = 0,\ 0 < p(u) < \infty$ if $0 < u < \infty$, $p$ is non-decreasing, right-continuous on $[0, \infty)$ and $\lim_{u\to\infty} p(u) = \infty$. A function $\varphi$ is called an $N$-function if it has the representation as $\varphi(t) = \int_0^t p(u)du$ for $t \geq 0$. We know that $\varphi$ is a convex function.

**Definition 1** ([1, 6])**.** Let $\varphi$ be an $N$-function and $\alpha > 0$. We define $\varphi(\alpha L)$ by

$$\varphi(\alpha L) := \left\{ f : \int_{-\pi}^{\pi} \varphi(\alpha|f(x)|)dx < \infty \right\}.$$

For $f \in \varphi(\alpha L)$ and $\varepsilon > 0,\ V(f; \alpha, \varepsilon)$ is defined by

$$V(f; \alpha, \varepsilon) := \left\{ g \in \varphi(\alpha L) : \int_{-\pi}^{\pi} \varphi(\alpha|f(x) - g(x)|)dx < \varepsilon \right\}.$$

An Orlicz space $L^*_\varphi$ is defined by $L^*_\varphi := \bigcup_{\alpha>0} \varphi(\alpha L)$. For $f \in L^*_\varphi$, the Luxemburg-Nakano norm (L-N norm for short) $\|f\|_{(\varphi)}$ of $f$ is defined by

$$\|f\|_{(\varphi)} := \inf \left\{ \lambda > 0 : \int_{-\pi}^{\pi} \varphi(\lambda^{-1}|f(x)|)dx < 1 \right\}.$$

A function $\varphi$ is said to satisfy the $\Delta_2$-condition if there exist $C > 0$ and $t_0 > 0$ such that

$$\varphi(2t) \leq C\varphi(t) \qquad \text{for every} \quad t \geq t_0.$$

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Hereafter we assume that a function $\varphi$ is an $N$-function and does not satisfy the $\Delta_2$-condition. Without the $\Delta_2$-condition we know the following: If $0<\alpha<\beta$, then for every $C>0$ and $t>0$, there exists $t_0>t$ such that $C\varphi(\alpha t_0)<\varphi(\beta t_0)$. By the convexity of $\varphi$, the inequality that $\varphi(s|f(x)|+t|g(x)|))\le s\varphi(|f(x)|)+t\varphi(|g(x)|)$ for $s+t=1, 0<s,t<1$ is often used.

**3 Ortho-convergence (r-convergence).** In this section, we see that a Cauchy sequence with L-N norm r-converges (Definition 5) in the Orlicz space, on the other hand there exists r-convergent and not a Cauchy sequence with L-N norm . We denote $\{x\in\mathbf{T} : |f(x)|>\varepsilon\}$ by $[|f|>\varepsilon]$, and $\Big|[|f|>\varepsilon]\Big|$ means the Lebesgue measure of $[|f|>\varepsilon]$.

**Proposition 1.** *Let $\{V_n\}=\{V_n(f_n;\alpha_n,\varepsilon_n)\}$ be a sequence satisfying that: for every non-negative integers $n$ and $m$ with $n<m$,*

$$0<\alpha_n\le\alpha_m,\ \varepsilon_n>\varepsilon_m>0 \text{ and } \lim_{n\to\infty}\varepsilon_n=0.$$

*If $f_m\in V_n$ for each $n$ and for every $m>n$, then every sequence $\{g_n : g_n\in V_n\}$ is a Cauchy sequence in measure.*

*Proof.* For every $\varepsilon>0$, for $n$ and $m$ with $n<m$,

$$\begin{aligned}
&\varphi\left(\frac{\alpha_1\varepsilon}{4}\right)\Big|[|g_n-g_m|>\varepsilon]\Big|\le\varphi\left(\frac{\alpha_n\varepsilon}{4}\right)\Big|[|g_n-g_m|>\varepsilon]\Big|\\
&=\int_{[|g_n-g_m|>\varepsilon]}\varphi\left(\frac{\alpha_n\varepsilon}{4}\right)dx\le\int_{-\pi}^{\pi}\varphi\left(\frac{\alpha_n}{4}|g_n(x)-g_m(x)|\right)dx\\
&\le\int_{-\pi}^{\pi}\varphi\left(\frac{\alpha_n}{4}(|f_n(x)-g_n(x)|+|f_n(x)-g_m(x)|)\right)dx\\
&\le\int_{-\pi}^{\pi}\varphi\left(\frac{\alpha_n}{2}|f_n(x)-g_n(x)|\right)dx+\int_{-\pi}^{\pi}\varphi\left(\frac{\alpha_n}{2}|f_n(x)-g_m(x)|\right)dx\\
&\le\int_{-\pi}^{\pi}\varphi(\alpha_n|f_n(x)-g_n(x)|)dx+\int_{-\pi}^{\pi}\varphi(\alpha_n|f_n(x)-f_m(x)|)dx\\
&\qquad\qquad+\int_{-\pi}^{\pi}\varphi(\alpha_m|f_m(x)-g_m(x)|)dx<2\varepsilon_n+\varepsilon_m<3\varepsilon_n.
\end{aligned}$$

Then, we have that $\lim_{m,n\to\infty}\Big|[|g_n-g_m|>\varepsilon]\Big|=0$ for every $\varepsilon>0$, thus $\{g_n : g_n\in V_n\}$ is a Cauchy sequence in measure. □

**Theorem 1.** *If $\{f_n\}$ is a Cauchy sequence with L-N norm for non-negative integer $n$, then there exists a sequence $\{V_n\}=\{V_n(f_n;\ \alpha_n,\varepsilon_n)\}$ such that every sequence $\{g_n : g_n\in V_n\}$ is a Cauchy sequence in measure.*

*Proof.* Let $\{f_n\}$ be a Cauchy sequence with L-N norm. Then we can find a sequence $\{\varepsilon_n\}$ satisfying the following properties: there exists $\varepsilon_{n_0}<1$ and if $n_0<n<m$, then $\varepsilon_{n_0}>\varepsilon_n>\varepsilon_m>0$, $\|f_n-f_m\|_{(\varphi)}<\varepsilon_n$, and $\lim_{n\to\infty}\varepsilon_n=0$. Moreover there exists $\gamma>0$ as follows: there exists $n_1\ge n_0$, $\|f_n\|_{(\varphi)}\le\gamma$ for every $n\ge n_1$. Then, we have

$$\int_{-\pi}^{\pi}\varphi\left(\frac{|f_n(x)|}{\varepsilon_n+\gamma}\right)dx\le\int_{-\pi}^{\pi}\varphi\left(\frac{|f_n(x)|}{\|f_n\|_{(\varphi)}}\right)dx\le 1.$$

We set $\alpha_n$ to $\min\left\{\dfrac{1}{\varepsilon_n+\gamma}, 1\right\}$. Then we have $f_n \in \varphi(\alpha_n L)$ and $0 < \alpha_n \leq \alpha_m \leq 1$ for $n_1 \leq n < m$. Let $\{V_n\}$ be $\{V_n(f_n; \alpha_n, \varepsilon_n)\}$. For $m > n \geq n_1$, by $\|f_n - f_m\|_{(\varphi)} < \varepsilon_n < 1$, if $\|f_n - f_m\|_{(\varphi)} \neq 0$, then we have

$$\begin{aligned}\frac{1}{\|f_n - f_m\|_{(\varphi)}} \int_{-\pi}^{\pi} \varphi(\alpha_n |f_n(x) - f_m(x)|)dx &\leq \int_{-\pi}^{\pi} \varphi\left(\frac{\alpha_n |f_n(x) - f_m(x)|}{\|f_n - f_m\|_{(\varphi)}}\right)dx \\ &\leq \int_{-\pi}^{\pi} \varphi\left(\frac{|f_n(x) - f_m(x)|}{\|f_n - f_m\|_{(\varphi)}}\right)dx \leq 1.\end{aligned}$$

Then $\displaystyle\int_{-\pi}^{\pi} \varphi(\alpha_n(|f_n(x) - f_m(x)|))dx < \varepsilon_n$. If $\|f_n - f_m\|_{(\varphi)} = 0$, then $\displaystyle\int_{-\pi}^{\pi} \varphi(\alpha_n |f_n(x) - f_m(x)|)dx < \varepsilon$ for every $\varepsilon > 0$. Moreover $f_m \in \varphi(\alpha_m L) \subset \varphi(\alpha_n L)$. This shows that for each $n \geq n_1$, for every $m > n$, $f_m \in V_n(f_n; \alpha_n, \varepsilon_n)$. Therefore $\{g_n : g_n \in V_n\}$ is a Cauchy sequence in measure by Proposition 1. □

We show that the converse of Theorem 1 does not hold.

**Proposition 2.** *There exists not a Cauchy sequence $\{f_n\}$ with L-N norm satisfying the following: there exists a sequence $\{V_n\} = \{V_n(f_n;\ \alpha_n, \varepsilon_n)\}$ in Proposition 1 such that every sequence $\{g_n : g_n \in V_n\}$ is a Cauchy sequence in measure.*

*Proof.* Let $\varepsilon_n$ be $2^{-n}$, $0 < \alpha_n < \alpha_m < 1$ for $n < m$ and $\lim_{n\to\infty} \alpha_n = \beta \leq 1$. Since $\varphi$ does not satisfy the $\Delta_2$-condition, for $\alpha_1, \beta$, there exists $t_1$ such that $0 < t_1$, $1 < \varphi(\alpha_1 t_1)$ and $2^2\varphi(\alpha_1 t_1) < \varphi(\beta t_1)$. Next, for $\alpha_2, \beta$, there exists $t_2$ such that $t_1 < t_2$ and $2^3\varphi(\alpha_2 t_2) < \varphi(\beta t_2)$. By repeating this process, we can find a sequence $\{t_n\}$ such that,

$$0 < t_1 < t_2 < \cdots < t_n < \cdots,\ 1 < \varphi(\alpha_1 t_1),\ 2^{n+1}\varphi(\alpha_n t_n) < \varphi(\beta t_n).$$

We choose a sequence of sets $\{E_n\}$ as follows: for each $n \in \mathbb{N}$

$$E_n \subset [-\pi, \pi],\ |E_n| = \frac{1}{2^{n+1}\varphi(\alpha_n t_n)},\ E_n \cap E_m = \emptyset\ (m \neq n).$$

We can construct a function $f_n$ that $f_n(x) = t_n \chi_{E_n}(x)$ for every $x \in [-\pi, \pi]$ and set $V_n = V_n(f_n; \alpha_n, 2^{-n})$ . Where $\chi_{E_n}$ is a characteristics function of $E_n$. Since, for each $n$ if $m > n$, then

$$\int_{-\pi}^{\pi} \varphi(\alpha_n |f_n(x) - f_m(x)|)dx = \varphi(\alpha_n t_n)|E_n| + \varphi(\alpha_n t_m)|E_m| < \frac{1}{2^n},$$

we have that $f_m \in V_n(f_n; \alpha_n, 2^{-n})$ and $\{g_n : g_n \in V_n\}$ is a Cauchy sequence in measure by Proposition 1. However, for every $m > n$,

$$\int_{-\pi}^{\pi} \varphi(\beta |f_n(x) - f_m(x)|)dx \geq \int_{E_n} 2^{n+1}\varphi(\alpha_n t_n)dx + \int_{E_m} 2^{m+1}\varphi(\alpha_m t_m)dx = 2,$$

then $\|f_n - f_m\|_{(\varphi)} \geq \dfrac{1}{\beta} \geq 1$. It shows that $\{f_n\}$ is not a Cauchy sequence with L-N norm
. □

We recall the definitions of a ranked space and ortho-convergence (r-convergence).

**Definition 2** ([3, 4, 5])**.** Let $X$ be a non-empty set such that, for every $f \in X$, there exists an associated non-empty family $\mathcal{U}(f)$, consisting of subsets of $X$, called preneighborhood of $f$, denoted by $U(f)$, $V(f)$, and so on. Define $\mathcal{U} = \bigcup_{f \in X} \mathcal{U}(f)$. Then the space $X$ is called a ranked space if, for each integer $n \in \{0\} \cup \mathbb{N}$, there exists an associated subfamily $\mathcal{U}_n$ of $\mathcal{U}$ satisfying the following conditions (A) and (a):

(A) If $U(f) \in \mathcal{U}(f)$, then $f \in U(f)$.

(a) for every $f \in X$, for every $U(f) \in \mathcal{U}(f)$ and for every $n \in \{0\} \cup \mathbb{N}$, there exists a $V(f)$ such that:

(a.1) $V(f) \subset U(f)$,

(a.2) $V(f) \in \mathcal{U}_m$ with some $m \geq n$.

$U(f) \in \mathcal{U}_n$ is said to be of rank $n$ in a ranked space.

**Definition 3** ([1])**.** For every $f \in L^*_\varphi$, a preneighborhood of $f$ is defined as follows: for $f \in \varphi(\alpha L)$ and $\varepsilon > 0$,

$$U(f) := U(f; \alpha, \varepsilon) = \left\{ g \in \varphi(\alpha L) : \int_{-\pi}^{\pi} \varphi(\alpha |f(x) - g(x)|) dx < \varepsilon \right\}.$$

And define, for $\alpha > 0$ such that $f \in \varphi(\alpha L)$:

$$\mathcal{U}(f) := \{U(f; \alpha, \varepsilon) : \alpha > 0, \varepsilon > 0\}, \qquad \mathcal{U} := \bigcup_{f \in L^*_\varphi} \mathcal{U}(f),$$

$$\mathcal{U}_n := \{U(f; \alpha, 2^{-n}) : \alpha > 0\} \text{ for each } n \in \{0\} \cup \mathbb{N}.$$

Then an Orlicz space $L^*_\varphi$ is considered as a ranked space.

**Definition 4** ([3, 4, 5])**.** For a sequence of functions $\{f_n\}$, a sequence of preneighborhoods $\{U_n\} = \{U_n(f_n; \alpha_n, \varepsilon_n)\}$ is said to be a fundamental sequence if it satisfies the following conditions:

(1) $U_n \supset U_m$, for $n < m$,

(2) $\varepsilon_n > \varepsilon_m > 0$, for $n < m$ and $\lim_{n \to \infty} \varepsilon_n = 0$.

A fundamental sequence $\{U_n(f_n; \alpha_n, \varepsilon_n)\}$ is said to be of center $f$ if $f_n = f$ for every $n$.

**Definition 5** ([3, 4, 5])**.** A sequence $\{f_n\}$ is said to be ortho-convergent (r-convergent) to $f$, if there exists a fundamental sequence $\{U_n(f; \alpha_n, \varepsilon_n)\}$ of center $f$, satisfying the following conditions: for every $n \in \{0\} \cup \mathbb{N}$, there exists a $n_0$ such that $f_m \in U_n(f; \alpha_n, \varepsilon_n)$ for every $m \geq n_0$.
For sequences of sets $\{U_n\}$ and $\{V_n\}$, $\{U_n\} \succ \{V_n\}$ means that for every $U_n \in \{U_n\}$, there exists $V_m \in \{V_n\}$ such that $U_n \supset V_m$.

Hereafter we consider the Orlicz space as a ranked space.

**Proposition 3.** *Let a sequence* $\{V_n\} = \{V_n(f_n; \alpha_n, \varepsilon_n)\}$ *in Proposition* 1*. Then every sequence* $\{g_n : g_n \in V_n\}$ *r-converges.*

*Proof.* Since $\{f_n\}$ is a Cauchy sequence in measure by Proposition 1, there exist a function $f$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ for almost everywhere $x \in \mathbf{T}$. Since by Fatou's lemma

$$\int_{-\pi}^{\pi} \varphi(\alpha_n|f_n(x) - f(x)|)dx \le \liminf_{k\to\infty} \int_{-\pi}^{\pi} \varphi(\alpha_n|f_n(x) - f_{n_k}(x)|)dx \le \varepsilon_n,$$

then,

$$\int_{-\pi}^{\pi} \varphi\left(\frac{\alpha_n}{2}|f(x)|\right) dx \le \frac{1}{2}\left(\int_{-\pi}^{\pi} \varphi(\alpha_n|f_n(x) - f(x)|)dx + \int_{-\pi}^{\pi} \varphi(\alpha_n|f_n(x)|)dx\right) < \infty.$$

Therefore we have $f \in \varphi(\alpha_n L/2)$ for every $n \in \{0\} \cup \mathbb{N}$. Let $\{U_n\} = \{U_n(f; \alpha_n/2, \varepsilon_n)\}$. Then $U_n \supset U_m$ for $n < m$, since $\alpha_n \le \alpha_m$ and $\varepsilon_n > \varepsilon_m$. Therefore $\{U_n\}$ is a fundamental sequence of center $f$. For every $g \in V_n$, we have $g \in \varphi(\alpha_n L/2)$ and

$$\int_{-\pi}^{\pi} \varphi\left(\frac{\alpha_n}{2}|f(x) - g(x)|\right) dx \le \frac{1}{2}\left(\int_{-\pi}^{\pi} \varphi(\alpha_n|f(x) - f_n(x)|)dx + \int_{-\pi}^{\pi} \varphi(\alpha_n|f_n(x) - g(x)|)dx\right) < \varepsilon_n.$$

Then $U_n \supset V_n$, we have $\{U_n\} \succ \{V_n\}$. This shows that every sequence $\{g_n : g_n \in V_n\}$ r-converges to $f$. □

**Theorem 2.** *A Cauchy sequence with L-N norm r-converges to the same point. However there exists an r-convergent and not a Cauchy sequence with L-N norm.*

*Proof.* Let $\{f_n\}$ be a Cauchy sequence with L-N norm. Then there exists a sequence $\{V_n\} = \{V_n(f_n; \alpha_n, \varepsilon_n)\}$ by Theorem 1, and a sequence $\{f_n\}$ r-converges to a function $f \in L^*_\varphi$ by Proposition 3. If there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ for almost everywhere, then $\lim_{k\to\infty} \|f_{n_k} - f\|_{(\varphi)} = 0$. Therefore we see that $\lim_{n\to\infty} \|f_n - f\|_{(\varphi)} = 0$. The example that an r-convergent and not a Cauchy sequence with L-N norm is showed in Proposition 2. □

By [1, Corollary 5.13], we know that $\alpha_n \le \alpha_m$ if $U_n \supset U_m$ for a fundamental sequence $\{U_n(f; \alpha_n, \varepsilon_n)\}$. This shows that a sequence $\{g_n : g_n \in U_n\}$ converges in measure to $f$.

**Theorem 3.** *Let $\{U_n\} = \{U_n(f; \alpha_n, \varepsilon_n)\}$ be a fundamental sequence. Then every sequence $\{g_n : g_n \in U_n\}$ converges in measure to $f$.*

*Proof.* Since $\{U_n\}$ is a fundamental sequence, $U_n \supset U_m$ if $n < m$. We have $\alpha_n \le \alpha_m$ by [1, Corollary 5.13]. For every $\varepsilon > 0$, for every $n \in \mathbb{N}$, we have similarly in Proposition 1 that

$$\varphi(\alpha_1\varepsilon)\Big|[|f - g_n| > \varepsilon]\Big| \le \varphi(\alpha_n\varepsilon)\Big|[|f - g_n| > \varepsilon]\Big| \le \int_{-\pi}^{\pi} \varphi(\alpha_n|f(x) - g_n(x)|)dx < \varepsilon_n.$$

Then we have that $\{g_n : g_n \in U_n\}$ converges in measure to $f$. □

By Theorem 3 we have that if $\{f_n\}$ r-converges, then converges in measure. However there exists a $\{f_n\}$ that converges in measure, and does not r-converge.

**Proposition 4.** *There exists a sequence which converges in measure, however does not r-converge.*

*Proof.* For each $n \in 0 \cup \mathbb{N}$, we set $E_n \subset [-\pi, \pi]$, $|E_n| = 1/2^n$, $E_n \cap E_m = \emptyset$ $(m \neq n)$, and $t_n = 2^n$. We define a function $f_n$ that $f_n(x) = t_n \chi_{E_n}(x)$ for every $x \in [-\pi, \pi]$. Then for every $0 < \varepsilon < 1$, $\Big|[|f_n| > \varepsilon]\Big| = 1/2^n$. This shows that a sequence $\{f_n\}$ converges in measure to 0. On the other hand, for every fundamental sequence $\{U_n(0; \alpha_n, \varepsilon_n)\}$ of center 0, we have

$$\int_{-\pi}^{\pi} \varphi(\alpha_n |f_n(x)|)dx = \frac{\varphi(2^n \alpha_n)}{2^n} \geq \varphi(\alpha_n) \geq \varphi(\alpha_1),$$

thus $\{f_n\}$ does not r-converge to 0. □

**4 Convergence in a fundamental sequence that its centers are not a unique function.** In this section, we see that for every fundamental sequence $\{U_n(f_n; \alpha_n, \varepsilon_n)\}$ such that $f_n \neq f_m$ for $n \neq m$, every sequence $\{g_n : g_n \in U_n\}$ r-converges in $L^*_\varphi$. We know that if $\beta \geq 2\alpha > 0$, $\varepsilon_1 > \varepsilon_2 > 0$ and $f \in \varphi(\beta L)$, then $U(f; \alpha, \varepsilon_1) \supset U(h; \beta, \varepsilon_2)$ for every function $h \in U(f; \beta, \varepsilon_2)$. Because if $g \in U(h; \beta, \varepsilon_2)$, then

$$\begin{aligned}
\int_{-\pi}^{\pi} \varphi(\alpha|f(x) - g(x)|)dx &\leq \int_{-\pi}^{\pi} \varphi\left(\frac{1}{2}(2\alpha|f(x) - h(x)| + 2\alpha|h(x) - g(x)|)\right) dx \\
&\leq \frac{1}{2}\int_{-\pi}^{\pi} \varphi(2\alpha|f(x) - h(x)|)dx + \frac{1}{2}\int_{-\pi}^{\pi} \varphi(2\alpha|h(x) - g(x)|)dx \\
&\leq \frac{1}{2}\int_{-\pi}^{\pi} \varphi(\beta|f(x) - h(x)|)dx + \frac{1}{2}\int_{-\pi}^{\pi} \varphi(\beta|h(x) - g(x)|)dx < \varepsilon_1.
\end{aligned}$$

We shall show that if $\beta > \alpha > 0$, $\varepsilon > 0$ and $f \in \varphi(\beta L)$, then a function $h$ exists such that $U(f; \alpha, \varepsilon) \supset U(f + h; \beta, \varepsilon)$. First, we shall show $U(0; \alpha, \varepsilon) \supset U(h; \beta, \varepsilon)$.

**Lemma 1.** *Let $\beta > \alpha > 0$. Then for every $b > 0$ there exists $a = a(\alpha, \beta, b) > 0$ such that for every function $g$,*

$$\int_{[g(x)\geq b]} \varphi(\alpha|g(x) + a|)dx \leq \int_{[g(x)\geq b]} \varphi(\beta|g(x)|)dx,$$
$$\int_{[g(x)\leq -b]} \varphi(\alpha|g(x) - a|)dx \leq \int_{[g(x)\leq -b]} \varphi(\beta|g(x)|)dx.$$

*Similarly for every $a > 0$, there exists $b = b(\alpha, \beta, a) > 0$ satisfying the above inequalities.*

*Proof.* For $b > 0$ let $0 < a \leq \dfrac{(\beta - \alpha)b}{\alpha}$. Then

$$\begin{aligned}
\int_{[g(x)\geq b]} \varphi(\alpha|g(x) + a|)dx &\leq \int_{[g(x)\geq b]} \varphi\left(\alpha\Big(g(x) + \frac{\beta - \alpha}{\alpha} b\Big)\right) dx \\
&\leq \int_{[g(x)\geq b]} \varphi\left(\alpha\Big(g(x) + \frac{\beta - \alpha}{\alpha} g(x)\Big)\right) dx = \int_{[g(x)\geq b]} \varphi(\beta|g(x)|)dx, \\
\int_{[g(x)\leq -b]} \varphi(\alpha|g(x) - a|)dx &\leq \int_{[g(x)\leq -b]} \varphi\left(\alpha\Big(-g(x) + \frac{\beta - \alpha}{\alpha} b\Big)\right) dx \\
&\leq \int_{[g(x)\leq -b]} \varphi\left(\alpha\Big(-g(x) - \frac{\beta - \alpha}{\alpha} g(x)\Big)\right) dx = \int_{[g(x)\leq -b]} \varphi(\beta(-g(x)))dx.
\end{aligned}$$

For every $a > 0$, let $b \geq \dfrac{\alpha a}{\beta - \alpha}$. Then we have $a \leq \dfrac{(\beta-\alpha)b}{\alpha}$. Similarly we can prove the above inequalities. □

**Lemma 2.** *Let $0 < \alpha < \beta$, $f \in \varphi(\beta L)$, and a function $h$ satisfy the following conditions: there exists $a > 0$ such that $\|h\|_{L^\infty} \leq a$. Then we have $f + h \in \varphi(\alpha L)$.*

*Proof.* If $f$ be a bounded function it is clear. Let $f$ be not a bounded function and there exists $a > 0$ such that $\|h\|_{L^\infty} \leq a$. Then by the latter part of Lemma 1, $b > 0$ exists such that $b \geq a$, and

$$\begin{aligned}\int_{[f(x)\geq b]} \varphi(\alpha|f(x)+h(x)|)dx &\leq \int_{[f(x)\geq b]} \varphi(\alpha|f(x)+a|)dx \\ &\leq \int_{[f(x)\geq b]} \varphi(\beta|f(x)|)dx < \infty, \\ \int_{[f(x)\leq -b]} \varphi(\alpha|f(x)+h(x)|)dx &\leq \int_{[f(x)\leq -b]} \varphi(\alpha|f(x)-a|)dx \\ &\leq \int_{[f(x)\leq -b]} \varphi(\beta|f(x)|)dx < \infty.\end{aligned}$$

Therefore $f + h \in \varphi(\alpha L)$. □

The following proofs of Proposition 5 and Theorem 4 are given by the referee.

**Proposition 5.** *If $\beta > \alpha > 0$, $\varepsilon > 0$, then there exists $a > 0$ such that $U(0; \alpha, \varepsilon) \supset U(h; \beta, \varepsilon)$ for every function $h$ with $\|h\|_{L^\infty} \leq a$.*

*Proof.* Let $a$ be $a > 0$, $\varphi\left(\dfrac{\alpha\beta}{\beta-\alpha}a\right) < \dfrac{\varepsilon}{3\pi}$ and $h$ be $\|h\|_{L^\infty} \leq a$. Then $h \in \varphi(\beta L)$. If $g \in U(h; \beta, \varepsilon)$, then by the convexity of $\varphi$ and $\dfrac{\alpha}{\beta} < 1$, $\dfrac{\beta-\alpha}{\beta} < 1$ that

$$\begin{aligned}&\int_{-\pi}^{\pi} \varphi(\alpha|g(x)|)dx \leq \int_{-\pi}^{\pi} \varphi(\alpha|g(x)-h(x)| + \alpha|h(x)|)dx \\ &= \int_{-\pi}^{\pi} \varphi\left(\frac{\alpha}{\beta}(\beta|g(x)-h(x)|) + \frac{\beta-\alpha}{\beta}\frac{\alpha\beta}{\beta-\alpha}|h(x)|\right)dx \\ &\leq \frac{\alpha}{\beta}\int_{-\pi}^{\pi} \varphi(\beta|h(x)-g(x)|)dx + \frac{\beta-\alpha}{\beta}\int_{-\pi}^{\pi} \varphi\left(\frac{\alpha\beta}{\beta-\alpha}|h(x)|\right)dx \\ &\leq \frac{\alpha}{\beta}\varepsilon + \frac{\beta-\alpha}{\beta}\frac{\varepsilon}{3\pi}2\pi < \varepsilon.\end{aligned}$$

We have $U(h; \beta, \varepsilon) \subset U(0; \alpha, \varepsilon)$, □

Next, we shall show that $U(f; \alpha, \varepsilon) \supset U(f + h; \beta, \varepsilon)$.

**Theorem 4.** *Let $\gamma > \beta > \alpha > 0$, $\varepsilon > 0$ and $f \in \varphi(\gamma L)$. Then there exists $a > 0$ such that $U(f; \alpha, \varepsilon) \supset U(f + h; \beta, \varepsilon)$ for every function $h$ with $\|h\|_{L^\infty} \leq a$.*

*Proof.* Let $a$ be $a > 0$, $\varphi\left(\dfrac{\alpha\beta}{\beta-\alpha}a\right) < \dfrac{\varepsilon}{3\pi}$ and $h$ be $\|h\|_{L^\infty} \leq a$. Since $f \in \varphi(\gamma L)$, $f + h \in \varphi(\beta L)$ by Lemma 2. If $g \in U(f + h; \beta, \varepsilon)$, then similarly in Proposition 5

$$\int_{-\pi}^{\pi} \varphi(\alpha|f(x)-g(x)|)dx = \int_{-\pi}^{\pi} \varphi(\alpha|f(x)+h(x)-g(x)-h(x)|)dx$$

$$\le \int_{-\pi}^{\pi} \varphi(\alpha|f(x)+h(x)-g(x)|+\alpha|h(x)|)dx$$
$$= \int_{-\pi}^{\pi} \varphi\left(\frac{\alpha}{\beta}(\beta|f(x)+h(x)-g(x)|)+\frac{\beta-\alpha}{\beta}\frac{\alpha\beta}{\beta-\alpha}|h(x)|\right)dx$$
$$\le \frac{\alpha}{\beta}\int_{-\pi}^{\pi} \varphi(\beta|f(x)+h(x)-g(x)|)dx+\frac{\beta-\alpha}{\beta}\int_{-\pi}^{\pi}\varphi\left(\frac{\alpha\beta}{\beta-\alpha}|h(x)|\right)dx$$
$$\le \frac{\alpha}{\beta}\varepsilon+\frac{\beta-\alpha}{\beta}\frac{\varepsilon}{3\pi}2\pi<\varepsilon.$$

Since $g \in \varphi(\beta L) \subset \varphi(\alpha L)$, we have $U(f+h;\beta,\varepsilon) \subset U(f;\alpha,\varepsilon)$. □

We know that $U(f;\alpha,\varepsilon) \supset U(f;\beta,\varepsilon_0)$ implies $\alpha \le \beta$ for every $\varepsilon,\ \varepsilon_0 > 0$, [1, Corollary 5.13]. We shall show that $U(f;\alpha,\varepsilon) \supset U(g;\beta,\varepsilon_0)$ implies $\alpha \le \beta$ by showing a contraposition.

**Theorem 5.** *Let $f \in \varphi(\alpha L),\ g \in \varphi(\beta L),\ 0<\varepsilon,\ 0<\varepsilon_0,\ 0<\alpha$ and $0<\beta$. If $U(f;\alpha,\varepsilon) \supset U(g;\beta,\varepsilon_0)$, then we have $\alpha \le \beta$.*

*Proof.* We shall show that if $\beta < \alpha$, then $U(f;\alpha,\varepsilon) \not\supset U(g;\beta,\varepsilon_0)$. For $g$ and $f$ we can find a positive real number $a$ and real number $s$ such that $\Big|[g(x)-f(x)>s]\cap[|g(x)|<a]\Big|>0$. And for $\varepsilon_0$, there exists $m \in \mathbb{N}$ such that $2^m\Big|[g(x)-f(x)>s]\cap[|g(x)|<a]\Big|>\varepsilon_0$. For $\gamma$ such that $\beta<\gamma<\alpha$, since $\varphi$ does not satisfy the $\Delta_2$-condition we have a sequence $\{t_k : k=1,2,\dots,l\}$ such that,

$$\max\left\{0,-s,\frac{-s\alpha}{\alpha-\gamma}\right\}<t_1<t_2<\cdots<t_l,\ 1<\varphi(\beta t_1),\ \varepsilon<l\varepsilon_0,$$
$$2^{k+m}\varphi(\beta t_k)<\varphi(\gamma t_k)\ (k=1,2,\dots,l).$$

We choose a sequence of sets $\{E_k : k=1,2,\dots,l\}$ as follows:

$$|E_k|=\frac{\varepsilon_0}{2^{k+m}\varphi(\beta t_k)},\ \bigcup_{k=1}^{l}E_k \subset [g(x)-f(x)>s]\cap[|g(x)|<a],$$
$$E_i\cap E_j=\emptyset (i\ne j).$$

Let $h(x)=\sum_{k=1}^{l} t_k\chi_{E_k}(x)$ for every $x\in[-\pi,\pi]$. Then we have

$$\int_{-\pi}^{\pi}\varphi(\beta h(x))dx=\sum_{k=1}^{l}\int_{E_k}\varphi(\beta t_k)dx=\sum_{k=1}^{l}\frac{\varepsilon_0}{2^{k+m}}<\varepsilon_0.$$

Since

$$\int_{-\pi}^{\pi}\varphi(\beta|g(x)+h(x)|)dx=\int_{[|g(x)|\le a]}\varphi(\beta|g(x)+h(x)|)dx$$
$$+\int_{[|g(x)|>a]}\varphi(\beta|g(x)|)dx<\infty,$$

then $g+h \in U(g;\beta,\varepsilon_0)$. By $\alpha|s+t_k| = \alpha(s+t_k) > \gamma t_k$,

$$\int_{-\pi}^{\pi} \varphi(\alpha|g(x)+h(x)-f(x)|)dx \geq \sum_{k=1}^{l}\int_{E_k} \varphi(\alpha(s+t_k))dx$$
$$\geq \sum_{k=1}^{l}\int_{E_k} \varphi(\gamma t_k)dx \geq \sum_{k=1}^{l}\int_{E_k} 2^{k+m}\varphi(\beta t_k) \geq \sum_{k=1}^{l}\varepsilon_0 > \varepsilon.$$

Thus we have that $g+h \notin U(f;\alpha,\varepsilon)$ and $U(f;\alpha,\varepsilon) \not\supset U(g;\beta,\varepsilon_0)$. Therefore if $U(f;\alpha,\varepsilon) \supset U(g;\beta,\varepsilon_0)$, then we have $\alpha \leq \beta$. □

We shall show that for every fundamental sequence $\{U_n(f_n;\alpha_n,\varepsilon_n)\}$, every sequence $\{g_n : g_n \in U_n\}$ is a Cauchy sequence in measure and r-converges in $L^*_\varphi$.

**Theorem 6.** *For every fundamental sequence $\{U_n(f_n;\alpha_n,\varepsilon_n)\}$, every sequence $\{g_n : g_n \in U_n\}$ is a Cauchy sequence in measure and r-converges to an $f \in \varphi(\alpha_n L/2)$.*

*Proof.* By Theorem 5, $U_n \supset U_m$ for $n < m$ implies that $\alpha_n \leq \alpha_m$. Similarly in Proposition 1 we have that $\lim\limits_{m,n\to\infty} \Big|[|g_n - g_m| > \varepsilon]\Big| = 0$ for every $\varepsilon > 0$, thus $\{g_n : g_n \in U_n\}$ is a Cauchy sequence in measure. So $\{f_n\}$ is Cauchy sequence in measure. Similarly in Proposition 3, there exist a function $f \in \varphi(\alpha_n L/2)$ such that $\lim\limits_{k\to\infty} f_{n_k}(x) = f(x)$ for almost everywhere $x \in \mathbf{T}$ for a subsequence $\{f_{n_k}\}$ of $\{f_n\}$, and there exists a fundamental sequence $\{V_n\} = \{V_n(f;\alpha_n/2,\varepsilon_n)\}$ of center $f$ such that $\{V_n\} \succ \{U_n\}$. This show that every sequence $\{g_n : g_n \in U_n\}$ r-converges to an $f \in L^*_\varphi$. □

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# EXISTENCE AND UNIQUENESS FOR NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS IN REAL LOCALLY COMPLETE SPACES

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Abstract. We extend existence and uniqueness results of [4] for nonlinear integro-differential equations of Volterra type between real locally complete vector spaces.

**1 Introduction** In [4] and [10], existence and uniqueness results are obtained for nonlinear integro-differential equations of Volterra types of the form

$$x' = H(t, x, Kx), \quad x(0) = x_0, \qquad (1)$$

where $H$ takes on values in a Banach space over $\mathbb{R}$, and $Kx$ is an integral operator depending on a continuous function $K$ having values in the same Banach space. Recently, such as in [1] and [8], generalizations of certain nonlinear problems have been extended to more general locally convex vector spaces or algebras. In this paper we extend the main results of [4] to the case in which the values are in a generalization of Banach spaces, specifically, locally complete vector spaces. Appropriate definitions are given below.

Throughout this paper, we consider a locally convex vector space over the field $\mathbb{R}$ of real numbers. We will denote such spaces by $(E, \mathcal{T})$, where $\mathcal{T}$ denotes the topology. Basic properties of locally convex spaces can be found in [3], [6], and [9]. Locally convex spaces are generalizations of normed spaces, and our interest here is a class of locally convex spaces that are generalizations of Banach spaces, as described next.

**1.1 Locally complete spaces.** The definition of a locally complete space relies on some information about certain kinds of bounded sets. Also, we need a way to construct linear subspaces that are normed spaces.

**Definition 1.1.** *In $(E, \mathcal{T})$ a set $A$ is:*

- **Bounded** *if, given any neighborhood $U$ of the origin, there exists a positive number $a = a_U$ such that $A \subset a \cdot U = \{a \cdot x : x \in U\}$.*
- *A* **disk** *if $A$ is both convex and balanced; i.e., if*

$$(\forall x, y \in A)(\forall s, t \in \mathbb{R}) \ni |s| + |t| \leq 1, \quad sx + ty \in A.$$

The unit ball of any normed space represents a set that is a bounded disk.

**Definition 1.2.** *Let $B$ be a bounded disk in $(E, \mathcal{T})$. Denote by $E_B$, the linear span of $B$. We equip $E_B$ with the normed topology given by the Minkowski sublinear functional of $B$ (see [9, p. 161]), defined by:*

$$(\forall x \in E_B) \quad \|x\|_B = \inf\{t \geq 0 : x \in t \cdot B\}.$$

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Given an arbitrary bounded set in a locally convex space, we can form the intersection of all disks that are closed and bounded, and which contain the set. It turns out that this type of construction always leads to a set that remains bounded, c.f. [6; 7.3.4, p. 135]. Hence, without loss of generality, we may assume that a bounded set is a closed, bounded disk.
The next definition represents the tool we need in order to work with normed spaces within (possibly not even metrizable) locally convex spaces. Our definition comes from [7; 5.1.29, p. 158].

**Definition 1.3.** *A locally convex space $(E, \mathcal{T})$ satisfies the* **strict Mackey convergence condition** *if for every bounded set $B$, there exists a closed, bounded disk $D$ such that $B \subset D$, and the topology of normed space $(E_D, \|\cdot\|_D)$ is equivalent to the topology $\mathcal{T}$ on $B$.*

Most of the typical spaces that occur in applications satisfy the strict Mackey convergence condition, including all metrizable locally convex spaces, countable products of such spaces, and certain inductive limits such as the space $\mathcal{D} = (\mathcal{D}, \mathcal{T}_{\mathcal{D}})$, the space of test functions from distribution theory. For details and more information about the strict Mackey convergence condition, see [7; Section 5.1, p. 158 - 159]. Finally, we define locally complete spaces next.

**Definition 1.4.** *A locally convex space $(E, \mathcal{T})$ is* **locally complete** *if for every closed bounded disk $B$, the normed space $(E_B, \|\cdot\|_B)$ is complete; i.e., a Banach space.*

In [5; 2.14, p. 20] locally complete spaces are also given the name $c^\infty-$, or *convenient* spaces. In several references, such as [2, 7.1, p. 275], the definition of a $c^\infty-$ space additionally requires the space to be bornological (i.e., any linear map from $E$ to an arbitrary locally convex space $F$ is continuous if and only if it maps bounded sets to bounded sets). We will use the definition of $c^\infty-$ spaces from [5; 2.14, p. 20]; i.e., locally complete spaces. It should be noted that the structures of $c^\infty-$ (locally complete) spaces have become important in recent years due to the use of such spaces in nonlinear distribution theory. More details about applications of these spaces can be found in [5], and the references therein.
Detailed information about locally complete spaces can be found in [7; Chapter 5]. For our purposes, the main facts are that the collection of locally complete spaces properly contains Banach spaces, and that every complete locally convex space is also locally complete, c.f. [7; Chapter 5] . The following example illustrates a space that is strictly more general than a Banach space, and we will apply our results to this example at the end of this paper.

**Example 1.5.** *Let $\mathcal{D} = (\mathcal{D}, \mathcal{T}_{\mathcal{D}})$ denote the space of test functions from distribution theory.*

The following facts hold for $\mathcal{D}$, with relevant references given in each case:

(D1) The space $\mathcal{D}$ can be expressed as an increasing countable union of complete metrizable locally convex spaces $(E_n, d_n)$, where $d_n$ represents the topology of the metric. Moreover, the topology $\mathcal{T} = \mathcal{T}_{\mathcal{D}}$ is the so - called inductive limit topology. See [6 ; 12.1.1, p. 289].

(D2) Given any bounded subset $A$ of $\mathcal{D}$, there exists a closed, bounded disk $D$ such that $A \subset D$, and the topology $\mathcal{T}_{\mathcal{D}}$ coincides with the normed topology $\|\cdot\|_D$ on $A$. See [6; 12.1.4, p. 290]. Thus, $\mathcal{D}$ satisfies the strict Mackey convergence condition.

(D3) $\mathcal{D}$ is complete. See [3; Ex 6, p. 165].

(D4) $\mathcal{D}$ is not metrizable. See [6; 12.1.5, p. 291].

Thus, $\mathcal{D}$ is a locally complete space that is not a Banach space.
We can now state the complete problem for this paper, and the main results.

**2 An existence and uniqueness theorem for functions between locally complete spaces.** We consider equation (1) given in the introduction with assumptions similar to those in [4], where the values are taken in a locally complete space. For completeness, we state the problem here. For the notation used below, $C[A,B]$ denotes the space of continuous functions from a set $A$ to a set $B$, and $B(a,r)$ denotes the open ball centered at $a$ of radius $r$ in a normed space. Finally, for a scalar $\alpha$, the notation $\alpha \cdot B(a,1)$ is equivalent to the statement: $\{x : \|x-a\| < \alpha\}$.
**Problem 1.** Consider the first order nonlinear integro-differential equation of Volterra type

$$x' = H(t,x,Kx), \quad x(0) = x_0. \tag{1}$$

Here, $x : \mathbb{R} \to E$, where $E = (E, \mathcal{T})$ is a locally complete space over $\mathbb{R}$, and $Kx$ is the operator defined by

$$(Kx)(t) = \int_0^T K(t,s,x(s))ds, \tag{2}$$

with $K$ and $H$ satisfying the following:

$$K \in C[\mathbb{R}^2 \times E, E], \quad H \in C[\mathbb{R} \times E \times E, E].$$

The differentiation in $E = (E, \mathcal{T})$ is defined as done for general locally convex spaces, such as in [11], or [2; 10.2, p. 279].

The following is inspired by Theorem 3.1 of [4]. Our result here is for functions from a locally complete space to the same locally complete space that satisfy some boundedness conditions outlined below. For consistency, we have chosen to use notation that coincides as closely as possible to that of [4].
**Theorem 1.** Assume there exists a closed, bounded disk $B \subset E$ such that for $J = [0,T]$, and some $K_0 > 0, H_0 > 0$,

$$K\left(J^2 \times B\right) \subset K_0 \cdot B; \tag{A1}$$

$$H\left(J \times B \times K_0 \cdot B\right) \subset H_0 \cdot B, \tag{A2}$$

$K$ is Lipschitz in the third argument with respect to the norm $\|\cdot\|_B$ on $E_B$; in particular,

$$(\exists k_1 > 0) \ni \ \|K(t,s,u) - K(t,s,\overline{u})\|_B \leq k_1 \|u - \overline{u}\|_B, \tag{A3}$$

on $J^2 \times B$, $H$ is locally Lipschitz; that is, for any $(t,x,y) \in J \times B \times K_0 \cdot B$, there exist $\delta = \delta(t,x,y) > 0$, $L = L(t,x,y) > 0$, and neighborhoods $U_x$ of $x$ and $U_y$ of $y$ within $J \times B \times K_0 \cdot B$ such that in $(E_B, \|\cdot\|_B)$,

$$\|H(t,x_1,y_1) - H(t,x_2,y_2)\|_B \leq L\left(\|x_1 - x_2\|_B + \|y_1 - y_2\|_B\right), \tag{A4}$$

for $(t,x_1,y_1), (t,x_2,y_2) \in J \times U_x \times U_y$. *Then*:
**(a)**: There exists $\eta > 0$ such that equation (1) has a unique solution on $J_0 = [0,\eta]$ in $(E_B, \|\cdot\|_B)$.

**(b)**: The sequence of approximations that converge to the unique solution in $(E_B, \|\cdot\|_B)$ from part (a) converges to the solution with respect to the topology $\mathcal{T}$ of $E$.

*Proof*: We start by rewriting some of the assumptions in terms of norms. By assumption (A1), $K\left(J^2 \times B\right) \subset K_0 \cdot B$ implies that on $J^2 \times B$,

$$\|K(t,s,x)\|_B \le K_0. \quad (A1)$$

The assumption (A2) implies that on $B^\circ$, that is, on $\{x \in E_B : \|x\|_B < 1\}$, and on $K_0 \cdot B = \{x \in E_B : \|x\|_B \le K_0\}$ we have

$$\|H(t,x,y)\|_B \le H_0. \quad (A2)$$

We will prove that $\eta = \min\left\{T, \frac{1}{2H_0}\right\}$ is the desired value.

For $t = 0$, $x = x_0$, $y = 0$, choose $\sigma_1,\ \gamma_1,\ \overline{\gamma_1} > 0$ such that $\sigma_1 H_0 < \overline{\gamma_1}$ , $\sigma_1 K_0 < \gamma_1$, and for which the Lipschitz inequality of (A4) holds on $R_1 = Rx_0$, where

$$R_1 = Rx_0 = [0,\sigma_1] \times B\left(x_0, \gamma_1\right) \times B\left(0, \overline{\gamma_1}\right).$$

By known methods such as Schauder's fixed point theorem or successive approximations, a unique solution $x(t)$ of (1) can be found on $[0,\sigma_1]$ as a limit of a sequence $(x_m) = (x_m(t))$, with respect to the norm $\|\cdot\|_B$. We now enlarge the interval of solution as follows. Let

$$x_{\sigma_1} = x_0 + \int_0^{\sigma_1} H(s, x(s), (Kx)(s))ds.$$

For $t = \sigma_1$ and $x = x_{\sigma_1}$, let

$$y_{\sigma_1} = \int_0^{\sigma_1} K(t,s,x(s))ds.$$

There exists $R_2 = R_{x_{\sigma_1}}$ for which the Lipschitz inequality of (A4) holds, given by

$$R_2 = [\sigma_1, \sigma_2] \times B\left(x_{\sigma_1}, \gamma_2\right) \times B\left(y_{\sigma_1}, \gamma_2\right),$$

where

$$\left(\sigma_2 - \sigma_1\right) H_0 < \gamma_2 \left(\sigma_2 - \sigma_1\right) K_0 < \overline{\gamma_2},$$

and such that the Lipschitz inequality of (A4) holds on $R_2$. It follows that we can prove the existence of a unique solution $x(t)$ on $[0, \sigma_1 + \sigma_2]$. We again denote the sequence of successive approximations by $(x_m)$. Let $S$ be the set of all unique solutions $x(t)$ to (1) on an interval $[0,\alpha]$ for $\alpha \le T$. It is easy to prove that a partial ordering of $S$ is given by set inclusion of intervals, and that Zorn's Lemma applies. Thus, we conclude that there is a maximal element, that is, there exists a unique solution to (1), on $[0,\eta]$. This proves part (a). To prove part (b), by [7; 3.2.2, p. 82], the topology of the norm $\|\cdot\|_B$ is stronger than the topology $\mathcal{T}$ on the vector space $E_B$. We conclude that if $(x_m)$ converges to the unique solution on $[0,\eta]$ with respect to $\|\cdot\|_B$, then $(x_n)$ converges to the unique solution $x = x(t)$ on $[0,\eta]$ with respect to $\mathcal{T}$ as well. $\square$

**Remark.** We proved this result by using the unit ball $B$ of $(E_B, \|\cdot\|_B)$. In general, one can prove Theorem 1 on a ball of radius $N$ in $(E_B, \|\cdot\|_B)$. In this case, the details follow as in the proof of Thm 3.1 of [4], with $\eta = \min\left\{T, \frac{N}{2H_0}\right\}$.

**3 Existence and uniqueness under a Lyapunov - dissipative condition** The result that follows generalizes Theorem 3.2 of [4] to locally complete spaces, under the assumptions of our previous theorem.

Within the context of a locally complete space $E$ of Theorem 1, we say that $H(t,x,y)$ satisfies a **Lyapunov - dissipative condition** if items (i) - (iii) below are satisfied:

(i)

$$\text{(A5)} \qquad V \in C\left[J \times B \times B, \mathbb{R}^+\right], \quad V(t,x,x) \equiv 0, \quad V(t,x,y) > 0,$$

if $x \neq y$ , for

$$(t,x),(t,y) \in J \times B^\circ, \quad K_0 \cdot B \subset B \subset E_B,$$

with $L > 0$ such that

$$|V(t,x,y) - V(t,x_1,y_1)| \leq L \cdot \left(\|x - x_1\|_B + \|y - y_1\|_B\right).$$

$(i)'$ If $(x_n)$ and $(y_m)$ are sequences in $B$ such that $lim_{m,n\to\infty} V(t,x_n,y_m) = 0$, then $lim_{m,n\to\infty}(x_n - y_m) = 0$ in the topology of $E$.

(ii) The following derivative relation holds:

$$D\left(V(t,x,y)\right) = \lim_{h\to 0^+} \frac{1}{h}\left\{V(t,x,y) - V(t-h, x - hH(t,x,Kx), y - hH(t,y,Ky)\right\},$$

and we have

$$D\left(V(t,x,y)\right) \leq g\left(t,\ V(t,x,y),\ \int_0^t S(t,s,V(s,x(s),y(s))ds\right),$$

for $t \in J$ and $x,y \in C[J,B]$, with $S \in C\left[J \times J \times \mathbb{R}^+, \mathbb{R}\right]$, $|S(t,s,V)| \leq S_0$ on $J \times J \times \mathbb{R}^+$; moreover, $S$ satisfies condition (L4) of [4].

(iii) The function $g$ satisfies: $g \in U^*$ of [4] with respect to $S$ and $t_0 = 0$.

**Theorem 3.1.** *Assume the hypotheses of Theorem 1, in particular, assumptions (A1) - (A3). Further, assume the Lyapunov dissipative condition (A5) and that the space* $(E,\mathcal{T})$ *satisfies the strict Mackey convergence condition. Then there exists* $\eta > 0$ *such that equation (1) has a unique solution on* $J_0 = [0,\eta]$.

*Proof:* As in the proof of Theorem 3.2 in [4], we construct a sequence $\{x_n(t)\}$ of $\varepsilon_n-$approximations on the interval $J_0 = [0,\eta]$, where $0 < \varepsilon_n < 1$ and $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$. To finish the proof, it will suffice to prove that the sequence converges to a continuous function $x(t)$ in the topology $\mathcal{T}$ of $E$; the proof that $x(t)$ is the unique solution follows from the same arguments as the proof of Theorem 3.2 in [4].

By [4; Thm 2.2, p. 94] and the assumption that $g \in U^*$, the arguments of Step II of [4; Thm 3.2, p. 101] apply to conclude that

$$\lim_{n\to\infty, m\to\infty} \left[V(t,x_n(t),x_m(t)\right] = 0,$$

in the topology $\mathcal{T}$ of $E$, for any $t \in J_0$. By assumption of the strict Mackey convergence condition, there is a closed, bounded disk $D$, with $B \subset D$, and for which $\mathcal{T}_D$ is equivalent to the normed topology of $(E_D, \|\cdot\|_D)$, on the set $B$. By local completeness, $(E_D, \|\cdot\|_D)$ is a Banach space, and we may apply the arguments from Step II of [4; p. 101], to conclude that $(x_n(t))$ is uniformly Cauchy in $(E_D, \|\cdot\|_D)$. Hence, $(x_n(t))$ converges in the space $(E_D, \|\cdot\|_D)$ to a continuous function $x(t)$. Finally, by the equivalence of the norm $\|\cdot\|_D$ to the topology of $E$ on the set $B$, we conclude that $(x_n(t))$ converges to $x(t)$ in the space $(E, \mathcal{T})$. □

**Remark 3.2.** *In view of items (D1) - D(4), the results of Theorem 3.1 hold for the space* $\mathcal{D} = (\mathcal{D}, \mathcal{T}_{\mathcal{D}})$*, the space of test functions from distribution theory.*

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# ON NERI'S MEAN FIELD EQUATION WITH HYPERBOLIC SINE VORTICITY DISTRIBUTION

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Abstract. Motivated by equations derived by Neri, Pointin-Lundgren and Joyce-Montgomery in the context of Onsager's statistical theory of vortices, we study the blow-up properties of some sinh-Gordon type extensions of the standard mean field equation with exponential nonlinearity on two-dimensional compact surfaces.

**1 Introduction and main results** Our study is motivated by the equation:

$$
(1.1)\qquad \begin{cases} -\Delta_g v = \lambda \left[ \dfrac{te^v - (1-t)e^{-v}}{\int_\Omega [te^v + (1-t)e^{-v}]\, dv_g} - \dfrac{1}{|\Omega|} \dfrac{\int_\Omega [te^v - (1-t)e^{-v}]\, dv_g}{\int_\Omega [te^v + (1-t)e^{-v}]\, dv_g} \right] & \text{in } \Omega, \\ \displaystyle\int_\Omega v\, dv_g = 0, \end{cases}
$$

where $(\Omega, g)$ is a compact orientable two-dimensional Riemannian manifold without boundary, $dv_g$ denotes the volume element on $\Omega$, $|\Omega|$ denotes the volume of $\Omega$, $t \in [0,1]$ and $\lambda > 0$ is a constant. Equation (1.1) is a special case of the mean field equation derived by Neri [10] in the context of the statistical mechanics description of two-dimensional turbulence, as initiated by Onsager [15] and further developed by Joyce and Montgomery [7], Pointin and Lundgren [16]. This special case captures the main features of the interaction between the positive part and the negative part of the exponential nonlinearity.

More precisely, Neri's equation is given by

$$
(1.2)\qquad \begin{cases} -\Delta_g v = \lambda \dfrac{\int_I \alpha(e^{\alpha v} - \frac{1}{|\Omega|}\int_\Omega e^{\alpha v} dv_g)\, \mathcal{P}(d\alpha)}{\iint_{I\times\Omega} e^{\alpha v}\, \mathcal{P}(d\alpha) dv_g} & \text{in } \Omega \\ \displaystyle\int_\Omega v = 0, \end{cases}
$$

where $v$ corresponds to the stream function and $\mathcal{P} = \mathcal{P}(d\alpha)$, $\alpha \in I = [-1,1]$, is the probability distribution of the relative circulations of the vortices, which are assumed to be independent identically distributed random variables. Equation (1.2) reduces to (1.1) when $\mathcal{P}$ is of the "hyperbolic sine type", namely

$$
(1.3)\qquad \mathcal{P}(d\alpha) = t\delta_1(d\alpha) + (1-t)\delta_{-1}(d\alpha).
$$

The mathematical analysis for equation (1.2) is quite recent. An existence result for solutions to equation (1.2) under Dirichlet boundary conditions was obtained by Neri himself in [10]. Results on the blow-up properties of solution sequences to (1.2), as well as the

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corresponding optimal Moser-Trudinger inequality are contained in our previous work [18], where the following more general problem is considered:

$$(1.4)\qquad \begin{cases} -\Delta_g v = \lambda \displaystyle\int_I V(\alpha,x,v)e^{\alpha v}\,\mathcal{P}(d\alpha) - \frac{\lambda}{|\Omega|}\int_{I\times\Omega} V(\alpha,x,v)e^{\alpha v}\,\mathcal{P}(d\alpha)dv_g & \text{in } \Omega \\ \displaystyle\int_\Omega v\,dv_g = 0. \end{cases}$$

On the other hand, a mean field equation similar to (1.2) may be derived by the method of [7, 16], assuming that $\mathcal{P}$ is the probability measure which determines the distribution of the relative circulations, see [4, 19]. Under such assumptions, the stream function $v$ satisfies:

$$(1.5)\qquad \begin{cases} -\Delta v = \lambda \displaystyle\int_I \alpha \left( \frac{e^{\alpha v}}{\int_\Omega e^{\alpha v}\,dv_g} - \frac{1}{|\Omega|} \right) \mathcal{P}(d\alpha) & \text{in } \Omega \\ \displaystyle\int_\Omega v\,dv_g = 0. \end{cases}$$

A blow-up analysis for (1.5) was obtained in [11] together with a related Moser-Trudinger type inequality, which is seen to be dual to the logarithmic Hardy-Littlewood-Sobolev inequality of Shafrir-Wolansky [21] when $\mathcal{P}$ is atomic. The optimal constant in the above mentioned Moser-Trudinger inequality was recently obtained in [17]. We note that the general equation (1.4) contains equation (1.2) and equation (1.5) as special cases.

The above mentioned results were motivated by the mathematical analysis carried out by Ohtsuka and Suzuki in [13, 14] for equation (1.5) in the special hyperbolic sine assumption (1.3). In such a case, (1.5) takes the form:

$$(1.6)\qquad \begin{cases} -\Delta_g v = \lambda t \left( \dfrac{e^{v}}{\int_\Omega e^{v}\,dv_g} - \dfrac{1}{|\Omega|} \right) - \lambda(1-t) \left( \dfrac{e^{-v}}{\int_\Omega e^{-v}\,dv_g} - \dfrac{1}{|\Omega|} \right) & \text{in } \Omega \\ \displaystyle\int_\Omega v\,dv_g = 0. \end{cases}$$

In particular, the articles [13, 14] contain some refined blow-up results for (1.6) whose extension to the general equation (1.4) appears to be difficult. Therefore, as a first step it is natural to try to extend such refined blow-up results to the special "sinh case" of equation (1.2), namely to equation (1.1). Indeed, this is the main objective in this note.

Before stating our main results, we note that under the assumption $\mathcal{P}(d\alpha) = \delta_1(d\alpha)$, the Dirac measure concentrated at $\alpha = 1$, both equation (1.2) and equation (1.5) reduce to the well known mean field equation

$$(1.7)\qquad \begin{cases} -\Delta_g v = \lambda \left( \dfrac{e^{v}}{\int_\Omega e^{v}\,dv_g} - \dfrac{1}{|\Omega|} \right) & \text{in } \Omega \\ \displaystyle\int_\Omega v\,dv_g = 0, \end{cases}$$

which has been extensively studied in recent years in connection with the Nirenberg problem, chemotaxis, Chern-Simons vortex theory as well as statistical hydrodynamics. See, e.g., [8, 20] and the references therein. Therefore, equation (1.1) and equation (1.6) may be viewed as physically relevant sinh-Gordon type extensions of the standard mean field equation (1.7).

Towards our objectives, we shall take the point of view of studying the slightly more general problem

$$
(1.8)\qquad \begin{cases} -\Delta_g v = \lambda\left[t\dfrac{e^v}{\mathcal{I}_1(v)} - (1-t)\dfrac{e^{-v}}{\mathcal{I}_2(v)}\right] - \lambda\kappa(v) & \text{in } \Omega \\ \displaystyle\int_\Omega v\,dv_g = 0, \end{cases}
$$

where $\mathcal{I}_1$, $\mathcal{I}_2$, are real functionals defined on $\mathcal{E}$ and where the constant $\kappa(v)$ is defined by $\kappa(v) = \kappa_1(v) - \kappa_2(v)$,

$$
(1.9)\qquad \kappa_1(v) = \frac{t}{|\Omega|}\int_\Omega \frac{e^v}{\mathcal{I}_1(v)}\,dv_g, \qquad \kappa_2(v) = \frac{1-t}{|\Omega|}\int_\Omega \frac{e^{-v}}{\mathcal{I}_2(v)}\,dv_g
$$

so that the integral of the r.h.s. in (1.8) is zero. We denote

$$
\mathcal{E} = \left\{ v \in H^1(\Omega) : \int_\Omega v\,dv_g = 0 \right\}.
$$

We make the following assumption on $\mathcal{I}_1, \mathcal{I}_2$:

($\mathcal{I}$) There exists $C_1 > 0$ such that $t\int_\Omega e^v\,dv_g \le C_1\mathcal{I}_1(v)$ and $(1-t)\int_\Omega e^{-v}\,dv_g \le C_1\mathcal{I}_2(v)$ for all $v \in \mathcal{E}$.

It follows from assumption ($\mathcal{I}$) that

$$
(1.10)\qquad |\kappa(v)| \leqslant \frac{C_1}{|\Omega|}
$$

for all $v \in \mathcal{E}$. For later convenience, we also observe that in view of Jensen's inequality we have $\mathcal{I}_1(v) \ge tC_1^{-1}|\Omega|$ and $\mathcal{I}_2(v) \ge (1-t)C_1^{-1}|\Omega|$ for all $v \in \mathcal{E}$. In particular, in the "strictly hyperbolic" case $t \in (0,1)$ there exists $c_0 > 0$ such that

$$
(1.11)\qquad \mathcal{I}_1(v) \ge c_0, \qquad \mathcal{I}_2(v) \ge c_0
$$

for all $v \in \mathcal{E}$. Clearly, (1.8) reduces to Neri's "sinh" case (1.1) when

$$
\mathcal{I}_1(v) = \mathcal{I}_2(v) = \int_\Omega [te^v + (1-t)e^{-v}]\,dv_g
$$

and it reduces to (1.6) when

$$
\mathcal{I}_1(v) = \int_\Omega e^v\,dv_g, \qquad \mathcal{I}_2(v) = \int_\Omega e^{-v}\,dv_g.
$$

Thus, by extending the results by Ohtsuka and Suzuki [13, 14] to equation (1.8), we conclude that equation (1.1) and equation (1.6) share analogous blow-up properties.

In order to state our main results, we denote by $G = G(x,y)$ the Green's function associated to $-\Delta_g$ on $\Omega$. Namely, $G$ is defined by

$$
(1.12)\qquad \begin{cases} -\Delta_g G(\cdot,y) = \delta_y - \dfrac{1}{|\Omega|} & \text{in } \Omega \\ \displaystyle\int_\Omega G(\cdot,y)\,dv_g(y) = 0. \end{cases}
$$

We shall be mainly concerned with blow-up sequences to (1.8). More precisely, we consider solution sequences $v_n$ to equation (1.8) with $\lambda = \lambda_n$ and $\lambda_n \to \lambda_0$. We define the measures $\mu_{1,n}, \mu_{2,n} \in \mathcal{M}(\Omega)$ by

$$\mu_{1,n} = \lambda_n t \frac{e^{v_n}}{\mathcal{I}_1(v_n)}, \qquad \mu_{2,n} = \lambda_n (1-t) \frac{e^{-v_n}}{\mathcal{I}_2(v_n)}. \tag{1.13}$$

By assumption $(\mathcal{I})$ we may assume that $\mu_{i,n} \stackrel{*}{\rightharpoonup} \mu_i \in \mathcal{M}(\Omega)$ weakly in the sense of measures, $i = 1, 2$. We set

$$u_{1,n} = G * \mu_{1,n}, \qquad u_{2,n} = G * \mu_{2,n}. \tag{1.14}$$

Then, $v_n = u_{1,n} - u_{2,n}$. We note that $(u_{1,n}, u_{2,n})$ satisfies a Liouville-type system as extensively analyzed in [2, 3, 21]:

$$\begin{cases} -\Delta u_{1,n} = \lambda \left( t \dfrac{e^{u_{1,n} - u_{2,n}}}{\mathcal{I}_1(v_n)} - \kappa_1(v_n) \right) & \text{in } \Omega \\ -\Delta u_{2,n} = \lambda \left( (1-t) \dfrac{e^{u_{2,n} - u_{1,n}}}{\mathcal{I}_2(v_n)} - \kappa_2(v_n) \right) & \text{in } \Omega \\ \displaystyle\int_\Omega u_{1,n} \, dv_g = 0 = \int_\Omega u_{2,n} \, dv_g. \end{cases}$$

We define as usual the blow-up sets:

$$\mathcal{S}_\pm = \{ p \in \Omega : \exists\, p_{\pm,n} \to p \text{ s.t. } v_n(p_{\pm,n}) \to \pm\infty ) \}$$

and we denote $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$. By adapting to our case some results for the general equation (1.4) as contained in Theorem 2.1, Theorem 2.2 and Theorem 4.1 in [18], we derive the following results for (1.8), which extend the alternatives as discovered in [1, 9].

**Theorem 1.1** ([18], Brezis-Merle type alternative)**.** *Let $\lambda_n \to \lambda_0$ and let $\{v_n\}$ be a sequence of solutions to* (1.8) *with $\lambda = \lambda_n$. Then, the following alternative holds:*

1) *Compactness:* $\limsup_{n\to\infty} \|v_n\| < +\infty$. *We have $\mathcal{S}_+ \cup \mathcal{S}_- = \emptyset$ and there exist a solution $v \in \mathcal{E}$ to* (1.8) *with $\lambda = \lambda_0$ and a subsequence $\{v_{n_k}\}$ such that $v_{n_k} \to v$ in $\mathcal{E}$.*

2) *Concentration:* $\limsup_{n\to\infty} \|v_n\| = +\infty$. *We have $\mathcal{S} \neq \emptyset$ and*

$$\mu_1 = \sum_{p \in \mathcal{S}_+} m_+(p) \delta_p + r_1 \, dv_g$$
$$\mu_2 = \sum_{p \in \mathcal{S}_-} m_-(p) \delta_p + r_2 \, dv_g$$

*where $\delta_p$ denotes the Dirac delta centered at $p \in \mathcal{S}$, the constants $m_\pm(p)$ satisfy the "minimum mass property"*

$$m_\pm(p) \geqslant 4\pi, \tag{1.15}$$

*and $r_i \in L^1(\Omega) \cap L^\infty_{loc}(\Omega \setminus \mathcal{S})$. Moreover the following facts hold:*

*2-i) If there exists $p_0 \in \mathcal{S}_\pm \setminus \mathcal{S}_\mp$ then $m_\pm(p_0) = 8\pi$ and $m_\mp(p_0) = 0$.*

*2-ii) For every $p_0 \in \mathcal{S}_+ \cap \mathcal{S}_-$, we have the quadratic identity*

$$8\pi[m_+(p_0) + m_-(p_0)] = [m_+(p_0) - m_-(p_0)]^2 \tag{1.16}$$

*and*

$$m_+(p_0) + m_-(p_0) \geqslant 4(3 + \sqrt{5})\pi. \tag{1.17}$$

Our first aim is to improve the minimum mass (1.15). We obtain the following.

**Theorem 1.2.** *Let $\lambda_n \to \lambda_0$ and let $\{v_n\}$ be a sequence of solutions to* (1.8) *with $\lambda = \lambda_n$. In the conclusion of Theorem 1.1, Alternative 2), the lower bound* (1.15) *is improved as follows:*

$$m_\pm(p) \geqslant 8\pi \tag{1.18}$$

*for any $p \in \mathcal{S}_\pm$. Consequently,* (1.17) *is also improved:*

$$m_+(p_0) + m_-(p_0) \geqslant 24\pi.$$

Clearly, when $p_0 \in \mathcal{S}_\pm \setminus \mathcal{S}_\mp$, Theorem 1.2 follows by Theorem 1.1, 2-*i*). Therefore, in the proof of Theorem 1.2 we need only consider the two-sided blow-up case $p_0 \in \mathcal{S}_+ \cap \mathcal{S}_-$. This result is obtained by performing the rescaling argument as in [13] using a Brezis-Merle type lemma and the classification of solutions to Liouville type equations in $\mathbb{R}^2$ obtained by Chen and Lin in [5, 6].

Our second result concerns residual vanishing in the one-sided blow-up case.

**Theorem 1.3** (One-sided concentration, residual vanishing)**.** *Let $\lambda_n \to \lambda_0$ and let $\{v_n\}$ be a sequence of solutions to* (1.8) *with $\lambda = \lambda_n$. Suppose $p_0 \in \mathcal{S}_+ \setminus \mathcal{S}_-$. Then,*

$$r_1 \equiv 0, \qquad \mu_1 = 8\pi \sum_{p \in \mathcal{S}_+} \delta_p,$$

*and $\mu_{1,n} \to 0$ in $L^\infty(\omega)$ for every $\omega \Subset \Omega \setminus \mathcal{S}_+$. Consequently, $u_{1,n} \to u_1$ in $H^1_{loc}(\Omega \setminus \mathcal{S}_+)$, where $u_1$ is given by*

$$u_1 = 8\pi \sum_{p \in \mathcal{S}_+} G(\cdot, p).$$

*Moreover, there exists $u_2 \in \mathcal{E}$ and a subsequence $\{u_{2,n}\}$ such that $u_{2,n} \to u_2$ in $\mathcal{E}$ and*

$$\begin{cases} -\Delta u_2 = \beta\lambda_0 \left[ K(x)e^{u_2} - \dfrac{1}{|\Omega|} \displaystyle\int_\Omega K(x)e^{u_2} \right] & \text{in } \Omega, \\ \displaystyle\int_\Omega u_2 = 0, \end{cases} \tag{1.19}$$

*where $\beta \in [0, \frac{1}{|\Omega|}]$ and $K(x) = e^{-\sum_{p \in \mathcal{S}_+} 8\pi G(x,p)}$. Analogous results hold in the converse case $p_0 \in \mathcal{S}_- \setminus \mathcal{S}_+$.*

Our third result concerns the location of the blow-up points in the one-sided blow-up case. In order to state it, we define a suitable local chart centered at the blow-up point $p_0$. More precisely, given $p_0 \in \mathcal{S}$ we denote by $(\Psi, \mathcal{U})$ an iso-thermal chart satisfying $\overline{\mathcal{U}} \cap \mathcal{S} = \{p_0\}$, $\Psi(\mathcal{U}) = \mathcal{O} \subset \mathbb{R}^2$ and

$$\Psi(p_0) = 0, \qquad g(X) = e^{\xi(X)}(dX_1^2 + dX_2^2), \qquad \xi(0) = 0, \tag{1.20}$$

where $X = (X_1, X_2)$ are Euclidean coordinates on $\mathcal{O}$. Then, the Laplace-Beltrami operator $\Delta_g$ is mapped to the operator $e^{-\xi(X)}\Delta_X$ on $\mathcal{O}$, where $\Delta_X = \partial^2_{X_1^2} + \partial^2_{X_2^2}$. We denote by $H_\Psi(x, y)$ the regular part of the Green's function $G(x, y)$ relative to the chart $(\mathcal{U}, \Psi)$, i.e.,

$$H_\Psi(x,y) = G(x,y) + \frac{1}{2\pi}\log(|\Psi(x) - \Psi(y)|) \tag{1.21}$$

for $x, y \in \mathcal{U}$. By $G_{\mathcal{O}}(X, Y)$ we denote the Green's function of $-\Delta_X$ on $\mathcal{O}$ with Dirichlet boundary conditions and by $H_{\mathcal{O}}(X, Y)$ we indicate its regular part. Namely,

$$\begin{cases} -\Delta_X G_{\mathcal{O}}(X,Y) = \delta_Y & \text{in } \mathcal{O} \\ G_{\mathcal{O}}(X,Y) = 0 & \text{on } \partial\mathcal{O} \end{cases}$$

and

$$H_{\mathcal{O}}(X,Y) = G_{\mathcal{O}}(X,Y) + \frac{1}{2\pi}\log(|X-Y|). \tag{1.22}$$

With this notation, we have the following.

**Theorem 1.4.** *Let $\lambda_n \to \lambda_0$ and let $\{v_n\}$ be a sequence of solutions to (1.8) with $\lambda = \lambda_n$. Suppose that there exists $p_0 \in \mathcal{S}_+ \setminus \mathcal{S}_-$. Then, the following relation holds in the iso-thermal chart (1.20) centered at $p_0$:*

(1.23)
$$\nabla_X \Big(8\pi H_\Psi(\Psi^{-1}(X), p_0) + 8\pi \sum_{p\in\mathcal{S}_+\setminus\{p_0\}} G(\Psi^{-1}(X), p) - u_2(\Psi^{-1}(X)) + \xi(X)\Big)\Big|_{X=0} = 0,$$

*where $u_2$ is given by (1.19). Analogous results hold in the converse case $p_0 \in \mathcal{S}_- \setminus \mathcal{S}_+$.*

We note that (1.23) coincides with the relation obtained in [14], Theorem 1.2, thus confirming that the hyperbolic sine mean field equations (1.1) and (1.6) exhibit analogous blow-up properties.

**Notation** For the sake of simplicity, in the local coordinate patch $\mathcal{O} \subset \mathbb{R}^2$ defined in (1.20) we denote $\nabla = \nabla_X$ and $\Delta = \Delta_X$.

**2 Proof of Theorem 1.2** Since Theorem 1.2 is already known in the one-sided blow-up case $p_0 \in \mathcal{S}_\pm \setminus \mathcal{S}_\mp$, without loss of generality we assume $t \in (0, 1)$. In particular, we assume that the lower bound (1.11) holds. Similarly to [13], the main observation towards obtaining our improved blow-up results is the local reduction of equation (1.8) to the following Liouville system:

$$\begin{cases} -\Delta w_1 = V_1 e^{w_1} - V_2 e^{w_2} & \text{in } \mathcal{O} \\ -\Delta w_2 = -V_1 e^{w_1} + V_2 e^{w_2} & \text{in } \mathcal{O}, \end{cases} \tag{2.1}$$

where $0 \leqslant V_i \leqslant C$ and $\int_{\mathcal{O}} e^{w_i} \leqslant C$, with $C$ a constant independent of $w_i$, $i = 1, 2$, to which the blow-up analysis developed in [13] may be applied, see Lemma 2.1 below. Indeed, we take a coordinate patch $(\Psi, \mathcal{U})$ as defined in (1.20). In particular, identifying $v(X) = v(\Psi^{-1}(X))$ for any function $v$ defined on $\Omega$, we have that a solution $v$ to equation (1.8) satisfies

$$-\Delta v = \lambda\left[\frac{te^v}{\mathcal{I}_1(v)} - \frac{(1-t)e^{-v}}{\mathcal{I}_2(v)}\right]e^\xi - \lambda\kappa(v)e^\xi \qquad \text{in } \mathcal{O}.$$

We define $h_\xi$ by

$$\begin{cases} -\Delta h_\xi = e^\xi & \text{in } \mathcal{O}, \\ h_\xi = 0 & \text{on } \partial\mathcal{O}. \end{cases} \tag{2.2}$$

Correspondingly, we define $w_i : \mathcal{O} \to \mathbb{R}$ by setting

$$w_1 = v - \log \mathcal{I}_1(v) + \lambda\kappa(v)h_\xi,$$
$$w_2 = -v - \log \mathcal{I}_2(v) - \lambda\kappa(v)h_\xi.$$

Then,

$$\frac{e^v}{\mathcal{I}_1(v)} = e^{w_1 - \lambda\kappa(v)h_\xi}, \qquad \frac{e^{-v}}{\mathcal{I}_2(v)} = e^{w_2 + \lambda\kappa(v)h_\xi}.$$

It follows that $w_1$ satisfies the equation

$$\begin{aligned} -\Delta w_1 = & -\Delta v + \kappa(v)(-\Delta)h_\xi = \lambda\left[\frac{te^v}{\mathcal{I}_1(v)} - \frac{(1-t)e^{-v}}{\mathcal{I}_2(v)}\right]e^\xi \\ = & \lambda t\, e^{\xi - \lambda\kappa(v)h_\xi} e^{w_1} - \lambda(1-t)\, e^{\xi + \lambda\kappa(v)h_\xi} e^{w_2} \end{aligned}$$

in $\mathcal{O}$. Setting

$$V_1 = \lambda t\, e^{\xi - \lambda\kappa(v)h_\xi}, \qquad V_2 = \lambda(1-t)\, e^{\xi + \lambda\kappa(v)h_\xi} \tag{2.3}$$

we conclude that $w_1$ satisfies the first equation in (2.1). Similarly, $w_2$ satisfies the second equation in (2.1) with $V_i$ defined by (2.3), $i = 1, 2$.

Now, we consider a solution sequence $v_n$ to (1.8) with $\lambda = \lambda_n \to \lambda_0$. Similarly, we set

$$w_{1,n} = v_n - \log \mathcal{I}_1(v_n) + \lambda_n\kappa(v_n)h_\xi,$$
$$w_{2,n} = -v_n - \log \mathcal{I}_2(v_n) - \lambda_n\kappa(v_n)h_\xi,$$

where $h_\xi$ is the function defined in (2.2) and, as before, we identify $v_n(X) = v_n(\Psi^{-1}(X))$. For later use, we note that in view of (1.11) we have

$$w_{1,n} + w_{2,n} = -\log \mathcal{I}_1(v_n) - \log \mathcal{I}_2(v_n) \leqslant -2\log c_0 \tag{2.4}$$

for some $c_0 > 0$. In view of the arguments above, we conclude that $(w_{1,n}, w_{2,n})$ satisfies the Liouville system

$$\begin{cases} -\Delta w_{1,n} = V_{1,n}e^{w_{1,n}} - V_{2,n}e^{w_{2,n}} & \text{in } \mathcal{O} \\ -\Delta w_{2,n} = -V_{1,n}e^{w_{1,n}} + V_{2,n}e^{w_{2,n}} & \text{in } \mathcal{O}, \end{cases} \tag{2.5}$$

where

$$V_{1,n} = \lambda_n t\, e^{\xi - \lambda_n\kappa(v_n)h_\xi}, \qquad V_{2,n} = \lambda_n(1-t)\, e^{\xi + \lambda_n\kappa(v_n)h_\xi} \tag{2.6}$$

and

$$0 \leqslant V_{i,n} \leqslant C, \qquad \int_{\mathcal{O}} e^{w_{i,n}} \leqslant C \tag{2.7}$$

$i = 1, 2$, for some $C > 0$ independent of $n$. In view of estimate (1.10), we may assume that $\kappa(v_n) \to \kappa_0$ and consequently

$$V_{1,n} \to V_1 = \lambda_0 t\, e^{\xi - \lambda_0 \kappa_0 h_\xi},$$
$$V_{2,n} \to V_2 = \lambda_0 (1-t)\, e^{\xi + \lambda_0 \kappa_0 h_\xi},$$

uniformly on $\overline{\mathcal{O}}$. We define

$$\mathcal{S}_i^0 = \{X \in \mathcal{O} : \exists X_n \to X \text{ s.t. } w_{i,n}(X_n) \to +\infty\}.$$

We recall the following Brezis-Merle type Lemma for (2.5) from [13], Lemma 2.1.

**Lemma 2.1** ([13]). *Suppose $\{w_{1,n}, w_{2,n}\}_n$ is a solution sequence to the Liouville system* (2.5)*, satisfying* (2.7)*. Then, up to subsequences, exactly one of the following alternatives holds true.*

1. *Both $\{w_{1,n}\}_n$ and $\{w_{2,n}\}_n$ are locally uniformly bounded in $\mathcal{O}$.*
2. *There is $i \in \{1, 2\}$ such that $\{w_{i,n}\}_n$ is uniformly bounded in $\mathcal{O}$ and $\{w_{j,n}\}_n \to -\infty$ locally uniformly in $\mathcal{O}$ for $j \neq i$.*
3. *Both $w_{1,n} \to -\infty$ and $w_{2,n} \to -\infty$ locally uniformly in $\mathcal{O}$.*
4. *For the blow-up sets $\mathcal{S}_1^0$, $\mathcal{S}_2^0$ defined for this subsequence, we have $\mathcal{S}_1^0 \cup \mathcal{S}_2^0 \neq \emptyset$ and $\sharp(\mathcal{S}_1^0 \cup \mathcal{S}_2^0) < +\infty$. Furthermore, for each $i \in \{1, 2\}$, either $\{w_{i,n}\}_n$ is locally uniformly bounded in $\mathcal{O} \setminus (S_1^0 \cup \mathcal{S}_2^0)$ or $w_{i,n} \to -\infty$ locally uniformly in $\mathcal{O} \setminus (S_1^0 \cup \mathcal{S}_2^0)$. Here, if $\mathcal{S}_i^0 \setminus (S_1^0 \cap \mathcal{S}_2^0) \neq \emptyset$ then $w_{i,n} \to -\infty$ locally uniformly in $\mathcal{O} \setminus (S_1^0 \cup \mathcal{S}_2^0)$, and for each $x_0 \in \mathcal{S}_i^0$ there exists $m_i(x_0) \geqslant 4\pi$ such that*

$$V_{i,n}(x) e^{w_{i,n}} \rightharpoonup \sum_{x_0 \in \mathcal{S}_i^0} m_i(x_0) \delta_{x_0} \qquad * \text{-weakly in } \mathcal{M}(\mathcal{O}).$$

We will start by giving the proof of the following preliminary lemma, which relies on arguments from [14].

**Lemma 2.2.** *Let $p_0 \in \mathcal{S}_+ \cap \mathcal{S}_-$. There exists a sequence $x_{1,n} \to p_0$ and a sequence $x_{2,n} \to p_0$ such that:*

*i)* $v_n(x_{1,n}) \to +\infty, \qquad v_n(x_{1,n}) - \log \mathcal{I}_1(v_n) \to +\infty,$

*ii)* $-v_n(x_{2,n}) \to +\infty, \qquad -v_n(x_{2,n}) - \log \mathcal{I}_2(v_n) \to +\infty.$

*Proof.* We prove only relation *i*). The proof of *ii*) is similar. Recall that

$$\mu_{1,n} = \lambda_n t \frac{e^{v_n}}{\mathcal{I}_1(v_n)}, \tag{2.8}$$

and that by Theorem 1.1 we have

$$\mu_{1,n} \overset{*}{\rightharpoonup} \mu_1 = \sum_{p \in \mathcal{S}_+} m_+(p) \delta_p + r_1\, dv_g. \tag{2.9}$$

Since $\mu_1$ is singular in $p_0$, $\mu_{1,n}$ is $L^\infty$-unbounded around $p_0 \in \mathcal{S}_+$. Hence, we can assume

$$\lim_{n\to\infty} \sup_{B(p_0, r_0)} (v_n - \log \mathcal{I}_1(v_n)) = +\infty, \qquad \forall r_0 > 0.$$

Hence, we find $r_0 > 0$ such that $\overline{B(p_0,r_0)} \cap \mathcal{S}_+ = \{p_0\}$ and a sequence of points $x_{1,n} \in \overline{B(p_0,r_0)}$, such that

$$v_n(x_{1,n}) - \log \mathcal{I}_1(v_n) = \max\left\{v_n(x) - \log \mathcal{I}_1(v_n) : x \in \overline{B(p_0,r_0)}\right\} \to +\infty.$$

Moreover, in view of (1.11), we have

$$v_n(x_{1,n}) - \log \mathcal{I}_1(v_n) \leqslant v_n(x_{1,n}) - \log c_0,$$

and therefore we also have $v_n(x_{1,n}) \to +\infty$. It remains to prove that $x_{1,n} \to p_0$. Suppose the contrary. Then up to subsequence, we may assume $x_{1,n} \to \bar{p} \neq p_0$, $\bar{p} \in B(p_0,r_0)$ and hence $\bar{p}$ is not in $\mathcal{S}_+$. This means $\limsup_{n\to\infty} v_n(x_{1,n}) < +\infty$, a contradiction. □

In view of Lemma 2.2 there exist $\{x_{1,n}\}$ and $\{x_{2,n}\}$ such that $x_{1,n} \to p_0$, $v_n(x_{1,n}) \to +\infty$, $x_{2,n} \to p_0$ and $-v_n(x_{2,n}) \to +\infty$, and furthermore

$$\begin{aligned} X_{1,n} = \Psi(x_{1,n}) \to 0 \quad &\text{and} \quad w_{1,n}(X_{1,n}) \to +\infty \\ X_{2,n} = \Psi(x_{2,n}) \to 0 \quad &\text{and} \quad w_{2,n}(X_{2,n}) \to +\infty. \end{aligned}$$

In particular, $0 \in \mathcal{S}_i^0$, $i = 1,2$, and

$$\mathcal{S}_1^0 = \Psi(\mathcal{U} \cap \mathcal{S}_+) = \{0\} = \Psi(\mathcal{U} \cap \mathcal{S}_-) = \mathcal{S}_2^0. \tag{2.10}$$

On the other hand, since

$$V_{1,n}e^{w_{1,n}} = \lambda_n t \frac{e^{v_n}}{\mathcal{I}_1(v_n)}\, e^{\xi}$$

and

$$V_{2,n}e^{w_{2,n}} = \lambda_n (1-t) \frac{e^{-v_n}}{\mathcal{I}_2(v_n)}\, e^{\xi},$$

recalling that $\xi(0) = 0$, from (2.9) we derive that

$$\begin{aligned} V_{1,n}e^{w_{1,n}} &\overset{*}{\rightharpoonup} m_+(p_0)\delta_0(dX) + s_1(X)\,dX, \\ V_{2,n}e^{w_{2,n}} &\overset{*}{\rightharpoonup} m_-(p_0)\delta_0(dX) + s_2(X)\,dX, \end{aligned} \tag{2.11}$$

$*$-weakly in $\mathcal{M}(\bar{\mathcal{O}})$, where $s_i(X) = r_i(\Psi^{-1}(X))e^{\xi(X)}$, $s_i \in L^1(\mathcal{O}) \cap L^\infty_{loc}(\bar{\mathcal{O}} \setminus \{0\})$, $i = 1,2$, and $\min(m_+(p_0), m_-(p_0)) \geqslant 4\pi$. In view of (2.10) there exist $Y_{1,n}, Y_{2,n} \in \mathcal{O}$, $Y_{1,n}, Y_{2,n} \to 0$ such that

$$\begin{aligned} w_{1,n}(Y_{1,n}) &= \sup_{\mathcal{O}} w_{1,n} \to +\infty \\ w_{2,n}(Y_{2,n}) &= \sup_{\mathcal{O}} w_{2,n} \to +\infty. \end{aligned}$$

*Proof of Theorem 1.2.* We define the rescaling parameters

$$\begin{aligned} \varepsilon_{1,n} &= e^{-w_{1,n}(Y_{1,n})/2} \\ \varepsilon_{2,n} &= e^{-w_{2,n}(Y_{2,n})/2}. \end{aligned}$$

Correspondingly, we rescale the Liouville system (2.5) two times.

Namely, we first rescale (2.5) around $Y_{1,n}$ with respect to $\varepsilon_{1,n}$. We define the expanding domain

$$\mathcal{O}_n^1 = \{X \in \mathbb{R}^2 :\ Y_{1,n} + \varepsilon_{1,n} X \in \mathcal{O}\}$$

and we define $\widetilde{w}^1_{1,n}, \widetilde{w}^1_{2,n} : \mathcal{O}^1_n \to \mathbb{R}$ by setting

$$\begin{aligned}\widetilde{w}^1_{1,n}(X) =& w_{1,n}(Y_{1,n} + \varepsilon_{1,n}X) - w_{1,n}(Y_{1,n})\\ \widetilde{w}^1_{2,n}(X) =& w_{2,n}(Y_{1,n} + \varepsilon_{1,n}X) - w_{1,n}(Y_{1,n}).\end{aligned}$$

We note that $\widetilde{w}^1_{1,n}$ is a standard rescaling. Then, $\widetilde{w}^1_{1,n}, \widetilde{w}^1_{2,n}$ is a solution for the Liouville system

$$\begin{cases} -\Delta\widetilde{w}^1_{1,n} =& \widetilde{V}^1_{1,n} e^{\widetilde{w}^1_{1,n}} - \widetilde{V}^1_{2,n} e^{\widetilde{w}^1_{2,n}}\\ -\Delta\widetilde{w}^1_{2,n} =& -\widetilde{V}^1_{1,n} e^{\widetilde{w}^1_{1,n}} + \widetilde{V}^1_{2,n} e^{\widetilde{w}^1_{2,n}}\end{cases} \tag{2.12}$$

in $\mathcal{O}^1_n$, where $\widetilde{V}^1_{1,n}(X) = V_{1,n}(Y_{1,n} + \varepsilon_{1,n}X)$ and $\widetilde{V}^1_{2,n}(X) = V_{2,n}(Y_{1,n} + \varepsilon_{1,n}X)$. We note that (2.4) implies that

$$\widetilde{w}^1_{1,n} + \widetilde{w}^1_{2,n} = -\log\mathcal{I}_1(v_n) - \log\mathcal{I}_2(v_n) - 2w_{1,n}(Y_{1,n}) \to -\infty. \tag{2.13}$$

Similarly, we rescale (2.5) around $Y_{2,n}$ with respect to $\varepsilon_{2,n}$. We define the expanding domain

$$\mathcal{O}^2_n = \{X \in \mathbb{R}^2 :\ Y_{2,n} + \varepsilon_{2,n}X \in \mathcal{O}\}$$

and we define $\widetilde{w}^2_{1,n}, \widetilde{w}^2_{2,n} : \mathcal{O}^2_n \to \mathbb{R}$ by setting

$$\begin{aligned}\widetilde{w}^2_{1,n}(X) =& w_{1,n}(Y_{2,n} + \varepsilon_{2,n}X) - w_{2,n}(Y_{2,n})\\ \widetilde{w}^2_{2,n}(X) =& w_{2,n}(Y_{2,n} + \varepsilon_{2,n}X) - w_{2,n}(Y_{2,n}).\end{aligned}$$

We note that $\widetilde{w}^2_{2,n}$ is a standard rescaling. Then, $\widetilde{w}^2_{1,n}, \widetilde{w}^2_{2,n}$ is a solution for the Liouville system

$$\begin{cases} -\Delta\widetilde{w}^2_{1,n} =& \widetilde{V}^2_{1,n} e^{\widetilde{w}^2_{1,n}} - \widetilde{V}^2_{2,n} e^{\widetilde{w}^2_{2,n}}\\ -\Delta\widetilde{w}^2_{2,n} =& -\widetilde{V}^2_{1,n} e^{\widetilde{w}^2_{1,n}} + \widetilde{V}^2_{2,n} e^{\widetilde{w}^2_{2,n}}\end{cases} \tag{2.14}$$

in $\mathcal{O}^2_n$, where $\widetilde{V}^2_{1,n}(X) = V_{1,n}(Y_{2,n}+\varepsilon_{2,n}X)$ and $\widetilde{V}^2_{2,n}(X) = V_{2,n}(Y_{2,n}+\varepsilon_{2,n}X)$. Furthermore, as above,

$$\widetilde{w}^2_{1,n} + \widetilde{w}^2_{2,n} = -\log\mathcal{I}_1(v_n) - \log\mathcal{I}_2(v_n) - 2w_{2,n}(Y_{2,n}) \to -\infty. \tag{2.15}$$

We observe that

$$0 \leqslant \widetilde{V}^k_{i,n}(X) \leqslant C, \qquad \int_{\mathcal{O}^k_n} e^{\widetilde{w}^k_{i,n}}\,dX \leqslant C$$

for $i, k = 1, 2$, for some $C > 0$ independent of $X$ and $n$. Therefore, the Brezis-Merle alternative Lemma 2.1 may be applied locally to the Liouville systems (2.12)–(2.14).

At this point, the remaining part of the proof is completely analogous to [13], and therefore we just outline it briefly. In view of the identities (2.13)–(2.15), we rule out Alternative 1. On the other hand, since $\widetilde{w}^1_{1,n}(0) = 0 = \widetilde{w}^2_{2,n}(0)$, we rule out Alternative 3. Let $\widetilde{w}_k : \mathbb{R}^2 \to \mathbb{R}$ be such that $\widetilde{w}^k_{k,n} \to \widetilde{w}_k$ uniformly on compact subsets of $\mathbb{R}^2$. In view of (2.15), the rescaled functions $\widetilde{w}^2_{1,n}$ and $\widetilde{w}^2_{2,n}$ cannot converge locally uniformly to a locally bounded function, and therefore the residual term in the rescaled equations is necessarily either zero, or a finite sum of negative Dirac masses. It follows that the limit equations for $\widetilde{w}^k_{k,n}$ are one of the following.

Either the standard Liouville equation

$$-\Delta\tilde{w}_k = V_k(0)e^{\tilde{w}_k} \qquad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^{\tilde{w}_k} < +\infty,$$

$k = 1, 2$, in which case, in view of [5], we derive

$$m_k \geqslant \int_{\mathbb{R}^2} V_k(0)e^{\tilde{w}_k} = 8\pi,$$

where for convenience we denote $m_1 = m_+(p_0)$, $m_2 = m_-(p_0)$.

Or, the singular Liouville equation

$$-\Delta\tilde{w}_k = V_k(0)e^{\tilde{w}_k} - \sum_{X_0\in\mathcal{S}} \alpha(X_0)\delta_{X_0} \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^{\tilde{w}_k} < +\infty, \tag{2.16}$$

where $\mathcal{S} \subset \mathbb{R}^2$ is a finite set and $\alpha(X_0) \geqslant 4\pi$ for any $X_0 \in \mathcal{S}$, in which case we derive

$$m_k \geqslant \int_{\mathbb{R}^2} V_k(0)e^{\tilde{w}_k} > 4\pi + \sum_{x_0\in\mathcal{S}} \alpha(x_0) > 8\pi$$

in view of [6]. In particular, we conclude that $m_k \geqslant 8\pi$ and (1.18) is established. □

**3 Proof of Theorem 1.3** We derive the proof of Theorem 1.3 by adapting arguments from [12], Theorem 2.1–(III). See also [18], Theorem 4.1. Under the assumptions of Theorem 1.3, since $p_0 \in \mathcal{S}_+ \setminus \mathcal{S}_-$, we necessarily have $t > 0$. therefore, assumption $(\mathcal{I})$ implies that

$$\mathcal{I}_1(v) \geq c_1 \int_\Omega e^v \tag{3.1}$$

for some $c_1 > 0$, independent of $v \in \mathcal{E}$. In order to prove Theorem 1.3 it clearly suffices to show the following.

**Lemma 3.1.** *Under the assumptions of Theorem 1.3, we have*

$$\lim_{n\to\infty} \mathcal{I}_1(v_n) = +\infty.$$

*Proof.* We denote $G^T(\cdot, p_0) = \min\{T, G(\cdot, p_0)\}$. We estimate:

$$u_{1,n} = G * \lambda_n t \,\frac{e^{v_n}}{\mathcal{I}_1(v_n)} \geq G^T * \lambda_n t \,\frac{e^{v_n}}{\mathcal{I}_1(v_n)} \to G^T * \sum_{p\in\mathcal{S}_+} (n_{+,p}\delta_p + r_1).$$

In particular, recalling that in view of Theorem 1.1, case 2–i) we have $n_{+,p_0} = 8\pi$, we have

$$u_{1,n}(x) \geq 8\pi G^T(x, p_0) - C.$$

In a local coordinate patch $(\psi, \mathcal{U})$, identifying $u_{1,n}(X) = u_{1,n}(\psi^{-1}(X))$, we derive that

$$e^{u_{1,n}(X)} \geq c\left[\frac{1}{|X|^4}\right]^T,$$

where for a general function $f$ defined on $\Psi(\mathcal{U})$ we denote $f^T = \min\{T, f\}$. In particular, for any $T > 0$ we have

$$\liminf_{n\to\infty} e^{u_{1,n}(X)} \geq c\left[\frac{1}{|X|^4}\right]^T.$$

By Fatou's lemma, we derive

$$\liminf_{n\to\infty}\int_{\Psi(\mathcal{U})} e^{u_{1,n}(X)}\,dX \geq c\int_{\Psi(\mathcal{U})}\left[\frac{1}{|X|^4}\right]^T dX.$$

Since $T$ is arbitrary, we conclude that $\lim_{n\to\infty}\int_{\Psi(\mathcal{U})} e^{u_{1,n}(X)}\,dX = +\infty$. Finally, since $u_{2,n}$ is bounded, using (3.1) we conclude that

$$\lim_{n\to\infty}\mathcal{I}_1(v_n) \geq c_1\int_\Omega e^v dv_g \geq c_2\int_{\Psi(\mathcal{U})} e^{u_{1,n}(X)}\,dX = +\infty.$$

□

**4 Proof of Theorem 1.4** We recall a particular case of Lemma 4.1 of [12] that will be useful in the sequel. Here $\mathcal{O}$ denotes a bounded domain in $\mathbb{R}^2$ with smooth boundary, and $0\in\mathcal{O}$ is assumed without loss of generality.

**Lemma 4.1** ([12])**.** *Let $f_n$ be a sequence in $W^{1,\infty}(\mathcal{O})$ satisfying*

$$\nabla f_n \to \mathbf{F} \qquad \text{in } L^\infty(\mathcal{O})^2$$

*for some $\mathbf{F}\in C(\mathcal{O})^2$. Moreover, suppose that $\{v_n\}\subset W_0^{1,2}(\mathcal{O})$ is a solution sequence to*

$$\begin{cases} -\Delta v_n = e^{v_n+f_n} & \text{in } \mathcal{O}\\ v_n = 0 & \text{on } \partial\mathcal{O}\end{cases} \tag{4.1}$$

*and that*

$$e^{v_n+f_n}\to 8\pi\delta_0 \qquad *-\text{ weakly in } \mathcal{M}(\bar{\mathcal{O}}).$$

*Then,*

$$\mathbf{F}(0) = -8\pi\nabla_X H_{\mathcal{O}}(X,0)|_{X=0}\,.$$

Let $p_0\in\mathcal{S}_+\setminus\mathcal{S}_-$ and let $(\mathcal{U},\Psi)$ satisfy (1.20) around $p_0$. We may assume that $\partial\mathcal{O}$ is smooth. We identify $u_{1,n}(X) = u_{1,n}(\Psi^{-1}(X))$, where $u_{1,n}$ is defined in (1.14). Then,

$$-\Delta u_{1,n} = \lambda_n\left(\frac{te^{u_{1,n}-u_{2,n}}}{\mathcal{I}_1(v_n)} - \kappa_1(v_n)\right)e^{\xi(X)} \qquad \text{in } \mathcal{O}. \tag{4.2}$$

Setting

$$\tilde\mu_{1,n} = \lambda_n t\frac{e^{u_{1,n}-u_{2,n}}}{\mathcal{I}_1(v_n)}e^{\xi} \tag{4.3}$$

we have $\tilde\mu_{1,n}\overset{*}{\rightharpoonup}8\pi\delta_0$ in $\mathcal{M}(\bar{\mathcal{O}})$. Moreover, by the assumption on $\mathcal{U}$ we also have

$$\limsup_{n\to\infty}\|u_{1,n}\|_{W^{2-\frac{1}{r},r}(\partial\mathcal{O})} < +\infty$$

for any $r\in(2,+\infty)$. Hence, the function $\tilde h_n(X)$ defined by

$$\begin{cases}\Delta\tilde h_n = 0 & \text{in } \mathcal{O}\\[2ex] \tilde h_n = u_{1,n} & \text{on } \partial\mathcal{O},\end{cases} \tag{4.4}$$

satisfies

$$\limsup_{n\to\infty} \|\tilde{h}_n\|_{W^{2,r}(\mathcal{O})} < +\infty.$$

It follows that we may assume

$$\tilde{h}_n \to \tilde{h}_\infty \qquad \text{in } C^1(\bar{\mathcal{O}}).$$

Now, we set

$$\tilde{u}_{1,n} = u_{1,n} + \lambda_n \kappa_1(v_n) h_\xi - \tilde{h}_n$$

where $h_\xi$ is defined by (2.2). Then, $\tilde{u}_{1,n}$ satisfies

$$\text{(4.5)} \qquad \begin{cases} -\Delta \tilde{u}_{1,n} = \lambda_n t \frac{e^{u_{1,n}-u_{2,n}}}{\mathcal{I}_1(v_n)} e^\xi = \lambda_n t \frac{\exp\{-\lambda_n\kappa_1(v_n)h_\xi + \tilde{h}_n - u_{2,n} + \xi\}}{\mathcal{I}_1(v_n)} e^{\tilde{u}_{1,n}} & \text{in } \mathcal{O} \\ \tilde{u}_{1,n} = 0 & \text{on } \partial\mathcal{O}. \end{cases}$$

That is, $\tilde{u}_{1,n}$ satisfies (4.1) with $f = f_n$ given by

$$f_n = -\lambda_n \kappa_1(v_n) h_\xi + \tilde{h}_n + \log\lambda_n - u_{2,n} - \log(\mathcal{I}_1(v_n)) + \xi.$$

Now we are ready to complete the proof of Theorem 1.4.

*Proof of Theorem 1.4.* By taking subsequences, we may assume that $\kappa_1(v_n) \to \kappa_1$. Then, we also have $f_n \in W^{1,\infty}(\mathcal{O})$,

$$\nabla f_n \to \nabla(-\lambda_0\kappa_1 h_\xi + \tilde{h}_\infty - u_2 + \xi) \qquad \text{in } L^\infty(\mathcal{O})^2$$

and

$$\nabla(-\lambda_0\kappa_1 h_\xi + \tilde{h}_\infty - u_2 + \xi) \qquad \text{belongs to } C(\mathcal{O})^2.$$

Hence, applying Lemma 4.1 to the sequence $\{\tilde{u}_{1,n}\}$ we conclude that

$$\text{(4.6)} \qquad \nabla(-\lambda_0\kappa_1 h_\xi + \tilde{h}_\infty - u_2 + \xi)\Big|_{X=0} = -8\pi \nabla_X H_{\mathcal{O}}(X,0)|_{X=0}\,.$$

We claim that

$$\text{(4.7)} \qquad -\lambda_0\kappa_1 h_\xi + \tilde{h}_\infty = 8\pi \sum_{p\in\mathcal{S}_+} G(\Psi^{-1}(\cdot),p) - 8\pi G_{\mathcal{O}}(\cdot,0).$$

To see this, we note that (1.13) implies

$$\mu_{1,n}(\Omega) \to 8\pi\, \sharp\mathcal{S}_+.$$

On the other hand, by definition of $\kappa_1$ we also have

$$\mu_{1,n}(\Omega) \to \lambda_0\kappa_1|\Omega|.$$

Hence, we conclude that

$$\text{(4.8)} \qquad \lambda_0\kappa_1|\Omega| = 8\pi\,\sharp\mathcal{S}_+.$$

It follows that $w = -\lambda_0\kappa_1 h_\xi + \tilde{h}_\infty$ satisfies the Dirichlet problem

$$\text{(4.9)} \qquad \begin{cases} -\Delta w = -\dfrac{8\pi\,\sharp\mathcal{S}_+}{|\Omega|}\, e^\xi & \text{in } \mathcal{O} \\ w = 8\pi \displaystyle\sum_{p\in\mathcal{S}_+} G(\Psi^{-1}(\cdot),p) & \text{on } \partial\mathcal{O}. \end{cases}$$

By uniqueness, we conclude that $w = 8\pi \sum_{p\in\mathcal{S}_+} G(\Psi^{-1}(\cdot), p) - 8\pi G_{\mathcal{O}}(\cdot, 0)$ so that (4.7) is established. Finally, we note that by (1.21) with $y = p_0$ we obtain

$$H_\Psi(\Psi^{-1}(X), p_0) = G(\Psi^{-1}(X), p_0) + \frac{1}{2\pi}\log|X|.$$

Therefore, using (1.22), we have

$$G(\Psi^{-1}(X), p_0) - G_{\mathcal{O}}(X, 0) = H_\Psi(\Psi^{-1}(X), p_0) - H_{\mathcal{O}}(X, 0)$$

so that

$$\sum_{p\in\mathcal{S}_+} G(\Psi^{-1}(\cdot), p) - G_{\mathcal{O}}(\cdot, 0) = \sum_{p\in\mathcal{S}_+\setminus\{p_0\}} G(\Psi^{-1}(\cdot), p) + H_\Psi(\Psi^{-1}(\cdot), p_0) - H_{\mathcal{O}}(\cdot, 0). \tag{4.10}$$

Combining (4.6), (4.7) and (4.10), we derive the asserted necessary condition (1.23). □

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# ON DEGREE OF NON-CONVEXITY OF FUZZY SETS

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Abstract. In the present paper, the convexity of fuzzy sets is generalized based on conjunctive aggregation functions, and the degree of the non-convexity of fuzzy sets is considered as an application of the generalized convexity. Then, the properties of the generalized convexity of fuzzy sets, and the properties of the degree of the non-convexity of fuzzy sets with respect to operations are investigated.

**1. Introduction.** The concept of fuzzy sets has been primarily introduced for representing sets containing uncertainty or vagueness by Zadeh [7] as fuzzy set theory. Then, fuzzy set theory has been applied in various areas of decision making theory including economics and optimization, etc., widely. We consider fuzzy sets on $\mathbb{R}^n$, and identify each fuzzy set on $\mathbb{R}^n$ with its membership function. The convexity of a fuzzy set is defined by the quasiconcavity of its membership function. Quasiconcavity of functions is defined using the minimum operation. Due to the importance in economics and optimization, etc., several generalizations of quasiconcavity of functions have been introduced and investigated; see [6] and the references therein. In [3], the quasiconcavity of membership functions is generalized by allowing arbitrary conjunctive aggregation functions instead of the minimum operation. In [5], the degree of the non-quasiconcavity of membership functions is proposed as an application of the generalized quasiconcavity. Since the convexity of a fuzzy set is defined by the quasiconcavity of its membership function, the generalized quasiconcavity of membership functions can be regarded as the generalized convexity of fuzzy sets, and the degree of the non-quasiconcavity of membership functions can be regarded as the non-convexity of fuzzy sets.

In the present paper, the properties of the generalized convexity of fuzzy sets, and the properties of the degree of the non-convexity of fuzzy sets with respect to operations are investigated.

The remainder of the present paper is organized as follows. In Section 2, some properties of continuous conjunctive aggregation functions are presented. In Section 3, some properties of fuzzy sets with respect to operations are presented. In Section 4, some properties of the generalized convexity of fuzzy sets, and some properties of the degree of the non-convexity of fuzzy sets with respect to operations are presented. Finally, conclusions are presented in Section 5.

**2. Aggregation functions** In this section, the properties of continuous conjunctive aggregation functions are investigated. Conjunctive aggregation functions are used in order to generalize the convexity of fuzzy sets. For details of aggregation functions, see [1,6].

For $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$, we set $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}, [a, b[= \{x \in \mathbb{R} : a \leq x < b\}, ]a, b] = \{x \in \mathbb{R} : a < x \leq b\}$, and $]a, b[= \{x \in \mathbb{R} : a < x < b\}$.

First, aggregation functions are defined.

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**Definition 1.** (See [1].) Let $G : [0,1]^2 \to [0,1]$. The function $G$ is called an aggregation function if the following two axioms are satisfied: (G1) if $x_i, y_i \in [0,1], x_i \le y_i, i = 1,2$, then $G(x_1,x_2) \le G(y_1,y_2)$ (monotonicity), and (G2) $G(0,0) = 0$ and $G(1,1) = 1$ (boundary condition).

Next, the definition of a property of aggregation functions is given.

**Definition 2.** (See [1].) Let $G : [0,1]^2 \to [0,1]$ be an aggregation function. The aggregation function $G$ is said to be conjunctive if $G(x,y) \le \min\{x,y\}$ for any $x,y \in [0,1]$.

Next, the definition of a relationship between two aggregation functions is given.

**Definition 3.** (See [4].) Let $G, G' : [0,1]^2 \to [0,1]$ be aggregation functions. $G$ is said to dominate $G'$ ($G \gg G'$) if $G(G'(x_1,y_1),\ G'(x_2,y_2)) \ge G'(G(x_1,\ x_2),\ G(y_1,\ y_2))$ for any $x_i, y_i \in [0,1], i = 1,2$.

The concept of the domination for aggregation functions is closely related to the preservation of $T$-transitivity in aggregating fuzzy relations, where $T$ is a triangular norm; see [4].

In order to measure the non-convexity of fuzzy sets in Section 4, we consider continuous conjunctive aggregation functions $G^{(p)} : [0,1]^2 \to [0,1], p \in [1,\infty[$ defined as

$$G^{(p)}(x,y) = [\min\{x,y\}]^p \quad \text{for } x,y \in [0,1] \tag{1}$$

for each $p \in [1,\infty[$. The larger $p$ is, the larger the difference between $G^{(p)}$ and $\min = G^{(1)}$ is.

The following proposition shows a property of continuous conjunctive aggregation function.

**Proposition 1.** Let $G : [0,1]^2 \to [0,1]$ be a continuous conjunctive aggregation function, and let $A, B \subset [0,1]$. Then,

$$\sup_{x\in A, y\in B} G(x,y) = G(\sup A, \sup B),$$

where $\sup\emptyset = 0$ for $\emptyset \subset [0,1]$.

**Proof.** If $A = \emptyset$ or $B = \emptyset$, then

$$\sup_{x\in A, y\in B} G(x,y) = 0 = G(\sup A, \sup B).$$

Assume that $A \ne \emptyset$ and $B \ne \emptyset$. Since $G(x,y) \le G(\sup A, \sup B)$ for any $x \in A$ and any $y \in B$ by the monotonicity of $G$, we have

$$\sup_{x\in A, y\in B} G(x,y) \le G(\sup A, \sup B).$$

Suppose that

$$\sup_{x\in A, y\in B} G(x,y) < G(\sup A, \sup B).$$

We set $\alpha = \sup A$ and $\beta = \sup B$. Note that $\alpha > 0$ and $\beta > 0$. We set

$$\varepsilon = \frac{G(\alpha,\beta) - \sup\limits_{x\in A, y\in B} G(x,y)}{2} > 0.$$

By the monotonicity of $G$ and the continuity of $G$ at $(\alpha, \beta)$, there exists $\delta$ such that $0 < \delta < \min\{\alpha, \beta\}$ and $G(\alpha, \beta) - \varepsilon < G(x, y) \leq G(\alpha, \beta)$ for any $(x, y) \in ]\alpha - \delta, \alpha] \times ]\beta - \delta, \beta]$. By the definitions of $\alpha$ and $\beta$, there exists $(x_0, y_0) \in (]\alpha - \delta, \alpha] \times ]\beta - \delta, \beta]) \cap (A \times B)$. Then, it follows that

$$G(x_0, y_0) \leq \sup_{x \in A, y \in B} G(x, y) < G(\alpha, \beta) - \varepsilon < G(x_0, y_0),$$

which is a contradiction. □

The following proposition shows a relationship between $\min = G^{(1)}$ and $G^{(p)}, p \in [1, \infty[$, where $G^{(p)}, p \in [1, \infty[$ are the continuous conjunctive aggregation functions defined by (1).

**Proposition 2.** $\min = G^{(1)} \gg G^{(p)}$ for any $p \in [1, \infty[$.

**Proof.** Let $x_i, y_i \in [0, 1], i = 1, 2$, and let $z$ be the minimum among $x_i, y_i, i = 1, 2$. Then, we have

$$\min\{G^{(p)}(x_1, y_1), G^{(p)}(x_2, y_2)\} = z^p = G^{(p)}(\min\{x_1, x_2\}, \min\{y_1, y_2\}).$$

□

**3. Fundamental properties of fuzzy sets** In this section, the properties of fuzzy sets with respect to operations are investigated.

We consider fuzzy sets on $\mathbb{R}^n$, and identify a fuzzy set $\widetilde{a}$ on $\mathbb{R}^n$ with its membership function $\widetilde{a} : \mathbb{R}^n \to [0, 1]$. Let $\mathcal{F}(\mathbb{R}^n)$ be the set of all fuzzy sets on $\mathbb{R}^n$.

For $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ and $\alpha \in ]0, 1]$, the set

$$[\widetilde{a}]_\alpha = \{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{a}(\boldsymbol{x}) \geq \alpha\}$$

is called the $\alpha$-level set of $\widetilde{a}$.

For a crisp set $S \subset \mathbb{R}^n$, the function $c_S : \mathbb{R}^n \to \{0, 1\}$ defined as

$$c_S(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in S, \\ 0 & \text{if } \boldsymbol{x} \notin S \end{cases}$$

for each $\boldsymbol{x} \in \mathbb{R}^n$ is called the indicator function of $S$. Whenever we consider $c_S$ as a fuzzy set, $c_S : \mathbb{R}^n \to \{0, 1\}$ is interpreted as $c_S : \mathbb{R}^n \to [0, 1]$.

A fuzzy set $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ can be represented as

$$\widetilde{a} = \sup_{\alpha \in ]0,1]} \alpha c_{[\widetilde{a}]_\alpha}, \tag{2}$$

which is well-known as the decomposition theorem; see [2].

A fuzzy set $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ is said to be convex if $\widetilde{a}(\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y}) \geq \min\{\widetilde{a}(\boldsymbol{x}), \widetilde{a}(\boldsymbol{y})\}$ for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and any $\lambda \in ]0, 1[$. That is, $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ is said to be convex if $\widetilde{a}$ is a quasiconcave function.

We set

$$\mathcal{S}(\mathbb{R}^n) = \{\{S_\alpha\}_{\alpha \in ]0,1]} : S_\alpha \subset \mathbb{R}^n, \alpha \in ]0, 1], \text{ and } S_\beta \supset S_\gamma \text{ for } \beta, \gamma \in ]0, 1] \text{ with } \beta < \gamma\},$$

and define $M : \mathcal{S}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ as

$$M(\{S_\alpha\}_{\alpha \in ]0,1]}) = \sup_{\alpha \in ]0,1]} \alpha c_{S_\alpha}$$

for each $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$. For $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$ and $\boldsymbol{x} \in \mathbb{R}^n$, it follows that

$$M(\{S_\alpha\}_{\alpha\in]0,1]})(\boldsymbol{x}) = \sup_{\alpha\in]0,1]} \alpha c_{S_\alpha}(\boldsymbol{x}) = \sup\{\alpha \in ]0,1] : \boldsymbol{x} \in S_\alpha\},$$

where $\sup\emptyset = 0$ for $\emptyset \subset ]0,1]$. The decomposition theorem (2) can be represented as

$$\widetilde{a} = M(\{[\widetilde{a}]_\alpha\}_{\alpha\in]0,1]})$$

for $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$.

When $\widetilde{a} = M(\{S_\alpha\}_{\alpha\in]0,1]})$ for $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ and $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$, $\widetilde{a}$ is called the fuzzy set generated by $\{S_\alpha\}_{\alpha\in]0,1]}$, and $\{S_\alpha\}_{\alpha\in]0,1]}$ is called the generator of $\widetilde{a}$.

The following proposition shows a relationship between the inclusion relation of two generators of two fuzzy sets and the inclusion relation of the two fuzzy sets.

**Proposition 3.** Let $\{S_\alpha\}_{\alpha\in]0,1]}, \{T_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$. If $S_\alpha \subset T_\alpha$ for any $\alpha \in ]0,1]$, then $M(\{S_\alpha\}_{\alpha\in]0,1]}) \leq M(\{T_\alpha\}_{\alpha\in]0,1]})$.

**Proof.** For any $\boldsymbol{x} \in \mathbb{R}^n$, it follows that

$$\{\alpha \in ]0,1] : \boldsymbol{x} \in S_\alpha\} \subset \{\alpha \in ]0,1] : \boldsymbol{x} \in T_\alpha\},$$

and that

$$\begin{aligned} M(\{S_\alpha\}_{\alpha\in]0,1]})(\boldsymbol{x}) &= \sup\{\alpha \in ]0,1] : \boldsymbol{x} \in S_\alpha\} \\ &\leq \sup\{\alpha \in ]0,1] : \boldsymbol{x} \in T_\alpha\} = M(\{T_\alpha\}_{\alpha\in]0,1]})(\boldsymbol{x}). \end{aligned}$$

□

The following proposition shows a relationship between a generator of a fuzzy set and level sets of the fuzzy set.

**Proposition 4.** Let $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$, and let $\widetilde{a} = M(\{S_\alpha\}_{\alpha\in]0,1]})$. In addition, let $\alpha \in ]0,1]$. Then,

$$[\widetilde{a}]_\alpha = \bigcap_{\beta\in]0,\alpha[} S_\beta.$$

**Proof.** It follows that

$$\begin{aligned} \boldsymbol{x} \in [\widetilde{a}]_\alpha \quad &\Leftrightarrow \quad \widetilde{a}(\boldsymbol{x}) = \sup\{\beta \in ]0,1] : \boldsymbol{x} \in S_\beta\} \geq \alpha \\ &\Leftrightarrow \quad \boldsymbol{x} \in S_\beta, \beta \in ]0,\alpha[ \\ &\Leftrightarrow \quad \boldsymbol{x} \in \bigcap_{\beta\in]0,\alpha[} S_\beta. \end{aligned}$$

□

The following definitions are addition and scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$ by Zadeh's extension principle. See [2] for Zadeh's extension principle.

**Definition 4.** For $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$, we define $\widetilde{a} + \widetilde{b}, \lambda\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ as

$$(\widetilde{a} + \widetilde{b})(\boldsymbol{x}) = \sup_{\boldsymbol{x}=\boldsymbol{y}+\boldsymbol{z}} \min\left\{\widetilde{a}(\boldsymbol{y}), \widetilde{b}(\boldsymbol{z})\right\}, \quad (\lambda\widetilde{a})(\boldsymbol{x}) = \sup_{\boldsymbol{x}=\lambda\boldsymbol{y}} \widetilde{a}(\boldsymbol{y})$$

for each $\boldsymbol{x} \in \mathbb{R}^n$, respectively.

The following proposition shows a property of level sets of fuzzy sets with respect to scalar multiplication.

**Proposition 5.** Let $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$, and let $\lambda \in \mathbb{R}$. In addition, let $\alpha \in ]0,1]$. Then, $[\lambda\widetilde{a}]_\alpha \supset \lambda[\widetilde{a}]_\alpha$.

**Proof.** Let $\boldsymbol{x} \in \lambda[\widetilde{a}]_\alpha$. Then, there exists $\boldsymbol{y}_0 \in [\widetilde{a}]_\alpha$ such that $\boldsymbol{x} = \lambda\boldsymbol{y}_0$. Since $\widetilde{a}(\boldsymbol{y}_0) \geq \alpha$, it follows that $(\lambda\widetilde{a})(\boldsymbol{x}) \geq \widetilde{a}(\boldsymbol{y}_0) \geq \alpha$. Therefore, we have $\boldsymbol{x} \in [\lambda\widetilde{a}]_\alpha$. □

The following proposition shows a relationship between scalar multiplication of fuzzy sets and generators of the fuzzy sets.

**Proposition 6.** Let $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$, and let $\widetilde{a} = M(\{S_\alpha\}_{\alpha\in]0,1]})$. In addition, let $\lambda \in \mathbb{R}$. Then,

$$\lambda\widetilde{a} = M(\{\lambda S_\alpha\}_{\alpha\in]0,1]}) = \sup_{\alpha\in]0,1]} \alpha c_{\lambda S_\alpha}.$$

**Proof.** For each $\alpha \in ]0,1]$, it follows that $[\widetilde{a}]_\alpha = \bigcap_{\beta\in]0,\alpha[} S_\beta \supset S_\alpha$ from Proposition 4, and that $[\lambda\widetilde{a}]_\alpha \supset \lambda[\widetilde{a}]_\alpha \supset \lambda S_\alpha$ from Proposition 5. Thus, it follows that $\lambda\widetilde{a} \geq M(\{\lambda S_\alpha\}_{\alpha\in]0,1]})$ from Proposition 3 and the decomposition theorem (2). Suppose that there exists $\boldsymbol{x}_0 \in \mathbb{R}^n$ such that $(\lambda\widetilde{a})(\boldsymbol{x}_0) > M(\{\lambda S_\alpha\}_{\alpha\in]0,1]})(\boldsymbol{x}_0)$. We set $\gamma = M(\{\lambda S_\alpha\}_{\alpha\in]0,1]})(\boldsymbol{x}_0)$. Then, since $(\lambda\widetilde{a})(\boldsymbol{x}_0) = \sup_{\boldsymbol{x}_0=\lambda\boldsymbol{y}} \widetilde{a}(\boldsymbol{y}) > \gamma$, there exists $\boldsymbol{y}_0 \in \mathbb{R}^n$ such that $\boldsymbol{x}_0 = \lambda\boldsymbol{y}_0$ and $\widetilde{a}(\boldsymbol{y}_0) > \gamma$. We set $\eta = \widetilde{a}(\boldsymbol{y}_0) > \gamma$. It follows that $\boldsymbol{y}_0 \in [\widetilde{a}]_\eta = \bigcap_{\beta\in]0,\eta[} S_\beta$ from Proposition 4, and that $\boldsymbol{x}_0 = \lambda\boldsymbol{y}_0 \in \lambda S_\beta$ for any $\beta \in ]0,\eta[$. Therefore, we have $\gamma = M(\{\lambda S_\alpha\}_{\alpha\in]0,1]})(\boldsymbol{x}_0) = \sup\{\alpha \in ]0,1] : \boldsymbol{x}_0 \in \lambda S_\alpha\} \geq \eta > \gamma$, which is a contradiction. □

The following proposition shows the properties of scalar multiplication of fuzzy sets.

**Proposition 7.** Let $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$, and let $\lambda, \mu \in \mathbb{R}$.

(i) $(\lambda\mu)\widetilde{a} = \lambda(\mu\widetilde{a})$.

(ii) $1\widetilde{a} = \widetilde{a}$.

**Proof.**
(i) From the decomposition theorem (2) and Proposition 6, we have

$$\begin{aligned}
(\lambda\mu)\widetilde{a} &= M\left(\{(\lambda\mu)[\widetilde{a}]_\alpha\}_{\alpha\in]0,1]}\right) \\
&= M\left(\{\lambda(\mu[\widetilde{a}]_\alpha)\}_{\alpha\in]0,1]}\right) \\
&= \lambda M\left(\{\mu[\widetilde{a}]_\alpha\}_{\alpha\in]0,1]}\right) \\
&= \lambda(\mu\widetilde{a}).
\end{aligned}$$

(ii) From the decomposition theorem (2) and Proposition 6, we have

$$\begin{aligned}
1\widetilde{a} &= M\left(\{1[\widetilde{a}]_\alpha\}_{\alpha\in]0,1]}\right) \\
&= M\left(\{[\widetilde{a}]_\alpha\}_{\alpha\in]0,1]}\right) \\
&= \widetilde{a}.
\end{aligned}$$

□

**4. Generalized convexity and degree of non-convexity** In this section, the properties of the generalized convexity of fuzzy sets, and the properties of the degree of the non-convexity of fuzzy sets with respect to operations are investigated.

The following definition is a generalization of the convexity of fuzzy sets by allowing arbitrary conjunctive aggregation functions instead of the minimum operation, and is first proposed in [3] as the generalized quasiconcavity of membership functions. We consider the generalized quasiconcavity of membership functions as the generalized convexity of fuzzy sets.

**Definition 5.** (See [3].) Let $G : [0,1]^2 \to [0,1]$ be a conjunctive aggregation function, and let $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$. The fuzzy set $\widetilde{a}$ is said to be $G$-convex if

$$\widetilde{a}(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \geq G(\widetilde{a}(\boldsymbol{x}), \widetilde{a}(\boldsymbol{y}))$$

for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and any $\lambda \in ]0,1[$.

For a conjunctive aggregation function $G : [0,1]^2 \to [0,1]$ and $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$, if $\widetilde{a}$ is convex, then $\widetilde{a}$ is $G$-convex from Definition 5.

The following proposition shows the properties of the $G$-convexity of fuzzy sets with respect to operations.

**Proposition 8.** Let $G : [0,1]^2 \to [0,1]$ be a continuous conjunctive aggregation function, and let $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R}^n)$. In addition, let $\lambda \in \mathbb{R}$.

(i) Assume that $\min = G^{(1)} \gg G$. If $\widetilde{a}$ and $\widetilde{b}$ are $G$-convex, then $\widetilde{a} + \widetilde{b}$ is $G$-convex.

(ii) If $\widetilde{a}$ is $G$-convex, then $\lambda\widetilde{a}$ is $G$-convex. When $\lambda \neq 0$, if $\lambda\widetilde{a}$ is $G$-convex, then $\widetilde{a}$ is $G$-convex.

**Proof.**
(i) Let $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}^n$, and let $\mu \in ]0,1[$. Fix any $\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{y}', \boldsymbol{z}' \in \mathbb{R}^n$ such that $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$ and $\boldsymbol{x}' = \boldsymbol{y}' + \boldsymbol{z}'$. Since $\mu\boldsymbol{x} + (1-\mu)\boldsymbol{x}' = (\mu\boldsymbol{y} + (1-\mu)\boldsymbol{y}') + (\mu\boldsymbol{z} + (1-\mu)\boldsymbol{z}')$, it follows that

$$\begin{aligned}
&(\widetilde{a} + \widetilde{b})(\mu\boldsymbol{x} + (1-\mu)\boldsymbol{x}') \\
&\geq \min\{\widetilde{a}(\mu\boldsymbol{y} + (1-\mu)\boldsymbol{y}'), \widetilde{b}(\mu\boldsymbol{z} + (1-\mu)\boldsymbol{z}')\} \\
&\geq \min\{G(\widetilde{a}(\boldsymbol{y}), \widetilde{a}(\boldsymbol{y}')), G(\widetilde{b}(\boldsymbol{z}), \widetilde{b}(\boldsymbol{z}'))\} \quad \text{(from the } G\text{-convexity of } \widetilde{a} \text{ and } \widetilde{b}) \\
&\geq G(\min\{\widetilde{a}(\boldsymbol{y}), \widetilde{b}(\boldsymbol{z})\}, \min\{\widetilde{a}(\boldsymbol{y}'), \widetilde{b}(\boldsymbol{z}')\}) \quad (\text{from } \min = G^{(1)} \gg G).
\end{aligned}$$

By the arbitrariness of $\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{y}', \boldsymbol{z}'$, we have

$$\begin{aligned}
&(\widetilde{a} + \widetilde{b})(\mu\boldsymbol{x} + (1-\mu)\boldsymbol{x}') \\
&\geq \sup_{\substack{\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z} \\ \boldsymbol{x}' = \boldsymbol{y}' + \boldsymbol{z}'}} G(\min\{\widetilde{a}(\boldsymbol{y}), \widetilde{b}(\boldsymbol{z})\}, \min\{\widetilde{a}(\boldsymbol{y}'), \widetilde{b}(\boldsymbol{z}')\}) \\
&= G\left(\sup_{\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}} \min\{\widetilde{a}(\boldsymbol{y}), \widetilde{b}(\boldsymbol{z})\}, \sup_{\boldsymbol{x}' = \boldsymbol{y}' + \boldsymbol{z}'} \min\{\widetilde{a}(\boldsymbol{y}'), \widetilde{b}(\boldsymbol{z}')\}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(from Proposition 1)} \\
&= G((\widetilde{a} + \widetilde{b})(\boldsymbol{x}), (\widetilde{a} + \widetilde{b})(\boldsymbol{x}')).
\end{aligned}$$

(ii) Assume that $\widetilde{a}$ is $G$-convex. Let $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}^n$, and let $\mu \in ]0,1[$. Fix any $\boldsymbol{y}, \boldsymbol{y}' \in \mathbb{R}^n$ such that $\boldsymbol{x} = \lambda \boldsymbol{y}$ and $\boldsymbol{x}' = \lambda \boldsymbol{y}'$. Since $\mu \boldsymbol{x} + (1-\mu)\boldsymbol{x}' = \lambda(\mu \boldsymbol{y} + (1-\mu)\boldsymbol{y}')$, it follows that

$$(\lambda\widetilde{a})(\mu \boldsymbol{x} + (1-\mu)\boldsymbol{x}') \geq \widetilde{a}(\mu \boldsymbol{y} + (1-\mu)\boldsymbol{y}') \geq G(\widetilde{a}(\boldsymbol{y}), \widetilde{a}(\boldsymbol{y}'))$$

from the $G$-convexity of $\widetilde{a}$. By the arbitrariness of $\boldsymbol{y}, \boldsymbol{y}'$, we have

$$\begin{aligned} &(\lambda\widetilde{a})(\mu \boldsymbol{x} + (1-\mu)\boldsymbol{x}') \\ &\geq \sup_{\substack{\boldsymbol{x} = \lambda \boldsymbol{y} \\ \boldsymbol{x}' = \lambda \boldsymbol{y}'}} G(\widetilde{a}(\boldsymbol{y}), \widetilde{a}(\boldsymbol{y}')) \\ &= G\left( \sup_{\boldsymbol{x}=\lambda\boldsymbol{y}} \widetilde{a}(\boldsymbol{y}), \sup_{\boldsymbol{x}'=\lambda\boldsymbol{y}'} \widetilde{a}(\boldsymbol{y}') \right) \quad \text{(from Proposition 1)} \\ &= G((\lambda\widetilde{a})(\boldsymbol{x}), (\lambda\widetilde{a})(\boldsymbol{x}')). \end{aligned}$$

The latter assertion follows from the former assertion and Proposition 7. □

Let $G : [0,1]^2 \to [0,1]$ be a conjunctive aggregation function. Assume that $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ is $G$-convex. Then, the larger the difference between $G$ and $\min = G^{(1)}$ is, the larger the allowable non-convexity of $\widetilde{a}$ is.

Now, the degree of the non-convexity of fuzzy sets is defined. The degree of the non-convexity of fuzzy sets is first proposed in [5] as the degree of the non-quasiconcavity of membership functions. We consider the degree of the non-quasiconcavity of membership functions as the degree of the non-convexity of fuzzy sets.

**Definition 6.** (See [5].) For $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$, the value

$$D(\widetilde{a}) = \min\{p \in [1,\infty[: \widetilde{a} \text{ is } G^{(p)}\text{-convex. }\} - 1 \tag{3}$$

is called the degree of the non-convexity of $\widetilde{a}$, where $\min \emptyset = \infty$ for $\emptyset \subset [1,\infty[$, and $G^{(p)}, p \in [1,\infty[$ are the continuous conjunctive aggregation functions defined by (1).

In (3), if $\{p \in [1,\infty[: \widetilde{a} \text{ is } G^{(p)}\text{-convex. }\} \neq \emptyset$, then the minimum is attained; see [5]. The degree of the non-convexity of $\widetilde{a}$ defined by (3) means that $\widetilde{a}$ is convex when $D(\widetilde{a}) = 0$, and that the larger $D(\widetilde{a})$ is, the larger the non-convexity of $\widetilde{a}$ is.

The following proposition shows the properties of the degree of the non-convexity of fuzzy sets.

**Proposition 9.** (See [5].) Let $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$.

(i) $\widetilde{a}$ is convex if and only if $D(\widetilde{a}) = 0$.

(ii) $\widetilde{a}$ is not $G^{(p)}$-convex for any $p \in [1, D(\widetilde{a}) + 1[$.

(iii) $\widetilde{a}$ is $G^{(p)}$-convex for any $p \in [D(\widetilde{a}) + 1, \infty[$.

The following example illustrates the degree of the non-convexity of fuzzy sets.

**Example 1.** (See [5].) For each $\alpha \in \left[0, \frac{1}{2}\right[$, we define $\widetilde{a}_\alpha \in \mathcal{F}(\mathbb{R})$ as

$$\widetilde{a}_\alpha(x) = \begin{cases} 0 & \text{if } x \in ]-\infty, 0] \cup [6, \infty[, \\ \frac{1}{2}x & \text{if } x \in [0,1], \\ \alpha \sin 4x\pi + \frac{1}{2} & \text{if } x \in [1,2], \\ \frac{1}{2}x - \frac{1}{2} & \text{if } x \in [2,3], \\ -\frac{1}{2}x + \frac{5}{2} & \text{if } x \in [3,4], \\ \alpha \sin 4x\pi + \frac{1}{2} & \text{if } x \in [4,5], \\ -\frac{1}{2}x + 3 & \text{if } x \in [5,6] \end{cases}$$

(Figure 1). Then, $D(\widetilde{a}_\alpha) = \frac{\log\left(\frac{1}{2}-\alpha\right)}{\log\left(\frac{1}{2}+\alpha\right)} - 1$ (Figure 2).

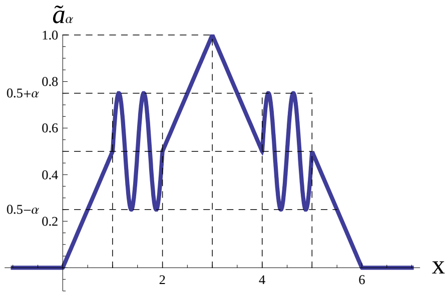


Figure 1. $\widetilde{a}_\alpha(x)$ $(\alpha = 0.25)$.

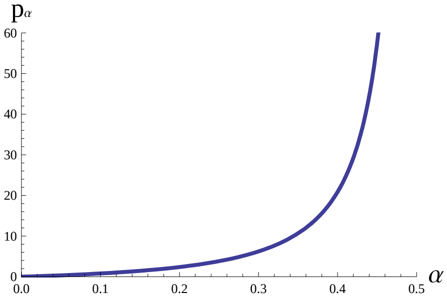


Figure 2. $p_\alpha = \frac{\log\left(\frac{1}{2}-\alpha\right)}{\log\left(\frac{1}{2}+\alpha\right)} - 1,\ \alpha \in \left[0, \frac{1}{2}\right[$.

The following proposition shows the properties of the degree of the non-convexity of fuzzy sets with respect to operations.

**Proposition 10.** Let $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R}^n)$, and let $\lambda \in \mathbb{R}$.

(i) $D(\widetilde{a} + \widetilde{b}) \le \max\{D(\widetilde{a}), D(\widetilde{b})\}$.

(ii) $D(\lambda\widetilde{a}) \le D(\widetilde{a})$. When $\lambda \neq 0$, $D(\lambda\widetilde{a}) = D(\widetilde{a})$.

**Proof.**
(i) From Proposition 9 (iii), $\widetilde{a}$ is $G^{(p)}$-convex for $p \in [D(\widetilde{a}) + 1, \infty[$, and $\widetilde{b}$ is $G^{(p)}$-convex for $p \in [D(\widetilde{b}) + 1, \infty[$. Thus, $\widetilde{a}$ and $\widetilde{b}$ are $G^{(p)}$-convex for $p \in [\max\{D(\widetilde{a}) + 1, D(\widetilde{b}) + 1\}, \infty[$. From Propositions 2 and 8 (i), $\widetilde{a} + \widetilde{b}$ is $G^{(p)}$-convex for $p \in [\max\{D(\widetilde{a}) + 1, D(\widetilde{b}) + 1\}, \infty[$. Therefore, we have

$$D(\widetilde{a} + \widetilde{b}) \le \max\{D(\widetilde{a}), D(\widetilde{b})\}.$$

(ii) From the former assertion of Proposition 8 (ii), we have

$$D(\lambda\widetilde{a}) \le D(\widetilde{a}).$$

From the latter assertion of Propositions 8 (ii), we have

$$D(\lambda\widetilde{a}) = D(\widetilde{a}).$$

when $\lambda \neq 0$. □

The following example illustrates the degree of the non-convexity of fuzzy sets with respect to operations.

**Example 2.** Consider $\widetilde{a}_\alpha \in \mathcal{F}(\mathbb{R})$ for $\alpha \in \left]0, \frac{1}{2}\right[$ defined in Example 1. We set $p_\alpha = \frac{\log\left(\frac{1}{2}-\alpha\right)}{\log\left(\frac{1}{2}+\alpha\right)} - 1$. Then, $p_\alpha > 0$ and $D(\widetilde{a}_\alpha) = p_\alpha$. We set $\widetilde{\mathbb{R}} = c_{\mathbb{R}} \in \mathcal{F}(\mathbb{R})$ and $\widetilde{0} = c_{\{0\}} \in \mathcal{F}(\mathbb{R})$.

(i) Let $\widetilde{a} = \widetilde{a}_\alpha$, and let $\widetilde{b} = \widetilde{\mathbb{R}}$. Since $\widetilde{a} + \widetilde{b} = \widetilde{\mathbb{R}}$, $D(\widetilde{a}) = p_\alpha$, and $D(\widetilde{b}) = 0$, we have

$$D(\widetilde{a} + \widetilde{b}) = 0 < p_\alpha = \max\{D(\widetilde{a}), D(\widetilde{b})\}.$$

(ii) Let $\widetilde{a} = \widetilde{a}_\alpha$, and let $\widetilde{b} = \widetilde{0}$. Since $\widetilde{a} + \widetilde{b} = \widetilde{a}$, $D(\widetilde{a}) = p_\alpha$, and $D(\widetilde{b}) = 0$, we have

$$D(\widetilde{a}+\widetilde{b}) = p_\alpha = \max\{D(\widetilde{a}), D(\widetilde{b})\}.$$

(iii) Let $\widetilde{a} = \widetilde{a}_\alpha$, and let $\lambda = 0$. Since $\lambda\widetilde{a} = \widetilde{0}$, $D(\widetilde{a}) = p_\alpha$, and $D(\lambda\widetilde{a}) = 0$, we have

$$D(\lambda\widetilde{a}) = 0 < p_\alpha = D(\widetilde{a}).$$

**5. Conclusions** We dealt with the $G$-convexity of fuzzy sets, which was a generalization

of the convexity by allowing arbitrary conjunctive aggregation functions instead of the minimum operation. Then, the properties of the $G$-convexity of fuzzy sets with respect to operations were investigated. The degree of the non-convexity of fuzzy sets was considered as an application of the $G$-convexity. Then, the properties of the degree of the non-convexity of fuzzy sets with respect to operations were investigated.

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# EXTENSIONS OF TSALLIS RELATIVE OPERATOR ENTROPY AND OPERATOR VALUED DISTANCE

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ABSTRACT. As a continuity of [4], we give a generalized Tsallis relative operator entropy $T_{t,r}(A|B) = \dfrac{A \sharp_{t,r} B - A}{t}$, where $A \sharp_{t,r} B = A^{\frac{1}{2}}\{(1-t)I + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\}^{\frac{1}{r}}$, $t \in [0,\ 1]$, $r \in [-1,\ 1]$, a path defined by the operator power mean. The relative operator entropy $S(A|B)$ is generalized in [4] as $S_r(A|B)$. In this note, we give a more expanded form $S_{t,r}(A|B)$, the derivative of the path $A \sharp_{t,r} B$ at $t \in [0,\ 1]$.

**1 Introduction.** Throughout this note, a bounded linear operator $T$ on a Hilbert space $H$ is positive if $(Tx,\ x) \geq 0$ for $x \in H$ and we denote $T \geq 0$, and if $T$ is invertible and positive, we denote $T > 0$.

Let $A$ and $B$ are positive invertible operators. In [4], we treated a path

$$A \natural_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}},\ t \in \mathbf{R}.$$

If $t \in [0,\ 1]$, then we use the notation $A \sharp_t B$ to distinguish as an operator mean, and considered $S_t(A|B)$ as a tangent at $t$ of this path, which is introduced by Furuta [3],

$$S_t(A|B) = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

We called this, according to [8], the generalized relative operator entropy because if $t = 0$, then $S_0(A|B) = A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = S(A|B)$, the relative operator entropy.

The Tsallis relative operator entropy is given by

$$T_t(A|B) = \frac{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}} - A}{t} = \frac{A \sharp_t B - A}{t}, \quad \text{for} \quad 0 < t < 1.$$

This is introduced by Yanagi, Kuriyama and Furuichi [8], and $\lim_{t\to 0} T_t(A|B) = S(A|B)$. In [4], we showed the following essential relation for $S_t(A|B)$ and $T_t(A|B)$:

$$(*) \qquad S(A|B) \leq T_t(A|B) \leq S_t(A|B) \leq -T_{1-t}(B|A) \leq -S(B|A) = S_1(A|B).$$

As a continuation of [4], let's consider the following path with two variables:

For $t \in [0,\ 1],\ r \in [-1,\ 1]$,

$$A \sharp_{t,r} B = A^{\frac{1}{2}}\{(1-t)I + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\}^{\frac{1}{r}} A^{\frac{1}{2}} = A \natural_{\frac{1}{r}} (A \nabla_t (A \natural_r B)).$$

This is called the operator power mean or quasi-arithmetic operator mean and the corresponding function is

$$p(t,r) = \{1 - t + ta^r\}^{\frac{1}{r}}.$$

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This compounds the arithmetic, geometric and harmonic interpolations, that is,

$$A \natural_{t,1} B = (1-t)A + tB = A \nabla_t B, \quad \text{arthmetic interpolation},$$

$$A \natural_{t,0} B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}} = A \natural_t B, \quad \text{geometric interpolation},$$

$$A \natural_{t,-1} B = ((1-t)A^{-1} + tB^{-1})^{-1} = A \Delta_t B, \quad \text{harmonic interpolation}.$$

Under the commutativity of $A$ and $B$, Uhlmann treated the geometric interpolation [7]. In this note, we show similar relations to $(*)$ for the path $A \natural_{t,r} B$.

**2 Generalized Tsallis relative operator entropy $T_{t,r}(A|B)$.** First we review the properties of $A \natural_{t,r} B$ (cf.[5]);

$$A \natural_{0,r} B = A, \quad A \natural_{1,r} B = B \quad \text{and} \quad \lim_{r\to 0} A \natural_{t,r} B = A \natural_t B.$$

So we have

$$\lim_{r\to 0} \frac{A \natural_{t,r} B - A}{t} = \frac{A \natural_t B - A}{t} = T_t(A|B)$$

and

$$\lim_{t\to 0} \frac{A \natural_{t,r} B - A}{t} = \frac{A \natural_r B - A}{r} = T_r(A|B);$$

the Tsallis relative operator entropies [8].

We here propose to define a generalized Tsallis relative operator entropy as follows:

**Definition 1.** For $A > 0$ and $B > 0$, we call $T_{t,r}(A|B)$ the generalized Tsallis entropy, which is given by

$$T_{t,r}(A|B) = \frac{A \natural_{t,r} B - A}{t}, \quad t \in (0,\ 1], \quad r \in [-1,\ 1].$$

We show the relations among $T_{t,r}(A|B)$, $T_r(A|B)$, $T_t(A|B)$ and $S(A|B)$ by the following figure:

$$\begin{array}{ccc} T_{t,r}(A|B) & \longrightarrow_{t\to 0} & T_r(A|B) \\ \downarrow_{r\to 0} & & \downarrow_{r\to 0} \\ T_t(A|B) & \longrightarrow_{t\to 0} & S(A|B) \end{array}$$

The next is an essential property of $A \natural_{t,r} B$.

**Lemma 1.** *Let $A > 0,\ B > 0$ and $t \in [0,\ 1],\ r \in [-1,\ 1]$. Then*

$$A \natural_{t,r} B = B \natural_{1-t,r} A.$$

The following has already shown in [5], but now we can easily obtain by this Lemma.

**Corollary 2.** *Let $A > 0,\ B > 0$ and $t \in [0,\ 1],\ r \in [-1,\ 1]$. Then*

$$\lim_{t\to 1} \frac{A \natural_{t,r} B - B}{t-1} = -T_r(B|A) \quad \text{and} \quad \lim_{r\to 0} \frac{A \natural_{t,r} B - B}{t-1} = -T_{1-t}(B|A).$$

These relations are also given by the following figure:

$$\begin{array}{ccc} \frac{A \natural_{t,r} B-B}{t-1} = \frac{B \natural_{1-t,r} A-B}{t-1} & \longrightarrow_{t\to 1} & -T_r(B|A) \\ \downarrow_{r\to 0} & & \downarrow_{r\to 0} \\ -T_{1-t}(B|A) & \longrightarrow_{t\to 1} & -S(B|A) = S_1(A|B). \end{array}$$

**3 Derivative of the path $A \natural_{t,r} B$.** Since the corresponding function to $A \natural_{t,r} B$ is

$$p(t,\ r) = (1 - t + ta^r)^{\frac{1}{r}} = \{1 + t(a^r - 1)\}^{\frac{1}{r}},\ t \in [0,\ 1],\ r \in [-1,\ 1]$$

and

$$\frac{\partial}{\partial t} p(t,\ r) = \{1 + t(a^r - 1)\}^{\frac{1}{r} - 1} \cdot \frac{a^r - 1}{r},$$

on the analogy to $S_t(A|B)$ in [3, 4], replacing $a$ by $A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ and multiplying both sides by $A^{\frac{1}{2}}$, we give the following definition.

**Definition 2.** For $A > 0,\ B > 0$, we give $S_{t,r}(A|B)$ as follows:

$$S_{t,r}(A|B) = A^{\frac{1}{2}} \left( \{1 + t((A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^r - 1)\}^{\frac{1}{r} - 1} \cdot \frac{(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^r - I}{r} \right) A^{\frac{1}{2}}.$$

We would like to call this an expanded relative operator entropy, because we have the following properties.

**Proposition 3.** *For $A > 0,\ B > 0$ and $t \in [0,\ 1],\ r \in [-1,\ 1]$, the followings hold:*

$$\lim_{r \to 0} S_{t,r}(A|B) = S_t(A|B), \tag{1}$$

$$S_{0,r}(A|B) = T_r(A|B), \tag{2}$$

$$S_{1,r}(A|B) = -T_r(B|A), \tag{3}$$

$$\lim_{r \to 0} S_{0,r}(A|B) = S(A|B), \tag{4}$$

$$\lim_{r \to 0} S_{1,r}(A|B) = S_1(A|B). \tag{5}$$

*Proof.* Since

$$\lim_{r \to 0} \{1 + t(a^r - 1)\}^{\frac{1}{r} - 1} \cdot \frac{a^r - 1}{r} = a^t \log a,$$

we have

$$\begin{aligned} &\lim_{r \to 0} A^{\frac{1}{2}} \left( \{I + t((A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^r - I)\}^{\frac{1}{r} - 1} \cdot \frac{(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^r - I}{r} \right) A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^t (\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}} = S_t(A|B). \end{aligned}$$

Moreover,

$$\frac{\partial}{\partial t} p(t,\ r)|_{t=0} = \frac{a^r - 1}{r}, \qquad \frac{\partial}{\partial t} p(t,\ r)|_{t=1} = \frac{a - a^{1-r}}{r}$$

and their limits are known that

$$\lim_{r \to 0} \frac{a^r - 1}{r} = \log a, \qquad \lim_{r \to 0} \frac{a - a^{1-r}}{r} = a \log a,$$

so we have

$$\lim_{r\to 0} A^{\frac{1}{2}}\left(\frac{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r-1}{r}\right)A^{\frac{1}{2}} = A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = S(A|B),$$

and

$$\begin{aligned}&\lim_{r\to 0} A^{\frac{1}{2}}\left(\frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-r}}{r}\right)A^{\frac{1}{2}}\\ &= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = S_1(A|B).\end{aligned}$$

□

## 4 Relations between $T_{t,r}(A|B)$ and $S_{t,r}(A|B)$.

In [4], we have shown

$$(*) \qquad S(A|B) \le T_t(A|B) \le S_t(A|B) \le -T_{1-t}(B|A) \le -S(B|A) = S_1(A|B).$$

So it is natural to see an extended relations of $(*)$ by the terms of $T_{t,r}(A|B)$ and $S_{t,r}(A|B)$. As a result, we can extend $(*)$ to the following $(\star)$ in which $(*)$ is the case where $r = 0$.

**Theorem 4.** *For $A > 0$, $B > 0$ and $t \in [0,\ 1]$, $r \in [-1,\ 1]$, the following holds.*

$$(\star) \qquad S_{0,r}(A|B) \le T_{t,r}(A|B) \le S_{t,r}(A|B) \le -T_{1-t,r}(B|A) \le S_{1,r}(A|B).$$

*Proof.* (1) First we show $S_{0,r}(A|B) \le T_{t,r}(A|B)$. Since $S_{0,r}(A|B) = T_r(A|B)$, we have only to show that

$$\frac{\{1+t(a^r-1)\}^{\frac{1}{r}}-1}{t} \ge \frac{a^r-1}{r}.$$

Let

$$f(t) = \{1+t(a^r-1)\}^{\frac{1}{r}} - 1 - t\frac{a^r-1}{r}.$$

Then we have

$$\frac{d}{dt}f(t) = \{1+t(a^r-1)\}^{\frac{1}{r}-1}\cdot\frac{a^r-1}{r} - \frac{a^r-1}{r} = \frac{\partial}{\partial t}p(t,r) - \frac{a^r-1}{r}$$

and

$$\frac{\partial^2}{\partial t^2}p(t,r) = \{1+t(a^r-1)\}^{\frac{1}{r}-2}\cdot\frac{(a^r-1)^2(1-r)}{r^2} \ge 0.$$

So $\frac{\partial}{\partial t}p(t,r)$ is an increasing function for $t \in [0,\ 1]$ and $\frac{\partial}{\partial t}p(t,r)|_{t=0} = \frac{a^r-1}{r}$. Since $\frac{df(t)}{dt} \ge 0$ by $\frac{df(0)}{dt} = 0$, we have $f(t)$ is increasing for $t \in [0,\ 1]$ and $f(0) = 0$, so we have $\{1+t(a^r-1)\}^{\frac{1}{r}} - 1 \ge t\frac{a^r-1}{r}$, that is, $\frac{\{1+t(a^r-1)\}^{\frac{1}{r}}-1}{t} \ge \frac{a^r-1}{r}$ and $S_{0,r}(A|B) \le T_{t,r}(A|B)$.

(2) Second, we show $T_{t,r}(A|B) \le S_{t,r}(A|B) \le -T_{t,r}(B|A)$. It is sufficient to show

$$\frac{\{1+t(a^r-1)\}^{\frac{1}{r}}-1}{t} \le \{1+t(a^r-1)\}^{\frac{1}{r}-1}\cdot\frac{a^r-1}{r}.$$

Since
$$\frac{d^2}{dt^2}f(t) = \frac{\partial^2}{\partial t^2}p(t,\ r) \geq 0, \quad \text{for} \quad t \in [0,\ 1] \quad \text{and} \quad t \in [-1,\ 1],$$
$f(t)$ and $p(t,r)$ are convex for $t$ on $[0,\ 1]$.

Since a function $f(t)$ is convex, then, for $a,\ h > 0$,
$$f(a) = f(\frac{a}{a+h}(a+h) + \frac{h}{a+h}0) \leq \frac{a}{a+h}f(a+h) + \frac{h}{a+h}f(0),$$
so that
$$\frac{f(a+h) - f(a)}{h} \geq \frac{f(a) - f(0)}{a}.$$
Hence we have
$$\frac{p(t+h,\ r) - p(t,\ r)}{h} \geq \frac{p(t,\ r) - p(0,\ r)}{t}, \text{ for } \ \forall h > 0,$$
that is,
$$\frac{(1+(t+h)(a^r-1))^{\frac{1}{r}} - (1+t(a^r-1))^{\frac{1}{r}}}{h} \geq \frac{(1+t(a^r-1))^{\frac{1}{r}} - 1}{t}, \text{ for } \forall h > 0,$$
and
$$\begin{aligned} & \lim_{h \to +0} \frac{(1+(t+h)(a^r-1))^{\frac{1}{r}} - (1+t(a^r-1))^{\frac{1}{r}}}{h} \\ = \quad & \{1+t(a^r-1)\}^{\frac{1}{r}-1} \cdot \frac{a^r-1}{r} \geq \frac{(1+t(a^r-1))^{\frac{1}{r}} - 1}{t}. \end{aligned}$$
So we can conclude
$$S_{t,r}(A|B) \geq T_{t,r}(A|B).$$

(3) Next we show $S_{t,r}(A|B) \leq -T_{1-t,r}(B|A)$. Since
$$\begin{aligned} T_{1-t,r}(B|A) &= \frac{B \ \natural_{1-t,r} \ A - B}{1-t} = \frac{A \ \natural_{t,r} \ B - B}{1-t} \\ &= \frac{A^{\frac{1}{2}}\left(\{I + t((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r - I)\}^{\frac{1}{r}} - (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\right)A^{\frac{1}{2}}}{1-t}, \end{aligned}$$
the function corresponding to $T_{1-t,r}(B|A)$ is $\dfrac{p(t,\ r) - a}{1-t}$.
For a convex function $f(t)$,
$$f(t) = f(\frac{1-t}{1-t+h}(t-h) + \frac{h}{1-t+h} \cdot 1) \leq \frac{1-t}{1-t+h}f(t-h) + \frac{h}{1-t+h}f(1).$$
So we have
$$\frac{f(t) - f(t-h)}{h} \leq \frac{f(1) - f(t)}{1-t}.$$
For $p(t,\ r) = (1+t(a^r-1))^{\frac{1}{r}}$,
$$\frac{p(t,\ r) - p(t-h,\ r)}{h} \leq \frac{p(1,\ r) - p(t,\ r)}{1-t},$$

that is,

$$\frac{\{1+t(a^r-1)\}^{\frac{1}{r}}-\{1+(t-h)(a^r-1)\}^{\frac{1}{r}}}{h} \le \frac{a-\{1+t(a^r-1)\}^{\frac{1}{r}}}{1-t}.$$

Since

$$\lim_{h\to 0}\frac{\{1+t(a^r-1)\}^{\frac{1}{r}}-\{1+(t-h)(a^r-1)\}^{\frac{1}{r}}}{h} = \frac{\partial}{\partial t}p(t,\ r) = \{1+t(a^r-1)\}^{\frac{1}{r}-1}\cdot\frac{a^r-1}{r},$$

we have

$$\{1+t(a^r-1)\}^{\frac{1}{r}-1}\cdot\frac{a^r-1}{r} \le \frac{a-\{1+t(a^r-1)\}^{\frac{1}{r}}}{1-t}.$$

So we conclude $S_{t,r}(A|B) \le -T_{1-t,r}(B|A)$.

(4) Finally, we see $-T_{1-t,r}(B|A) \le S_{1,r}(A|B)$, it is sufficient to show that

$$-\frac{\{1+t(a^r-1)\}^{\frac{1}{r}}-a}{1-t} \le \frac{a-a^{1-r}}{r}.$$

This is equivalent to

$$\{1+t(a^r-1)\}^{\frac{1}{r}}-a \ge -(1-t)\frac{a-a^{1-r}}{r}.$$

Let

$$g(t) = \{1+t(a^r-1)\}^{\frac{1}{r}}-a+(1-t)\frac{a-a^{1-r}}{r},$$

then for $t\in[0,\ 1]$,

$$\frac{d}{dt}g(t) = \{1+t(a^r-1)\}^{\frac{1}{r}-1}\cdot\frac{a^r-1}{r}-\frac{a-a^{1-r}}{r}.$$

As we showed above $\frac{\partial}{\partial t}p(t,r) = \{1+t(a^r-1)\}^{\frac{1}{r}-1}\cdot\frac{a^r-1}{r}$ is an increasing function for $t\in[0,\ 1]$ and $\frac{d}{dt}g(t)|_{t=1}=0$. Hence $\frac{d}{dt}g(t)\le 0$ for $t\in[0,\ 1]$. So the function $g(t)$ is a decreasing function and $g(1)=0$, that is, $g(t)\ge 0$ for $t\in[0,\ 1]$. Hence we have the conclusion $-T_{1-t,r}(B|A) \le S_{1,r}(A|B)$. □

*Remark* 1. In [2, 5], we had shown the convexity of interpolational operator mean and the power mean is a typical one, that is,

$$(A\ \sharp_{t,r}\ B)\ \sharp_{\frac{1}{2},r}\ (A\ \sharp_{s,r}\ B) = A\ \sharp_{\frac{t+s}{2},r}\ B.$$

**5 Operator valued distances $S(A|B)$ and $T_r(A|B)$.** Essential properties of relative operator entropy between $A$ and the path $A\ \natural_t\ B,\ \ t\in\mathbf{R}$, and Tsallis relative operator entropy between $A$ and $A\ \sharp_{t,r}\ B,\ \ t\in[0,\ 1]$ and $r\in[-1,\ 1]$ are already shown in [5] as follows:

(i) $S(A|A\ \natural_t\ B) = A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}(A\ \natural_t\ B)A^{-\frac{1}{2}})A^{\frac{1}{2}} = A^{\frac{1}{2}}(\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t)A^{\frac{1}{2}} = tS(A|B)$

and

$$\text{(ii)} \qquad T_r(A|A \sharp_{t,r} B) = \frac{A \sharp_r (A \sharp_{t,r} B) - A}{r} = t\frac{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{\frac{1}{2}})^r A^{-\frac{1}{2}} - A}{r} = tT_r(A|B).$$

These equalities (i) and (ii) suggest us that $S(A|X)$ and $T_r(A|X)$ are regarded as operator valued distances from the basic point $A$ to $X$ (along with the corresponding paths). The following properties are easily obtained from (i);

$$S(A|A) = 0, \qquad S(A|B) = 0 \quad \Longleftrightarrow \quad A = B,$$

$$S(A|B \natural_t A) = S(A|A \natural_{1-t} B) = (1-t)S(A|B),$$

$$S(A|A \natural_{t+s} B) = S(A|A \natural_t B) + S(A|A \natural_s B),$$

$$S(A|A \natural_{ts} B) = tS(A|A \natural_s B),$$

$$S(A|A \natural_t B) + S(A|B \natural_t A) = S(A|B).$$

and similar properties are obtained from (ii);

$$T_r(A|A) = 0, \qquad T_r(A|B) = 0 \quad \Longleftrightarrow \quad A = B,$$

$$T_r(A|B \sharp_{t,r} A) = (1-t)T_r(A|B),$$

$$T_r(A|A \sharp_{\frac{t+s}{2},r} B) = T_r(A|A \sharp_{\frac{t}{2},r} B) + T_r(A|A \sharp_{\frac{s}{2},r} B), \quad t,\ s \in [0,\ 1],$$

$$T_r(A|A \sharp_{t,r} B) + T_r(A|B \sharp_{t,r} A) = T_r(A|B).$$

So, comparing the distances $S(A|A \sharp_t B)$ and $S(A|A \sharp_{t,r} B)$, we have the next.

**Theorem 5.** *Let $A > 0,\ B > 0$ and $t \in [0,\ 1],\ r \in [0,\ 1]$. Then the following holds:*

$$tT_{-r}(A|B) \le S(A|A \sharp_{t,-r} B) \le tS(A|B) = S(A|A \sharp_t B) \le S(A|A \sharp_{t,r} B) \le tT_r(A|B).$$

*Proof.* Since $(A|A \sharp_t B) = tS(A|B)$ by Theorem 4 in [5],

$$\begin{aligned}
S(A|A \sharp_{t,r} B) &= A^{\frac{1}{2}}(\log\{(1-t)I + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\}^{\frac{1}{r}})A^{\frac{1}{2}} \\
&= \frac{1}{r}A^{\frac{1}{2}}(\log\{(1-t)I + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\})A^{\frac{1}{2}} \\
&\leq \frac{1}{r}A^{\frac{1}{2}}((1-t)I + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r - I)A^{\frac{1}{2}} \\
&= \frac{t}{r}A^{\frac{1}{2}}((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r - I)A^{\frac{1}{2}} = \frac{t}{r}(A \sharp_r B - A) = tT_r(A|B).
\end{aligned}$$

$$\begin{aligned}
S(A|A \sharp_{t,r} B) &= A^{\frac{1}{2}}(\log\{(1-t)I + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\}^{\frac{1}{r}})A^{\frac{1}{2}} \\
&= \frac{1}{r}A^{\frac{1}{2}}(\log\{(1-t)I + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\})A^{\frac{1}{2}} \\
&\geq \frac{1}{r}A^{\frac{1}{2}}((1-t)\log I + t(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}}))A^{\frac{1}{2}} \\
&= \frac{t}{r}A^{\frac{1}{2}}(\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r)A^{\frac{1}{2}} \\
&= tA^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = tS(A|B) = S(A|A \sharp_t B).
\end{aligned}$$

*and*

$$\begin{aligned}
S(A|A \sharp_{t,-r} B) &= A^{\frac{1}{2}}(\log\{(1-t)I + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-r}\}^{-\frac{1}{r}})A^{\frac{1}{2}} \\
&= -\frac{1}{r}A^{\frac{1}{2}}(\log\{(1-t)I + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-r}\})A^{\frac{1}{2}} \\
&\leq -\frac{1}{r}A^{\frac{1}{2}}((1-t)\log I + t(\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-r})A^{\frac{1}{2}} \\
&= -\frac{t}{r}A^{\frac{1}{2}}(\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-r})A^{\frac{1}{2}} \\
&= tA^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = tS(A|B) = S(A|A \sharp_t B).
\end{aligned}$$

$$\begin{aligned}
S(A|A \sharp_{t,-r} B) &= A^{\frac{1}{2}}(\log\{(1-t)I + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-r}\}^{-\frac{1}{r}})A^{\frac{1}{2}} \\
&= -\frac{1}{r}A^{\frac{1}{2}}(\log\{(1-t)I + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-r}\})A^{\frac{1}{2}} \\
&\geq -\frac{1}{r}A^{\frac{1}{2}}((1-t)I + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-r} - I)A^{\frac{1}{2}} \\
&= -\frac{t}{r}A^{\frac{1}{2}}((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-r} - I)A^{\frac{1}{2}} = -\frac{t}{r}(A \natural_{-r} B - A) = tT_{-r}(A|B).
\end{aligned}$$

$\square$

If we put $r = 0$, then the distances from $A$ to $A \nabla_t B$, $A \sharp_t B$ and $A \Delta_t B$ are the following relations.

**Corollary 6.** *For $A > 0,\ B > 0$ and $t \in [0,\ 1]$, the following holds:*

$$t(A - AB^{-1}A) \leq S(A|A \Delta_t B) \leq S(A|A \sharp_t B) \leq S(A|A \nabla_t B) \leq t(B - A).$$

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# A CHARACTERIZATION OF $\omega_1$-STRONGLY COUNTABLE-DIMENSIONAL SPACES IN TERMS OF $K$-APPROXIMATIONS

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**Abstract.** We give a characterization of $\omega_1$-strongly countable-dimensional metrizable spaces in terms of $K$-approximations. A characterization of locally finite-dimensional metrizable spaces is also obtained.

**1 Introduction** The purpose of this paper is to characterize a class of $\omega_1$-strongly countable-dimensional metrizable spaces in terms of $K$-approximations. A concept of a $K$-approximation is due to Dydak-Mishra-Shukla.

**Definition 1.1.** ([1; Definition 1.1]) Let $X$ be a normal space, let $K$ be a metric simplicial complex (i.e., a simplicial complex equipped with the metric topology) and let $f : X \to K$ be a continuous mapping. A continuous mapping $g : X \to K$ is a *$K$-approximation* of $f$ provided for each simplex $\Delta$ of $K$ and each $x \in X$, $f(x) \in \Delta$ implies $g(x) \in \Delta$. $g$ is an *$n$-dimensional* (respectively, *finite-dimensional*) *$K$-approximation* of $f$ if it is a *$K$-approximation* and $g(X) \subset K^{(n)}$ (respectively, $g(X) \subset K^{(m)}$ for some $m$).

Dydak-Mishra-Shukla gave a characterization of $n$-dimensional spaces in terms of $K$-approximations. If every finite open cover of a normal space $X$ has a finite open refinement of order $\leq n+1$, then $X$ has *covering dimension* $\leq n$, $\dim X \leq n$.

**Theorem 1.2.** ([1; Theorem 2.2]) *Let $n$ be an integer. For a paracompact space $X$ the following conditions are equivalent:*

($a$) $\dim X \leq n$.

($b$) *For every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is an $n$-dimensional $K$-approximation $g$ of $f$.*

($c$) *For every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is an $n$-dimensional $K$-approximation $g$ of $f$ such that $g|f^{-1}(K^{(n)}) = f|f^{-1}(K^{(n)})$.*

Also, Dydak-Mishra-Shukla characterized finitistic spaces. A normal space $X$ is *finitistic* if every open cover of $X$ has an open refinement of finite order.

**Theorem 1.3.** ([1; Theorem 2.1]) *For a paracompact space $X$ the following conditions are equivalent:*

($a$) *$X$ is finitistic.*

($b$) *For every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is a finite-dimensional $K$-approximation $g$ of $f$.*

($c$) *For every integer $m \geq -1$, every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is a finite-dimensional $K$-approximation $g$ of $f$ such that $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.*

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In [5], Y. Hattori extended Theorem 1.2 to strong large transfinite dimensional spaces. A normal space $X$ is said to have *strong large transfinite dimension* if $X$ has both large transfinite dimension and strong small transfinite dimension (see Definition 2.3). For a space $X$ we denote $\mathcal{D}(X) = \{D \mid D$ is a closed discrete subset of $X\}$.

**Theorem 1.4.** ([5; Theorem]) *For a metrizable space $X$ the following conditions are equivalent:*

($a$) *$X$ has a strong large transfinite dimension.*

($b$) *There is a function $\varphi : \mathcal{D}(X) \to \omega$ such that for every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is a $K$-approximation $g$ of $f$ such that $g(D) \subset K^{(\varphi(D))}$ for each $D \in \mathcal{D}(X)$.*

($c$) *For every integer $m \geq -1$, there is a function $\psi : \mathcal{D}(X) \to \omega$ such that for every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is a finite-dimensional $K$-approximation $g$ of $f$ such that $g(D) \subset K^{(\psi(D))}$ for each $D \in \mathcal{D}(X)$ and $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.*

A normal space $X$ is *strongly countable-dimensional* if $X$ can be represented as a countable union of closed finite-dimensional subspaces.

**Theorem 1.5.** ([5; Corollary]) *For a paracompact space $X$ the following conditions are equivalent:*

($a$) *$X$ is a strongly countable-dimensional space.*

($b$) *There is a function $\varphi : X \to \omega$ such that for every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is a $K$-approximation $g$ of $f$ such that $g(x) \in K^{(\varphi(x))}$ for each $x \in X$.*

($c$) *For every integer $m \geq -1$, there is a function $\psi : X \to \omega$ such that for every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is a $K$-approximation $g$ of $f$ such that $g(x) \in K^{(\psi(x))}$ for each $x \in X$ and $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.*

## 2 Characterizations

In this section, we give a characterization of $\omega_1$-strongly countable-dimensional metrizable spaces in terms of $K$-approximations. A characterization of locally finite-dimensional metrizable spaces is also obtained.

A notion of a locally finite-dimensional space is well known (cf. [2]).

**Definition 2.1.** A metrizable space $X$ is *locally finite-dimensional* if for every point $x \in X$ there exists an open subspace $U$ of $X$ such that $x \in U$ and $\dim U < \infty$.

The first infinite ordinal number is denoted by $\omega$ and $\omega_1$ is the first uncountable ordinal number. Z. Shmuely introduced and studied $\omega_1$-strongly countable-dimensional spaces ([8]).

**Definition 2.2.** A metrizable space $X$ is called an *$\omega_1$-strongly countable-dimensional space* if $X = \bigcup\{P_\xi \mid 0 \leq \xi < \xi_0\}$, $\xi_0 < \omega_1$, where $P_\xi$ is an open subset of $X - \bigcup\{P_\eta \mid 0 \leq \eta < \xi\}$ and $\dim P_\xi < \infty$.

For a metrizable space $X$ and a non-negative integer $n$, we put

$$P_n(X) = \bigcup\{U \mid U \text{ is an open subspace of } X \text{ and } \dim U \leq n\}.$$

We notice that for each ordinal number $\alpha$, we can put $\alpha = \lambda(\alpha) + n(\alpha)$, where $\lambda(\alpha)$ is a limit ordinal number or 0 and $n(\alpha)$ is a non-negative integer. Strong small transfinite dimension is studied by Y. Hattori (cf. [3]).

**Definition 2.3.** Let $X$ be a metrizable space and $\alpha$ either an ordinal number $\geq 0$ or the integer $-1$. Then *strong small transfinite dimension sind* of $X$ is defined as follows:

(1) $\operatorname{sind} X = -1$ if and only if $X = \emptyset$.

(2) $\operatorname{sind} X \leq \alpha$ if $X$ is expressed in the form $X = \bigcup\{P_\xi \mid \xi < \alpha\}$, where $P_\xi = P_{n(\xi)}(X - \bigcup\{P_\eta \mid \eta < \lambda(\xi)\})$.

Furthermore, if $\operatorname{sind} X$ is defined, we say that $X$ has *strong small transfinite dimension*.

Clearly, a metrizable space $X$ is locally finite-dimensional if and only if $\operatorname{sind} X \leq \omega$ (cf. [2; Proposition 5.5.3]). And $X$ is $\omega_1$-strongly countable-dimensional if and only if there is a $\xi_0 < \omega_1$ such that $\operatorname{sind} X \leq \xi_0$.

Let $X$ be a metrizable space, let $\alpha$ be an ordinal number and let $\mathcal{F} = \{X_\beta \mid 0 \leq \beta \leq \alpha\}$ be a family of subsets of $X$. We say that $\mathcal{F}$ is a *closed $\alpha$-sequence in $X$* if

(f-1) $X_\beta$ is closed in $X$ for $\beta \leq \alpha$,

(f-2) $X_0 = X$,

(f-3) $X_\beta \supset X_{\beta'}$ for $\beta \leq \beta' \leq \alpha$,

(f-4) $X_\beta = \bigcap\{X_{\beta'} \mid \beta' < \beta\}$ if $\beta$ is a limit.

The *power set* of $X$ shall be denoted by $\mathcal{P}(X)$.

Let $N : X \to \mathcal{P}(X)$ be a function and let $\mathcal{F} = \{X_\beta \mid 0 \leq \beta \leq \alpha\}$ be a closed $\alpha$-sequence in $X$. We say that $N$ is an *$\mathcal{F}$-neighborhood function* provided that $N(x)$ is an open neighborhood of $x$ in $X_{\beta(x)}$ for each $x \in X$, where $\beta(x) = \max\{\beta \mid x \in X_\beta,\ 0 \leq \beta \leq \alpha\}$.

**Remark 2.4.** ([6; Remark 2.5]) Let $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ be a closed $\alpha$-sequence in $X$. Then we shall show that for every point $x$ of $X$, there is a maximum element $\beta(x)$ of $\{\beta \mid x \in X_\beta\}$. Indeed, if $x \in X_{\lambda(\alpha)}$, then $\beta(x) = \max\{\beta \mid x \in X_\beta,\ \lambda(\alpha) \leq \beta \leq \alpha\}$. Now, we suppose that $x \notin X_{\lambda(\alpha)}$, there is a minimum element $\beta_0 > 0$ of $\{\beta \mid x \notin X_\beta\}$. Assume that $\beta_0$ is limit. By the condition (f-4), $x \in \bigcap\{X_\beta \mid \beta < \beta_0\} = X_{\beta_0}$. This contradicts the choice of $\beta_0$. Therefore $\beta_0$ is not limit and hence $\beta(x) = \beta_0 - 1$.

Theorem 2.8 is a main theorem. Thus we characterize the class of $\omega_1$-strongly countable-dimensional metrizable spaces in terms of $K$-approximations. To prove this theorem, we need Theorem 2.5.

**Theorem 2.5.** *Let $\alpha$ be an ordinal number with $\alpha < \omega_1$ and let $n$ be a non-negative integer. The following conditions are equivalent for a metrizable space $X$:*

($a$) $\operatorname{sind} X \leq \omega\alpha + n$.

($b$) *There are a closed $\alpha$-sequence $\mathcal{F} = \{X_\beta \mid 0 \leq \beta \leq \alpha\}$ in $X$, an $\mathcal{F}$-neighborhood function $N : X \to \mathcal{P}(X)$ and a function $\varphi : X \to \omega$ satisfing the following conditions: $X_\alpha = \emptyset$ if $n = 0$, $\varphi(X_\alpha) = n - 1$, and for every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is a $K$-approximation $g$ of $f$ such that $g(N(x)) \subset K^{(\varphi(x))}$ for each $x \in X$.*

($c$) *There are a closed $\alpha$-sequence $\mathcal{F} = \{X_\beta \mid 0 \leq \beta \leq \alpha\}$ in $X$, an $\mathcal{F}$-neighborhood function $N : X \to \mathcal{P}(X)$ and a function $\psi : X \to \omega$ satisfing the following conditions: $X_\alpha = \emptyset$ if $n = 0$, $\psi(X_\alpha) = n - 1$, and for every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is a $K$-approximation $g$ of $f$ such that $g(N(x)) \subset K^{(\psi(x))}$ for each $x \in X$ and $g|f^{-1}(K^{(n-1)}) = f|f^{-1}(K^{(n-1)})$.*

To prove this theorem, we need the following lemmas. Essentially, the following lemma is the same as [4; Lemma 1.5]. By a minor modification in the proof of [4; Lemma 1.5], we obtain the following lemma.

**Lemma 2.6.** ([4; Lemma 1.5], [7; Lemma 1]) *Let $n$ be a non-negative integer and let $\{F_m \mid m = 0,\ 1, ...\}$ be a closed cover of a metrizable space $X$ such that $\dim F_m \leq (n-1)+m$, $F_m \subset F_{m+1}$ for $m = 0,\ 1, ....$ Then for every open cover $\mathcal{U}$ of $X$, there are a sequence $\mathcal{V}_1$, $\mathcal{V}_2$, ... of discrete families of open subsets of $X$ and an open cover $\mathcal{W}$ of $X$ which satisfy the following conditions:*

(1) $\bigcup\{\mathcal{V}_k \mid k \in \mathbb{N}\}$ *is a cover of* $X$.

(2) $\bigcup\{\mathcal{V}_k \mid k \in \mathbb{N}\}$ *refines* $\mathcal{U}$.

(3) *If $W \in \mathcal{W}$ satisfies $W \cap F_m \neq \emptyset$, then $W$ meets at most one member of $\mathcal{V}_k$ for $k \leq (n+0)+(n+1)+...+(n+m)$ and meets no member of $\mathcal{V}_k$ for $k > (n+0)+(n+1)+...+(n+m)$.*

**Lemma 2.7.** ([1; Corollary 1.7]) *Let $f : X \to K$ be a map from a normal space $X$ to a metric simplicial complex $K$ so that $f(A) \subset K^{(n)}$ for some subset $A$ of $X$. There is a $K$-approximation $g$ of $f$ so that $g|U$ is an $n$-dimensional $K$-approximation of $f|U$ for some open neighborhood $U$ of $A$ in $X$ and $g|A = f|A$.*

Proof of Theorem 2.5. (a) $\Rightarrow$ (b) : Let $\mathrm{sind}X \leq \omega\alpha + n$. We use the construction in [6; Theorem 2.4]. We put

$$Y_\gamma = X - \bigcup\{P_\xi \mid \xi < \gamma\} \quad \text{for} \quad \gamma \leq \omega\alpha + n$$

and

$$X_\beta = Y_{\omega\beta} \quad \text{for} \quad \beta \leq \alpha.$$

Clearly, $\mathcal{F} = \{X_\beta \mid 0 \leq \beta \leq \alpha\}$ is a closed $\alpha$-sequence in $X$ satisfing $X_\alpha = \emptyset$ if $n = 0$.

Notice that $P_{\omega\beta+m}$ is an open subset of $X_\beta$ such that $P_{\omega\beta+m} \subset P_{\omega\beta+(m+1)}$ for $m = 0, 1, ....$ Also $P_{\omega\alpha+(n-1)}$ is a closed subset of $X$. Hence for each $\beta \leq \alpha$ there is a family $\{W_{\omega\beta+m} \mid m = 0,\ 1, ...\}$ of open subsets of $X_\beta$ such that

(1) $\overline{W_{\omega\beta+m}} \subset P_{\omega\beta+m}$,

(2) $\overline{W_{\omega\beta+m}} \subset W_{\omega\beta+(m+1)}$,

(3) $\bigcup_{m=0}^\infty W_{\omega\beta+m} = \bigcup_{m=0}^\infty P_{\omega\beta+m}$.

Since $\{\beta \mid 0 \leq \beta < \alpha\}$ is countable, there is a mapping $h$ from $\omega$ onto $\{\beta \mid 0 \leq \beta < \alpha\}$. For each $m = 0, 1, ...$, we put

$$V_0 = P_{\omega\alpha+(n-1)},$$

$$V_1 = P_{\omega\alpha+(n-1)} \cup W_{\omega h(1)+(n-1)+1},$$

$$V_2 = P_{\omega\alpha+(n-1)} \cup W_{\omega h(1)+(n-1)+2} \cup W_{\omega h(2)+(n-1)+2},$$

...

$$V_m = P_{\omega\alpha+(n-1)} \cup W_{\omega h(1)+(n-1)+m} \cup W_{\omega h(2)+(n-1)+m} \cup ... \cup W_{\omega h(m)+(n-1)+m},$$

...

Then $V_0, V_1, ...$ are subsets of $X$ satisfing the following conditions:

(4) $\overline{V_m} \subset V_{m+1}$.

(5) $\dim \overline{V_m} \leq (n-1) + m$.

(6) $X = \bigcup_{m=0}^\infty V_m$.

Let $x \in X$. Put $n_0 = \min\{m \mid x \in V_m\}$.

Clearly, if $n_0 = 0$ then $x \in V_0 = P_{\omega\alpha+(n-1)} = X_\alpha$. Now we shall show that if $n_0 > 0$, then $x \in W_{\omega\beta(x)+(n-1)+n_0} \subset X_{\beta(x)}$. By the definition of $n_0$, $x \in V_{n_0} = P_{\omega\alpha+(n-1)} \cup W_{\omega h(1)+(n-1)+n_0} \cup W_{\omega h(2)+(n-1)+n_0} \cup ... \cup W_{\omega h(n_0)+(n-1)+n_0}$. Since $x \notin P_{\omega\alpha+(n-1)}$ by

$n_0 > 0$, there is a natural number $i \le n_0$ such that $x \in W_{\omega h(i)+(n-1)+n_0}$. Hence $x \in W_{\omega h(i)+(n-1)+n_0} \subset P_{\omega h(i)+(n-1)+n_0} \subset X_{h(i)} - X_{h(i)+1}$. On the other hand, since $\beta(x) = \max\{\beta \mid x \in X_\beta\} < \alpha$, $x \in X_{\beta(x)} - X_{\beta(x)+1}$. Hence $h(i) = \beta(x)$ and hence $x \in W_{\omega\beta(x)+(n-1)+n_0} \subset X_{\beta(x)}$.

We put

$$N(x) = \begin{cases} P_{\omega\alpha+(n-1)}, & \text{if } n_0 = 0, \\ W_{\omega\beta(x)+(n-1)+n_0}, & \text{if } n_0 > 0. \end{cases}$$

Since $N(x)$ is an open neighborhood of $x$ in $X_{\beta(x)}$, $N : X \to \mathcal{P}(X)$ is an $\mathcal{F}$-neighborhood function.

Put $\varphi(x) = (n+0) + (n+1) + ... + (n+n_0) - 1$. Then $\varphi(X_\alpha) = n - 1$.

The latter half of the proof is similar to the proof of [5; Theorem]. Let $K$ be a metric simplicial complex and let $f : X \to K$ be a continuous mapping. By Lemma 2.6, there are a sequence $\mathcal{U}_1, \mathcal{U}_2, ...$ of discrete families of open subsets of $X$ and an open cover $\mathcal{W}$ of $X$ which satisfy the following conditions:

(7) $\bigcup_{k=1}^\infty \mathcal{U}_k$ is a cover of $X$.

(8) $\bigcup_{k=1}^\infty \mathcal{U}_k$ refines $\{f^{-1}(St(v,K)) \mid v \in K^{(0)}\}$.

(9) If $W \in \mathcal{W}$ satisfies $W \cap \overline{V_m} \ne \emptyset$, then $W$ meets at most one member of $\mathcal{U}_k$ for $k \le (n+0)+(n+1)+...+(n+m)$ and meets no member of $\mathcal{U}_k$ for $k > (n+0)+(n+1)+...+(n+m)$.

Then $\mathcal{U} = \bigcup_{k=1}^\infty \mathcal{U}_k$ is a locally finite open cover of $X$ by (7) and (9). For each $U \in \mathcal{U}$ there is $v(U) \in K^{(0)}$ such that $U \subset f^{-1}(St(v(U),K))$ by (8). For each $v \in K^{(0)}$ we put $Q_v = \bigcup\{U \in \mathcal{U} \mid v(U) = v\}$, and $\mathcal{Q} = \{Q_v \mid v \in K^{(0)}\}$. Then $\mathcal{Q}$ is a locally finite open cover of $X$ such that $Q_v \subset f^{-1}(St(v,K))$ for each $v \in K^{(0)}$. Let $\{\kappa_v \mid v \in K^{(0)}\}$ be a partition of unity subordinated to $\mathcal{Q}$. We define $g : X \to K$ as $g(x) = \sum_{v \in K^{(0)}} \kappa_v(x)v$. Then $g$ is a $K$-approximation of $f$.

Now, let $x \in X$. Notice that $N(x) \subset V_{n_0} \subset \overline{V_{n_0}}$. By (9), $\mathrm{ord}_y\, \mathcal{Q} \le \mathrm{ord}_y \mathcal{U} \le \varphi(x) + 1$ for each $y \in N(x)$. Hence $g(y) \in K^{(\varphi(x))}$ for each $y \in N(x)$ and hence $g(N(x)) \subset K^{(\varphi(x))}$.

(b) $\Rightarrow$ (a) : We shall show that for every $\beta \le \alpha$

$$(10) \quad X - \bigcup\{P_\xi \mid \xi < \omega\beta\} \subset X_\beta.$$

The validity of (10) is clear for $\beta = 0$. To prove (10) by transfinite induction we assume (10) for $\gamma < \beta$. Let $x \notin X_\beta$. Notice that $\beta(x) < \beta$.

If $x \in \bigcup\{P_\xi \mid \xi < \omega\beta(x)\}$, then $x \in \bigcup\{P_\xi \mid \xi < \omega\beta\}$ by $\beta(x) < \beta$.

We shall also show that if $x \in X - \bigcup\{P_\xi \mid \xi < \omega\beta(x)\}$, then $x \in \bigcup\{P_\xi \mid \xi < \omega\beta\}$. Since $N(x)$ is an open neighborhood of $x$ in $X_{\beta(x)}$, by the induction hypothesis, $N(x) \cap (X - \bigcup\{P_\xi \mid \xi < \omega\beta(x)\})$ is an open neighborhood of $x$ in $X - \bigcup\{P_\xi \mid \xi < \omega\beta(x)\}$. There is an open neighborhood $V(x)$ of $x$ in $X - \bigcup\{P_\xi \mid \xi < \omega\beta(x)\}$ such that

$$\overline{V(x)} \subset N(x) \cap (X - \bigcup\{P_\xi \mid \xi < \omega\beta(x)\}).$$

Let $\mathcal{U}$ be a finite open cover of $\overline{V(x)}$. Given $U \in \mathcal{U}$, choose an open subset $\tilde{U}$ of $X$ such that $\tilde{U} \cap \overline{V(x)} = U$. Put $\tilde{\mathcal{U}} = \{\tilde{U} \mid U \in \mathcal{U}\} \cup \{X - \overline{V(x)}\}$. We index a covering $\tilde{\mathcal{U}}$ as $\tilde{\mathcal{U}} = \{U_v \mid v \in S\}$. We use the proof of [1; Theorem 2.1]. Choose a partition of unity $\{\alpha_v \mid v \in S\}$ of $X$ with $\alpha_v^{-1}(0,1] \subset U_v$ for all $v \in S$ and notice that $f(y) = \sum_{v \in S} \alpha_v(y)v$ defines a map $f : X \to K$, where $K$ is the full complex with $S$ as its set of vertices. Then, by (b), there is a $K$-approximation $g$ of $f$ such that $g(N(y)) \subset K^{(\varphi(y))}$ for each $y \in X$. Notice that $g^{-1}(St(v,K)) \subset U_v$ for all $v \in S$ and $\tilde{\mathcal{V}} = \{g^{-1}(St(v,K)) \mid v \in S\}$ is an open cover of $X$. In particular, $g(\overline{V(x)}) \subset g(N(x)) \subset K^{(\varphi(x))}$. Then $\mathcal{V} = \{\tilde{V} \cap \overline{V(x)} \mid \tilde{V} \in \tilde{\mathcal{V}}\}$ is a finite open cover of $\overline{V(x)}$ such that $\mathcal{V}$ is a refinement of $\mathcal{U}$ and $\sup\{\mathrm{ord}_y \mathcal{V} \mid y \in \overline{V(x)}\} \leq \varphi(x)+1$. Hence

$$(11) \quad \dim V(x) \leq \dim \overline{V(x)} \leq \varphi(x)$$

and hence

$$x \in V(x) \subset P_{\varphi(x)}(X - \bigcup\{P_\xi \mid \xi < \omega\beta(x)\}) = P_{\omega\beta(x)+\varphi(x)}$$

$$\subset \bigcup\{P_\xi \mid \xi < \omega(\beta(x)+1)\} \subset \bigcup\{P_\xi \mid \xi < \omega\beta\}.$$

Thus, (10) holds.

In particular,

$$(12) \quad X - \bigcup\{P_\xi \mid \xi < \omega\alpha\} \subset X_\alpha.$$

We shall show that

$$(13) \quad X - \bigcup\{P_\xi \mid \xi < \omega\alpha\} \subset \bigcup\{P_\xi \mid \omega\alpha \leq \xi < \omega\alpha + n\}.$$

If $n = 0$ then $X_\alpha = \emptyset$, and hence $X - \bigcup\{P_\xi \mid \xi < \omega\alpha\} = \emptyset$ by (12).

Assume that $n > 0$. Let $x \in X - \bigcup\{P_\xi \mid \xi < \omega\alpha\}$. Since $x \in X_\alpha$ by (12), $\beta(x) = \alpha$. Hence $N(x)$ is an open neighborhood of $x$ in $X_\alpha$. By (12), $N(x) \cap (X - \bigcup\{P_\xi \mid \xi < \omega\alpha\})$ is an open neighborhood of $x$ in $X - \bigcup\{P_\xi \mid \xi < \omega\alpha\}$. There is an open neighborhood $V(x)$ of $x$ in $X - \bigcup\{P_\xi \mid \xi < \omega\alpha\}$ such that

$$\overline{V(x)} \subset N(x) \cap (X - \bigcup\{P_\xi \mid \xi < \omega\alpha\}).$$

By the proof of (11), $\dim V(x) \leq \dim \overline{V(x)} \leq \varphi(x)$. Furthermore $\varphi(x) = n-1$ by $x \in X_\alpha$. Hence,

$$x \in V(x) \subset P_{\varphi(x)}(X - \bigcup\{P_\xi \mid \xi < \omega\alpha\}) = P_{\omega\alpha+\varphi(x)}$$

$$\subset \bigcup\{P_\xi \mid \omega\alpha \leq \xi \leq \omega\alpha + \varphi(x)\} \subset \bigcup\{P_\xi \mid \omega\alpha \leq \xi < \omega\alpha + n\}.$$

Thus, (13) holds.

Therefore $X = \bigcup\{P_\xi \mid 0 \leq \xi < \omega\alpha + n\}$ and hence $\mathrm{sind}\, X \leq \omega\alpha + n$.

(b) $\Rightarrow$ (c) : The proof is similar to the proof of [5; Theorem]. For completeness, we give the proof. Let $\varphi : X \to \omega$ be as in (b). We put $\psi(x) = \max\{n-1, \varphi(x)\}$ for each $x \in X$. Let $K$ be a metric simplicial complex and let $f : X \to K$ be a continuous mapping. By Lemma 2.7, there are an open subset $U$ of $X$ and a $K$-approximation $g_1$ of $f$ such that $f^{-1}(K^{(n-1)}) \subset U$, $g_1|f^{-1}(K^{(n-1)}) = f|f^{-1}(K^{(n-1)})$ and $g_1|U$ is an $(n-1)$-dimensional $K$-approximation of $f|U$. Then, by (b), there is a $K$-approximation $g_2$ of $g_1$ such that $g_2(N(x)) \subset K^{(\varphi(x))}$ for each $x \in X$. Let $\kappa : X \to [0,1]$ be a continuous mapping such that $\kappa(f^{-1}(K^{(n-1)})) = 1$ and $\kappa(X - U) = 0$. We define $g(x) = \kappa(x)g_1(x) + (1-\kappa(x))g_2(x)$ for each $x \in X$. Then $g$ is a $K$-approximation of $f$ and $g(N(x)) \subset K^{(\psi(x))}$ for each $x \in X$.

(c) $\Rightarrow$ (b) is obvious. □

By Theorem 2.5, we obtain the Main Theorem 2.8 and Theorem 2.9.

**Theorem 2.8.** *The following conditions are equivalent for a metrizable space $X$:*

($a$) *$X$ is an $\omega_1$-strongly countable-dimensional space.*

($b$) *There are an ordinal number $\alpha < \omega_1$, a closed $\alpha$-sequence $\mathcal{F} = \{X_\beta \mid 0 \leq \beta \leq \alpha\}$ in $X$, an $\mathcal{F}$-neighborhood function $N : X \to \mathcal{P}(X)$ and a function $\varphi : X \to \omega$ satisfing the following condition: For every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is a $K$-approximation $g$ of $f$ such that $g(N(x)) \subset K^{(\varphi(x))}$ for each $x \in X$.*

($c$) *There are an ordinal number $\alpha < \omega_1$, a closed $\alpha$-sequence $\mathcal{F} = \{X_\beta \mid 0 \leq \beta \leq \alpha\}$ in $X$ and an $\mathcal{F}$-neighborhood function $N : X \to \mathcal{P}(X)$, and for every integer $m \geq -1$ there is a function $\psi : X \to \omega$ satisfing the following condition: For every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is a $K$-approximation $g$ of $f$ such that $g(N(x)) \subset K^{(\psi(x))}$ for each $x \in X$ and $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.*

Proof. (b) $\Rightarrow$ (a) : By the proof of (13) of Theorem 2.5, $X - \bigcup\{P_\xi \mid \xi < \omega\alpha\} \subset \bigcup\{P_\xi \mid \omega\alpha \leq \xi < \omega\alpha + \omega\}$. Hence $\operatorname{sind} X \leq \omega\alpha + \omega$, and hence $X$ is an $\omega_1$-strongly countable-dimensional space.

The implications (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (b) are obvious by Theorem 2.5. □

Notice that if $N : X \to \mathcal{P}(X)$ is an $\{X\}$-neighborhood function then $N(x)$ is an open neighborhood of $x$ in $X$ for each $x \in X$.

**Theorem 2.9.** *The following conditions are equivalent for a metrizable space $X$:*

($a$) *$X$ is a locally finite-dimensional space.*

($b$) *There are an $\{X\}$-neighborhood function $N : X \to \mathcal{P}(X)$ and a function $\varphi : X \to \omega$ satisfing the following condition: For every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is a $K$-approximation $g$ of $f$ such that $g(N(x)) \subset K^{(\varphi(x))}$ for each $x \in X$.*

($c$) *There is an $\{X\}$-neighborhood function $N : X \to \mathcal{P}(X)$, and for every integer $m \geq -1$ there is a function $\psi : X \to \omega$ satisfing the following condition: For every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is a $K$-approximation $g$ of $f$ such that $g(N(x)) \subset K^{(\psi(x))}$ for each $x \in X$ and $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.*

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# A COMMENT ON A STATISTICAL PROOF OF THE CONCAVITY ON THE EFFICIENT FISHER INFORMATION

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Abstract. The efficient Fisher information is defined by the Schur complement in the Fisher information. We statistically prove the concavity of the matrix function on the efficient Fisher information more sophisticated way than the previous one [2].

**1 Introduction** Let $X$ be arbitrary random vector with the parameter $\theta$, which is distributed with a distribution $F(x)$ with respect to some probability measure on $\mathbf{R}^s$. Note that $s = p + q$ and the parameter is partitioned by $\theta = (\theta^{(1)\,T}, \theta^{(2)\,T})^T$, where the notation $^T$ stands for the transpose and the dimension of $\theta^{(1)}$ is $p$. Let $f(x;\theta)$ be the probability density function with an expectation $\boldsymbol{\mu} = \boldsymbol{\mu}(\theta)$ and a covariance matrix $\boldsymbol{\Sigma}$ which is positive definite. Let the score function be partitioned as follows:

$$\frac{\partial \log f(x;\theta)}{\partial \theta} \;=\; \boldsymbol{J}(\theta) \;=\; \begin{pmatrix} \boldsymbol{J}_1(\theta) \\ \boldsymbol{J}_2(\theta) \end{pmatrix},$$

where $\boldsymbol{J}_i(\theta)$ are derived by $\theta^{(i)}$, $(i = 1, 2)$, so that we have the Fisher information which is partitioned as follows:

$$\boldsymbol{I}_X(\theta) \;=\; E_\theta\left(\boldsymbol{J}(\theta)\boldsymbol{J}(\theta)^T\right) \;=\; \begin{pmatrix} \boldsymbol{I}_{11,X}(\theta) & \boldsymbol{I}_{12,X}(\theta) \\ \boldsymbol{I}_{21,X}(\theta) & \boldsymbol{I}_{22,X}(\theta) \end{pmatrix}.$$

Assume that the Fisher information is positive definite. For the Fisher information, the monotonicity and the additivity hold as follows: If $T = T(X)$ is a statistic with density function $g(t;\theta)$, then the Fisher information $\boldsymbol{I}_T(\theta)$ on $T$ satisfies

$$\boldsymbol{I}_X(\theta) \;\geq\; \boldsymbol{I}_T(\theta). \tag{1}$$

If $X, Y$ are independent random variables then $\boldsymbol{I}_{(X,Y)}(\theta) = \boldsymbol{I}_X(\theta) + \boldsymbol{I}_Y(\theta)$.

[1](page 28) introduced the efficient Fisher information matrix on $\theta^{(1)}$ in $X$ by

$$\widehat{\boldsymbol{I}}_{1,X}(\theta) \;=\; \boldsymbol{I}_{11,X} - \boldsymbol{I}_{12,X}\boldsymbol{I}_{22,X}^{-1}\boldsymbol{I}_{21,X}, \tag{2}$$

which is known as the Schur complement of $\boldsymbol{I}_{22,X}$ in $\boldsymbol{I}_X$. [2] showed that the monotonicity and the superadditivity on (2) hold as follows: If $T = T(X)$ is a statistic with $\boldsymbol{I}_T(\theta)$ positive definite, then $\widehat{\boldsymbol{I}}_{1,T}(\theta) \leq \widehat{\boldsymbol{I}}_{1,X}(\theta)$. If $X, Y$ are independent random variables then

$$\widehat{\boldsymbol{I}}_{1,X}(\theta) + \widehat{\boldsymbol{I}}_{1,Y}(\theta) \;\leq\; \widehat{\boldsymbol{I}}_{1,(X,Y)}(\theta). \tag{3}$$

Although they tried to show the statistical proof of the concavity of the efficient Fisher information, the inequality (14) in page 347 was a little complicated or misleading to complete their proof. Here we prove it more sophisticated way than theirs, and our proof is easier to understand statistically.

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**2 Concavity on the efficient Fisher information** The convexity of the matrix function $\phi(\boldsymbol{A}) = \boldsymbol{A}^{-1}$ on the set of positive definite symmetric matrices $\{\boldsymbol{A}_j\}$ is

$$\sum_{j=1}^{n} w_j \phi(\boldsymbol{A}_j) \ \geq \ \phi\left(\sum_{j=1}^{n} w_j \boldsymbol{A}_j\right),$$

where the weights $w_j$ are nonnegative and $\sum_{j=1}^{n} w_j = 1$. On the other hand, the concavity of the matrix function $\phi(\boldsymbol{A}) = (\boldsymbol{A}^{11})^{-1}$ is

$$\sum_{j=1}^{n} w_j \phi(\boldsymbol{A}_j) \ \leq \ \phi\left(\sum_{j=1}^{n} w_j \boldsymbol{A}_j\right).$$

For an analytic proof, see [3](page 678).

For both the above concavity and the convexity of the matrix functions, we shall prove them by the view of statistics simpler and wider than the previous way in [2]. Let $X_1, \ldots, X_n$ be independent $s$-variate normal random vectors with

$$E_\theta(X_j) \ = \ v_j\theta, \quad V_\theta(X_j) \ = \ \boldsymbol{\Sigma}_j^{-1}, \quad (j = 1, \ldots, n),$$

where $v_j$ are known, $\sum_{j=1}^{n} v_j^2 = 1$, and $\boldsymbol{\Sigma}_j$ are positive definite. Setting $X = (X_1, \ldots, X_n)$, $v_j^2 = w_j$ gives the Fisher information matrices as follows:

$$\boldsymbol{I}_{X_j} \ = \ w_j \boldsymbol{\Sigma}_j, \quad \boldsymbol{I}_X \ = \ \sum_{j=1}^{n} \boldsymbol{I}_{X_j} \ = \ \sum_{j=1}^{n} w_j \boldsymbol{\Sigma}_j.$$

Since $S = \sum_{j=1}^{n} v_j X_j$ is a statistics of $X$ and is distributed with the normal distribution with

$$E_\theta(S) \ = \ \sum_{j=1}^{n} v_j^2 \theta \ = \ \theta, \quad V_\theta(S) \ = \ \sum_{j=1}^{n} w_j \boldsymbol{\Sigma}_j^{-1},$$

so that the Fisher information on $S$ is

$$\boldsymbol{I}_S \ = \ \left(\sum_{j=1}^{n} w_j \boldsymbol{\Sigma}_j^{-1}\right)^{-1}.$$

The monotonicity (1) on the Fisher information with respect to $S$ and $X$ gives

$$\sum_{j=1}^{n} w_j (\boldsymbol{\Sigma}_j^{-1})^{-1} \ = \ \boldsymbol{I}_X \quad \geq \quad \boldsymbol{I}_S \ = \ \left(\sum_{j=1}^{n} w_j \boldsymbol{\Sigma}_j^{-1}\right)^{-1},$$

so that the convexity of the matrix function holds. On the other hand, since the matrices $\boldsymbol{\Sigma}_j$ and $\boldsymbol{\Sigma}_j^{-1}$ are partitioned by

$$\boldsymbol{\Sigma}_j \ = \ \begin{pmatrix} \boldsymbol{\Sigma}_{11,j} & \boldsymbol{\Sigma}_{12,j} \\ \boldsymbol{\Sigma}_{21,j} & \boldsymbol{\Sigma}_{22,j} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Sigma}_j^{-1} \ = \ \begin{pmatrix} \boldsymbol{\Sigma}_j^{11} & \boldsymbol{\Sigma}_j^{12} \\ \boldsymbol{\Sigma}_j^{21} & \boldsymbol{\Sigma}_j^{22} \end{pmatrix},$$

the superadditivity (3) on the efficient Fisher information gives

$$\sum_{j=1}^{n} w_j \left(\boldsymbol{\Sigma}_j^{11}\right)^{-1} \ = \ \sum_{j=1}^{n} \widehat{\boldsymbol{I}}_{1,X_j} \quad \leq \quad \widehat{\boldsymbol{I}}_{1,X} \ = \ \left(\left(\sum_{j=1}^{n} w_j \boldsymbol{\Sigma}_j\right)^{11}\right)^{-1},$$

so that the concavity of the matrix function holds.

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# HOPF HYPERSURFACES WITH $\eta$-PARALLEL RICCI TENSORS IN A NONFLAT COMPLEX SPACE FORM

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Abstract. We give a classification theorem of Hopf hypersurfaces $M^{2n-1}$ with $\eta$-parallel Ricci tensors in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$. There exist non-homogeneous Hopf hypersurfaces $M^3$ with $\eta$-parallel Ricci tensors in $\widetilde{M}_2(c), c \neq 0$. Note that these real hypersurfaces do not have $\eta$-parallel shape operators in this ambient space.

**1 Introduction** We denote by $\widetilde{M}_n(c)$ a complex $n$-dimensional nonflat complex space form of constant holomorphic sectional curvature $c(\neq 0)$. That is, $\widetilde{M}_n(c)$ is holomorphically congruent to either a complex projective space of constant holomorphic sectional curvature $c(> 0)$ or a complex hyperbolic space of constant holomorphic sectional curvature $c(< 0)$.

The study of real hypersurfaces isometrically immersed into $\widetilde{M}_n(c)$ is one of the most interesting objects in differential geometry. There are many nice results in this field (cf [6]). For $n \geqq 3$ some results can be proved affirmatively, but for $n = 2$ it is difficult to get those same results or there exist counter examples to them.

The classification theorem of Hopf hypersurfaces $M$ (namely real hypersurfaces $M$ such that the characteristic vector $\xi$ of $M$ is principal at its each point) with $\eta$-parallel Ricci tensors in $\widetilde{M}_n(c)$ is one of such results. We here review the definition of the $\eta$-parallelism for a tensor field $T$ of type $(1, 1)$ on a real hypersurface $M$ in $\widetilde{M}_n(c)$. $T$ is called *$\eta$-parallel* if $g((\nabla_X T)Y, Z) = 0$ for all vectors $X, Y$ and $Z$ which are orthogonal to $\xi$ on $M$.

The purpose of this paper is to prove the following two theorems.

**Theorem 1.** *Let $M$ be a connected Hopf hypersurface in $\mathbb{C}P^n(c), n \geqq 2$. Suppose that $M$ has $\eta$-parallel Ricci tensor. Then $M$ is either locally congruent to one of homogeneous real hypersurfaces of types* $(\mathrm{A}_1)$, $(\mathrm{A}_2)$ *and* (B) *in $\mathbb{C}P^n(c), n \geqq 2$, or a non-homogeneous real hypersurface with $A\xi = 0$ in $\mathbb{C}P^2(c)$. This non-homogeneous real hypersurface $M$ is locally congruent a tube of radius $\pi/(2\sqrt{c}\,)$ over a non-totally geodesic complex curve which does not have the principal curvatures $\pm\sqrt{c}/2$ in $\mathbb{C}P^2(c)$.*

**Theorem 2.** *Let $M$ be a connected Hopf hypersurface in $\mathbb{C}H^n(c), n \geqq 2$. Suppose that $M$ has $\eta$-parallel Ricci tensor. Then $M$ is either locally congruent to one of homogeneous real hypersurfaces of types* $(\mathrm{A}_0)$, $(\mathrm{A}_{1,0})$, $(\mathrm{A}_{1,1})$, $(\mathrm{A}_2)$ *and* (B) *in $\mathbb{C}H^n(c), n \geqq 2$, or a non-homogeneous real hypersurface with $A\xi = 0$ in $\mathbb{C}H^2(c)$.*

In [7], the classification problem of Hopf hypersurfaces with $\eta$-parallel Ricci tensors in $\widetilde{M}_n(c), n \geqq 2$ was discussed. However, there are some serious gaps in that paper. Non-homogeneous real hypersurfaces $M^3$ with $A\xi = 0$ in $\widetilde{M}_2(c)(= \mathbb{C}P^2(c)$ or $\mathbb{C}H^2(c))$ are counter examples to results of [7].

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**2 Preliminaries** Let $M^{2n-1}$ be a real hypersurface immersed into a nonflat complex space form $\widetilde{M}_n(c)$ through an isometric immersion with a unit normal local vector field $\mathcal{N}$. The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$ and $\nabla$ of $M$ are related by the following formulas of Gauss and Weingarten:

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N}, \tag{2.1}$$

$$\widetilde{\nabla}_X \mathcal{N} = -AX \tag{2.2}$$

for arbitrary vector fields $X$ and $Y$ on $M$, where $g$ is the Riemannian metric of $M$ induced from the ambient space $\widetilde{M}_n(c)$ and $A$ is the shape operator of $M$ in $\widetilde{M}_n(c)$. An eigenvector of the shape operator $A$ is called a *principal curvature vector* of $M$ in $\widetilde{M}_n(c)$ and an eigenvalue of $A$ is called a *principal curvature* of $M$ in $\widetilde{M}_n(c)$. We set $V_\lambda = \{v \in TM \mid Av = \lambda v\}$ which is called the principal distribution associated to the principal curvature $\lambda$.

It is well-known that $M$ has an almost contact metric structure induced from the Kähler structure $(J, g)$ of the ambient space $\widetilde{M}_n(c)$. That is, we have a quadruple $(\phi, \xi, \eta, g)$ defined by

$$g(\phi X, Y) = g(JX, Y), \quad \xi = -J\mathcal{N} \quad \text{and} \quad \eta(X) = g(\xi, X) = g(JX, \mathcal{N}).$$

Then they satisfy

$$\phi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1 \quad \text{and} \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vectors $X, Y \in TM$. It is known that these equations imply that $\phi\xi = 0$ and $\eta(\phi X) = 0$. In the following, we call $\phi$, $\xi$ and $\eta$ *the structure tensor*, *the characteristic vector* and *the contact form* on $M$, respectively.

It follows from (2.1), (2.2), $\widetilde{\nabla} J = 0$ and $JX = \phi X + \eta(X)\mathcal{N}$ that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.3}$$

$$\nabla_X \xi = \phi AX. \tag{2.4}$$

Denoting the curvature tensor of $M$ by $R$, we have the equation of Gauss given by

$$\begin{aligned} g((R(X,Y)Z, W) = {} & (c/4)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ & + g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) - 2g(\phi X, Y)g(\phi Z, W)\} \\ & + g(AY, Z)g(AX, W) - g(AX, Z)g(AY, W). \end{aligned} \tag{2.5}$$

The following is called the equation of Codazzi.

$$(\nabla_X A)Y - (\nabla_Y A)X = (c/4)(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi). \tag{2.6}$$

We usually call $M$ a *Hopf hypersurface* if the characteristic vector $\xi$ of $M$ is a principal curvature vector at each point of $M$. The following lemma clarifies fundamental properties of principal curvatures of a Hopf hypersurface $M$ in $\widetilde{M}_n(c)$ (cf. [6]).

**Lemma 1.** *Let $M$ be a Hopf hypersurface of a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$. Then the following hold.*

1. *If a nonzero vector $v \in TM$ orthogonal to $\xi$ satisfies $Av = \lambda v$, then $(2\lambda - \delta)A\phi v = (\delta\lambda + (c/2))\phi v$, where $\delta$ is the principal curvature associated with $\xi$. In particular, when $c > 0$, we have $A\phi v = \big((\delta\lambda + (c/2))/(2\lambda - \delta)\big)\phi v$.*

2. *The principal curvature $\delta$ associated with $\xi$ is constant locally on $M$.*

In $\mathbb{C}P^n(c)$ $(n \geqq 2)$, a connected Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following (see [4, 8]):

($A_1$) A geodesic sphere of radius $r$, where $0 < r < \pi/\sqrt{c}$;

($A_2$) A tube of radius $r$ around a totally geodesic $\mathbb{C}P^\ell(c)$ $(1 \leqq \ell \leqq n-2)$, where $0 < r < \pi/\sqrt{c}$;

(B) A tube of radius $r$ around a complex hyperquadric $\mathbb{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c})$;

(C) A tube of radius $r$ around the Segre embedding of $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c})$ and $n$ $(\geqq 5)$ is odd;

(D) A tube of radius $r$ around the Plücker embedding of a complex Grassmannian $\mathbb{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c})$ and $n = 9$;

(E) A tube of radius $r$ around a Hermitian symmetric space SO(10)/U(5), where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 15$.

These real hypersurfaces are said to be of types ($A_1$), ($A_2$), (B), (C), (D) and (E). Unifying real hypersurfaces of types ($A_1$) and ($A_2$), we call them hypersurfaces of type (A). The numbers of distinct principal curvatures of these real hypersurfaces are $2, 3, 3, 5, 5, 5$, respectively.

In $\mathbb{C}H^n(c)$ $(n \geqq 2)$, a connected Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following (see [1]):

($A_0$) A horosphere in $\mathbb{C}H^n(c)$;

($A_{1,0}$) A geodesic sphere of radius $r$, where $0 < r < \infty$;

($A_{1,1}$) A tube of radius $r$ around a totally geodesic hypersurface $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$;

($A_2$) A tube of radius $r$ around a totally geodesic $\mathbb{C}H^\ell(c)$ $(1 \leqq \ell \leqq n-2)$, where $0 < r < \infty$;

(B) A tube of radius $r$ around a totally real totally geodesic $\mathbb{R}H^n(c/4)$, where $0 < r < \infty$.

These real hypersurfaces are said to be of types ($A_0$), ($A_{1,0}$), ($A_{1,1}$), ($A_2$) and (B). A real hypersurface of type (B) with radius $r = (1/\sqrt{|c|}\,)\log_e(2+\sqrt{3}\,)$ has two distinct constant principal curvatures $\lambda_1 = \delta = \sqrt{3|c|}/2$ and $\lambda_2 = \sqrt{|c|}/(2\sqrt{3}\,)$. Except for this real hypersurface, the numbers of distinct principal curvatures of Hopf hypersurfaces with constant principal curvatures are $2, 2, 2, 3, 3$, respectively. Unifying real hypersurfaces of types ($A_0$), ($A_{1,0}$), ($A_{1,1}$) and ($A_2$), we call them hypersurfaces of type (A).

**3 Proof of Theorems 1 and 2** First of all we recall the following two classification theorems of Hopf hypersurfaces with $\eta$-parallel shape operators in a nonflat complex space form.

**Theorem A** ([5]). *Let $M$ be a connected Hopf hypersurface of $\mathbb{C}P^n(c), n \geqq 2$. Then $M$ has $\eta$-parallel shape operator if and only if $M$ is locally congruent to one of homogeneous real hypersurfaces of types* ($A_1$), ($A_2$) *and* (B) *in $\mathbb{C}P^n(c)$.*

**Theorem B** ([7]). *Let $M$ be a connected Hopf hypersurface of $\mathbb{C}H^n(c), n \geqq 2$. Then $M$ has $\eta$-parallel shape operator if and only if $M$ is locally congruent to one of homogeneous real hypersurfaces of types* (A$_0$), (A$_{1,0}$), (A$_{1,1}$), (A$_2$) *and* (B) *in $\mathbb{C}H^n(c)$.*

The Ricci tensor $S$ of an arbitrary real hypersurface $M$ in $\widetilde{M}_n(c)$ is expressed as (see (2.5)):

$$SX = (c/4)\big((2n+1)X - 3\eta(X)\xi\big) + (\text{trace } A)AX - A^2X. \tag{3.1}$$

By (3.1) we get the following equation on an arbitrary real hypersurface $M$ in the ambient space $\widetilde{M}_n(c)$:

$$\begin{aligned} g((\nabla_X S)Y, Z) &= (X(\text{trace } A))g(AY, Z) + (\text{trace } A)g((\nabla_X A)Y, Z) \\ &\quad - g((\nabla_X A^2)Y, Z) = 0 \quad \text{for } X, Y, Z(\perp \xi) \in TM. \end{aligned} \tag{3.2}$$

*Remark* 1. It follows from

$$g((\nabla_X A^2)Y, Z) = g((\nabla_X A)Y, AZ) + g(AY, (\nabla_X A)Z)$$

and Equation (3.2) that if a Hopf hypersurface $M$ in a nonflat complex space form $\widetilde{M}_n(c)$ has $\eta$-parallel shape operator $A$ and trace $A$ is constant on $M$, then $M$ has $\eta$-parallel Ricci tensor $S$.

We shall prove Theorem 1. Without loss of generality, we may set $c = 4$.

We first study the case of $n \geqq 3$. Our discussion here is essentially due to [7]. We suppose that the Ricci tensor $S$ of our Hopf hypersurface $M$ is $\eta$-parallel. Then, for a unit vector $Y(\perp \xi)$ with $AY = \lambda Y$, putting $h = \text{trace } A$ in Equation (3.2), we find

$$\lambda(Xh) + h(X\lambda) - X(\lambda^2) = 0 \quad \text{for any } X(\perp \xi), \tag{3.3}$$

which means that $X(\lambda h - \lambda^2) = 0$ for any $X(\perp \xi)$. On the other hand, for any $Y(\perp \xi)$ such that $AY = \lambda Y$ we have $(\nabla_\xi A)Y = (\xi\lambda)Y + (\lambda I - A)\nabla_\xi Y$. Thus, from (2.6) we get $\xi\lambda = g((\nabla_\xi A)Y, Y) = g((\nabla_Y A)\xi, Y) = 0$. This, together with Lemma 1(2), implies that $\xi h = 0$. Hence, the function $\lambda h - \lambda^2$ is constant on $M$. Thus, for any principal curvatures $\lambda$, $\mu$ with $AX = \lambda X(\perp \xi)$, $AY = \mu Y(\perp \xi)$, we can put

$$\lambda h - \lambda^2 = a \tag{3.4}$$

and

$$\mu h - \mu^2 = b. \tag{3.5}$$

We here consider both cases of $a = b$ and $a \neq b$. It follows from Lemma 1(1), (3.4) and (3.5) that

$$(2h\delta - \delta^2 - 4b)\lambda^2 - \{(\delta^2 - 4)h + 4\delta - 4b\delta\}\lambda - (2\delta h + b\delta^2 + 4) = 0.$$

This, combined with $h\lambda = \lambda^2 + a$ (see (3.4)), yields the following algebraic equation:

$$2\delta\lambda^4 - (2\delta^2 + 4b - 4)\lambda^3 + 2(a\delta + 2b\delta - 3\delta)\lambda^2 - (a\delta^2 - 4a + b\delta^2 + 4)\lambda - 2a\delta = 0. \tag{3.6}$$

Except the case of $\delta = 0$ and $a = b = 1$, $\lambda$ satisfies the algebraic equation (3.6) with constant coefficients, so that $\lambda$ is constant locally on $M$.

We next consider the case that $\delta = 0$ and $a = b = 1$. Note that in this case coefficients in Equation (3.6) are all vanishing. So, from Lemma 1 we must consider the case that $M$ has at most three distinct principal curvatures $\delta = 0$ with multiplicity 1, $\lambda$ with multiplicity, say $k$, and $1/\lambda$ with multiplicity $2n-2-k$ having the equation $h = \lambda + (1/\lambda)$ (see (3.4)). When $k$ satisfies either $k = 0$ or $2n-2-k = 0$, $\lambda$ must satisfy $\lambda = 1/\lambda$, so that $\lambda = 1$ or $-1$.

We shall show that the case where $k \geqq 1$ and $2n-2-k \geqq 1$ does not occur. It follows from $h = \lambda + (1/\lambda)$ that

$$(k-1)\lambda^2 + 2n - 3 - k = 0.$$

Since $k-1 \geqq 0$ and $2n-3-k \geqq 0$, this equation holds if and only if $k = 1$ and $n = 2$, which contradicts to the assumption $n \geqq 3$. Therefore our Hopf hypersurface with $\eta$-parallel Ricci tensor in a nonflat complex space form must be homogeneous in $\mathbb{C}P^n(4)$.

In order to prove our Theorem, we shall show that our real hypersurface satisfying the assumption has $\eta$-parallel shape operator. Let $M$ be homogeneous in $\mathbb{C}P^n(4)$. We take three principal curvature vectors $X \in V_\lambda^0,\ Y \in V_\mu^0,\ Z \in V_\nu^0$. Here, for example $V_\lambda^0$ is defined by $V_\lambda^0 = \{X \in TM | AX = \lambda X, X \perp \xi\}$. Note that Codazzi equation (2.6) shows that $g((\nabla_X A)Y, Z)$ is symmetric for all $X, Y, Z \in T^0M$. We have

$$\begin{aligned} g((\nabla_X A)Y, Z) &= g(\nabla_X(AY) - A\nabla_X Y, Z) \\ &= g((\mu I - A)\nabla_X Y, Z) = (\mu - \nu)g(\nabla_X Y, Z). \end{aligned} \tag{3.7}$$

On the other hand, it follows from (3.2) that

$$\begin{aligned} g((\nabla_X S)Y, Z) &= h \cdot g((\nabla_X A)Y, Z) - g((\nabla_X A^2)Y, Z) \\ &= h \cdot g((\nabla_X A)Y, Z) - g((\nabla_X A)Y, AZ) - g(AY, (\nabla_X A)Z) \\ &= (h - \mu - \nu)g((\nabla_X A)Y, Z), \end{aligned}$$

which, together with the assumption that the Ricci tensor $S$ is $\eta$-parallel, shows

$$(h - \mu - \nu)g((\nabla_X A)Y, Z) = 0. \tag{3.8}$$

When $\mu = \nu$, Equation (3.7) implies $g((\nabla_X A)Y, Z) = 0$. So, in the following it suffices to consider the case that $\mu \neq \nu$.

When $h - \mu - \nu \neq 0$, Equation (3.8) yields $g((\nabla_X A)Y, Z) = 0$. Thus it remains to consider the case that $h - \mu - \nu = 0$. Changing $X$ and $Y$ in (3.8), we get

$$(h - \lambda - \nu)g((\nabla_Y A)X, Z) = 0. \tag{3.9}$$

If $\lambda \neq \mu$, then $h - \mu - \nu = 0$ implies $h - \lambda - \nu \neq 0$. This, combined with (3.9), yields $g((\nabla_X A)Y, Z) = g((\nabla_Y A)X, Z) = 0$. If $\lambda = \mu$, Equation (3.7) gives $g((\nabla_X A)Y, Z) = g((\nabla_Z A)X, Y) = 0$. Therefore we can see that our Hopf hypersurface $M$ has $\eta$-parallel shape operator. So $M$ is locally congruent to one of homogeneous real hypersurfaces of types $(\mathrm{A}_1), (\mathrm{A}_2)$ and (B) (see Theorem A).

Conversely, let $M$ be of either type $(\mathrm{A}_1)$, type $(\mathrm{A}_2)$ or type (B). Then, from Theorem A and Remark 1 we see easily that $M$ has $\eta$-parallel Ricci tensor.

We next study the case of $n = 2$. When $\delta \neq 0$, Equation (3.6) means that $\lambda$ is constant locally. Hence $M$ is a Hopf hypersurface with constant principal curvatures with $A\xi \neq 0$ in $\mathbb{C}P^2(4)$. In this case, $M$ is of either type $(\mathrm{A}_1)$ of radius $r (\neq \pi/4)$ or type (B) of each radius $r \in (0, \pi/4)$ in $\mathbb{C}P^2(4)$.

Next, let consider the case that $\delta = 0$ and tr $A$ is constant locally. Then $M$ is a Hopf hypersurface with constant principal curvatures with $A\xi = 0$ in $\mathbb{C}P^2(4)$. In this case, $M$ is of type $(\mathrm{A}_1)$ of radius $\pi/4$ in $\mathbb{C}P^2(4)$.

In these two cases, $M$ has $\eta$-parallel shape operator. Then we can see that $M$ has $\eta$-parallel Ricci tensor by Theorem A and Remark 1.

We finally consider the case that $\delta = 0$ and $h$ is a non-constant function on $M$. We shall verify that every non-homogeneous real hypersurface $M$ with $A\xi = 0$ in $\mathbb{C}P^2(4)$ has $\eta$-parallel Ricci tensor. Such a real hypersurface $M$ has three distinct principal curvatures $\delta = 0, \lambda$ and $1/\lambda$, where $\lambda$ is a non-constant smooth function on $M$. We take two unit vectors $X$ and $Y$ with $AX = \lambda X$ and $AY = (1/\lambda)Y$. Using Equation (3.2) and $h = \lambda + (1/\lambda)$ repeatedly, we obtain the following:

$$\begin{aligned} g((\nabla_X S)X, X) &= \Big(X\lambda - \frac{X\lambda}{\lambda^2}\Big)g(AX, X) + \Big(\lambda + \frac{1}{\lambda}\Big)g((\nabla_X A)X, X) \\ &\quad - g((\nabla_X A)X, AX) - g(AX, (\nabla_X A)X) \\ &= \lambda\Big(X\lambda - \frac{X\lambda}{\lambda^2}\Big) + \Big(\frac{1}{\lambda} - \lambda\Big)g((\nabla_X A)X, X) \\ &= \lambda\Big(X\lambda - \frac{X\lambda}{\lambda^2}\Big) + (X\lambda)\Big(\frac{1}{\lambda} - \lambda\Big) = 0, \end{aligned}$$

$$\begin{aligned} g((\nabla_X S)Y, Y) &= \Big(X\lambda - \frac{X\lambda}{\lambda^2}\Big)g(AY, Y) + \Big(\lambda + \frac{1}{\lambda}\Big)g((\nabla_X A)Y, Y) \\ &\quad - g((\nabla_X A)Y, AY) - g(AY, (\nabla_X A)Y) \\ &= \frac{1}{\lambda}\Big(X\lambda - \frac{X\lambda}{\lambda^2}\Big) + \Big(\lambda - \frac{1}{\lambda}\Big)g((\nabla_X A)Y, Y) \\ &= \frac{1}{\lambda}\Big(X\lambda - \frac{X\lambda}{\lambda^2}\Big) - \frac{X\lambda}{\lambda^2}\Big(\lambda - \frac{1}{\lambda}\Big) = 0, \end{aligned}$$

and

$$\begin{aligned} g((\nabla_X S)X, Y) &= (Xh)g(AX, Y) + h \cdot g((\nabla_X A)X, Y) \\ &\quad - g((\nabla_X A)X, AY) - g(AX, (\nabla_X A)Y) \\ &= \Big(h - \lambda - \frac{1}{\lambda}\Big)g((\nabla_X A)X, Y) = 0. \end{aligned}$$

We have similarly the following:

$$g((\nabla_Y S)Y, Y) = g((\nabla_Y S)X, Y) = g((\nabla_Y S)X, X) = 0.$$

These, combined with the symmetry of $S$, show that $M$ has $\eta$-parallel Ricci tensor.

The rest of the proof is to guarantee the existence of a non-homogeneous real hypersurface $M$ with $A\xi = 0$ in $\mathbb{C}P^2(4)$. To do this, we recall the following fact due to [2]:

**Fact.** 1. *Every tube $M$ of sufficiently small constant radius around each Kähler submanifold of $\mathbb{C}P^n(c)$ is a Hopf hypersurface in this ambient space. However, in general $M$ has singular points, namely $M$ is not smooth at these points.*

2. *Let $M^{2n-1}$ be a Hopf hypersurface with $A\xi = \delta\xi$ of $\mathbb{C}P^n(c), n \geqq 2$. Suppose that all principal curvatures of $M$ in the ambient space $\mathbb{C}P^n(c)$ have constant multiplicities on $M$. Then $M$ is locally congruent to a tube of constant radius $r(> 0)$ around a certain Kähler submanifold $N$ of $\mathbb{C}P^n(c)$. Moreover, $\delta = \sqrt{c}\,\cot(\sqrt{c}\,r)$ and all other principal curvatures $\lambda$ of $M$ are expressed as either $\lambda = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$, $\lambda = (-\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$ or $\lambda = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2 \pm \theta)$, where $(\pm\sqrt{c}/2)\cot\theta$ are principal curvatures of the Kähler submanifold $N$.*

This fact means that every non-homogeneous real hypersurface $M$ with $A\xi = 0$ in $\mathbb{C}P^2(4)$ is locally congruent a tube of radius $\pi/4$ over a non-totally geodesic complex curve in the ambient space $\mathbb{C}P^2(4)$. Here, note that to delete singular points of $M$ we have only to consider the complex curve without having the principal curvatures $\pm 1$ in $\mathbb{C}P^2(4)$.

Therefore we obtain the desired conclusion of Theorem 1. □

We next prove Theorem 2. Without loss of generality, we may set $c = -4$. We first investigate the case of $n \geqq 3$. The "only if" part is obvious from Theorem B and Remark 1. So we shall prove the "if" part.

We suppose that the Ricci tensor $S$ of our Hopf hypersurface $M$ (with $A\xi = \delta\xi$) is $\eta$-parallel. By the same discussion as that in the proof of Theorem 1 we also have Equations (3.4) and (3.5).

We first take a unit vector $X \in V_\lambda^0$ with $2\lambda - \delta \neq 0$. Then $A\phi X = \mu\phi X$ with $\mu = (\delta\lambda - 2)/(2\lambda - \delta)$ (see Lemma 1). This, together with (3.5), yields

$$(2\delta h - \delta^2 - 4b)\lambda^2 + \{4\delta + 4b\delta - (\delta^2 + 4)h\}\lambda + 2\delta h - 4 - b\delta^2 = 0. \tag{3.10}$$

It follows from (3.4) and (3.10) that

$$2\delta\lambda^4 - 2(\delta^2 + 2b + 2)\lambda^3 + 2\delta(a + 2b + 3)\lambda^2 - (a\delta^2 + b\delta^2 + 4a + 4)\lambda + 2a\delta = 0, \tag{3.11}$$

which corresponds to Equation(3.6). Except the case of $\delta = 0$ and $a = b = -1$, $\lambda$ satisfies the algebraic equation (3.11) with constant coefficients, so that $\lambda$ is constant locally on $M$.

We next consider the case of $\delta = 0$ and $a = b = -1$. Note that in this case coefficients in Equation (3.11) are all vanishing. Here, from Lemma 1 we must consider the case that that $M$ has at most three distinct principal curvatures $\delta = 0$ with multiplicity $1, \lambda$ with multiplicity , say $k$, and $-1/\lambda$ with multiplicity $2n-2-k$ having the equation $h = \lambda-(1/\lambda)$ (see (3.4)). When $k$ satisfies either $k = 0$ or $2n-2-k = 0$, $\lambda$ must satisfy $\lambda = -1/\lambda$, which is a contradiction.

So we only to study the case that $k \geqq 1$ and $2n - 2 - k \geqq 1$. Since $h = \lambda - (1/\lambda)$, we have

$$(k - 1)\lambda^2 - (2n - 3 - k) = 0.$$

We note that $k \neq 1$, since $n \geqq 3$. Hence, $\lambda$ is also constant locally on $M$. However, there does not exist such a Hopf hypersurface with three constant principal curvatures $\delta = 0, \lambda, -1/\lambda$ in $\mathbb{C}H^n(-4)$ (see [6]).

We finally consider the case of $2\lambda - \delta = 0$ at some point of $M$. We shall verify that $2\lambda - \delta$ vanishes identically on $M$. Assume that $2\lambda - \delta \neq 0$ at some point $x_0 \in M$, and set $y_0 = (2\lambda - \delta)(x_0)$. Let $\widetilde{N}$ be the subset of those points $x \in M$ such that $(2\lambda - \delta)(x) = y_0$. Clearly $\widetilde{N}$ is a non-empty closed subset of $M$. It is also open, since the discussion in the case of $2\lambda - \delta \neq 0$ means that the function $2\lambda - \delta$ is constantly equal to $y_0 \neq 0$ on some neighborhood of each point $x \in \widetilde{N}$. Since $M$ is connected, we find that $\widetilde{N} = M$, which is a contradiction. So we find that $\lambda = \delta/2$ on $M$.

Therefore we can see that every principal curvature $\lambda$ of $M$ is constant locally, so that $M$ is locally congruent to one of homogeneous real hypersurfaces of types ($A_0$), ($A_{1,0}$), ($A_{1,1}$), ($A_2$) and (B).

We next study the case of $n = 2$. By the same discussion as that in the proof of Theorem 1, we can see that our real hypersurface $M$ satisfying the assumption is locally congruent to either a homogeneous Hopf hypersurface or a non-homogeneous Hopf hypersurface with $A\xi = 0$ in the ambient space $\mathbb{C}H^2(-4)$.

At the end of this paper we explain briefly the construction of a non-homogeneous real hypersurface $M^3$ with $A\xi = 0$ in $\mathbb{C}H^2(-4)$. In [3], T.A. Ivey and P.J. Ryan construct the class of Hopf hypersurfaces in $\mathbb{C}H^2(-4)$ with $A\xi = \delta\xi$ and $0 \leqq \delta \leqq 2$. Moreover, they show that every such Hopf hypersurface for $\delta < 2$ can be characterized in terms of Weierstrass-type data which take the form of a pair of embedded contact curves in a unit sphere $S^3$ (for details, see Theorem 2 in [3]).

Hence we obtain the desired conclusion.

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# NUMERICAL RADIUS OF MOORE-PENROSE INVERSE

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Abstract. We studied in [4] the result by Stampfli that an invertible operator $A$ is unitary if the numerical radius $w(A) \leqq 1$ and $w(A^{-1}) \leqq 1$. In this note, instead of the invertibility we consider the Moore-Penrose inverse $A^\dagger$ of $A$ and we characterize bounded operators $A$ satisfying that $w(A) \leqq 1, w(A^\dagger) \leqq 1$ and $AA^\dagger = A^\dagger A$.

**Moore-Penrose inverse and numerical radius** For a bounded linear operator $A \in B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, the numerical radius $w(A)$ of $A$ is defined as

$$w(A) = \sup\{|\langle A\xi, \xi\rangle| : \|\xi\| = 1\}.$$

We refer to [3] for basic properties of numerical radius. If $A$ is invertible and satisfies that $w(A) \leqq 1$ and $w(A^{-1}) \leqq 1$, then $A$ is unitary. This is proved by Stampfli in [5, 6], and an alternative proof is given by Sano-Uchiyama in [4]. See also [1].

In this note, we study the general case: instead of the invertibility of $A$, we consider the Moore-Penrose inverse $A^\dagger$ of $A$, and we observe the conditions that $w(A) \leqq 1, w(A^\dagger) \leqq 1$.

Before presenting our theorem, we recall the definition of the Moore-Penrose inverse. For $A \in \mathcal{B}(\mathcal{H})$, we denote the null (or kernel) space of $A$ by $\mathcal{N}(A)$ and the range space of $A$ by $\mathcal{R}(A)$. The Moore-Penrose inverse of $A$, denoted by $A^\dagger$, is defined on $\mathcal{R}(A) \oplus \mathcal{R}(A)^\perp$ by $A^\dagger Ay = y$ for $y \in \mathcal{N}(A)^\perp$ and $A^\dagger|_{\mathcal{R}(A)^\perp} = 0$. $A^\dagger$ is densely-defined and unbounded in general.

It is known that $\mathcal{R}(A)$ is closed if and only if $A^\dagger$ is bounded. In this case, $A^\dagger|_{\mathcal{R}(A)}$ is the bounded inverse of $A|_{\mathcal{N}(A)^\perp}$. For the readers' convenience, we give a proof; the only if part follows from Banach inversion theorem by thinking of $A|_{\mathcal{N}(A)^\perp}$ as one-to-one operator from $\mathcal{N}(A)^\perp$ onto $\mathcal{R}(A)$. For the if part, take any sequence $\{Ax_n\}$ for $x_n \in \mathcal{N}(A)^\perp$ which converges to $y \in \mathcal{H}$. Since $A^\dagger$ is bounded, $\{A^\dagger(Ax_n)\} = \{x_n\}$ converges to $A^\dagger y =: x \in \mathcal{H}$. Hence, $\{Ax_n\}$ converges to $Ax$, or $y = Ax \in \mathcal{R}(A)$; that is, $\mathcal{R}(A)$ is closed.

These conditions are also equivalent to the closedness of $\mathcal{R}(A^*)$; let $A = V|A|$ be the polar decomposition of $A$. By symmetric argument, it suffices to show that $\mathcal{R}(A)$ is closed if $\mathcal{R}(A^*)$ is closed. We remark that $\mathcal{R}(A^*) = \mathcal{R}(|A|)$; for $A^* = |A|V^*$ implies $\mathcal{R}(A^*) \subseteqq \mathcal{R}(|A|)$ and $|A| = A^*V$ does the converse inclusion. Suppose that $\{Ax_n\}$ converges to $x$. Then $\{|A|x_n\} = \{V^*Ax_n\}$ converges $V^*x$. Since $\mathcal{R}(|A|)$ is closed, $V^*x = |A|y$ for some $y \in \mathcal{H}$. Thus, $\{Ax_n\} = \{V|A|x_n\}$ converges to $V|A|y = Ay$, or $x = Ay$. This means that $\mathcal{R}(A)$ is closed.

The following is a generalization of Stampfli's theorem.

**Theorem** Let $A \in B(\mathcal{H})$ be a bounded linear operator on $\mathcal{H}$. If $A$ satisfies that $w(A) \leqq 1, w(A^\dagger) \leqq 1$ and $AA^\dagger = A^\dagger A$, then $A$ is unitary on $\mathcal{N}(A)^\perp$.

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It is known in [3, Theorem 1.3-1] that the numerical radius is equivalent to the operator norm:

$$w(T) \leqq \|T\| \leqq 2w(T).$$

Hence, the assumption that $w(A^\dagger) \leqq 1$ means the boundedness of $A^\dagger$.

**Proof of Theorem.** The assumption $AA^\dagger = A^\dagger A$ yields that $\mathcal{R}(A) = \mathcal{R}(A^*) =: \mathcal{K}$, which are closed by the preceding remark. We consider the restriction of $A$ from $\mathcal{K}$ into itself. Since $A^\dagger|_{\mathcal{K}} = (A|_{\mathcal{K}})^{-1}$, $w((A|_{\mathcal{K}})^{-1}) = w(A^\dagger|_{\mathcal{K}}) = w(A^\dagger) \leqq 1$. Combining this with $w(A|_{\mathcal{K}}) = w(A) \leqq 1$ and applying Stampfli's theorem to $A|_{\mathcal{K}}$ and $(A|_{\mathcal{K}})^{-1}$, we conclude that $A|_{\mathcal{K}}$ is unitary on $\mathcal{K}$. ■

In fact, $AA^\dagger = A^\dagger A$ is equivalent to $\mathcal{R}(A) = \mathcal{R}(A^*)$ since $A^\dagger A = \mathrm{Proj}_{\mathcal{R}(A^*)}$ and $AA^\dagger = \mathrm{Proj}_{\mathcal{R}(A)}$ when $A^\dagger$ is bounded.

**Example** Let

$$X = \begin{pmatrix} r & \sqrt{1-r^2} \\ \sqrt{1-r^2} & -r \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

for $a > 0$ and $1 \geqq r > 0$. The Moore-Penrose inverse $X^\dagger$ of $X$ is given as

$$X^\dagger = \begin{pmatrix} 1/a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r & \sqrt{1-r^2} \\ \sqrt{1-r^2} & -r \end{pmatrix},$$

which can be checked by the characterization of $X^\dagger$: $XX^\dagger X = X, X^\dagger XX^\dagger = X^\dagger$, $(XX^\dagger)^* = XX^\dagger$, and $(X^\dagger X)^* = X^\dagger X$. We refer the reader to [2] for this characterization.

By direct computation, we see that

$$w(X) = \frac{1+r}{2}\ a, \quad w(X^\dagger) = \frac{1+r}{2}\ \frac{1}{a}.$$

Hence, $w(X) \leqq 1$ and $w(X^\dagger) \leqq 1$ if and only if

$$\frac{1+r}{2} \leqq a \leqq \frac{2}{1+r}.$$

Note that $XX^\dagger = X^\dagger X$ if and only if $r = 1$. If we take $r < 1$ and $a = 2/(1+r)$, then $w(X) \leqq 1$ and $w(X^\dagger) \leqq 1$, and $\|X\| = a = 2/(1+r) > 1$, which means that $X$ is not a contraction.

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# EXAMPLES ON IRRESOLVABILITY

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Abstract. We construct several examples of Hausdorff (resp. regular) open-hereditarily irresolvable not hereditarily irresolvable or hereditarily irresolvable not submaximal spaces. Also, examples of separable or countable (connected or not) irresolvable spaces are constructed.

**1 Introduction** The concepts of maximal, submaximal and irresolvable spaces were introduced by E. Hewitt in [12], while the concept of open-hereditarily irresolvable space was introduced by E. K. van Douwen in [27], and the concept of maximal connected space was introduced by J. P. Thomas in [25]. These properties have been widely studied in the last sixty years.

In the sequel all spaces are considered to be crowded (without isolated points).

**Definition 1.1.** A space $X$ is called:

1. Resolvable ([12]) if $X$ has two disjoint dense subsets, and it is called irresolvable ([12]) if it is not resolvable.

2. Open-hereditarily irresolvable ([27]), if every open subspace of $X$ is irresolvable.

3. Hereditarily irresolvable ([12]), if every subspace of $X$ is irresolvable.

Irresolvable spaces have been also studied by K. Kunen, A. Szymański and F. Tall in [17], by J. Dontchev, M. Ganster and D. Rose in [8], by O. T. Alas, M. Sanchis, M. G. Tkačenko, V. V. Tkachuk and R. G. Wilson in [1] and by W.W. Comfort and S. Garcia-Ferreira in [7] where a number of relevant references is provided, as well as a number of interesting open problems is listed.

**Definition 1.2.** A space $(X, \tau)$ is called:

1. Submaximal ([12]), if every dense subset of $X$ is open.

2. Maximal connected ([25]), if every finer topology than $\tau$ is not connected.

3. Maximal Hausdorff ([12]), if $\tau$ is maximal in the set of Hausdorff crowded topologies on $X$.

4. Maximal regular ([12]), if $\tau$ is maximal in the set of regular crowded topologies on $X$.

5. Extremally disconnected, if the closure of every open subset is open.

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Submaximal spaces as well as maximal topologies are studied in detail by N. Bourbaki [4], D. E. Cameron [5], E. K. van Douwen [27], A. V. Arhangel'skii and P. J. Collins [2], R. Levy and J. R. Porter [19] .

It is known that all maximal Hausdorff spaces are submaximal ([13], [15]) and that all maximal connected spaces are also submaximal ([6]). Since every subspace of a submaximal is submaximal ([4]) and every connected subspace of a maximal connected is maximal connected ([10]) it follows that every submaximal (connected or not) is hereditarily irresolvable, hence open-hereditarily irresolvable and hence irresolvable.

The examples constructed by D. Rose, K. Sizemore and B. Thurston in [23], by G. Bezhanishvili, R. Mines and P. J. Morandi in [3] and by E. K. van Douwen in [27] prove that none of the previous implications is reversible. We note that the Example 1.12 in [27] is a regular disconnected (or totally disconnected) space and the Example 1.9. in [27] is a regular extremally disconnected space.

In this paper we prove that every Hausdorff (resp. regular) space $S$ can be embedded as a closed nowhere dense subset in a open-hereditarily irresolvable Hausdorff (resp. regular) space $T$. The space $T$ is obtained by attaching to $S$ an auxiliary space $Z$ which is the cone constructed from a space $X$. Since the properties of the final space $T$ depend on the properties of $S$ and $Z$, it follows that the attachment of $Z$ to $S$ leads to several examples of all kinds of irresolvability.

We note that using spaces with appropriate properties, either for $S$ or for $Z$, the attachment presented in this paper can expand the known examples of different kinds of irresolvability so that the final spaces become in addition connected. In Remarks 3.2 and 3.3 we present several relevant examples. Moreover, by weakening the topology of the space $Z$, the space $Z$ itself leads to several examples of connected spaces on irresolvability.

**2 The auxiliary space $Z$.** In the sequel we will use the following Lemma 2.1 and Lemma 2.2 whose statements are well known.

**Lemma 2.1.**

(1) *Let $(X, \tau)$ be a Hausdorff space. The set of all topologies finer than $\tau$ having the same regular-open sets as $\tau$, has a regular-open maximal topology which is Hausdorff submaximal.*

(2) *If $(X, \tau)$ is Hausdorff connected (resp. countable connected), then the regular-open maximal topology is Hausdorff connected (resp. countable connected) submaximal.*

*Proof.* (1) This is proved in [20].
(2) It follows from the fact that the two topologies have the same regular-open sets. □

**Lemma 2.2.**

(1) *Every Hausdorff and maximal connected is submaximal.*
(2) *Every maximal Hausdorff is submaximal.*
(3) *Every Hausdorff submaximal is hereditarily irresolvable.*
(4) *Every maximal regular is hereditarily irresolvable.*

*Proof.* (1) By [6] a maximal connected space is submaximal.
(2) By [4] (Exercise 21 of §11) a Hausdorff space $X$ is maximal Hausdorff if and only if $X$ is submaximal and extremally disconnected.
(3) Obviously every submaximal is hereditarily irresolvable. Since by [4] (Exercise 22 of §8) every subspace of a submaximal space is submaximal, it follows that it is hereditarily irresolvable.
(4) Let $X$ be maximal regular. If $D$ is dense in $X$, then by [4] (Exercise 21 of §11) the subset $IntD$ is open-dense. Hence $X$ is irresolvable. If a subspace of $X$ contains isolated

points, then obviously it is irresolvable. If a subspace is crowded then by [27] it is maximal regular and hence irresolvable. □

We now consider the cone $Z$ constructed from an arbitrary topological space $X$. The space $Z$ will be used in the sequel as auxiliary space attached to a space $S$ with specific properties. The final space $T$ gives several examples of irresolvable spaces. For the cone and its applications see J. K. Kohli [16], S. Watson [28] and J. R. Porter [22].

Let $X$ be a topological space and let $X_i, i \in I$, be pairwise disjoint homeomorphic copies of $X$. We fix a point $x \in X$ and let $x_i$ be the copy of $x$ in $X_i$ for every $i \in I$. We set $Y = X \setminus \{x\}$ and $Y_i = X_i \setminus \{x_i\}, i \in I$. We identify the points $x_i$, for every $i \in I$ and we denote this common point by $z$.

The cone constructed from $X$ is the set $Z = \{z\} \cup (\cup_{i \in I} Y_i)$ with the following topology: Each copy $Y_i$ keeps the subspace topology of the space $X_i$, that is, for every $y_i \in Y_i, i \in I$ a basis of open neighborhoods of $y_i$ in $Z$ is the (homeomorphic) copy of a basis of open neighborhoods of $y$ in $Y$ whose copy in $Y_i$ is $y_i$. For the point $z$, a basis of open neighborhoods in $Z$ consists of the subsets $O_z = \{z\} \cup W$, where for every $i \in I$ the set $W \cap Y_i$ is an open deleted neighborhood of $x_i$ in $X_i$, that is the set $W \cap Y_i$ is the (homeomorphic) copy in $Y_i$ of a deleted open neighborhood of $x$ in $X$.

**Lemma 2.3.**

(1) *If $X$ is Hausdorff (resp. regular), then $Z$ is Hausdorff (resp. regular).*

(2) *If $X$ is submaximal, then $Z$ is submaximal.*

(3) *If $X$ is countable submaximal and the index set $I$ is countable, then $Z$ is countable submaximal.*

(4) *If $X$ is maximal Hausdorff, then $Z$ is Hausdorff submaximal not extremally disconnected.*

(5) *If $X$ is countable maximal Hausdorff and the index set $I$ is countable, then $Z$ is countable Hausdorff submaximal not extremally disconnected.*

(6) *If $X$ is maximal regular, then $Z$ is regular hereditarily irresolvable not extremally disconnected.*

(7) *If $X$ is countable maximal regular and the index set $I$ is countable, then $Z$ is countable regular hereditarily irresolvable not extremally disconnected.*

(8) *If $X$ is separable submaximal and the index set $I$ is countable, then $Z$ is separable submaximal.*

(9) *If $X$ is connected submaximal, then $Z$ is connected submaximal.*

(10) *If $X$ is separable connected submaximal and the index set $I$ is countable, then $Z$ is separable connected submaximal.*

(11) *If $X$ is countable connected submaximal and the index set $I$ is countable, then $Z$ is countable connected submaximal.*

*Proof.* (1) Let $X$ be Hausdorff and $a, b \in Z \setminus \{z\}$. If both $a, b$ belong to the same copy $Y_i$ for some $i \in I$, then since $Y_i$ is Hausdorff there exist in $Y_i$ disjoint open neighborhoods $U_a, U_b$ of $a, b$ respectively. If $a \in Y_i, b \in Y_j, i \neq j$ then the subspaces $Y_i, Y_j$ are disjoint open in $Z$ containing $a, b$ respectively. Let $a \in Z \setminus \{z\}$ and $b = z$. Then $a \in Y_i$ for some $i \in I$. Since $X_i$ is Hausdorff it follows that for the points $a, x_i$ there exist in $X_i$ open neighborhoods $U_a, U_{x_i}$ of $a, x_i$ respectively such that $U_a \cap U_{x_i} = \emptyset$. Therefore the sets $U_a$ and $\{z\} \cup W$ where $W \cap Y_i = W_{x_i} \setminus \{x_i\}$ are disjoint open sets in $Z$ containing $a, z$ respectively. Thus, $Z$ is Hausdorff.

Let $X$ be regular. Obviously, the space $Z$ is regular at every point $y_i \in Y_i$, for every $i \in I$. For the point $z$, let $O_z = \{z\} \cup W$ be an open neighborhood of $z \in Z$. By the definition of topology in $Z$ the set $W_{x_i} = W \cap Y_i$ is an open neighborhood of $x_i \in X_i$.

Since for every $i \in I$ the space $X_i$ is regular, it follows that for every $W_{x_i}$ there exist an open neighborhood $O_{x_i}$ of $x_i$ in $X_i$ such that $Cl_{X_i}(O_i \setminus \{x_i\}) \subseteq W_{x_i}$. Therefore the set $O_z = \{z\} \cup O$ such that $O \cap Y_i = O_{x_i}$ is an open neighborhood of the point $z \in Z$ such that $Cl_Z(\{z\} \cup O) \subseteq \{z\} \cup W$ that is, $Z$ is regular at $z$.
(2) Let $D$ be dense in $Z$, and $z \in D$. Since for every $i \in I$ the subset $D \cap Y_i$ is open-dense in $Y_i$, it follows that for the point $z$ in the subspace $\{z\} \cup Y_i$, there exists an open set $U_{i(D)}$ (depended on $D \cap Y_i$) containing $z$ and such that $U_{i(D)} \setminus \{z\} \subseteq D \cap Y_i$. Therefore the set $\{z\} \cup W$ for which $W \cap Y_i = U_{i(D)} \setminus \{z\}$, $\forall i \in I$, is an open set in $Z$ containing $z$ and included in $D$. That is, the point $z$ is an interior point of $D$. Therefore $D$ is open.
(4) By Lemma 2.2 (2), $X$ is submaximal. Hence $Z$ is submaximal. Since for the open subset $Y_i$ of $Z$ it holds that $Cl_Z Y_i = \{z\} \cup Y_i$, it follows that $Z$ is not extremally disconnected.
(6) Let $A$ be a subspace of $Z$. By Lemma 2.2 (4), $X$ is hereditarily irresolvable. Since for every $i \in I$ the subspace $\{z\} \cup Y_i$ is homeomorphic to $X$ it follows that $A$ is a disjoint union of hereditarily irresolvable subspaces. Hence, $A$ is irresolvable. That $Z$ is not extremally disconnected is proved as previously.

The remaining statements (3), (5), (7), (8), (9), (10), (11) are obvious. □

**3 The space $T$.** In [27] E. K. van Douwen constructs two maximal regular spaces. The first (Example 1.9) is not maximal Hausdorff while the second (Example 3.3) is countable and maximal Hausdorff. In [19] R. Levy and J. R. Porter construct uncountable Hausdorff (and Tychonoff) submaximal separable spaces. The first example of a connected submaximal Hausdorff space is constructed by K. Padmavally [21]. Maximal connected Hausdorff spaces are constructed by A. G. El'kin [9], J. A. Guthrie, H. E. Stone and M. L. Wage [11], and G. J. Kennedy and S. D. McCartan [14]. For countable connected Hausdorff spaces see the list of references in [26].

Since a submaximal space is hereditarily irresolvable (Lemma 2.2 (3)), it follows that in all cases the initial space $X$ is a hereditarily irresolvable space, implying that the space $Z$ is also hereditarily irresolvable. Therefore, as initial space $X$ it can be used any submaximal space of Lemma 2.1 or any hereditarily irresolvable space of Lemma 2.2, as well as any of the previous specific spaces.

**Proposition 3.1.** *Every Hausdorff (resp. regular) space $S$ can be embedded as a closed nowhere dense subset in a open-hereditarily irresolvable Hausdorff (resp. regular) space $T$. If in addition $S$ is separable, then $T$ is separable.*

*Proof.* Let $S$ be a Hausdorff (resp. regular) space. We consider a hereditarily irresolvable Hausdorff (resp. regular) space $X$ and we construct the space $Z$, the index set $I$ having the same cardinality as the set $S$. In the space $X$ we fix a point $a \neq x$ and let $a_i$ be the copy of $a$ in $Y_i$. Hence, for every $i \in I$, $a_i \neq x_i$ and therefore $a_i \neq z$, because by the construction of the space $Z$ the point $z$ is defined by identifying the points $x_i$. We attach the space $Z$ to the space $S$ identifying every point of $S$ with a point $a_i$ of $Z$.

On the set $T = S \cup (Z \setminus \{a_i : i \in I\})$ we define the following topology: The subset $Z \setminus \{a_i : i \in I\}$ keeps the subspace topology of $Z$. For every open subset $U$ of $S$ the subset $O_U$ of $T$ is open in $T$ if and only if $O_U = U \cup W(U)$ where $W(U) = \bigcup_{a_i \in U} U_{a_i}$, and $U_{a_i}$ is an open deleted neighborhood of $a_i$ in $Y_i$. It can be easily verified that this is a topology observing that if $U, V$ are open sets in $S$, then for the sets $O_U, O_V$ it holds that $O_U \cap O_V = (U \cap V) \cup W(U \cap V) = O_{U \cap V}$. Also, if $U_i, i \in I$ are open sets in $S$ then for the sets $O_{U_i}$ it holds that $\cup O_{U_i} = (\cup U_i) \cup W(\cup U_i) = O_{\cup U_i}$ because $\cup W(U_i) = \cup(\bigcup_{a_i \in U_i} U_{a_i}) = W(\cup U_i)$ .

We prove that $T$ is Hausdorff. Let $x, y \in T$. If $x, y \in Z \setminus S$ then the points $x, y$ either belong to a common $Y_i \setminus \{a_i\}$ for some $i \in I$ or $x \in Y_i \setminus \{a_i\}$ and $y \in Y_j \setminus \{a_j\}$, $i \neq j$, or $x \in Y_i \setminus \{a_i\}$ and $y = z$. The proof of these cases is the same as in Lemma 2.3 (1). Let

$x, y \in S$. Since $S$ is Hausdorff, there exist open subsets $U, V$ in $S$ containing the points $x, y$ respectively and such that $U \cap V = \emptyset$. Since for every $i \in I$ all copies $Y_i$ are pairwise disjoint it is obvious that we can choose $W(U)$ and $W(V)$ such that $W(U) \cap W(V) = \emptyset$. Hence the corresponding sets $O_U, O_V$ are the required open subsets.

Let $x \in T \setminus S$, $x \neq z$ and $s \in S$. Then $x \in Y_i$, for some $i \in I$. If $Y_i$ is attached to $s'$ and $s' \neq s$ then the proof is as in the previous case. If $Y_i$ is attached to $s$, that is $a_i = s$, then, since $Y_i$ is Hausdorff, it follows that there exist open sets $W_{a_i}$ and $V_x$ in $Y_i$ containing $s$ and $x$ respectively, and such that $W_{a_i} \cap V_x = \emptyset$. Hence if $U$ is an open set in $S$ containing $s$, then the corresponding set $O_U = U \cup W(U)$ for which $W(U) \cap Y_i = W_{a_i}$ and the set $V_x$ are the required open subsets.

It remains the case for the point $z$ and a point $s \in S$. Let $U$ be an open set in $S$ containing the point $s$. Let $Y_j$, $j \in I' \subset I$ be the copies whose points $a_j$ are attached to $U$. For the point $z$ and for every $a_j$ there exist open sets $W_j, W_{a_j}$ in the subspace $\{z\} \cup Y_j$ containing $z$ and $a_j$ respectively, and such that $W_j \cap W_{a_j} = \emptyset$. Hence the subset $O_z = \{z\} \cup W$ for which $W \cap Y_j = W_j$ and the subset $O_U = U \cup W(U)$ for which $W(U) \cap Y_j = U_{a_j}$ are the required open subsets.

We now prove that $T$ is regular. By the definition of the topology on $T$, it follows that $T$ is regular at every point of $T \setminus S$. Since $S$ is regular, for every $s \in S$ there exist open sets $U, V$ in $S$ such that

$$s \in V \subseteq Cl_S V \subseteq U.$$

We consider $W(Cl_S V)$, that is the subset of $W(U)$ for which $W(U) \cap Y_i$ is an open set in $Y_i$ containing those $a_i$ which are attached to the points of $Cl_S V$. Since each $Y_i$ is regular, then for every such open set there exists an open set $W_{a_i}$ in $Y_i$ containing $a_i$ and such that

$$W_{a_i} \subseteq Cl_{Y_i} W_{a_i} \subseteq W \cap Y_i.$$

Hence,

$$\bigcup_{a_i \in V} W_{a_i} \subseteq \bigcup_{a_i \in Cl_T V} W_{a_i} \subseteq \bigcup_{a_i \in Cl_T V} Cl_{Y_i} W_{a_i} \subseteq W,$$

and therefore

$$s \in O_V \subseteq Cl_T O_V \subseteq O_U,$$

that is, $T$ is regular. Obviously, the subset $T \setminus S$ is open. Since for every open set $U$ in $S$ it holds that $O_U \cap (T \setminus S) \neq \emptyset$, it follows that $T \setminus S$ is also dense. Hence $S$ is closed nowhere dense in $T$.

It remains to prove that $T$ is open-hereditarily irresolvable. Let $U$ be an open subspace of $T$. If $U$ is a subset of $T \setminus S$ then, since $Z$ is hereditarily irresolvable, is follows that $U$ is an irresolvable subspace of $T$. If the open set is of the form $O_U = U \cup W(U)$ then, by the definition of $O_U$ the subset $U$ is open in $S$ and nowhere dense in $T$ and the subset $W(U)$ is an open subset of $T$. Hence $W(U)$ is irresolvable and therefore $O_U$ is irresolvable. Hence $T$ is open-hereditarily irresolvable.

Finally, let $S$ be separable. Let $D$ be a countable dense subset of $S$, and $Z$ be as in (8) of Lemma 2.3. We attach $Z$ to $S$, identifying every point of $D$ with a point $a_i$, $i = 1, 2, \dots$ of $Z$. The topology on the set $T = S \cup (Z \setminus \{a_i : i \in \mathbb{N}\})$ is defined in exactly the same manner as above. Obviously $T$ is separable. That $T$ is open-hereditarily irresolvable is proved as previously. $\square$

The following remarks are consequences of Lemma 2.3 and the previous Proposition, indicating that none of the following implications
" submaximal $\Rightarrow$ hereditarily irresolvable $\Rightarrow$ open-hereditarily irresolvable $\Rightarrow$ irresolvable "

is reversible. We must mention that the two examples constructed by D. Rose, K. Sizemore and B. Thurston in [23] (Examples 2.5), the Example 3.2 constructed by G. Bezhanishvili, R. Mines and P. J. Morandi in [3] and the Examples 1.12 and 1.9 constructed by E. K. van Douwen in [27] give an answer to this. Specifically, the first example in [23] is a crowded $T_1$ hereditarily irresolvable not submaximal and the second is a crowded $T_1$ open-hereditarily irresolvable not hereditarily irresolvable. The example in [3] is a connected crowded $T_1$ irresolvable not open-hereditarily irresolvable. Obviously, the space $X$ in this example is Hausdorff (resp. regular) if both spaces $Y, Z$ used for the construction of $X$ are Hausdorff (resp. regular). We observe that in order to be connected it is needed both $Y, Z$ to be connected. The Example 1.12 in [27] is a regular disconnected (or totally disconnected) open-hereditarely irresolvable but not hereditarely irresolvable space. The Example 1.9. in [27] is maximal regular but not maximal Hausdorff. Hence by [4] (Exercise 21 of §11) it is extremally disconnected and by Lemma 2.2 (4) it is hereditarily irresolvable. Since a Hausdorff space is maximal Hausdorff if and only if it is extremally disconnected and submaximal, it follows that this space is not submaximal.

Remark 3.2 below is referred to open-hereditarily irresolvable not hereditarily irresolvable spaces. Specifically, the cases (3), (4), and part of (5) deal with connected spaces. Remark 3.3 is referred to hereditarily irresolvable not submaximal spaces, some of which are also connected. The final space $T$ is, in all cases, Hausdorff (resp. regular) if both $S$, $Z$ are Hausdorff (resp. regular). Using Lemma 2.3 and Proposition 3.1 we can expand the different kinds of irresolvability so that the spaces become in addition connected. In Remark 3.4 we examine whether the set of all dense subsets in these spaces, is a filter. In what follows $\mathcal{D}(X)$ denotes the set of all dense subsets of $X$.

*Remark* 3.2. (1) If $S$ is resolvable and $Z$ is as in (2), (4) or (6) of Lemma 2.3, then $T$ is open-hereditarily irresolvable but not hereditarily irresolvable.

(2) If $S$ is separable resolvable and $Z$ is as in (8) of Lemma 2.3 , then $T$ is in addition separable .

(3) If $S$ is a resolvable not necessarily connected space and $Z$ is as in (9) of Lemma 2.3 , then $T$ is in addition connected. If $S$ is separable (resp. countable) resolvable not necessarily connected space and $Z$ is as in (10) (resp. (11)) of Lemma 2.3, then $T$ is in addition separable (resp. countable) connected. The space $T$ is connected, either if $Z$ is attached to the whole of $S$ or to a countable dense subset $D$ of $S$, because the subset $D \cup (Z \setminus \{a_i : i \in \mathbb{N}\})$ is dense connected and therefore $Cl_T(D \cup (Z \setminus \{a_i : i \in \mathbb{N}\})) = T$ is connected.

(4) If $S$ is countable resolvable (not necessarily connected) and $Z$ is as in (11) of Lemma 2.3 , then $T$ is countable connected open-hereditarily irresolvable not hereditarily irresolvable.

(5) Consider [18] (Chapter I, §9) the set of rational numbers of the interval $[0, 1]$, written as irreducible fractions $\frac{p}{q}$. We set $D = \{(\frac{p}{q}, \frac{1}{q}) : p, q \in \mathbb{N}\}$. The subspace $S = D \cup [0, 1]$ (resp. $S = D \cup (\mathbb{Q} \cap [0, 1])$) of the plane is regular and the subset $D$ of isolated points is countable and dense. Obviously the subspace $[0, 1]$ (resp. $\mathbb{Q} \cap [0, 1]$) is resolvable (resp. countable resolvable). Hence, by Proposition 3.1 , the attachment of any hereditarily irresolvable space $Z$ of Lemma 2.3 to the subspace $D$ of $S$ leads to a space $T$ being in all cases open-hereditarily irresolvable but not hereditarily irresolvable.

Specifically, if $Z$ is as in (9), (10) or (11) of Lemma 2.3 , then $T$ is in addition connected, separable connected or countable connected (if $S = D \cup (\mathbb{Q} \cap [0, 1])$), respectively.

We observe that in all the previous cases the space $T$ is not submaximal since the closed nowhere dense subset $S$ of $T$ is not discrete.

We note that in all cases $\mathcal{D}(T)$ is a filter (as it was expected, see Remark 3.4) because if

$L, M$ are dense subsets of $T$ then since the subset $T \setminus S$ is open-dense it follows that both subsets $L \cap (T \setminus S)$ and $M \cap (T \setminus S)$ are dense in $T$. Since $T \setminus S$ is hereditarily irresolvable it follows that both subsets $Int_T(L \cap (T \setminus S))$ and $Int_T(M \cap (T \setminus S))$ are open-dense. Therefore the set $L \cap M$ is dense.

This construction is actually based on the construction of Example 1.12 in [27], with the following modification: instead of attaching to the set $D$ disjoint copies of spaces, we attach to $D$ the space $Z$ of Lemma 2.3. We note that attaching disjoint copies of spaces to $D$ the final space is not connected even if all copies are connected.

*Remark* 3.3. Let $S, Z$ be any hereditarily irresolvable spaces. We attach the space $Z$ to $S$ as in Proposition 3.1. The space $T$ is always hereditarily irresolvable not submaximal (even if both spaces $S$, $Z$ are submaximal). In order to prove this, we consider an open-dense subset $D$ of $Z$, not containing anyone of the points $a_i$ which are attached to $S$. Since $S$ is closed nowhere dense in $T$, it follows that $D$ is open-dense in $T$. Hence, if $s \in S$ then the set $D \cup \{s\}$ is dense in $T$ but not open. It is obvious that if $Z$ is as in (9), (10) or (11) of Lemma 2.3, then $T$ is in addition connected. Specifically, if $S$ is separable and $Z$ as in (10) then $T$ is separable connected. If $S$ is countable and $Z$ as in (11) then $T$ is countable connected.

Hereditarily irresolvable not submaximal spaces can also be constructed as follows: Let $(Z, \tau)$ be any submaximal space of Lemma 2.3. We weaken the topology on $(Z, \tau)$ changing the topology only at the point $z$ as follows: The subset $O_z$ is open in $Z$ containing $z$ if and only if $O_z = \{z\} \cup W$, where for every finite subset $I' \subset I$ and for every $i \in I'$, the subset $W \cap Y_i$ is an open deleted neighborhood of $x_i$ in $X_i$ while $W \cap Y_i = Y_i$, $\forall i \in I \setminus I'$. We denote this topology by $\tau^*$. It can be easily proved that $(Z, \tau^*)$ remains Hausdorff (resp. regular) if $(Z, \tau)$ is Hausdorff (resp. regular). The proof that in all cases $(Z, \tau^*)$ is hereditarily irresolvable is the same as in (6) of Lemma 2.3.

We prove that $(Z, \tau^*)$ is not submaximal. Let $D$ be a proper dense subset of $X$. We set $D_i = D \cap Y_i$. Since each $D_i$ is open-dense in $Y_i$ it follows that $\bigcup_{i \in I} D_i$ is open-dense in $Z$. But $\{z\} \cup (\bigcup_{i \in I} D_i)$ is not open because every open neighborhood of $z$ contains all but finite copies of $Y_i$. We observe that again the set $\mathcal{D}(Z)$ is a filter on $(Z, \tau^*)$.

Obviously, if $(Z, \tau)$ is as in (9), (10) or (11) of Lemma 2.3, hence connected, then since $\tau^* \subset \tau$ it follows that $(Z, \tau^*)$ is in addition connected, separable connected or countable connected, respectively.

*Remark* 3.4. The Example 3.2 in [3] has the additional property that on the space $X$ the set $\mathcal{D}(X)$ is not a filter. This can occur only to irresolvable not open-hereditarily irresolvable spaces. For if $X$ is submaximal then every dense subset is open and hence $\mathcal{D}(X)$ is a filter. If $X$ is hereditarily irresolvable then $\mathcal{D}(X)$ is a filter ([3] , Theorem 2.4). Finally, if $X$ is open-hereditarily irresolvable then $\mathcal{D}(X)$ is again a filter. In order to prove this it suffices to prove that for every dense subset $D$ of $X$ it holds that the set $IntD$ is open-dense. Obviously $IntD \neq \emptyset$. If $IntD$ is not dense then there exists an open set $U$ such that $U \cap IntD = \emptyset$. Obviously the sets $D \cap U$ and $U \setminus D$ are non-empty, dense in $U$, that is the open set $U$ is resolvable, which is a contradiction. Examples of irresolvable Hausdorff (resp. regular) spaces with additional properties and on which the set of all dense subsets is not a filter can be also constructed (using the space $Z$ of Lemma 2.2 and imitating the wedge construction of Example 3.2 in [3]).

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# SOME INEQUALITIES FOR GENERALIZED FRACTIONAL INTEGRAL OPERATORS ON GENERALIZED MORREY SPACES AND THEIR REMARKS

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Abstract.

Some inequalities for fractional integral operators on Morrey spaces are investigated by many researchers. In this paper, we extend and generalize some of them. More precisely, we show some inequalities concerning with $g \cdot T_\rho f$, where $T_\rho$ is the generalized fractional integral operator and $f$ and $g$ are functions which belong to generalized Morrey spaces. We also compare our results and present counterexamples showing that our results are sharp.

**1 Introduction** For $0 < \alpha < 1$, the fractional integral operator is defined by

$$I_\alpha f(x) = \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-n\alpha}} dy, \tag{1.1}$$

where $f$ is a suitable measurable function. A well-known fact is that $I_\alpha$ is bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ provided that $1 < p < 1/\alpha$ and $1/q = 1/p - \alpha$ (see [23, p.119] for example). The result is nowadays referred to as Hardy-Littlewood-Sobolev's theorem. In 1975, Adams extended this theorem to Morrey spaces ([1, Theorem 3.1]). Morrey spaces, which stem from the paper by Morrey ([12]), are normed spaces of locally integrable functions. Let $0 < p \le p_0 < +\infty$ and denote the ball centered at $x$ with radius $t$ by $B(x,t)$. Then the Morrey norm is defined by

$$\|f\|_{M_p^{p_0}} = \sup_{x \in \mathbf{R}^n, t>0} |B(x,t)|^{1/p_0 - 1/p} \left( \int_{B(x,t)} |f(y)|^p dy \right)^{1/p} \tag{1.2}$$

for a measurable function $f$ on $\mathbf{R}^n$. And the Morrey space is defined by

$$M_p^{p_0}(\mathbf{R}^n) = M_p^{p_0} = \{ f \in L^p_{\mathrm{loc}}(\mathbf{R}^n) \mid \|f\|_{M_p^{p_0}} < +\infty \}. \tag{1.3}$$

We remark that $M_p^p$ is the $L^p$ space. Adams proved that $I_\alpha$ is bounded from $M_p^{p_0}$ to $M_q^{q_0}$ provided that $1/q_0 = 1/p_0 - \alpha$ and $1/q = (1/p)(1 - p_0\alpha)$, that is, $p/p_0 = q/q_0$ (see Theorem 2.2 below). Since two parameters $p$ and $q$ seem to serve to measure different integrability in the definition of the Morrey norm $\| \cdot \|_{M_p^{p_0}}$, we can say that Adams' theorem is more precise than Hardy-Littlewood-Sobolev's theorem. However, in Adams' theorem, due to the smoothing effect, we suspect that there is no need to make full use of the property of $M_p^{p_0}$ in order that the image is contained in $M_q^{q_0}$. In [22], the authors realized this aspect by proposing a new inequality of intersection type. A typical result is as follows:

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**Theorem 1.1** ([22, Corollary 4.2]) *Let* $1 < p \le p_0 < +\infty$, $1 < q \le q_0 < +\infty$, $0 < \alpha < 1/p_0$,

$$\frac{1}{q_0} = \frac{1}{p_0} - \alpha, \quad \frac{1}{q} = \frac{1}{p}(1 - p_0\alpha). \tag{1.4}$$

*Then there exists a positive constant* $C$ *such that*

$$\|I_\alpha f\|_{M_q^{q_0}} \le C\|f\|_{M_p^{p_0}}^{1-p_0\alpha}\|f\|_{M_1^{p_0}}^{p_0\alpha}, \tag{1.5}$$

*where* $f \in M_p^{p_0}$.

In this paper, we show variations and generalizations of Theorem 1.1. For a function $\rho : (0, +\infty) \to (0, +\infty)$, the generalized fractional integral operator is defined by

$$T_\rho f(x) = \int_{\mathbf{R}^n} \frac{\rho(|x-y|)f(y)}{|x-y|^n} dy \tag{1.6}$$

for any suitable function on $f$ on $\mathbf{R}^n$. For $0 < p < +\infty$ and a function $\phi : (0, +\infty) \to (0, +\infty)$, the generalized Morrey norm is defined by

$$\|f\|_{L_p^\phi} = \sup_{x \in \mathbf{R}^n, t>0} \phi(t) \left( \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(y)|^p dy \right)^{1/p}. \tag{1.7}$$

And the generalized Morrey space is defined by

$$L_p^\phi(\mathbf{R}^n) = L_p^\phi = \{f \in L_{\mathrm{loc}}^p(\mathbf{R}^n) | \, \|f\|_{L_p^\phi} < +\infty\}. \tag{1.8}$$

We remark that, by Hölder's inequality, if $0 < p_1 \le p_2 < +\infty$ then $\|\cdot\|_{L_{p_1}^\phi} \le \|\cdot\|_{L_{p_2}^\phi}$ and $L_{p_2}^\phi \subset L_{p_1}^\phi$.

The purpose of this paper is to show three inequalities concerning with $g \cdot T_\alpha f$, where $f$ and $g$ are functions which belong to generalized Morrey spaces (Theorems 3.1, 3.9, and 3.16). They extend and generalize a known result (Proposition 2.9). Some inequalities concerning with Proposition 2.9 are known ([17], [19], [20], [24] and so on). We state the details in Section 2.

The plan of this paper is as follows. In Section 2, we describe some known results and explain background of our theorems. In Section 3, we formulate our three theorems and their corollaries. Sections 4, 5, and 6 are devoted to the proof of our theorems. In Section 7, we compare our corollaries. Finally, in Section 8, we state counterexamples concerning our corollaries.

Throughout this paper the letter $C$ stands for a constant not necessarily the same at each occurrence.

## 2 Background

In this section, we describe some known results and explain background of our theorems.

Boundedness of fractional integral operators on Morrey spaces was shown in [18] as Spanne's unpublished result as the following Theorem 2.1 states.

**Theorem 2.1** ([18, Theorem 5.4]) *Let* $1 < p \le p_0 < +\infty$, $1 < q \le q_0 < +\infty$, $0 < \alpha < 1/p_0$,

$$\frac{1}{q_0} = \frac{1}{p_0} - \alpha, \quad \frac{1}{q} = \frac{1}{p} - \alpha. \tag{2.1}$$

*Then there exists a positive constant $C$ such that*

$$\|I_\alpha f\|_{M_q^{q_0}} \le C\|f\|_{M_p^{p_0}}, \tag{2.2}$$

*where $f \in M_p^{p_0}$.*

Adams strengthened Theorem 2.1 as the following Theorem 2.2 states.

**Theorem 2.2** ([1, Theorem 3.1]) *Let $1 < p \le p_0 < +\infty$, $1 < q \le q_0 < +\infty$, $0 < \alpha < 1/p_0$,*

$$\frac{1}{q_0} = \frac{1}{p_0} - \alpha, \quad \frac{1}{q} = \frac{1}{p}(1 - p_0\alpha). \tag{2.3}$$

*Then there exists a positive constant $C$ such that*

$$\|I_\alpha f\|_{M_q^{q_0}} \le C\|f\|_{M_p^{p_0}}, \tag{2.4}$$

*where $f \in M_p^{p_0}$.*

**Remark 2.3** Since $p_0/p \ge 1$, Theorem 2.2 extends Theorem 2.1. In [2], the authors reproved Theorem 2.2 by using the Hardy-Littlewood maximal operator and proved Theorem 2.1 as its corollary.

**Remark 2.4** It is known that Theorem 2.2 is optimal, that is, the operator $I_\alpha : M_p^{p_0} \to M_q^{q_0}$ is unbounded for $1/q_0 = 1/p_0 - \alpha$ and $q > p/(1 - p_0\alpha)$ ([17, Theorem 10]).

We note Hölder's inequality on Morrey spaces.

**Lemma 2.5** (Hölder's inequality on Morrey spaces, [15, Corollary 4.3] for example) *Let $1 < p \le p_0 < +\infty$, $1 < q \le q_0 < +\infty$, $1 < r \le r_0 < +\infty$,*

$$\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0}, \quad \frac{1}{r} = \frac{1}{q} + \frac{1}{p}. \tag{2.5}$$

*Then*

$$\|gf\|_{M_r^{r_0}} \le \|g\|_{M_q^{q_0}}\|f\|_{M_p^{p_0}}, \tag{2.6}$$

*where $f \in M_p^{p_0}$ and $g \in M_q^{q_0}$.*

For some function $g$, we obtain some inequalities for $g \cdot I_\alpha f$. Using Theorem 2.1 and applying Lemma 2.5 to $g \cdot I_\alpha f$, we have

**Proposition 2.6** *Let $1 < p \le p_0 < +\infty$, $1 < q \le q_0 < +\infty$, $1 < r \le r_0 < +\infty$, $0 < \alpha < 1/p_0$,*

$$\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \alpha, \quad \frac{1}{r} = \frac{1}{q} + \frac{1}{p} - \alpha, \tag{2.7}$$

*and assume $g \in M_q^{q_0}$. Then there exists a positive constant $C$ such that*

$$\|g \cdot I_\alpha f\|_{M_r^{r_0}} \le C\|g\|_{M_q^{q_0}}\|f\|_{M_p^{p_0}}, \tag{2.8}$$

*where $f \in M_p^{p_0}$.*

As a special case $r = p$ and $r_0 = p_0$ in Proposition 2.6, we have

**Corollary 2.7** *Let* $1 < p \le p_0 < +\infty$, $0 < \alpha < 1/p_0$, $g \in M^{1/\alpha}_{1/\alpha}$. *Then there exists a positive constant* $C$ *such that*

$$\|g \cdot I_\alpha f\|_{M^{p_0}_p} \le C\|g\|_{M^{1/\alpha}_{1/\alpha}} \|f\|_{M^{p_0}_p}, \tag{2.9}$$

*where* $f \in M^{p_0}_p$.

**Remark 2.8** We remark that $M^{1/\alpha}_{1/\alpha}$ is the $L^{1/\alpha}$ space by the definition. If we use Theorem 2.2 instead of Theorem 2.1, we can obtain an inequality for $g \cdot I_\alpha f$ with $g$ which belongs to the Morrey space (see Corollary 2.10).

Using Theorem 2.2 and applying Lemma 2.5 to $g \cdot I_\alpha f$, we have

**Proposition 2.9** *Let* $1 < p \le p_0 < +\infty$, $1 < q \le q_0 < +\infty$, $1 < r \le r_0 < +\infty$, $0 < \alpha < 1/p_0$,

$$\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \alpha, \quad \frac{1}{r} = \frac{1}{q} + \frac{1}{p}(1 - p_0\alpha), \tag{2.10}$$

*and assume* $g \in M^{q_0}_q$. *Then there exists a positive constant* $C$ *such that*

$$\|g \cdot I_\alpha f\|_{M^{r_0}_r} \le C\|g\|_{M^{q_0}_q} \|f\|_{M^{p_0}_p}, \tag{2.11}$$

*where* $f \in M^{p_0}_p$.

As a special case $r_0 = p_0$ and $r = p$ in Proposition 2.9, we have

**Corollary 2.10** *Let* $1 < p \le p_0 < +\infty$, $0 < \alpha < 1/p_0$, $g \in M^{1/\alpha}_{p/(p_0\alpha)}$. *Then there exists a positive constant* $C$ *such that*

$$\|g \cdot I_\alpha f\|_{M^{p_0}_p} \le C\|g\|_{M^{1/\alpha}_{p/(p_0\alpha)}} \|f\|_{M^{p_0}_p}, \tag{2.12}$$

*where* $f \in M^{p_0}_p$.

In [17], Olsen obtained another inequality for $g \cdot I_\alpha f$.

**Theorem 2.11** ([17, Theorem 2]) *Let* $\Omega$ *be a bounded domain and* $1 < p \le p_0 < +\infty$, $1 < q \le q_0 < +\infty$, $1 < r \le r_0 < +\infty$, $1/q_0 \le \alpha < 1/p_0$, $q > r$,

$$\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \alpha, \quad \frac{1}{r} = \frac{1}{q_0} + \frac{1}{p} - \alpha, \tag{2.13}$$

*and assume* $g \in M^{q_0}_q$. *Then there exists a constant* $C = C_\Omega$ *depending on* $\Omega$ *and parameters above such that*

$$\|g \cdot I_\alpha f\|_{M^{r_0}_r} \le C\|g\|_{M^{q_0}_q} \|f\|_{M^{p_0}_p}, \tag{2.14}$$

*for all positive and measurable functions* $f$ *and* $g$ *such that the support of* $f$ *is contained in* $\Omega$.

**Remark 2.12** In Proposition 2.9, the condition $1/r = 1/p + 1/q - (p_0/p)\alpha$ is equivalent to $1/r - 1/q = (1/p)(1 - p_0\alpha)$. Hence under the assumption $1/r = 1/p + 1/q - (p_0/p)\alpha$, the condition $\alpha < 1/p_0$ is equivalent to $q > r$.

**Remark 2.13** In Theorem 2.11, the condition $1/q_0 \le \alpha$ is equivalent to $r \ge p$ under the assumption $1/r = 1/p + 1/q_0 - \alpha$.

As a special case $r = p$ in Theorem 2.11, we have

**Corollary 2.14** ([17, Corollary 3]) *Let $\Omega$ be a bounded domain and $1 < p \le p_0 < +\infty$, $0 < \alpha < 1/p_0$, $1 < q \le 1/\alpha < +\infty$, $q > p$ and assume $g \in M_q^{1/\alpha}$. Then there exists a constant $C = C_\Omega$ depending on $\Omega$ and parameters above such that*

$$\|g \cdot I_\alpha f\|_{M_p^{p_0}} \le C \|g\|_{M_q^{1/\alpha}} \|f\|_{M_p^{p_0}}, \tag{2.15}$$

*for all positive and measurable functions $f$ and $g$ such that the support of $f$ is contained in $\Omega$.*

**Remark 2.15** We compare Corollary 2.10 with Corollary 2.14. In (2.12), the Morrey norm of $g$ depends on $p$ and $p_0$ which appear in the Morrey norms of $f$ and $g \cdot I_\alpha f$. On the other hand, (2.15) holds for $g$ which belong to $M_q^{1/\alpha}$, where $q$ is not necessarily the same as $p/(p_0\alpha)$.

Sawano, Tanaka, and the author strengthened and generalized Theorem 2.11 ([19], [20]). Here, we recall a result for $I_\alpha$ from these works.

**Theorem 2.16** ([20, Proposition 1.8]) *Let $1 < p \le p_0 < +\infty$, $1 < q \le q_0 < +\infty$, $1 < r \le r_0 < +\infty$, $1/q_0 \le \alpha < 1/p_0$, $q > r$,*

$$\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \alpha, \quad \frac{1}{r} = \frac{1}{p} + \frac{p_0}{p}\left(\frac{1}{q_0} - \alpha\right), \tag{2.16}$$

*and assume $g \in M_q^{q_0}$. Then there exists a positive constant $C$ such that*

$$\|g \cdot I_\alpha f\|_{M_r^{r_0}} \le C \|g\|_{M_q^{q_0}} \|f\|_{M_p^{p_0}}, \tag{2.17}$$

*where $f \in M_p^{p_0}$.*

**Remark 2.17** In Theorem 2.16, the condition $1/r = 1/p + (p_0/p)(1/q_0 - \alpha)$ is equivalent to $1/r = (p_0/p)(1/p_0 + 1/q_0 - \alpha)$. Hence under the assumption $1/r_0 = 1/p_0 + 1/q_0 - \alpha$, the condition $1/r = 1/p + (p_0/p)(1/q_0 - \alpha)$ is equivalent to $r/r_0 = p/p_0$.

**Remark 2.18** ([20, Remark 1.9]) Since $p_0/p \ge 1$, Theorem 2.16 implies Theorem 2.11.

The case $r = p$ in Theorem 2.16 is the same as Corollary 2.14. The comparison between Proposition 2.9 and Theorem 2.16 was shown in [21, p.53]. For readers' convenience, we recall it as the following Comparison 2.19.

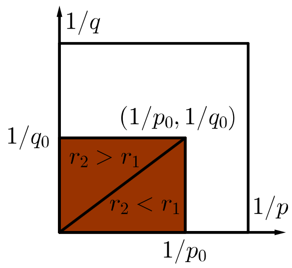
**Comparison 2.19** ([21]) We compare Proposition 2.9 with Theorem 2.16. If we use Proposition 2.9 for $f \in M_p^{p_0}$ and $g \in M_q^{q_0}$, we obtain $g \cdot I_\alpha f \in M_{r_1}^{R_1}$, where $r_1$ and $R_1$ satisfy

$$\frac{1}{R_1} = \frac{1}{q_0} + \frac{1}{p_0} - \alpha, \quad \frac{1}{r_1} = \frac{1}{q} + \frac{1}{p} - \frac{p_0}{p}\alpha. \tag{2.18}$$

If we use Theorem 2.16 for $f \in M_p^{p_0}$ and $g \in M_q^{q_0}$, we obtain $g \cdot I_\alpha f \in M_{r_2}^{R_2}$, where $r_2$ and $R_2$ satisfy

$$q > r_2, \quad \frac{1}{R_2} = \frac{1}{q_0} + \frac{1}{p_0} - \alpha, \quad \frac{1}{r_2} = \frac{1}{p} + \frac{p_0}{p}\left(\frac{1}{q_0} - \alpha\right). \tag{2.19}$$

We have $R_1 = R_2$ and
$$\frac{1}{r_2} - \frac{1}{r_1} = \frac{p_0}{pq_0} - \frac{1}{q} = \frac{qp_0 - pq_0}{pqq_0}.$$
If $pq_0/p_0 < q \le q_0$ then $r_2 < r_1$. By the conditions $r_2/R_2 = p/p_0 (= r_2/R_1)$ and $q > r_2$, we have $pR_1/p_0 < q$. We remark that $R_1 < q_0$ since $1/R_1 = 1/p_0 + 1/q_0 - \alpha$ and $1/p_0 > \alpha$. Hence there exists $q$ which satisfies $pR_1/p_0 < q < pq_0/p_0$ and it follows that if $pR_1/p_0 < q < pq_0/p_0$ then $r_2 > r_1$.




In the previous paper [24], the author extended and generalized Proposition 2.9 ([24, Theorem 3.1, Corollary 3.5]). In this paper, the author extends Proposition 2.9 to other directions (also extends [24, Corollary 3.5]) and generalizes Proposition 2.9.

On generalizations of boundedness of fractional integral operators on Morrey spaces, many results were shown ([3], [4], [5], [7], [8], [10], [13], [14], [19], [20], [24], [25] and so on). In this paper, the details are omitted (see [24, Section 2]). Recently, further results were investigated by many researchers ([6], [11] for example).

**3 Our Theorems** In this section, we state three theorems and their corollaries. For simplicity, we write $\|\cdot\|_{p,\phi}$ for $\|\cdot\|_{L_p^\phi}$ below.

**Theorem 3.1** *Let $k = 1, 2$. Suppose that we are given parameters $p_0$, $p_1$, $p_2$, $q_1$, $q_2$, $r$ satisfying $1 \le p_1 \le p_0 < +\infty$, $1 < p_2 \le p_0 < +\infty$, $1 < q_k < +\infty$, $1 < r < +\infty$. For $\phi_k(t)$ and $\rho(t)$, assume that there exist positive constants $C_1$, $C_2$, $C_3$, $C_4$, $\alpha$, $p_0$ and non-negative constants $\gamma$, $\delta$ with $0 < \alpha(1+\gamma)/(1-\delta) < 1/p_0$ such that*

$$\frac{1}{C_1} \le \frac{\phi_k(s)}{\phi_k(t)} \le C_1 \quad \textit{for} \quad \frac{1}{2} \le \frac{s}{t} \le 2, \tag{3.1}$$

$$\phi_1(t) \ge \frac{C_2 t^{n/p_0}}{1 + t^{n\delta/p_0}}, \tag{3.2}$$

$$\int_t^{+\infty} \frac{ds}{s\phi_2(s)^{p_2}} \le \frac{C_3}{\phi_2(t)^{p_2}}, \tag{3.3}$$

$$\rho(t) \le C_4 t^{n\alpha}(1 + t^{n\alpha\gamma}) \tag{3.4}$$

*for every $t > 0$. Assume also that*

$$\frac{1}{r} = \frac{1}{q_1} + \frac{1}{p_2}(1 - p_0\alpha) = \frac{1}{q_2} + \frac{1}{p_2}\left(1 - p_0\alpha \cdot \frac{1+\gamma}{1-\delta}\right), \tag{3.5}$$

$$\psi(t) = \eta_1(t)\phi_2(t)^{1-p_0\alpha} = \eta_2(t)\phi_2(t)^{1-p_0\alpha(1+\gamma)/(1-\delta)}, \tag{3.6}$$

*and* $g \in L^{\eta_1}_{q_1} \cap L^{\eta_2}_{q_2}$*. Then there exists a positive constant* $C$ *such that*

$$\begin{aligned} \|g\cdot T_\rho f\|_{r,\psi} \le C \Big( &\|g\|_{q_1,\eta_1}\|f\|^{1-p_0\alpha}_{p_2,\phi_2}\|f\|^{p_0\alpha}_{p_1,\phi_1} \\ &+ \|g\|_{q_2,\eta_2}\|f\|^{1-p_0\alpha(1+\gamma)/(1-\delta)}_{p_2,\phi_2}\|f\|^{p_0\alpha(1+\gamma)/(1-\delta)}_{p_1,\phi_1}\Big), \end{aligned} \tag{3.7}$$

*where* $f \in L^{\phi_1}_{p_1} \cap L^{\phi_2}_{p_2}$*.*

**Remark 3.2** The function $\phi_k(t) = t^{n/p_0}/\log(2+t)$ with $1 \le p_1 \le p_0 < +\infty$ and $1 < p_2 \le p_0 < +\infty$ satisfies (3.1), (3.2), and (3.3) for every $\delta > 0$ and the function $\rho(t) = t^{n\alpha}\log(2+t)$ with $0 < \alpha < 1$ satisfies (3.4) for every $\gamma > 0$.

Let $1 < s \le s_0 < +\infty$. As a special case $\gamma = \delta = 0$, $\phi_1(t) = t^{n/p_0}$, $\phi_2(t) = t^{n/s_0}$, and $\rho(t) = t^{n\alpha}$ in Theorem 3.1, we have

**Corollary 3.3** *Let* $1 \le p \le p_0 < +\infty$*,* $1 < q \le q_0 < +\infty$*,* $1 < r \le r_0 < +\infty$*,* $1 < s \le s_0 < +\infty$*,* $0 < \alpha < 1/p_0$*,*

$$\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{s_0}(1-p_0\alpha), \quad \frac{1}{r} = \frac{1}{q} + \frac{1}{s}(1-p_0\alpha), \tag{3.8}$$

*and assume* $g \in M^{q_0}_q$*. Then there exists a positive constant* $C$ *such that*

$$\|g\cdot I_\alpha f\|_{M^{r_0}_r} \le C\|g\|_{M^{q_0}_q}\|f\|^{1-p_0\alpha}_{M^{s_0}_s}\|f\|^{p_0\alpha}_{M^{p_0}_p}, \tag{3.9}$$

*where* $f \in M^{p_0}_p \cap M^{s_0}_s$*.*

The case $1 < p \le p_0 < +\infty$, $s_0 = p_0$, and $s = p$ in Corollary 3.3 is the same as Proposition 2.9. As a special case $s_0 = p_0$ and $s = r$ in Corollary 3.3, we have

**Corollary 3.4** ([24, Corollary 3.5]) *Let* $1 \le p \le p_0 < +\infty$*,* $1 < q \le q_0 < +\infty$*,* $1 < r \le \min\{p_0, r_0\} < +\infty$*,* $0 < \alpha < 1/p_0$*,*

$$\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \alpha, \quad \frac{1}{r} = \frac{1}{qp_0\alpha}, \tag{3.10}$$

*and assume* $g \in M^{q_0}_q$*. Then there exists a positive constant* $C$ *such that*

$$\|g\cdot I_\alpha f\|_{M^{r_0}_r} \le C\|g\|_{M^{q_0}_q}\|f\|^{1-p_0\alpha}_{M^{p_0}_r}\|f\|^{p_0\alpha}_{M^{p_0}_p}, \tag{3.11}$$

*where* $f \in M^{p_0}_{\max\{p,r\}}$*.*

As a special case $s_0 = r_0$ and $s = r$ in Corollary 3.3, we have

**Corollary 3.5** *Let* $1 \le p \le p_0 < +\infty$*,* $1 < r \le r_0 < +\infty$*,* $0 < \alpha < 1/p_0$*,* $g \in M^{r_0/(p_0\alpha)}_{r/(p_0\alpha)}$*. Then there exists a positive constant* $C$ *such that*

$$\|g\cdot I_\alpha f\|_{M^{r_0}_r} \le C\|g\|_{M^{r_0/(p_0\alpha)}_{r/(p_0\alpha)}}\|f\|^{1-p_0\alpha}_{M^{r_0}_r}\|f\|^{p_0\alpha}_{M^{p_0}_p}, \tag{3.12}$$

*where* $f \in M^{p_0}_p \cap M^{r_0}_r$*.*

**Remark 3.6** We compare Proposition 2.9 with Corollary 3.5. We consider the case $p_0 \neq r_0$ and $p \neq r$. Corollary 3.5 says that if $f$ belongs to $M_p^{p_0}$ and it also belongs to $M_r^{r_0}$ then $g \cdot I_\alpha f \in M_r^{r_0}$ for $g \in M_{r/(p_0\alpha)}^{r_0/(p_0\alpha)}$. We cannot take $q_0 = r_0/(p_0\alpha)$ and $q = r/(p_0\alpha)$ in Proposition 2.9 since $0 < \alpha < 1/p_0$.

In Corollary 3.5, changing roles of the letters $r_0$ and $r$ with $p_0$ and $p$ respectively, we have

**Corollary 3.7** *Let $1 < p \leq p_0 < +\infty$, $1 \leq r \leq r_0 < +\infty$, $0 < \alpha < 1/r_0$, $g \in M_{p/(r_0\alpha)}^{p_0/(r_0\alpha)}$. Then there exists a positive constant $C$ such that*

$$\|g \cdot I_\alpha f\|_{M_p^{p_0}} \leq C\|g\|_{M_{p/(r_0\alpha)}^{p_0/(r_0\alpha)}} \|f\|_{M_p^{p_0}}^{1-r_0\alpha} \|f\|_{M_r^{r_0}}^{r_0\alpha}, \tag{3.13}$$

*where $f \in M_p^{p_0} \cap M_r^{r_0}$.*

**Remark 3.8** We compare Corollary 2.10 with Corollary 3.7. We consider the case $p_0 > r_0$. If $p_0 > r_0$ then "$\alpha < 1/p_0$ implies $\alpha < 1/r_0$". Corollary 3.7 says that if $f$ belongs to $M_p^{p_0}$ and it also belongs to $M_r^{r_0}$ then $g \cdot I_\alpha f \in M_p^{p_0}$ for $g \in M_{p/(r_0\alpha)}^{p_0/(r_0\alpha)}$ under the assumption $0 < \alpha < 1/r_0$ which is weaker than $0 < \alpha < 1/p_0$. In this case, the space which includes $g$ is different from the one in Corollary 2.10.

**Theorem 3.9** *Let $j = 1, 2$, $k = 1, 2, 3$. Suppose that we are given parameters $p_1$, $p_2$, $p_3$, $q_1$, $q_2$, $r$, $u$ satisfying $1 < p_1 < +\infty$, $1 < p_3 \leq p_2 < u < +\infty$, $1 < q_j < +\infty$, $1 < r < +\infty$. For $\phi_k(t)$ and $\rho(t)$, assume that there exist positive constants $C_1$, $C_4$, $C_5$, $C_6$, $\alpha$, $\theta$ and a non-negative constant $\gamma$ with $\alpha < \theta$, $\theta(1+\gamma) \leq 1$ such that* (3.1) *and* (3.4) *hold and that*

$$\int_t^{+\infty} \frac{ds}{s\phi_k(s)^{p_k}} \leq \frac{C_5}{\phi_k(t)^{p_k}}, \tag{3.14}$$

$$\int_t^{+\infty} \frac{s^{n\alpha/\theta-1}}{\phi_2(s)} ds \leq C_6 \sum_{i=2,3} t^{(n\alpha/\theta)/(1-u/p_i)}, \tag{3.15}$$

*for every $t > 0$. Assume also that*

$$\frac{1}{r} = \frac{1}{q_1} + \frac{\theta}{u} + \frac{1-\theta}{p_1} = \frac{1}{q_2} + \frac{\theta(1+\gamma)}{u} + \frac{1-\theta(1+\gamma)}{p_1}, \tag{3.16}$$

$$\begin{aligned} \psi(t) &= \eta_1(t) \left( \sum_{i=2,3} \phi_2(t)^{p_i\theta/u} \right) \phi_1(t)^{1-\theta} \\ &= \eta_2(t) \left( \sum_{i=2,3} \phi_2(t)^{p_i\theta(1+\gamma)/u} \right) \phi_1(t)^{1-\theta(1+\gamma)}, \end{aligned} \tag{3.17}$$

*and $g \in L_{q_1}^{\eta_1} \cap L_{q_2}^{\eta_2}$. Then there exists a positive constant $C$ such that*

$$\begin{aligned} \|g \cdot T_\rho f\|_{r,\psi} \leq C \Big( &\|g\|_{q_1,\eta_1} \|f\|_{p_2,\phi_2}^{\theta} \|f\|_{p_1,\phi_1}^{1-\theta} \\ &+ \|g\|_{q_2,\eta_2} \|f\|_{p_2,\phi_2}^{\theta(1+\gamma)} \|f\|_{p_1,\phi_1}^{1-\theta(1+\gamma)} \Big), \end{aligned} \tag{3.18}$$

*where $f \in L_{p_1}^{\phi_1} \cap L_{p_2}^{\phi_2}$.*

**Remark 3.10** Let $1 < p \le p_0 < +\infty$. In Theorem 3.9, we can take $\theta = 1 - p_0\alpha$ under the assumption $0 \le \gamma \le 1$. Indeed, if $\theta = 1 - p_0\alpha$ then the assumption $\alpha < \theta$ is equivalent to $\alpha < 1/(p_0 + 1)$ and the assumption $\theta(1 + \gamma) \le 1$ is equivalent to $\alpha \ge \gamma/\{p_0(1 + \gamma)\}$. There exists $\alpha$ satisfying $\gamma/\{p_0(1 + \gamma)\} \le \alpha < 1/(p_0 + 1)$ since $p_0 > 1$ and $0 \le \gamma \le 1$.

**Remark 3.11** Let $1 < s \le s_0 < +\infty$, $0 \le \delta < 1$, $1/u = (1/p_2)(1 - s_0\alpha/\theta) = (1/p_3)[1 - s_0\alpha/\{\theta(1 - \delta)\}]$. Then the function $\phi_2(t) = t^{n/s_0}/(1 + t^{n\delta/s_0})$ satisfies (3.1), (3.14), and (3.15).

Let $1 < p \le p_0 < +\infty$. As a special case $\gamma = \delta = 0$, $\phi_1(t) = t^{n/p_0}$, $\phi_2(t) = t^{n/s_0}$, and $1/u = (1/s)(1 - s_0\alpha/\theta)$ in Theorem 3.9, we have

**Corollary 3.12** *Let $1 < p \le p_0 < +\infty$, $1 < q \le q_0 < +\infty$, $1 < r \le r_0 < +\infty$, $1 < s \le s_0 < +\infty$, $0 < s_0\alpha < \theta \le 1$,*

$$\frac{1}{r_0} = \frac{1}{q_0} + \frac{\theta}{s_0} + \frac{1-\theta}{p_0} - \alpha, \quad \frac{1}{r} = \frac{1}{q} + \frac{\theta}{s} - \frac{s_0}{s}\alpha + \frac{1-\theta}{p}, \tag{3.19}$$

*and assume $g \in M_q^{q_0}$. Then there exists a positive constant $C$ such that*

$$\|g \cdot I_\alpha f\|_{M_r^{r_0}} \le C\|g\|_{M_q^{q_0}} \|f\|_{M_s^{s_0}}^{\theta} \|f\|_{M_p^{p_0}}^{1-\theta}, \tag{3.20}$$

*where $f \in M_p^{p_0} \cap M_s^{s_0}$.*

The case "$s_0 = p_0$ and $s = p$" or $\theta = 1$ in Corollary 3.12 is the same as Proposition 2.9.

**Remark 3.13** In Corollary 3.12, we can replace $p$ and $p_0$ by $s$ and $s_0$ respectively. In this case, we cannot let $\theta = p_0\alpha$ to state its corollary since the corollary holds only for $\theta > p_0\alpha$.

Letting $\theta = 1 - p_0\alpha$ in Corollary 3.12, we have

**Corollary 3.14** *Let $1 < p \le p_0 < +\infty$, $1 < q \le q_0 < +\infty$, $1 < r \le r_0 < +\infty$, $1 < s \le s_0 < +\infty$, $0 < \alpha < 1/(p_0 + s_0)$,*

$$\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{s_0}(1 - p_0\alpha), \quad \frac{1}{r} = \frac{1}{q} + \frac{1}{s}(1 - p_0\alpha) + \left(\frac{p_0}{p} - \frac{s_0}{s}\right)\alpha, \tag{3.21}$$

*and assume $g \in M_q^{q_0}$. Then there exists a positive constant $C$ such that*

$$\|g \cdot I_\alpha f\|_{M_r^{r_0}} \le C\|g\|_{M_q^{q_0}} \|f\|_{M_s^{s_0}}^{1-p_0\alpha} \|f\|_{M_p^{p_0}}^{p_0\alpha}, \tag{3.22}$$

*where $f \in M_p^{p_0} \cap M_s^{s_0}$.*

Letting $\theta = 1/2$ in Corollary 3.12, we have

**Corollary 3.15** *Let $1 < p \le p_0 < +\infty$, $1 < q \le q_0 < +\infty$, $1 < r \le r_0 < +\infty$, $1 < s \le s_0 < +\infty$, $0 < \alpha < 1/(2s_0)$,*

$$\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{2}\left(\frac{1}{s_0} + \frac{1}{p_0}\right) - \alpha, \quad \frac{1}{r} = \frac{1}{q} + \frac{1}{2}\left(\frac{1}{s} + \frac{1}{p}\right) - \frac{s_0}{s}\alpha, \tag{3.23}$$

*and assume $g \in M_q^{q_0}$. Then there exists a positive constant $C$ such that*

$$\|g \cdot I_\alpha f\|_{M_r^{r_0}} \le C\|g\|_{M_q^{q_0}} \|f\|_{M_s^{s_0}}^{1/2} \|f\|_{M_p^{p_0}}^{1/2}, \tag{3.24}$$

*where $f \in M_p^{p_0} \cap M_s^{s_0}$.*

**Theorem 3.16** *Let $j = 1, 2$, $k = 1, 2, 3, 4$. Suppose that we are given parameters $p_1$, $p_2$, $p_3$, $p_4$, $q_1$, $q_2$, $u_1$, $u_2$ satisfying $1 < p_{j+2} \leq p_j < u_j < +\infty$, $1 < q_j < +\infty$, $1 < r < +\infty$. For $\phi_k(t)$ and $\rho(t)$, assume that there exist positive constants $C_1$, $C_4$, $C_5$, $C_7$, $C_8$, $\alpha$, $\epsilon_1$, $\epsilon_2$ and a non-negative constant $\gamma$ and with $\epsilon_2 \leq \alpha$, $\alpha + \epsilon_1 < 1$, $\alpha\gamma < \epsilon_1$, such that (3.1), (3.4), and (3.14) hold and that*

$$\int_t^{+\infty} \frac{s^{n(\alpha+\epsilon_1)-1}}{\phi_1(s)} ds \leq C_7 \sum_{i=1,3} t^{n(\alpha+\epsilon_1)/(1-u_1/p_i)}, \tag{3.25}$$

$$\int_t^{+\infty} \frac{s^{n(\alpha-\epsilon_2)-1}}{\phi_2(s)} ds \leq C_8 \sum_{i=2,4} t^{n(\alpha-\epsilon_2)/(1-u_2/p_i)}, \tag{3.26}$$

*for every $t > 0$. Assume also that*

$$\begin{aligned} \frac{1}{r} &= \frac{1}{q_1} + \frac{\epsilon_1}{\epsilon_1+\epsilon_2} \cdot \frac{1}{u_2} + \frac{\epsilon_2}{\epsilon_1+\epsilon_2} \cdot \frac{1}{u_1} \\ &= \frac{1}{q_2} + \frac{\epsilon_1-\alpha\gamma}{\epsilon_1+\epsilon_2} \cdot \frac{1}{u_2} + \frac{\epsilon_2+\alpha\gamma}{\epsilon_1+\epsilon_2} \cdot \frac{1}{u_1}, \end{aligned} \tag{3.27}$$

$$\begin{aligned} \psi(t) &= \eta_1(t) \left( \sum_{i=2,4} \phi_2(t)^{p_i\epsilon_1/\{u_2(\epsilon_1+\epsilon_2)\}} \right) \left( \sum_{i=1,3} \phi_1(t)^{p_i\epsilon_2/\{u_1(\epsilon_1+\epsilon_2)\}} \right) \\ &= \eta_2(t) \left( \sum_{i=2,4} \phi_2(t)^{p_i(\epsilon_1-\alpha\gamma)/\{u_2(\epsilon_1+\epsilon_2)\}} \right) \\ &\quad \cdot \left( \sum_{i=1,3} \phi_1(t)^{p_i(\epsilon_2+\alpha\gamma)/\{u_1(\epsilon_1+\epsilon_2)\}} \right), \end{aligned} \tag{3.28}$$

*and $g \in L_{q_1}^{\eta_1} \cap L_{q_2}^{\eta_2}$. Then there exists a positive constant $C$ such that*

$$\begin{aligned} \|g \cdot T_\rho f\|_{r,\psi} \leq C \Big( &\|g\|_{q_1,\eta_1} \|f\|_{p_2,\phi_2}^{\epsilon_1/(\epsilon_1+\epsilon_2)} \|f\|_{p_1,\phi_1}^{\epsilon_2/(\epsilon_1+\epsilon_2)} \\ &+ \|g\|_{q_2,\eta_2} \|f\|_{p_2,\phi_2}^{(\epsilon_1-\alpha\gamma)/(\epsilon_1+\epsilon_2)} \|f\|_{p_1,\phi_1}^{(\epsilon_2+\alpha\gamma)/(\epsilon_1+\epsilon_2)} \Big), \end{aligned} \tag{3.29}$$

*where $f \in L_{p_1}^{\phi_1} \cap L_{p_2}^{\phi_2}$.*

As a special case $\gamma = 0$, $p_{j+2} = p_j$, $j = 1, 2$, $\phi_1(t) = t^{n/p_0}$, $\phi_2(t) = t^{n/s_0}$, $1/u_1 = (1/p)\{1 - p_0(\alpha+\epsilon_1)\}$, and $1/u_2 = (1/s)\{1 - s_0(\alpha - \epsilon_2)\}$ in Theorem 3.16, we have

**Corollary 3.17** *Let $1 < p \leq p_0 < +\infty$, $1 < q \leq q_0 < +\infty$, $1 < r \leq r_0 < +\infty$, $1 < s \leq s_0 < +\infty$, $0 < \epsilon_2 \leq \alpha < \alpha + \epsilon_1 < 1/p_0$, $\alpha - \epsilon_2 < 1/s_0$,*

$$\frac{1}{r_0} = \frac{1}{q_0} + \frac{\epsilon_1}{\epsilon_1+\epsilon_2} \cdot \frac{1}{s_0} + \frac{\epsilon_2}{\epsilon_1+\epsilon_2} \cdot \frac{1}{p_0} - \alpha, \tag{3.30}$$

$$\frac{1}{r} = \frac{1}{q} + \frac{\epsilon_1}{\epsilon_1+\epsilon_2} \left\{ \frac{1}{s} - \frac{s_0}{s}(\alpha - \epsilon_2) \right\} + \frac{\epsilon_2}{\epsilon_1+\epsilon_2} \left\{ \frac{1}{p} - \frac{p_0}{p}(\alpha+\epsilon_1) \right\}, \tag{3.31}$$

*and assume $g \in M_q^{q_0}$. Then there exists a positive constant $C$ such that*

$$\|g \cdot I_\alpha f\|_{M_r^{r_0}} \leq C \|g\|_{M_q^{q_0}} \|f\|_{M_s^{s_0}}^{\epsilon_1/(\epsilon_1+\epsilon_2)} \|f\|_{M_p^{p_0}}^{\epsilon_2/(\epsilon_1+\epsilon_2)}, \tag{3.32}$$

*where $f \in M_p^{p_0} \cap M_s^{s_0}$.*

The case $s_0 = p_0$ and $s = p$ in Corollary 3.17 is the same as Proposition 2.9. Letting $\epsilon_1 = \epsilon_2 = \epsilon$ in Corollary 3.17, we have

**Corollary 3.18** *Let $1 < p \le p_0 < +\infty$, $1 < q \le q_0 < +\infty$, $1 < r \le r_0 < +\infty$, $1 < s \le s_0 < +\infty$, $0 < \epsilon \le \alpha < \alpha + \epsilon < 1/p_0$, $\alpha - \epsilon < 1/s_0$,*

$$\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{2}\left(\frac{1}{s_0} + \frac{1}{p_0}\right) - \alpha, \tag{3.33}$$

$$\frac{1}{r} = \frac{1}{q} + \frac{1}{2}\left\{\frac{1}{s} - \frac{s_0}{s}(\alpha - \epsilon) + \frac{1}{p} - \frac{p_0}{p}(\alpha + \epsilon)\right\}, \tag{3.34}$$

*and assume $g \in M_q^{q_0}$. Then there exists a positive constant $C$ such that*

$$\|g \cdot I_\alpha f\|_{M_r^{r_0}} \le C\|g\|_{M_q^{q_0}} \|f\|_{M_s^{s_0}}^{1/2} \|f\|_{M_p^{p_0}}^{1/2}, \tag{3.35}$$

*where $f \in M_p^{p_0} \cap M_s^{s_0}$.*

**4 Proof of Theorem 3.1** In this section, we prove Theorem 3.1. Let

$$L_c^\infty(\mathbf{R}^n) = \{f \in L^\infty(\mathbf{R}^n) |\, \mathrm{supp}\, f \text{ is compact}\}. \tag{4.1}$$

By the monotone convergence theorem, we may assume $f \in L_{p_1}^{\phi_1} \cap L_{p_2}^{\phi_2} \cap L_c^\infty$ without loss of generality in the proof. (See [10, p.1129] for the details.) For $0 \le \alpha < 1$ the fractional maximal operator is defined by

$$M_\alpha f(x) = \sup_{x \in B} \frac{1}{|B|^{1-\alpha}} \int_B |f(y)| dy, \tag{4.2}$$

where the supremum on the right side is taken over all balls $B \subset \mathbf{R}^n$ containing $x$. We denote $M_0$ by $M$, which is the Hardy-Littlewood maximal operator. To prove Theorem 3.1, we use a pointwise estimate by the Hardy-Littlewood maximal operator the following Lemma 4.1 states.

**Lemma 4.1** *Let $1 \le p_1 \le p_0 < +\infty$, $\alpha > 0$, $\gamma \ge 0$, $\delta \ge 0$, $0 < \alpha(1+\gamma)/(1-\delta) < 1/p_0$. Under the assumptions (3.1) for $k = 1$, (3.2), and (3.4), we have*

$$\begin{aligned} |T_\rho f(x)| \le C \Big( & Mf(x)^{1-p_0\alpha} \|f\|_{p_1,\phi_1}^{p_0\alpha} \\ & + Mf(x)^{1-p_0\alpha(1+\gamma)/(1-\delta)} \|f\|_{p_1,\phi_1}^{p_0\alpha(1+\gamma)/(1-\delta)} \Big), \end{aligned} \tag{4.3}$$

*where $f \in L_{p_1}^{\phi_1} \cap L_c^\infty$.*

In Lemma 4.1, the case "$\gamma = \delta = 0$ and $1 < p_1 \le p_0 < +\infty$" was shown by Chiarenza and Frasca ([2, p.277]).

**Remark 4.2** The condition (3.1) is called the doubling condition. Under the assumptions (3.1) and (3.3), the Hardy-Littlewood maximal operator is known to be bounded from $L_{p_2}^{\phi_2}$ to itself, where $1 < p_2 < +\infty$ (see [13, Theorem 1], also [4], [7]).

**Remark 4.3** If $\alpha > 0$, $\gamma \geq 0$, $\delta \geq 0$, $0 < \alpha(1+\gamma)/(1-\delta) < 1/p_0$, and (3.2) holds, then there exists a constant $C$ such that

$$\int_t^{+\infty} \frac{s^{n\alpha(1+\gamma)-1}}{\phi_1(s)}\, ds \leq C\frac{t^{n\alpha(1+\gamma)}}{\phi_1(t)}. \tag{4.4}$$

*Proof of Lemma 4.1.* We follow the argument in [2]. We write, for $\sigma > 0$ which will be determined later,

$$\begin{aligned} T_\rho f(x) &= \int_{|x-y|<\sigma} \frac{\rho(|x-y|)f(y)}{|x-y|^n}\, dy + \int_{|x-y|\geq\sigma} \frac{\rho(|x-y|)f(y)}{|x-y|^n}\, dy \\ &= I_1 + I_2. \end{aligned} \tag{4.5}$$

By using (3.4), we can estimate $I_1$ by

$$\begin{aligned} |I_1| &\leq C\sum_{j=-\infty}^{-1}\int_{2^j\sigma\leq|x-y|<2^{j+1}\sigma} \frac{|x-y|^{n\alpha}(1+|x-y|^{n\alpha\gamma})|f(y)|}{|x-y|^n}\, dy \\ &\leq C(\sigma^{n\alpha}+\sigma^{n\alpha+n\alpha\gamma})Mf(x). \end{aligned} \tag{4.6}$$

Again by using (3.4), we let

$$\begin{aligned} |I_2| &\leq C\left(\int_{|x-y|\geq\sigma} \frac{|f(y)|}{|x-y|^{n-n\alpha}}\, dy + \int_{|x-y|\geq\sigma} \frac{|f(y)|}{|x-y|^{n-n\alpha-n\alpha\gamma}}\, dy\right) \\ &= C(I_3+I_4). \end{aligned} \tag{4.7}$$

First we estimate $I_3$. We observe that

$$\begin{aligned} I_3 &= (n-n\alpha)\int_{\mathbf{R}^n}\left(\int_{|x-y|}^{+\infty} \chi_{B(x,\sigma)^c}(y)|f(y)|\frac{dl}{l^{n-n\alpha+1}}\right) dy \\ &= (n-n\alpha)\int_{\mathbf{R}^n}\left(\int_0^{+\infty} \chi_{B(x,l)\setminus B(x,\sigma)^c}(y)|f(y)|\frac{dl}{l^{n-n\alpha+1}}\right) dy \\ &\leq (n-n\alpha)\int_{\mathbf{R}^n}\left(\int_\sigma^{+\infty} \chi_{B(x,l)}(y)|f(y)|\frac{dl}{l^{n-n\alpha+1}}\right) dy \\ &= (n-n\alpha)\int_\sigma^{+\infty}\left(\frac{1}{l^{n-n\alpha+1}}\int_{B(x,l)}|f(y)|dy\right) dl \\ &\leq (n-n\alpha)\|f\|_{1,\phi_1}\int_\sigma^{+\infty}\frac{l^{n\alpha-1}}{\phi_1(l)} dl. \end{aligned}$$

Then by using the case $\gamma = 0$ in (4.4), we have

$$I_3 \leq C\frac{\sigma^{n\alpha}}{\phi_1(\sigma)}\|f\|_{1,\phi_1}.$$

Similarly, for $I_4$, by using (4.4), we have

$$I_4 \leq C\frac{\sigma^{n\alpha+n\alpha\gamma}}{\phi_1(\sigma)}\|f\|_{1,\phi_1}.$$

By the assumption (3.2), we have

$$
\begin{aligned}
|I_2| &\le C\frac{(1+\sigma^{n\alpha\gamma})\sigma^{n\alpha}}{\phi_1(\sigma)}\|f\|_{1,\phi_1} \\
&\le C(1+\sigma^{n\delta/p_0}+\sigma^{n\alpha\gamma}+\sigma^{n\delta/p_0+n\alpha\gamma})\sigma^{n\alpha-n/p_0}\|f\|_{1,\phi_1}. \qquad (4.8)
\end{aligned}
$$

Combining (4.6) with (4.8), we obtain

$$
\begin{aligned}
|T_\rho f(x)| \le C\,\{&(\sigma^{n\alpha}+\sigma^{n\alpha+n\alpha\gamma})Mf(x) \\
&+\left(\sigma^{n\alpha-n/p_0}+\sigma^{n\alpha+n\alpha\gamma-n/p_0+n\delta/p_0}\right)\|f\|_{1,\phi_1}\Big\}.
\end{aligned}
$$

We consider two cases. For the case $\|f\|_{1,\phi_1}/Mf(x) \le 1$, that is, $\|f\|_{1,\phi_1} \le Mf(x)$, we choose

$$\sigma = (\|f\|_{1,\phi_1}/Mf(x))^{p_0/n}.$$

Since $\sigma \le 1$, it follows that

$$|T_\rho f(x)| \le C\left(\frac{\|f\|_{1,\phi_1}}{Mf(x)}\right)^{p_0\alpha} Mf(x) = CMf(x)^{1-p_0\alpha}\|f\|_{1,\phi_1}^{p_0\alpha}.$$

For the case $\|f\|_{1,\phi_1}/Mf(x) \ge 1$, we choose

$$\sigma = (\|f\|_{1,\phi_1}/Mf(x))^{p_0/\{n(1-\delta)\}}.$$

In this case, it follows in a similar way that

$$|T_\rho f(x)| \le CMf(x)^{1-p_0\alpha(1+\gamma)/(1-\delta)}\|f\|_{1,\phi_1}^{p_0\alpha(1+\gamma)/(1-\delta)}.$$

Since $p_1 \ge 1$, we obtain the desired estimate. □

We also need the following Lemma 4.4. (We use only the case $k=2$ to prove Theorem 3.1. The case $k=3$ is needed to prove Theorems 3.9 and 3.16.)

**Lemma 4.4** (Hölder's inequality on generalized Morrey spaces, [15, Corollary 4.3] for example) *Let $k \in \mathbf{N}\cap[2,+\infty)$, $i = 0, 1, 2, \ldots, k$, $1 < p_i < +\infty$, and let $\phi_i(t)$ be functions which satisfy $0 < \phi_i(t) < +\infty$ for $t > 0$. Assume*

$$\frac{1}{p_0} = \sum_{j=1}^{k}\frac{1}{p_j}, \quad \phi_0 = \prod_{j=1}^{k}\phi_j. \qquad (4.9)$$

*Then*

$$\left\|\prod_{j=1}^{k} f_j\right\|_{p_0,\phi_0} \le \prod_{j=1}^{k}\|f_j\|_{p_j,\phi_j}, \qquad (4.10)$$

*where $f_j \in L_{p_j}^{\phi_j}$.*

Now we are ready to give

*Proof of Theorem 3.1.* By Lemma 4.1, we let

$$
\begin{aligned}
|T_\rho f(x)| &\le C\Big(Mf(x)^{1-p_0\alpha}\|f\|_{p_1,\phi_1}^{p_0\alpha} \\
&\quad +Mf(x)^{1-p_0\alpha(1+\gamma)/(1-\delta)}\|f\|_{p_1,\phi_1}^{p_0\alpha(1+\gamma)/(1-\delta)}\Big) \\
&= C(J_1(x)+J_2(x)).
\end{aligned}
$$

By the assumption, we have

$$\frac{1}{q_1/r} + \frac{1}{p_2/\{r(1-p_0\alpha)\}} = 1.$$

We write $B(x,t) = B$ for short. By Hölder's inequality, we have

$$\begin{aligned}
&\psi(t)\left(\frac{1}{|B|}\int_B |g(y)J_1(y)|^r\,dy\right)^{1/r}\\
&\le C\psi(t)\left(\frac{1}{|B|}\int_B |g(y)|^r|Mf(y)|^{r(1-p_0\alpha)}\,dy\right)^{1/r}\|f\|_{p_1,\phi_1}^{p_0\alpha}\\
&\le C\eta_1(t)\left(\frac{1}{|B|}\int_B |g(y)|^{q_1}\,dy\right)^{1/q_1}\\
&\quad\cdot\,\phi_2(t)^{1-p_0\alpha}\left(\frac{1}{|B|}\int_B |Mf(y)|^{p_2}dy\right)^{(1-p_0\alpha)/p_2}\|f\|_{p_1,\phi_1}^{p_0\alpha}\\
&\le C\|g\|_{q_1,\eta_1}\|f\|_{p_2,\phi_2}^{1-p_0\alpha}\|f\|_{p_1,\phi_1}^{p_0\alpha}.
\end{aligned}$$

In the last inequality we used the fact that $\|Mf\|_{p_2,\phi_2} \le C\|f\|_{p_2,\phi_2}$ holds under the assumptions (3.1) and (3.3) (see Remark 4.2). This implies

$$\|gJ_1\|_{r,\psi} \le C\|g\|_{q_1,\eta_1}\|f\|_{p_2,\phi_2}^{1-p_0\alpha}\|f\|_{p_1,\phi_1}^{p_0\alpha}.$$

In a similar way we have

$$\|gJ_2\|_{r,\psi} \le C\|g\|_{q_2,\eta_2}\|f\|_{p_2,\phi_2}^{1-p_0\alpha(1+\gamma)/(1-\delta)}\|f\|_{p_1,\phi_1}^{p_0\alpha(1+\gamma)/(1-\delta)}.$$

This completes the proof of Theorem 3.1. □

**5 Proof of Theorem 3.9** In this section, we prove Theorem 3.9. To prove Theorem 3.9, we use a pointwise estimate the following Lemma 5.1 states.

**Lemma 5.1** *Let $0 < \alpha < \theta \le \theta(1+\gamma) \le 1$. Under the assumption* (3.4)*, we have*

$$\begin{aligned}
|T_\rho f(x)| \le C\,\big(&I_{\alpha/\theta}(|f|)(x)^\theta Mf(x)^{1-\theta}\\
&+I_{\alpha/\theta}(|f|)(x)^{\theta(1+\gamma)}Mf(x)^{1-\theta(1+\gamma)}\big).
\end{aligned} \tag{5.1}$$

In Lemma 5.1, the case $\gamma = 0$ was shown by Hedberg ([9, p.508]).

*Proof of Lemma 5.1.* We follow the argument in [9]. As in the proof of Lemma 4.1, we use the statements (4.5), (4.6), and (4.7). For $I_3$, we have

$$\begin{aligned}
I_3 &\le \sigma^{n\alpha-n\alpha/\theta}\int_{|x-y|\ge\sigma}\frac{|f(y)|}{|x-y|^{n-n\alpha/\theta}}\,dy\\
&\le \sigma^{n\alpha-n\alpha/\theta}I_{\alpha/\theta}(|f|)(x).
\end{aligned} \tag{5.2}$$

Similarly, for $I_4$, we have

$$\begin{aligned}
I_4 &\le \sigma^{n\alpha-n\alpha/\theta+n\alpha\gamma}\int_{|x-y|\ge\sigma}\frac{|f(y)|}{|x-y|^{n-n\alpha/\theta}}\,dy\\
&\le \sigma^{n\alpha-n\alpha/\theta+n\alpha\gamma}I_{\alpha/\theta}(|f|)(x).
\end{aligned} \tag{5.3}$$

Hence, it follows that

$$\begin{aligned}|T_\rho f(x)| \le C \{&(\sigma^{n\alpha} + \sigma^{n\alpha+n\alpha\gamma}) Mf(x) \\ &+ \left(\sigma^{n\alpha-n\alpha/\theta} + \sigma^{n\alpha-n\alpha/\theta+n\alpha\gamma}\right) I_{\alpha/\theta}(|f|)(x)\right\}.\end{aligned}$$

For the case $I_{\alpha/\theta}(|f|)(x)/Mf(x) \le 1$, we choose

$$\sigma = (I_{\alpha/\theta}(|f|)(x)/Mf(x))^{\theta/(n\alpha)}. \tag{5.4}$$

Since $\sigma \le 1$, it follows that

$$|T_\rho f(x)| \le C I_{\alpha/\theta}(|f|)(x)^\theta Mf(x)^{1-\theta}.$$

For the case $I_{\alpha/\theta}(|f|)(x)/Mf(x) \ge 1$, we also choose $\sigma$ as (5.4). Since $\sigma \ge 1$, it follows that

$$|T_\rho f(x)| \le C I_{\alpha/\theta}(|f|)(x)^{\theta(1+\gamma)} Mf(x)^{1-\theta(1+\gamma)}.$$

Hence, we obtain the desired estimate. □
We also need the following Proposition 5.2.

**Proposition 5.2** (A special case of [4, Theorem A]) *Let* $1 < p < q < +\infty$, $0 < \alpha < 1$. *For* $\phi(t)$ *assume that there exist positive constants* $C_9$, $C_{10}$, $C_{11}$, *such that*

$$\frac{1}{C_9} \le \frac{\phi(s)}{\phi(t)} \le C_9 \quad \textit{for} \quad \frac{1}{2} \le \frac{s}{t} \le 2, \tag{5.5}$$

$$\int_t^{+\infty} \frac{ds}{s\phi(s)^p} \le \frac{C_{10}}{\phi(t)^p}, \tag{5.6}$$

$$\int_t^{+\infty} \frac{s^{n\alpha-1}}{\phi(s)} ds \le C_{11} t^{n\alpha/(1-q/p)}, \tag{5.7}$$

*for every* $t > 0$. *Then there exists a positive constant* $C$ *such that*

$$\|I_\alpha f\|_{q,\phi^{p/q}} \le C\|f\|_{p,\phi}, \tag{5.8}$$

*where* $f \in L_p^\phi$.

Now we are ready to give
*Proof of Theorem 3.9.* By Lemma 5.1, we let

$$\begin{aligned}|T_\rho f(x)| &\le C\left(I_{\alpha/\theta}(|f|)(x)^\theta Mf(x)^{1-\theta}\right. \\ &\qquad \left.+ I_{\alpha/\theta}(|f|)(x)^{\theta(1+\gamma)} Mf(x)^{1-\theta(1+\gamma)}\right) \\ &= C(J_3(x) + J_4(x)).\end{aligned}$$

By Proposition 5.2, we have

$$\|I_{\alpha/\theta}(|f|)\|_{u,\phi_2^{p_i/u}} \le C\|f\|_{p_i,\phi_2}, \tag{5.9}$$

where $i = 2, 3$. By the assumption, we have

$$\frac{1}{q_1/r} + \frac{1}{u/(r\theta)} + \frac{1}{p_1/\{r(1-\theta)\}} = 1.$$

By Hölder's inequality, we have

$$\begin{aligned}
&\psi(t)\left(\frac{1}{|B|}\int_B |g(y)J_3(y)|^r\,dy\right)^{1/r} \\
&\le C\psi(t)\left(\frac{1}{|B|}\int_B |g(y)|^r|I_{\alpha/\theta}(|f|)(y)|^{r\theta}|Mf(y)|^{r(1-\theta)}\,dy\right)^{1/r} \\
&\le C\eta_1(t)\left(\frac{1}{|B|}\int_B |g(y)|^{q_1}\,dy\right)^{1/q_1} \\
&\quad\cdot\left\{\sum_{i=2,3}\left(\phi_2(t)^{p_i/u}\right)^{\theta}\left(\frac{1}{|B|}\int_B |I_{\alpha/\theta}(|f|)(y)|^u\,dy\right)^{\theta/u}\right\} \\
&\quad\cdot\phi_1(t)^{1-\theta}\left(\frac{1}{|B|}\int_B |Mf(y)|^{p_1}dy\right)^{(1-\theta)/p_1} \\
&\le C\|g\|_{q_1,\eta_1}\left(\sum_{i=2,3}\|f\|_{p_i,\phi_2}^{\theta}\right)\|f\|_{p_1,\phi_1}^{1-\theta} \\
&\le C\|g\|_{q_1,\eta_1}\|f\|_{p_2,\phi_2}^{\theta}\|f\|_{p_1,\phi_1}^{1-\theta}.
\end{aligned}$$

This implies

$$\|gJ_3\|_{r,\psi} \le C\|g\|_{q_1,\eta_1}\|f\|_{p_2,\phi_2}^{\theta}\|f\|_{p_1,\phi_1}^{1-\theta}.$$

In a similar way we have

$$\|gJ_4\|_{r,\psi} \le C\|g\|_{q_2,\eta_2}\|f\|_{p_2,\phi_2}^{\theta(1+\gamma)}\|f\|_{p_1,\phi_1}^{1-\theta(1+\gamma)}.$$

This completes the proof of Theorem 3.9. □

**Remark 5.3** If we use Lemma 5.1, Hölder's inequality, and [4, Theorem B], we can obtain another inequality concerning Theorem 3.9. However, it is a generalization of Proposition 2.6.

**6 Proof of Theorem 3.16** In this section, we prove Theorem 3.16. To prove Theorem 3.16, we use a pointwise estimate the following Lemma 6.1 states.

**Lemma 6.1** *Let* $0 < \epsilon_2 \le \alpha < \alpha + \epsilon_1 < 1$, $\alpha\gamma < \epsilon_1$. *Under the assumption* (3.4), *we have*

$$\begin{aligned}
|T_\rho f(x)| \le C\Big(&M_{\alpha-\epsilon_2}f(x)^{\epsilon_1/(\epsilon_1+\epsilon_2)}M_{\alpha+\epsilon_1}f(x)^{\epsilon_2/(\epsilon_1+\epsilon_2)} \\
&+M_{\alpha-\epsilon_2}f(x)^{(\epsilon_1-\alpha\gamma)/(\epsilon_1+\epsilon_2)}M_{\alpha+\epsilon_1}f(x)^{(\epsilon_2+\alpha\gamma)/(\epsilon_1+\epsilon_2)}\Big).
\end{aligned} \tag{6.1}$$

Lemma 6.1 extends Lemma 5.1. In Lemma 6.1, the case $\gamma = 0$ and $\epsilon_1 = \epsilon_2$ was shown by Welland ([26, p.146]).

*Proof of Lemma 6.1.* We follow the argument in [26]. We use the statement (4.5). For $I_1$, by using (3.4), we let

$$|I_1| \le C\left(\sum_{j=-\infty}^{-1}\int_{2^j\sigma\le|x-y|<2^{j+1}\sigma}\frac{|f(y)|}{|x-y|^{n-n\alpha}}\,dy + \sum_{j=-\infty}^{-1}\int_{2^j\sigma\le|x-y|<2^{j+1}\sigma}\frac{|f(y)|}{|x-y|^{n-n\alpha-n\alpha\gamma}}\,dy\right)$$
$$= C(I_9+I_{10}). \tag{6.2}$$

For $I_9$, we have

$$I_9 \le \sum_{j=-\infty}^{-1}\frac{1}{(2^j\sigma)^{n-n\alpha}}\int_{|x-y|<2^{j+1}\sigma}|f(y)|\,dy$$
$$= \sum_{j=-\infty}^{-1}(2^j\sigma)^{n\epsilon_2}\frac{1}{(2^j\sigma)^{n-n\alpha+n\epsilon_2}}\int_{|x-y|<2^{j+1}\sigma}|f(y)|\,dy$$
$$\le C\sigma^{n\epsilon_2}M_{\alpha-\epsilon_2}f(x). \tag{6.3}$$

Similarly, for $I_{10}$, we have

$$I_{10} \le \sum_{j=-\infty}^{-1}(2^j\sigma)^{n\epsilon_2+n\alpha\gamma}\frac{1}{(2^j\sigma)^{n-n\alpha+n\epsilon_2}}\int_{|x-y|<2^{j+1}\sigma}|f(y)|\,dy$$
$$\le C\sigma^{n\epsilon_2+n\alpha\gamma}M_{\alpha-\epsilon_2}f(x). \tag{6.4}$$

Combining (6.3) with (6.4), we have

$$|I_1| \le C(\sigma^{n\epsilon_2}+\sigma^{n\epsilon_2+n\alpha\gamma})M_{\alpha-\epsilon_2}f(x).$$

For $I_2$, we use the statement (4.7). Then we have

$$I_3 \le \sum_{j=0}^{+\infty}\int_{2^j\sigma\le|x-y|<2^{j+1}\sigma}\frac{|f(y)|}{|x-y|^{n-n\alpha}}\,dy$$
$$\le \sum_{j=0}^{+\infty}(2^j\sigma)^{-n\epsilon_1}\frac{1}{(2^j\sigma)^{n-n\alpha-n\epsilon_1}}\int_{|x-y|<2^{j+1}\sigma}|f(y)|\,dy$$
$$\le C\sigma^{-n\epsilon_1}M_{\alpha+\epsilon_1}f(x).$$

Similarly,

$$I_4 \le \sum_{j=0}^{+\infty}(2^j\sigma)^{-n\epsilon_1+n\alpha\gamma}\frac{1}{(2^j\sigma)^{n-n\alpha-n\epsilon_1}}\int_{|x-y|<2^{j+1}\sigma}|f(y)|\,dy$$
$$\le C\sigma^{-n\epsilon_1+n\alpha\gamma}M_{\alpha+\epsilon_1}f(x).$$

Hence, it follows that

$$|T_\rho f(x)| \le C\left\{\left(\sigma^{n\epsilon_2}+\sigma^{n\epsilon_2+n\alpha\gamma}\right)M_{\alpha-\epsilon_2}f(x) + \left(\sigma^{-n\epsilon_1}+\sigma^{-n\epsilon_1+n\alpha\gamma}\right)M_{\alpha+\epsilon_1}f(x)\right\}.$$

For the case $M_{\alpha+\epsilon_1}f(x)/M_{\alpha-\epsilon_2}f(x) \le 1$, we choose

$$\sigma = (M_{\alpha+\epsilon_1}f(x)/M_{\alpha-\epsilon_2}f(x))^{1/\{n(\epsilon_1+\epsilon_2)\}}. \tag{6.5}$$

Then it follows that

$$|T_\rho f(x)| \le CM_{\alpha-\epsilon_2}f(x)^{\epsilon_1/(\epsilon_1+\epsilon_2)}M_{\alpha+\epsilon_1}f(x)^{\epsilon_2/(\epsilon_1+\epsilon_2)}.$$

For the case $M_{\alpha+\epsilon_1}f(x)/M_{\alpha-\epsilon_2}f(x) \ge 1$, we also choose $\sigma$ as (6.5). Then it follows that

$$|T_\rho f(x)| \le CM_{\alpha-\epsilon_2}f(x)^{(\epsilon_1-\alpha\gamma)/(\epsilon_1+\epsilon_2)}M_{\alpha+\epsilon_1}f(x)^{(\epsilon_2+\alpha\gamma)/(\epsilon_1+\epsilon_2)}.$$

Hence, we obtain the desired estimate. □

Now we are ready to give

*Proof of Theorem 3.16.* By Lemma 6.1, we let

$$\begin{aligned}|T_\rho f(x)| &\le C\Big(M_{\alpha-\epsilon_2}f(x)^{\epsilon_1/(\epsilon_1+\epsilon_2)}M_{\alpha+\epsilon_1}f(x)^{\epsilon_2/(\epsilon_1+\epsilon_2)}\\ &\qquad +M_{\alpha-\epsilon_2}f(x)^{(\epsilon_1-\alpha\gamma)/(\epsilon_1+\epsilon_2)}M_{\alpha+\epsilon_1}f(x)^{(\epsilon_2+\alpha\gamma)/(\epsilon_1+\epsilon_2)}\Big).\\ &= C(J_5(x)+J_6(x)).\end{aligned}$$

Since $M.f(x) \le CI.f(x)$, by Proposition 5.2 we have

$$\|M_{\alpha+\epsilon_1}f\|_{u_1,\phi_1^{p_i/u_1}} \le C\|f\|_{p_i,\phi_1}, \tag{6.6}$$

$$\|M_{\alpha-\epsilon_2}f\|_{u_2,\phi_2^{p_{i+1}/u_2}} \le C\|f\|_{p_{i+1},\phi_2}, \tag{6.7}$$

where $i = 1$, 3. By the assumption, we have

$$\frac{1}{q_1/r}+\frac{1}{u_2(\epsilon_1+\epsilon_2)/(\epsilon_1 r)}+\frac{1}{u_1(\epsilon_1+\epsilon_2)/(\epsilon_2 r)} = 1.$$

By Hölder's inequality, we have

$$\begin{aligned}&\psi(t)\left(\frac{1}{|B|}\int_B |g(y)J_5(y)|^r\,dy\right)^{1/r}\\ &\le C\psi(t)\left(\frac{1}{|B|}\int_B |g(y)|^r|M_{\alpha-\epsilon_2}f(y)|^{\epsilon_1 r/(\epsilon_1+\epsilon_2)}\right.\\ &\qquad \left.\cdot|M_{\alpha+\epsilon_1}f(y)|^{\epsilon_2 r/(\epsilon_1+\epsilon_2)}dy\right)^{1/r}\\ &\le C\eta_1(t)\left(\frac{1}{|B|}\int_B |g(y)|^{q_1}\,dy\right)^{1/q_1}\\ &\qquad \cdot\left(\sum_{i=2,4}\phi_2(t)^{p_i/u_2}\right)^{\epsilon_1/(\epsilon_1+\epsilon_2)}\left(\frac{1}{|B|}\int_B |M_{\alpha-\epsilon_2}f(y)|^{u_2}\,dy\right)^{\epsilon_1/\{u_2(\epsilon_1+\epsilon_2)\}}\\ &\qquad \cdot\left(\sum_{i=1,3}\phi_1(t)^{p_i/u_1}\right)^{\epsilon_2/(\epsilon_1+\epsilon_2)}\left(\frac{1}{|B|}\int_B |M_{\alpha+\epsilon_1}f(y)|^{u_1}\,dy\right)^{\epsilon_2/\{u_1(\epsilon_1+\epsilon_2)\}}\\ &\le C\|g\|_{q_1,\eta_1}\left(\sum_{i=2,4}\|f\|_{p_i,\phi_2}^{\epsilon_1/(\epsilon_1+\epsilon_2)}\right)\left(\sum_{i=1,3}\|f\|_{p_i,\phi_1}^{\epsilon_2/(\epsilon_1+\epsilon_2)}\right)\\ &\le C\|g\|_{q_1,\eta_1}\|f\|_{p_2,\phi_2}^{\epsilon_1/(\epsilon_1+\epsilon_2)}\|f\|_{p_1,\phi_1}^{\epsilon_2/(\epsilon_1+\epsilon_2)}.\end{aligned}$$

This implies

$$\|gJ_5\|_{r,\psi} \le C\|g\|_{q_1,\eta_1}\|f\|_{p_2,\phi_2}^{\epsilon_1/(\epsilon_1+\epsilon_2)}\|f\|_{p_1,\phi_1}^{\epsilon_2/(\epsilon_1+\epsilon_2)}.$$

In a similar way we have

$$\|gJ_6\|_{r,\psi} \le C\|g\|_{q_2,\eta_2}\|f\|_{p_2,\phi_2}^{(\epsilon_1-\alpha\gamma)/(\epsilon_1+\epsilon_2)}\|f\|_{p_1,\phi_1}^{(\epsilon_2+\alpha\gamma)/(\epsilon_1+\epsilon_2)}.$$

This completes the proof of Theorem 3.16. □

**7 Comparisons** In this section, we compare Corollaries 3.3, 3.12, and 3.17. In the following Comparisons, we let $1 < p \le p_0 < +\infty$, $1 < q \le q_0 < +\infty$, $1 < s \le s_0 < +\infty$, $0 < \epsilon_2 \le \alpha < \alpha + \epsilon_1 < 1/p_0$, $s_0\alpha < \theta < 1$, and $\alpha - \epsilon_2 < 1/s_0$. We also assume $1 < r_k \le R_k < +\infty$ for $k = 1, 2, 3$.

**Comparison 7.1** We compare Corollary 3.3 with Corollary 3.12. If we use Corollary 3.3 for $f \in M_p^{p_0} \cap M_s^{s_0}$ and $g \in M_q^{q_0}$, we obtain $g \cdot I_\alpha f \in M_{r_1}^{R_1}$, where $R_1$ and $r_1$ satisfy

$$\frac{1}{R_1} = \frac{1}{q_0} + \frac{1}{s_0}(1 - p_0\alpha), \quad \frac{1}{r_1} = \frac{1}{q} + \frac{1}{s}(1 - p_0\alpha). \tag{7.1}$$

If we use Corollary 3.12 for $f \in M_p^{p_0} \cap M_s^{s_0}$ and $g \in M_q^{q_0}$, we obtain $g \cdot I_\alpha f \in M_{r_2}^{R_2}$, where $R_2$ and $r_2$ satisfy

$$\frac{1}{R_2} = \frac{1}{q_0} + \frac{\theta}{s_0} + \frac{1-\theta}{p_0} - \alpha, \quad \frac{1}{r_2} = \frac{1}{q} + \frac{\theta}{s} - \frac{s_0}{s}\alpha + \frac{1-\theta}{p}. \tag{7.2}$$

Then we have

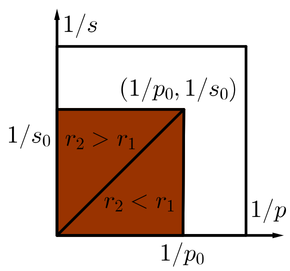
$$\begin{aligned}\frac{1}{R_2} - \frac{1}{R_1} &= \frac{1-\theta}{p_0} - \frac{1-\theta}{s_0} + \left(\frac{p_0}{s_0} - 1\right)\alpha \\ &= (1-\theta)\frac{s_0 - p_0}{p_0 s_0} - \frac{s_0 - p_0}{s_0}\alpha \\ &= \frac{s_0 - p_0}{s_0}\left(\frac{1-\theta}{p_0} - \alpha\right).\end{aligned}$$

If $s_0 = p_0$ or $\theta = 1 - p_0\alpha$ then $R_1 = R_2$. First we consider the case $s_0 = p_0$. In this case

$$\frac{1}{r_2} - \frac{1}{r_1} = \frac{1-\theta}{p} - \frac{1-\theta}{s} = (1-\theta)\left(\frac{1}{p} - \frac{1}{s}\right).$$

If $p < s$ then $r_2 < r_1$ and if $p > s$ then $r_2 > r_1$.

Case $s_0 = p_0$

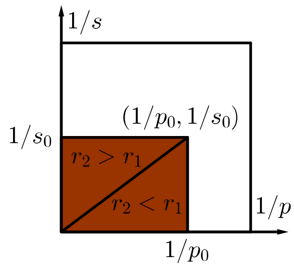


Next we consider the case $\theta = 1 - p_0\alpha$. In this case

$$\frac{1}{r_2} - \frac{1}{r_1} = \alpha\left(\frac{p_0}{p} - \frac{s_0}{s}\right).$$

If $p_0/p > s_0/s$ then $r_2 < r_1$ and if $p_0/p < s_0/s$ then $r_2 > r_1$.

Case $\theta = 1 - p_0\alpha$




**Comparison 7.2** We compare Corollary 3.3 with Corollary 3.17. If we use Corollary 3.3 for $f \in M_p^{p_0} \cap M_s^{s_0}$ and $g \in M_q^{q_0}$, we obtain $g \cdot I_\alpha f \in M_{r_1}^{R_1}$, where $R_1$ and $r_1$ satisfy (7.1). If we use Corollary 3.17 for $f \in M_p^{p_0} \cap M_s^{s_0}$ and $g \in M_q^{q_0}$, we obtain $g \cdot I_\alpha f \in M_{r_3}^{R_3}$, where $R_3$ and $r_3$ satisfy

$$\frac{1}{R_3} = \frac{1}{q_0} + \frac{\epsilon_1}{\epsilon_1+\epsilon_2}\cdot\frac{1}{s_0} + \frac{\epsilon_2}{\epsilon_1+\epsilon_2}\cdot\frac{1}{p_0} - \alpha \tag{7.3}$$

and

$$\frac{1}{r_3} = \frac{1}{q} + \frac{\epsilon_1}{\epsilon_1+\epsilon_2}\left\{\frac{1}{s} - \frac{s_0}{s}(\alpha-\epsilon_2)\right\} + \frac{\epsilon_2}{\epsilon_1+\epsilon_2}\left\{\frac{1}{p} - \frac{p_0}{p}(\alpha+\epsilon_1)\right\} \tag{7.4}$$

respectively. Then we have

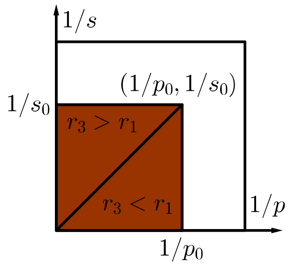
$$\begin{aligned}\frac{1}{R_3} - \frac{1}{R_1} &= \frac{\epsilon_2}{\epsilon_1+\epsilon_2}\left(\frac{1}{p_0} - \frac{1}{s_0}\right) - \left(1 - \frac{p_0}{s_0}\right)\alpha \\ &= \frac{\epsilon_2}{\epsilon_1+\epsilon_2}\cdot\frac{s_0-p_0}{p_0 s_0} - \frac{s_0-p_0}{s_0}\alpha \\ &= \frac{s_0-p_0}{s_0}\left(\frac{\epsilon_2}{\epsilon_1+\epsilon_2}\cdot\frac{1}{p_0} - \alpha\right).\end{aligned}$$

If $s_0 = p_0$ or $p_0\alpha = \epsilon_2/(\epsilon_1+\epsilon_2)$ then $R_1 = R_3$. First we consider the case $s_0 = p_0$. In this case

$$\begin{aligned}\frac{1}{r_3} - \frac{1}{r_1} &= \frac{\epsilon_2}{\epsilon_1+\epsilon_2}\left\{\frac{\epsilon_1}{\epsilon_2 s} - \frac{\epsilon_1 p_0}{\epsilon_2 s}(\alpha-\epsilon_2) - \frac{\epsilon_1+\epsilon_2}{\epsilon_2 s} + \frac{\epsilon_1+\epsilon_2}{\epsilon_2 s}p_0\alpha\right\} \\ &\quad + \frac{\epsilon_2}{\epsilon_1+\epsilon_2}\left\{\frac{1}{p} - \frac{p_0}{p}(\alpha+\epsilon_1)\right\} \\ &= \frac{\epsilon_2}{\epsilon_1+\epsilon_2}\cdot\frac{1}{s}\{p_0(\alpha+\epsilon_1)-1\} + \frac{\epsilon_2}{\epsilon_1+\epsilon_2}\cdot\frac{1}{p}\{1-p_0(\alpha+\epsilon_1)\} \\ &= \frac{\epsilon_2}{\epsilon_1+\epsilon_2}\{1-p_0(\alpha+\epsilon_1)\}\left(\frac{1}{p} - \frac{1}{s}\right).\end{aligned}$$

Since $1 - p_0(\alpha+\epsilon_1) > 0$, if $p < s$ then $r_3 < r_1$ and if $p > s$ then $r_3 > r_1$.

Case $s_0 = p_0$

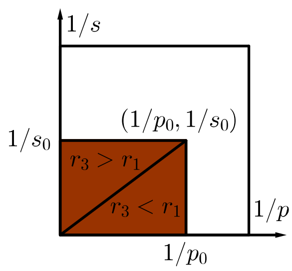



Next we consider the case $p_0\alpha = \epsilon_2/(\epsilon_1+\epsilon_2)$. In this case

$$\begin{aligned}
&\frac{1}{r_3}-\frac{1}{r_1}\\
&=\frac{\epsilon_2}{\epsilon_1+\epsilon_2}\left(\frac{\epsilon_1}{\epsilon_2 s}-\frac{s_0}{s}\cdot\frac{\epsilon_1}{\epsilon_1+\epsilon_2}\cdot\frac{1}{p_0}+\frac{s_0}{s}\epsilon_1-\frac{1}{s}\cdot\frac{\epsilon_1+\epsilon_2}{\epsilon_2}+\frac{1}{s}\right)\\
&\quad+\frac{\epsilon_2}{\epsilon_1+\epsilon_2}\left(\frac{1}{p}-\frac{1}{p}\cdot\frac{\epsilon_2}{\epsilon_1+\epsilon_2}-\frac{p_0}{p}\epsilon_1\right)\\
&=\frac{\epsilon_2}{\epsilon_1+\epsilon_2}\left\{-\frac{s_0}{s}\cdot\frac{\epsilon_1}{\epsilon_1+\epsilon_2}\cdot\frac{1}{p_0}+\left(1-\frac{\epsilon_2}{\epsilon_1+\epsilon_2}\right)\frac{1}{p}\right.\\
&\quad\left.+\left(\frac{s_0}{s}-\frac{p_0}{p}\right)\epsilon_1\right\}\\
&=\frac{\epsilon_2}{\epsilon_1+\epsilon_2}\left\{\left(\frac{p_0}{p}-\frac{s_0}{s}\right)\cdot\frac{1}{p_0}\cdot\frac{\epsilon_1}{\epsilon_1+\epsilon_2}-\left(\frac{p_0}{p}-\frac{s_0}{s}\right)\epsilon_1\right\}\\
&=\frac{\epsilon_1\epsilon_2}{\epsilon_1+\epsilon_2}\left(\frac{p_0}{p}-\frac{s_0}{s}\right)\left\{\frac{1}{p_0(\epsilon_1+\epsilon_2)}-1\right\}.
\end{aligned}$$

Since $\epsilon_1+\epsilon_2 \le \alpha+\epsilon_1 < 1/p_0$, we have $1/\{p_0(\epsilon_1+\epsilon_2)\}-1>0$. Hence if $p_0/p > s_0/s$ then $r_3 < r_1$ and if $p_0/p < s_0/s$ then $r_3 > r_1$.

Case $p_0\alpha = \epsilon_2/(\epsilon_1+\epsilon_2)$




**Comparison 7.3** We compare Corollary 3.12 with Corollary 3.17. If we use Corollary 3.12 for $f \in M_p^{p_0} \cap M_s^{s_0}$ and $g \in M_q^{q_0}$, we obtain $g \cdot I_\alpha f \in M_{r_2}^{R_2}$, where $R_2$ and $r_2$ satisfy (7.2). If we use Corollary 3.17 for $f \in M_p^{p_0} \cap M_s^{s_0}$ and $g \in M_q^{q_0}$, we obtain $g \cdot I_\alpha f \in M_{r_3}^{R_3}$, where

$R_3$ and $r_3$ satisfy (7.3) and (7.4) respectively. Then we have

$$\begin{aligned}\frac{1}{R_3}-\frac{1}{R_2}&=\frac{1}{p_0}\left(\theta+\frac{\epsilon_2}{\epsilon_1+\epsilon_2}-1\right)-\frac{1}{s_0}\left(\theta-\frac{\epsilon_1}{\epsilon_1+\epsilon_2}\right)\\&=\frac{s_0-p_0}{p_0 s_0}\left(\theta-\frac{\epsilon_1}{\epsilon_1+\epsilon_2}\right).\end{aligned}$$

If $s_0=p_0$ or $\theta=\epsilon_1/(\epsilon_1+\epsilon_2)$ then $R_2=R_3$. First we consider the case $s_0=p_0$. In this case

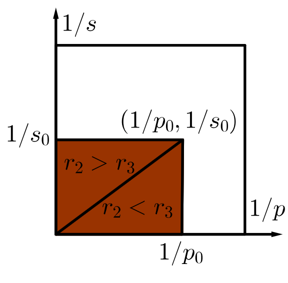
$$\begin{aligned}\frac{1}{r_3}-\frac{1}{r_2}&=\frac{\epsilon_1}{\epsilon_1+\epsilon_2}\left(\frac{\epsilon_2}{\epsilon_1 p}-\frac{\epsilon_2 p_0}{\epsilon_1 p}\alpha-\frac{p_0}{p}\epsilon_2-\frac{\epsilon_1+\epsilon_2}{\epsilon_1 p}+\frac{\epsilon_1+\epsilon_2}{\epsilon_1 p}\theta\right)\\&\quad+\frac{\epsilon_1}{\epsilon_1+\epsilon_2}\left(\frac{1}{s}-\frac{p_0}{s}\alpha+\frac{p_0}{s}\epsilon_2-\frac{\epsilon_1+\epsilon_2}{\epsilon_1 s}\theta+\frac{\epsilon_1+\epsilon_2}{\epsilon_1 s}p_0\alpha\right)\\&=\frac{\epsilon_1}{\epsilon_1+\epsilon_2}\cdot\frac{1}{p}\left\{\frac{\epsilon_1+\epsilon_2}{\epsilon_1}\theta-1-\frac{\epsilon_2}{\epsilon_1}p_0(\alpha+\epsilon_1)\right\}\\&\quad-\frac{\epsilon_1}{\epsilon_1+\epsilon_2}\cdot\frac{1}{s}\left\{\frac{\epsilon_1+\epsilon_2}{\epsilon_1}\theta-1-\frac{\epsilon_2}{\epsilon_1}p_0(\alpha+\epsilon_1)\right\}\\&=\frac{\epsilon_1}{\epsilon_1+\epsilon_2}\left\{\frac{\epsilon_1+\epsilon_2}{\epsilon_1}\theta-1-\frac{\epsilon_2}{\epsilon_1}p_0(\alpha+\epsilon_1)\right\}\left(\frac{1}{p}-\frac{1}{s}\right).\end{aligned}$$

If "$\theta>\{\epsilon_1+\epsilon_2 p_0(\alpha+\epsilon_1)\}/(\epsilon_1+\epsilon_2)$ and $p<s$" or "$\theta<\{\epsilon_1+\epsilon_2 p_0(\alpha+\epsilon_1)\}/(\epsilon_1+\epsilon_2)$ and $p>s$" then $r_3<r_2$. And if "$\theta>\{\epsilon_1+\epsilon_2 p_0(\alpha+\epsilon_1)\}/(\epsilon_1+\epsilon_2)$ and $p>s$" or "$\theta<\{\epsilon_1+\epsilon_2 p_0(\alpha+\epsilon_1)\}/(\epsilon_1+\epsilon_2)$ and $p<s$" then $r_3>r_2$. Next we consider the case $\theta=\epsilon_1/(\epsilon_1+\epsilon_2)$. In this case

$$\begin{aligned}\frac{1}{r_2}-\frac{1}{r_3}&=\frac{\epsilon_2}{\epsilon_1+\epsilon_2}\left(-\frac{\epsilon_1+\epsilon_2}{\epsilon_2}\cdot\frac{s_0}{s}\alpha+\frac{\epsilon_1 s_0}{\epsilon_2 s}\alpha-\frac{s_0}{s}\epsilon_1\right)\\&\quad+\frac{\epsilon_2}{\epsilon_1+\epsilon_2}\left(\frac{p_0}{p}\alpha+\frac{p_0}{p}\epsilon_1\right)\\&=\frac{\epsilon_2}{\epsilon_1+\epsilon_2}\left\{\left(\frac{p_0}{p}-\frac{s_0}{s}\right)\alpha+\left(\frac{p_0}{p}-\frac{s_0}{s}\right)\epsilon_1\right\}\\&=\frac{\epsilon_2}{\epsilon_1+\epsilon_2}\left(\frac{p_0}{p}-\frac{s_0}{s}\right)(\alpha+\epsilon_1).\end{aligned}$$

If $p_0/p>s_0/s$ then $r_2<r_3$ and if $p_0/p<s_0/s$ then $r_2>r_3$.

Case $\theta=\epsilon_1/(\epsilon_1+\epsilon_2)$




**Remark 7.4** We rename Corollaries 3.3, 3.12, and 3.17 Corollaries 1, 2, and 3 respectively. Our conclusion of Comparisons 7.1 through 7.3 is as follows: Let $k=1, 2, 3$. If we use

Corollary $k$ for $f \in M_p^{p_0} \cap M_s^{s_0}$ and $g \in M_q^{q_0}$, we obtain $g \cdot I_\alpha f \in M_{r_k}^{R_k}$. We consider the case $s_0 \neq p_0$ and $\epsilon_1 = \epsilon_2$. If $p_0\alpha = \theta = 1/2$ then $R_1 = R_2 = R_3$. In this case if $p_0/p > s_0/s$ then $r_2 < r_3 < r_1$ and if $p_0/p < s_0/s$ then $r_2 > r_3 > r_1$.

**8 Counterexamples** In this section we describe two counterexamples concerning Corollaries 3.3, 3.12, and 3.17 which were suggested by Nakai.

**Proposition 8.1** ([16]) *Let $1 < p < s < t < +\infty$, $0 < \theta < 1$. Then there exist sequences of functions $\{f_j\}, \{g_j\} \subset M_p^s(\mathbf{R}) \cap M_p^t(\mathbf{R})$ such that*

$$\lim_{j\to+\infty} \|f_j\|_{M_p^t} = +\infty \quad \textit{and} \quad \sup_j \|f_j\|_{M_p^s}^{\theta} \|f_j\|_{M_p^t}^{1-\theta} < +\infty, \tag{8.1}$$

*and*

$$\lim_{j\to+\infty} \|g_j\|_{M_p^s} = +\infty \quad \textit{and} \quad \sup_j \|g_j\|_{M_p^s}^{\theta} \|g_j\|_{M_p^t}^{1-\theta} < +\infty. \tag{8.2}$$

*Proof.* Let $a > 0$, $b > 0$, $h(x) = b\chi_{[0,a]}(x)$. Then

$$\|h\|_{M_p^s} = a^{1/s} b, \quad \|h\|_{M_p^t} = a^{1/t} b.$$

We choose $b$ so that

$$\|h\|_{M_p^s}^{\theta} \|h\|_{M_p^t}^{1-\theta} = (a^{1/s} b)^{\theta} (a^{1/t} b)^{1-\theta} = 1.$$

Then $b = a^{-\theta/s-(1-\theta)/t}$. In this case

$$\lim_{a\to 0} a^{1/t} b = \lim_{a\to 0} a^{(1/t-1/s)\theta} = +\infty$$

and

$$\lim_{a\to+\infty} a^{1/s} b = \lim_{a\to+\infty} a^{(1/s-1/t)(1-\theta)} = +\infty. \quad \square$$

**Proposition 8.2** ([16]) *Let $1 < p < q < s < +\infty$, $0 < \theta < 1$. Then there exists a sequence of functions $\{f_j\} \subset M_q^s(\mathbf{R}) \subset M_p^s(\mathbf{R})$ such that*

$$\lim_{j\to+\infty} \|f_j\|_{M_q^s} = +\infty \quad \textit{and} \quad \sup_j \|f_j\|_{M_p^s}^{\theta} \|f_j\|_{M_q^s}^{1-\theta} < +\infty. \tag{8.3}$$

*Proof.* There exist a positive constant $C_0$, a ball $B_0$, and a sequence of functions $\{h_j\} \subset L^\infty(\mathbf{R})$ such that

$$\sup_j \|h_j\|_{M_p^s} \leq C_0, \quad \operatorname{supp} h_j \subset B_0, \quad \text{and} \quad \lim_{j\to+\infty} \|h_j\|_{L^q} = +\infty.$$

(See [15, Lemma 4.10] and its proof.) Let $c_j = \|h_j\|_{M_q^s}$. Then $c_j \to +\infty$ as $j \to +\infty$. Let $f_j = c_j^{-(1-\theta)} h_j$. Then

$$\|f_j\|_{M_p^s}^{\theta} \|f_j\|_{M_q^s}^{1-\theta} = c_j^{-(1-\theta)} \|h_j\|_{M_p^s}^{\theta} \|h_j\|_{M_q^s}^{1-\theta} \leq C_0^{\theta}$$

and

$$\lim_{j\to+\infty} \|f_j\|_{M_q^s} = \lim_{j\to+\infty} c_j^{\theta} = +\infty. \quad \square$$

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# A NONLINEAR MODEL OF A SEARCH ALLOCATION GAME WITH FALSE CONTACTS

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Abstract. This paper deals with a two-person zero-sum search game called *search allocation game* (SAG) with a searcher and a target as players, taking account of false contacts. The searcher distributes his searching resource in a search space to detect the target and the target moves to evade the searcher. The searcher obtains a profit of target value on detection of target but expends cost for the search. The payoff of the game is the expected reward defined by obtained target value minus expended searching cost. The searcher's strategy is denoted by a distribution plan about where and when he distributes his searching resource and the target strategy is the selection of a path to follow from some options. In the search operation, any sensor cannot get rid of false contacts caused by signal processing noises and real objects similar to the true target under noisy environment. On their happening, they make the searcher waste some time for investigation and interrupt the search operation for a while. There have been few researches dealing with the SAG with the false contacts. In this paper, we model the game with false contacts by a stochastic process and discuss a general procedure to derive an equilibrium point through a nonlinear programming method for a searcher's best response to the target's behavior.

**1 Introduction** This paper deals with a search game called *search allocation game* (SAG) [5, 7], where a searcher distributes his searching resource in a search space expecting the detection of a target and the target moves to evade the searcher. This game is categorized in the so-called *search-and-evasion game* (SEG), where searchers and moving targets compete each other. In the search operation, any sensor cannot get rid of false contacts caused by signal processing noises and real objects similar to the true target under noisy environment. In this paper, we analyze optimal strategies of the searcher and the target, taking account of the occurrence of the false contacts.

We can see the original of the SEG in datum search game. An exposed position of targets is referred to as datum point and the datum is a generic of information about the target including the datum point. We call the search operation kicked off by the datum the datum search. In an early research of search theory, entitled "Search and Screening" [21], Koopman discussed an optimal datum search against a submarine moving in a randomized direction from a datum point on a plane.

Meinardi [22] dealt with a datum search game but focused on the diffusive motion of a target on a one-dimensional line and tried to obtain the target motion such that the distribution probability of the target is as uniformly as possible on the line. Most modelings of the datum search were taken in military operations such as anti-submarine warfare (ASW) of a submarine vs. ASW airplanes. Danskin [2] showed us optimal strategies of players in the datum search game, where a submarine chooses a fixed speed and course at first and keeps them through the game, and an AWS airplane selects a point to dip his active sound buoys each time. There are some other models of the datum search game applied to the

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ASW operation, such as Baston and Bostock [1] and Garnaev [4]. In their models, a submarine hides on a one-dimensional space and an ASW airplane tries to maximize the probability of destroying the submarine by dropping several depth charges. Those authors implicitly assumed that the searcher can distribute his searching resource, e.g. sonars or depth charges, wherever he likes because the mobility of ASW vehicles is superior to that of submarines. The model of the SAG requires the assumption on the searcher's superior mobility.

If there is not such superiority between the moving capabilities of searchers and targets, we'd better consider the moving strategy for the searcher as well as the target. Washburn [26] discussed the moving strategies of the searcher and the target in a multi-stage game, which had the payoff of the total traveling cost of the searcher until the coincidence of players' positions. Kikuta [18] adopted searching cost as the payoff of the game with a moving searcher and a hiding target. Assuming that the positions of two moving players determined a temporary payoff each time and its total gave a comprehensive payoff of the game, Eagle and Washburn [3] investigated a single-stage game.

The research on the SAG started with stationary target models, where the target strategy is to choose the position of hiding himself. For the SAG with stationary targets, Nakai [23] and Iida et al. [15] studied several types of payoffs such as the detection probability of target or the reward of searcher. Hohzaki and Iida [16, 11, 13] extended the stationary target model to the SAG with moving targets. Hohzaki and Iida [12] generalized the model further and proposed a general numerical algorithm to derive an equilibrium of the SAG. Most of the previous researches assume comparatively simple target motions such as the selection of a path from a limited number of options or the limited mobility to neighbor places from its current position. Washburn and Hohzaki [27], Hohzaki et al. [14] and Hohzaki [7] considered energy constraints on target motion and a large number of options for target paths to tackle more practical SAG models. Hohzaki [9] discussed a multi-stage SAG and Hohzaki [10] first invented a cooperative SAG game with a coalition of multiple searchers against the target.

The past researches surveyed so far handled only the detection of true targets. In the search operation, however, there inevitably occur false contacts caused by environmental noise, signal processing noise or real objects other than true targets. They often interrupt the search operation and make the searcher waste some time for their investigation, by which the searcher figures out if it is what he is looking for. The searcher usually takes a two-phase operation consisting of broad search and then investigation for contact signals. There have been several papers on the false contact search model but most of them discussed a one-sided optimization problem for the searcher. Stone [25] solved an optimal distribution problem of searching resource to minimize the expected total amount of searching time plus investigation time until detecting true targets, assuming that objects exist in the search space and they bring about false contacts. But he did not take account of the two-phase operation and therefore his problem is not different from the true-target model in essence. Kisi [19] considered the problem with noise-type false contacts. In his model, contacts occur according to Poisson distribution. In order to maximize the detection probability of the true target, the searcher has to make decision about how much time he should spend for investigation each time the contact occurs. Iida et al. [17, 20] refined the Kisi's study but their basic approach is the same as the Kisi's. In the studies, their assumption of Poisson distribution with the stationary occurrence rate of contacts makes the problem easy to be solved because they can regard the search as a renewal process. As seen above, the previous researches with false contacts discussed only the one-sided problem for the searcher and never dealt with the game from two-sided point of view.

Hohzaki [6, 8] are the first researches that tackled the SAG model with false contacts.

He assumed that the searcher's strategy, i.e. the distribution of searching resource, affects the occurrence probability of false contacts such that more sonars or sound hydrophones would much easily pick up signals caused from false targets or noises. But, as the payoff of the game, he did not deal with an exact function of the detection probability of true targets but a linear function of resources accumulated on the target path. This paper discusses the SAG with false contacts and adopt the precise expression of the expected reward as payoff, which is a general criterion of the search operation and includes the detection probability as a special case, in the practical two-phase operation model with broad search and investigation process. In this paper, we aim to derive an equilibrium for the game and clarify the characteristics of optimal strategies in a competitive search operation between the searcher and the target.

In the next section, we describe some assumptions about a SAG with a searcher and a target, and model the detection of a true target and false contacts in a stochastic process. In Section 3, we define the payoff of the game by the expected reward caused by the target detection and the expenditure of searching cost, which would be given by a nonlinear function of decision variables of the searcher's strategy. In Section 4, we devise a computational algorithm to obtain an optimal distribution of searching resource by the searcher given the target strategy. As our final result, we propose a numerical algorithm to derive an equilibrium point for our SAG in Section 5. We take some numerical examples to investigate some properties of optimal strategies of players and do some sensitivity analyses in Section 6.

**2 Description of Model and Instance of Events** We consider with a two-person zero-sum search game with false contacts, which a search and a target play.

A1. A search space consists of a discrete cell space $\boldsymbol{K} = \{1, \cdots, K\}$ and a discrete time space $\boldsymbol{T} = \{1, \cdots, T\}$ and it is denoted by $\boldsymbol{K} \times \boldsymbol{T}$.

A2. A target chooses a path running through the search space to evade a searcher. An entire set of paths is denoted by $\Omega$. A path $\omega \in \Omega$ is a mapping from $\boldsymbol{T}$ to $\boldsymbol{K}$, namely $\omega : \boldsymbol{T} \to \boldsymbol{K}$, and is assumed to pass through cell $\omega(t)$ at time $t \in \boldsymbol{T}$.

A3. The searcher wants to detect the target by distributing searching resource after time point $\tau$, that is, during a time duration $\widehat{\boldsymbol{T}} \equiv \{\tau, \cdots, T\} \subseteq \boldsymbol{T}$. The searcher can use $\Phi(t)$ resources at each time point $t \in \widehat{\boldsymbol{T}}$. Let us denote the amount of resource to be distributed in cell $i$ at time $t$ by $\varphi(i,t)$. But the distribution costs $c_0(i,t) > 0$ per unit resource.

If the target is in cell $i \in \boldsymbol{K}$ at time $t \in \widehat{\boldsymbol{T}}$, the distribution of $\varphi(i,t)$ resources there brings the searcher detection probability

$$1 - \exp\left(-\alpha_i \varphi(i,t)\right), \tag{1}$$

where parameter $\alpha_i$ indicates the efficiency of unit resource in cell $i$ for detection.

A4. There could exist two types of contact events: the detection of true target and the contact of false target or noise after the search begins. Each event occurs once at most at each time independent of the other event and the occurrence probability of false contact is $Q_t$ at time $t$. If there is no detection event, the search continues at the next time. If a contact happens, the searcher starts an investigation phase for the contact, which requires $t_f - 1$ time points. During the investigation phase, the searcher has to stop the search. The searcher gets the faultless diagnosis about the contact at the earlier time of the completion time of the inspection or the last time $T$. If the true-target detection is diagnosed, he gets a target value and terminates the search game. If only the false contact occurs, the search operation resumes after the investigation phase.

A5. The searcher is given target value $V(t)$ of the time $t$ when the contact of the true target occurs, after getting its diagnosis. The target value is nonnegative and decreases as time elapses, namely, for any time $t = 1, \dots, T-1$,

$$V(t) \geq V(t+1) \geq 0. \tag{2}$$

A6. The game ends on detection of the true target or at the final time point $T$.

A7. The payoff of the game is the reward on the searcher's side. The reward is defined by the target value at the detection time minus the distribution cost expended until the detection or only the distribution cost in the case of no detection. The searcher wants to maximize the payoff and the target desires to minimize it.

In Assumption A3, we denote a searcher's strategy of resource distribution by $\varphi = \{\varphi(i,t), i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}}\}$. As assumed in A2, the target chooses a path $\omega$, which is his pure strategy. We take a mixed strategy for him, that is, the target selects $\omega \in \Omega$ with probability $\pi(\omega)$. The respective feasible regions of the searcher and the target strategies are as follows:

$$\Psi \equiv \left\{ \varphi \,\middle|\, \sum_{i \in K} \varphi(i,t) \leq \Phi(t),\ t \in \widehat{\boldsymbol{T}},\ \ \varphi(i,t) \geq 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}} \right\}, \tag{3}$$

$$\Pi \equiv \left\{ \pi \,\middle|\, \sum_{\omega \in \Omega} \pi(\omega) = 1,\ \pi(\omega) \geq 0,\ \omega \in \Omega \right\}. \tag{4}$$

Corresponding to all combinations of players' strategies, there could occur four types of events or states every time after the beginning of the search: (1) detection event including true target detection, (2) false contact, (3) no-detection and (4) investigation state. We denote the four events by 'D', 'F', 'S' and 'I', respectively. The four types of events occurs in probabilistic manner during a period $\widehat{\boldsymbol{T}}$ with $T - \tau + 1$ time points. We are interested in the detection of true target and therefore we consider the enumeration of a sequence of events exclusive to the true detection, namely, events not including symbol 'D'. From Assumption A4, we can make every instance by the following rule.

(1) Put 'S' or 'F' at an initial time $\tau$.

(2) As a general combination of false contact and investigation, we can think of a 'F' followed by $t_f - 1$ successive 'I's. As some special cases of the combination, the time reaches the last time $T$ in the middle of investigation. To represent the situation, we put a 'F' followed by $y$ 'I's for $0 \leq y \leq t_f - 2$.

(3) At time points not assigned to the above two types of sequences, We put 'S'.

Figure 1 shows an instance of the sequence of contacts without any 'D' in the case of $T = 9$ and $t_f = 4$. The last inspection phase is truncated.

During a time period $[\tau, L]$ with the final time $L$, a set of instances without true detection, $A_L$, has the following cases. The first is the case that time is up to the last time point $L$ in the middle of investigation process after a false contact. Let the repetition number of investigation be $y$. $y$ would be in an interval $0 \leq y \leq t_f - 2$. The number of false contacts, $M$, could be $\lfloor (L - y - \tau)/t_f \rfloor$ at most because the contacts occur from time $\tau$ until just t= 1 2 3 4 5 6 7 8 T

--S-F-I-I-I-S-S-F-I→

Figure 1: An instance of contacts sequence

before the last occurrence time $L-y-1$. Therefore, $M$ must be $0 \leq M \leq \lfloor (L-\tau-y)/t_f \rfloor$. There are $M$ pairs of false contact and a sequent investigation process, and the last false contact, namely, the $M+1$-th false contact, occurs followed by $y$ investigation time points until the last time. At other time points not assigned to $M+1$ groups of the false contact, which we count $L-\tau-Mt_f-y$, event 'S' would be assigned. We can specify an instance of those 'S's by $x_j$ $(j=1,\ldots,M+1)$, where $x_j$ is the number of successive 'S's just before the $j$-th false contact. The numbers $\{x_j\}$ have to satisfy the following condition:

$$\sum_{j=1}^{M+1} x_j = L-\tau-Mt_f-y. \tag{5}$$

The total number of feasible sets for $\{x_j\}$ is ${}_{L-\tau-M(t_f-1)-y}C_M$.

The second is the case that the completion of the last investigation process is not intercepted by the last time $L$. Every investigation process with $t_f$ time points is put correctly inside of a whole time period of $L-\tau+1$. The number of false contacts, $M$, must be $0 \leq M \leq \lfloor (L-\tau+1)/t_f \rfloor$. An instance of 'S' occurrence at $L-\tau-Mt_f+1$ time points is also specified by $\{x_j, j=1,\ldots,M\}$ satisfying

$$\sum_{j=1}^{M+1} x_j = L-\tau-Mt_f+1.$$

The total number of such specification is ${}_{L-\tau-M(t_f-1)+1}C_M$. The second case is realized by applying $y=-1$ to the first case. Now we can construct an algorithm to enumerate all instances $A_L$ without any true detection as follows:

(i) Change $y$ among $-1,0,1,\ldots,t_f-2$.

(ii) For each $y$, set $M$ to each of $0,1,\ldots,\lfloor (L-y-\tau)/t_f \rfloor$.

(iii) For a combination of $y$ and $M$, enumerate all vectors $(x_j, j=1,\cdots,M+1)$ so as to satisfy Eq. (5).

A combination of $y$, $M$ and $\{x_j\}$ gives us an instance. In the next section, we calculate the occurrence probability of each instance and derive the expected payoff of the game.

**3 Derivation of Payoff Function** In the end of Section 2, we explain an algorithm to enumerate an entire set of instances without any detection of true target. In an instance, we use symbols 'S', 'F' and 'I' to represent events. Here we replace $\{S,F,I\}$ with $\sigma(t) \in \{1,-1,0\}$, respectively, to formulate the payoff of the game. We can express any instance of a sequence of events by a vector $(\sigma(t), t \in \widehat{\boldsymbol{T}})$.

Let us assume that the target takes a path $\omega$ and the searcher distributes his searching resource by a plan $\varphi$. Because event S denoted $\sigma(t)=1$ indicates that there is no detection of true target and no false contact, its occurrence probability is $(1-Q_t)\exp(-\alpha_{\omega(t)}\varphi(\omega(t),t))$

from Assumption A3 and A4. Similarly, the event of $\sigma(t) = -1$ is a false contact without any true contact and has its occurrence probability $Q_t \exp(-\alpha_{\omega(t)}\varphi(\omega(t), t))$. In the investigation process denoted by $\sigma(t) = 0$, no event happens because of no execution of the search. From the discussion above, we have the following occurrence probability for each event $\sigma(t) \in \{1, -1, 0\}$:

$$\left(1 - \sigma(t)Q_t - \delta_{\sigma(t),-1}\right) \exp\left(-|\sigma(t)|\alpha_{\omega(t)}\varphi(\omega(t), t)\right),$$

where $\delta_{i,j}$ is the Kronecker's delta which is 1 for $i = j$ and 0 for $i \neq j$. Using the expression, we obtain the detection probability of true target, $P(\varphi, \omega)$, during time period $[\tau, T]$ as the occurrence probability of instances exclusive to the enumerated ones by $A_T$.

$$P(\varphi, \omega) = 1 - \sum_{\sigma \in A_T} \left[\prod_{t=\tau}^{T} \left(1 - \sigma(t)Q_t - \delta_{\sigma(t),-1}\right)\right] \exp\left(-\sum_{t=\tau}^{T} |\sigma(t)|\alpha_{\omega(t)}\varphi(\omega(t), t)\right) \tag{6}$$

Especially in the case of a constant probability of false contact with $Q_t = Q$, Eq. (6) ends up to a simple expression.

$$P(\varphi, \omega) = 1 - \sum_{\sigma \in A_T} Q^{m(\sigma)}(1 - Q)^{n(\sigma)} \exp\left(-\sum_{t=\tau}^{T} |\sigma(t)|\alpha_{\omega(t)}\varphi(\omega(t), t)\right), \tag{7}$$

where, $m(\sigma)$ is the number of false contacts and $n(\sigma)$ is the number of no-contacts in the search.

Furthermore, we are going to derive the expected payoff of the game. ¿From the discussion above and Assumption A3, the occurrence probability of an instance $\sigma$, the detection probability of target and searching cost $C_t(\varphi, \sigma)$ during a period $[\tau, t]$ are given by the followings.

$$Q_t(\sigma) = \prod_{\zeta=\tau}^{t} \left(1 - \sigma(\zeta)Q_\zeta - \delta_{\sigma(\zeta),-1}\right) \tag{8}$$

$$P_t(\varphi, \omega, \sigma) = 1 - \exp\left(-\sum_{\zeta=\tau}^{t} |\sigma(\zeta)|\alpha_{\omega(\zeta)}\varphi(\omega(\zeta), \zeta)\right) \tag{9}$$

$$C_t(\varphi, \sigma) = \sum_{\zeta=\tau}^{t} |\sigma(\zeta)| \sum_{i \in K} c_0(i, \zeta)\varphi(i, \zeta) \tag{10}$$

The searcher gets a reward $V(t) - C_t(\varphi, \sigma)$ on detection of target at time $t \in \widehat{\boldsymbol{T}}$ but just loses searching cost $C_T(\varphi, \sigma)$ in the case of no detection. Taking account of both cases, we can evaluate the expected payoff by

$$\begin{aligned} R(\varphi, \omega) \quad &= \quad \sum_{t=\tau}^{T} \sum_{\sigma \in A_t} (V(t) - C_t(\varphi, \sigma))Q_t(\sigma) \left(P_t(\varphi, \omega, \sigma) - P_{t-1}(\varphi, \omega, \sigma)\right) \\ &\quad - \sum_{\sigma \in A_T} C_T(\varphi, \sigma)Q_T(\sigma) \left(1 - P_T(\varphi, \omega, \sigma)\right). \end{aligned}$$

To calculate $P_{t-1}(\varphi, \omega, \sigma)$, we need just the part $(\sigma(\tau), \cdots, \sigma(t-1))$ of an entire vector $\sigma \in A_t$. However we use notation $P_{t-1}(\varphi, \omega, \sigma)$ instead of $P_{t-1}(\varphi, \omega, \sigma|_{A_{t-1}})$ for the sake of simplicity. Let us transform the above expression into a simpler form, as follows.

$$\begin{aligned}
R(\varphi,\omega) &= \sum_{t=\tau}^{T}\sum_{\sigma\in A_t}(V(t)-C_t(\varphi,\sigma))Q_t(\sigma)P_t(\varphi,\omega,\sigma)\\
&\quad -\sum_{t=\tau-1}^{T-1}\sum_{\sigma\in A_{t+1}}(V(t+1)-C_{t+1}(\varphi,\sigma))Q_{t+1}(\sigma)P_t(\varphi,\omega,\sigma)\\
&\quad -\sum_{\sigma\in A_T}C_T(\varphi,\sigma)Q_T(\sigma)\left(1-P_T(\varphi,\omega,\sigma)\right)\\
&= \sum_{t=\tau}^{T-1}\left\{\sum_{\sigma\in A_t}(V(t)-C_t(\varphi,\sigma))Q_t(\sigma)P_t(\varphi,\omega,\sigma)\right.\\
&\quad \left.-\sum_{\sigma\in A_{t+1}}(V(t+1)-C_{t+1}(\varphi,\sigma))Q_{t+1}(\sigma)P_t(\varphi,\omega,\sigma)\right\}\\
&\quad +\sum_{\sigma\in A_T}(V(T)-C_T(\varphi,\sigma))Q_T(\sigma)P_T(\varphi,\omega,\sigma)\\
&\quad -\sum_{\sigma\in A_T}C_T(\varphi,\sigma)Q_T(\sigma)\left(1-P_T(\varphi,\omega,\sigma)\right) \qquad (11)
\end{aligned}$$

Any instance of enumeration $A_{t+1}$ is made by adding an event $\sigma(t+1)$ to an enumeration $A_t$. For the additional event, there are two cases. In the case of $\sigma(t+1)\in\{1,-1\}$, we can make an instance until time $t+1$ using a common sequence of instances $\{\sigma(\tau),\cdots,\sigma(t)\}$. In the case of $\sigma(t+1)=0$ of investigation, however, $\{\sigma(\tau),\cdots,\sigma(t)\}$ should be exclusive to the above one. In the former case, the following calculation is possible:

$$\begin{aligned}
\sum_{\sigma\in A_{t+1}|\sigma(t+1)\in\{1,-1\}}Q_{t+1}(\sigma) &= \sum_{\sigma\in A_{t+1}|\sigma(t+1)\in\{1,-1\}}Q_t(\sigma)\{(1-Q_{t+1})+Q_{t+1}\}\\
&= \sum_{\sigma\in A_{t+1}|\sigma(t+1)\in\{1,-1\}}Q_t(\sigma) \qquad (12)
\end{aligned}$$

from Eq. (8). We also have the similar result of

$$\sum_{\sigma\in A_{t+1}|\sigma(t+1)=0}Q_{t+1}(\sigma)=\sum_{\sigma\in A_{t+1}|\sigma(t+1)=0}Q_t(\sigma)$$

in the latter case. As a result, $\sum_{\sigma\in A_{t+1}}Q_{t+1}(\sigma)=\sum_{\sigma\in A_t}Q_t(\sigma)$ is given. Similarly, we have the following calculation

$$\begin{aligned}
&\sum_{\sigma\in A_{t+1}|\sigma(t+1)\in\{1,-1\}}C_{t+1}(\varphi,\sigma)Q_{t+1}(\sigma)\\
&= \sum_{\sigma\in A_{t+1}|\sigma(t+1)\in\{1,-1\}}\left(\sum_{i\in K}c_0(i,t+1)\varphi(i,t+1)+C_t(\varphi,\sigma)\right)Q_t(\sigma)\\
&\qquad \times\{(1-Q_{t+1})+Q_{t+1}\}\\
&= \sum_{\sigma\in A_{t+1}|\sigma(t+1)\in\{1,-1\}}\left(\sum_{i\in K}c_0(i,t+1)\varphi(i,t+1)+C_t(\varphi,\sigma)\right)Q_t(\sigma)
\end{aligned}$$

in the former case but we have $\sum_{\sigma\in A_{t+1}|\sigma(t+1)=0} C_t(\varphi,\sigma)Q_t(\sigma)$ in the latter case. We can unify both expressions into

$$\sum_{\sigma\in A_{t+1}}\left(|\sigma(t+1)|\sum_{i\in K}c_0(i,t+1)\varphi(i,t+1)+C_t(\varphi,\sigma)\right)Q_t(\sigma).$$

We apply the above to the first item of Eq. (11) to get the following result:

$$\begin{aligned}R(\varphi,\omega)=&\sum_{t=\tau}^{T-1}\sum_{\sigma\in A_{t+1}}\left\{(V(t)-V(t+1))+|\sigma(t+1)|\sum_{i\in K}c_0(i,t+1)\varphi(i,t+1)\right\}\\&\times Q_{t+1}(\sigma)P_t(\varphi,\omega,\sigma)+\sum_{\sigma\in A_T}V(T)Q_T(\sigma)P_T(\varphi,\omega,\sigma)-\sum_{\sigma\in A_T}C_T(\varphi,\sigma)Q_T(\sigma)\end{aligned}\tag{13}$$

Considering the monotonically non-increasingness of target value, the linearity of $C_t(\varphi,\sigma)$ for $\varphi$ and the concavity of $P_t(\varphi,\omega,\sigma)$ for $\varphi$, which are seen from condition (2), Eq. (10) and (9), respectively, the payoff $R(\varphi,\omega)$ becomes strictly concave for $\varphi$. We take expectation for the payoff $R(\varphi,\omega)$ by a mixed strategy of target $\pi$ to get

$$\begin{aligned}R(\varphi,\pi)&=\sum_{\omega\in\Omega}\pi(\omega)R(\varphi,\omega)\\&=\sum_{\omega\in\Omega}\pi(\omega)\left[\sum_{t=\tau}^{T-1}\sum_{\sigma\in A_{t+1}}\left\{(V(t)-V(t+1))+|\sigma(t+1)|\sum_{i\in K}c_0(i,t+1)\varphi(i,t+1)\right\}\right.\\&\quad\left.\times Q_{t+1}(\sigma)P_t(\varphi,\omega,\sigma)+\sum_{\sigma\in A_T}V(T)Q_T(\sigma)P_T(\varphi,\omega,\sigma)\right]-\sum_{\sigma\in A_T}C_T(\varphi,\sigma)Q_T(\sigma).\end{aligned}\tag{14}$$

The expression is linear for $\pi$ and strictly concave for $\varphi$. We often use the detection probability of target as a criterion in the search problem and we can obtain its expectation $P(\varphi,\pi)$ by applying parameters $c_0(i,t)=0$ and $V(t)=1$ to Eq. (14), as follows.

$$\begin{aligned}P(\varphi,\pi)&=\sum_{\omega\in\Omega}\pi(\omega)Q_T(\sigma)P_T(\varphi,\omega,\sigma)\\&=1-\sum_{\omega\in\Omega}\pi(\omega)\sum_{\sigma\in A_T}\left[\prod_{t=\tau}^{T}\left(1-\sigma(t)Q_t-\delta_{\sigma(t),-1}\right)\right]\exp\left(-\sum_{t=\tau}^{T}|\sigma(\omega(t))|\varphi(\omega(t),t)\right)\end{aligned}\tag{15}$$

**4 Maximization of the Expected Payoff** In this section, our purpose is to propose a computational algorithm to derive an optimal distribution of searching resource of maximizing the expected payoff given a target mixed strategy $\pi$.

**4.1 Necessary and sufficient conditions for optimal solution** The following maximization problem has a unique optimal solution because the expected payoff of Eq. (14) is strictly concave for $\varphi$, as shown in Section 3.

$$(P1)\ \max_{\varphi}\ R(\varphi,\pi)\quad s.t.\ \sum_{i\in K}\varphi(i,t)\le\Phi(t),\ t\in\widehat{\boldsymbol{T}},\quad \varphi(i,t)\ge 0,\ i\in\boldsymbol{K},\ t\in\widehat{\boldsymbol{T}}$$

We can derive necessary and sufficient conditions for the optimal solution of Problem $(P1)$ as its Karush-Kuhn-Tucker (KKT) conditions, using Lagrangean multipliers $\{\lambda(t),t\in\widehat{\boldsymbol{T}}\}$

and $\{\mu(i,t),\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}}\}$, and a Lagrangean function

$$L(\varphi;\lambda,\mu) = R(\varphi,\pi) + \sum_{t\in\widehat{\boldsymbol{T}}} \lambda(t)\left(\Phi(t) - \sum_{i\in K}\varphi(i,t)\right) + \sum_{i\in K,}\sum_{t\in\widehat{\boldsymbol{T}}} \mu(i,t)\varphi(i,t).$$

As the result, we have the following conditions:

$$\frac{\partial L}{\partial\varphi(i,t)} = \frac{\partial R}{\partial\varphi(i,t)} - \lambda(t) + \mu(i,t) = 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}} \tag{16}$$

$$\lambda(t) \geq 0,\ t \in \widehat{\boldsymbol{T}} \tag{17}$$

$$\mu(i,t) \geq 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}} \tag{18}$$

$$\sum_{i\in K}\varphi(i,t) \leq \Phi(t),\ t \in \widehat{\boldsymbol{T}} \tag{19}$$

$$\varphi(i,t) \geq 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}} \tag{20}$$

$$\lambda(t)\left(\Phi(t) - \sum_{i\in K}\varphi(i,t)\right) = 0,\ t \in \widehat{\boldsymbol{T}} \tag{21}$$

$$\mu(i,t)\varphi(i,t) = 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}}. \tag{22}$$

We express $\partial R/\partial\varphi(i,t)$ as a function of variable $\varphi(i,t)$ in an explicit manner and have

$$\frac{\partial R}{\partial\varphi(i,t)} = B(i,t)\exp(-\alpha_i\varphi(i,t)) + C(i,t). \tag{23}$$

In the expression above, we use the following notation:

$$\begin{aligned} B(i,t) \equiv \sum_{\omega\in\Omega_{it}} \pi(\omega) \Bigg[ \sum_{\zeta=t}^{T-1} \sum_{\sigma\in A_{\zeta+1}} & \left\{ -\Delta V(\zeta) + |\sigma(\zeta+1)| \sum_{i\in K} c_0(i,\zeta+1)\varphi(i,\zeta+1) \right\} \\ & \times |\sigma(t)|\alpha_i Q_{\zeta+1}(\sigma)\exp\left(-\sum_{\xi=\tau,\xi\neq t}^{\zeta} |\sigma(\xi)|\alpha_{\omega(\xi)}\varphi(\omega(\xi),\xi)\right) \\ & + \sum_{\sigma\in A_T} V(T)|\sigma(t)|\alpha_i Q_T(\sigma)\exp\left(-\sum_{xi=\tau,\xi\neq t}^{T} |\sigma(\xi)|\alpha_{\omega(\xi)}\varphi(\omega(\xi),\xi)\right)\Bigg] \end{aligned} \tag{24}$$

$$C(i,t) \equiv \sum_{\sigma\in A_t} |\sigma(t)|c_0(i,t)Q_t(\sigma)\sum_{\omega\in\Omega}\pi(\omega)P_{t-1}(\varphi,\omega,\sigma) - \sum_{\sigma\in A_T}|\sigma(t)|c_0(i,t)Q_T(\sigma) \tag{25}$$

with definitions of $\Omega_{it} \equiv \{\omega\in\Omega|\omega(t)=i\}$ and $\Delta V(\zeta) \equiv V(\zeta+1) - V(\zeta)$.

For an instance $\sigma$ with $\sigma(t) = 0$ at time $t$, the distribution plan of searching resouce, $\{\varphi(i,t), i \in \boldsymbol{K}\}$, cannot be executed at that time practically and then we count or enumerate only $\sigma(t)$ of $|\sigma(t)| = 1$ to calculate $B(i,t)$ and $C(i,t)$.

Anyway we can verify $C(i,t) < 0$ because of Eq. (12) and the following transformation.

$$\begin{aligned} &\sum_{\sigma\in A_T} |\sigma(t)|c_0(i,t)Q_T(\sigma) = |\sigma(t)|c_0(i,t)\sum_{\sigma\in A_{T-1}} Q_{T-1}(\sigma) = \cdots \\ &= |\sigma(t)|c_0(i,t)\sum_{\sigma\in A_t} Q_t(\sigma) > |\sigma(t)|c_0(i,t)\sum_{\sigma\in A_t} Q_t(\sigma)\sum_{\omega}\pi(\omega)P_{t-1}(\varphi,\omega,\sigma). \end{aligned}$$

Let us consider the properties of optimal distribution of searching resource.

First, the complementary slackness condition (21) tells us that an equation $\sum_{i\in K}\varphi(i,t)=\Phi(t)$ holds if $\lambda(t)>0$. Next we are going to derive necessary and sufficient conditions for $\varphi(i,t)>0$ or $\varphi(i,t)=0$.

If $\varphi(i,t)>0$, we have $\mu(i,t)=0$ from Eq. (22) and then $B(i,t)\exp(-\alpha_i\varphi(i,t))+C(i,t)=\lambda(t)$ from (16) and (23). It also indicates $B(i,t)+C(i,t)>\lambda(t)$.

In the case of $B(i,t)=0$ in Definition (24), there is no path running through Cell $i$ at $t$ or $\Omega_{it}=\emptyset$ and an optimal distribution should be $\varphi(i,t)=0$ by the following reason. If $\varphi(i,t)>0$, we have $C(i,t)=\lambda(t)$. But the equation contradicts $C(i,t)<0$ and $\lambda(t)\geq 0$.

Now let us assume $B(i,t)>0$. If $\varphi(i,t)=0$, it follows that $B(i,t)\exp(-\alpha_i\varphi(i,t))+C(i,t)=B(i,t)+C(i,t)\leq\lambda(t)$ from Eqs. (16) and (18). Conversely, we have $B(i,t)\exp(-\alpha_i\varphi(i,t))+C(i,t)< B(i,t)+C(i,t)\leq\lambda(t)$ for any $\varphi(i,t)>0$ if $B(i,t)+C(i,t)\leq\lambda(t)$ but the fact contradicts the inequality we derived from $\varphi(i,t)>0$ just before. Therefore, $\varphi(i,t)=0$ is equivalent to the condition $B(i,t)+C(i,t)\leq\lambda(t)$ in the case of $B(i,t)>0$. Noting that $B(i,t)+C(i,t)\leq\lambda(t)$ includes the condition $B(i,t)=0$, necessary and sufficient condition for $\varphi(i,t)=0$ is $B(i,t)+C(i,t)\leq\lambda(t)$.

According to the discussion so far, we classify the conditions that make an optimal distribution of resource $\varphi(i,t)$ positive or zero into the following two cases:

(i) If and only if $B(i,t)+C(i,t)>\lambda(t)$, an optimal distribution $\varphi^*(i,t)>0$ is given by

$$\varphi^*(i,t)=\frac{1}{\alpha_i}\ln\frac{B(i,t)}{\lambda(t)-C(i,t)}.$$

(ii) If and only if $B(i,t)+C(i,t)\leq\lambda(t)$, $\varphi(i,t)=0$.

We have a formula about an optimal distribution in both cases of (i) and (ii).

$$\varphi^*(i,t)=\frac{1}{\alpha_i}\left[\ln\frac{B(i,t)}{\lambda(t)-C(i,t)}\right]^+, \tag{26}$$

where we use notation $[x]^+\equiv\max\{0,x\}$. The total amount of resouces distributed optimally at time $t$ is given by

$$\sum_{\{i\in K|B(i,t)+C(i,t)>\lambda(t)\}}\frac{1}{\alpha_i}\left[\ln\frac{B(i,t)}{\lambda(t)-C(i,t)}\right]^+ \tag{27}$$

and it is monotonically decreasing for Lagrangean multiplier $\lambda(t)$ within $[0,\overline{\lambda_t}]$ and becomes zero for $\lambda(t)\geq\overline{\lambda_t}$, where $\overline{\lambda_t}\equiv\max_{i\in K}\{B(i,t)+C(i,t)\}$. Here let us make sure that $B(i,t)$ and $C(i,t)$ do not contain $\{\varphi(i,t),\ i\in\boldsymbol{K}\}$ distributed at $t$.

**4.2 A computational algorithm to derive an optimal solution** We repeat constructing an optimal distribution $\{\varphi(i,t),i\in\boldsymbol{K}\}$ of a time $t$ for every $t\in\widehat{\boldsymbol{T}}$ while keeping $\{\varphi(i,\zeta),i\in\boldsymbol{K}\}$ unchanged at any other time $\zeta\in\widehat{\boldsymbol{T}}$, from the properties of optimal solution discussed in Section 4.1.

Finding an optimal distribution of searching resource at the time $t$ proceeds as follows. First, we search for an optimal $\lambda(t)$. We apply $\lambda(t)=0$ to Eq. (26) and calculate $\{\varphi(i,t),i\in\boldsymbol{K}\}$. If $\sum_i\varphi(i,t)\leq\Phi(t)$, we have obtained an optimal solution at the time $t$. Otherwise, we find $\lambda(t)$ satisfying $\sum_{i\in K}\varphi(i,t)=\Phi(t)$ within $[0,\overline{\lambda_t}]$. The search is done by dichotomy using the monotonically decreasingness of the expression (27). We repeat the above procedure for optimal solution until the satisfaction of the KKT conditions (16)$\sim$(22) for every time

$t = \tau, \ldots, T$. We come to an optimal solution $\{\varphi^*(i,t), i \in \boldsymbol{K}, t \in \widehat{\boldsymbol{T}}\}$ if the newly-obtained solution does not change from old one for all $t \in \widehat{\boldsymbol{T}}$. Our algorithm always brings a larger expected reward than the previous one after every calculation at each time and therefore it certainly finishes on convergence to the largest reward. Our algorithm is constructed as follows.

An algorithm for an optimal distribution of searching resource: $\Gamma(\pi)$

(S1) Repeat Step (S2)∼(S3) for $t = \tau, \ldots, T$ until the convergence of solution.

(S2) Set $\lambda(t) = 0$ and calculate $\{\varphi(i,t), i \in \boldsymbol{K}\}$ by Eq. (26).

(S3) If $\sum_{i \in K} \varphi(i,t) \leq \Phi(t)$, end and go back to Step (S1). Otherwise, if $\sum_{i \in K} \varphi(i,t) > \Phi(t)$, execute the following algorithm to obtain an optimal multiplier $\lambda^*(t)$ and go back to Step (S1).

(i) Sort values of $B(i,t) + C(i,t)$ in the increasing order like $B(I_1,t) + C(I_1,t) \leq B(I_2,t) + C(I_2,t) \leq \ldots$ and renumber all cells $i \in \boldsymbol{K}$ in the form of $I_1, I_2, \ldots$. Set parameters $\xi = 1$ and $\lambda_0 = 0$.

(ii) Apply $\lambda(t) = B(I_\xi,t) + C(I_\xi,t)$ to Eq. (26) and calculate $\{\varphi(i,t), i \in \boldsymbol{K}\}$ as follows:

$$\varphi(i,t) = \left\{ \begin{array}{cl} 0, & i = 1, \ldots, I_\xi, \\ (1/\alpha_i) \ln \{B(i,t)/(\lambda(t) - C(i,t))\}, & i = I_{\xi+1}, \ldots, I_K \end{array} \right. \tag{28}$$

(iii) If $\sum_{i \in K} \varphi(i,t) \leq \Phi(t)$, find an optimal $\lambda^*(t)$ satisfying

$$\sum_{k=\xi+1}^{K} \frac{1}{\alpha_{I_k}} \ln \frac{B(I_k,t)}{\lambda^*(t) - C(I_k,t)} = \Phi(t)$$

during an interval $[\lambda_0, \lambda(t)]$ by dichotomy, using the monotonically decreasingness of the left-hand side of the above expression. Calculate an optimal solution $\{\varphi(i,t), i \in \boldsymbol{K}\}$ by substituting the $\lambda^*(t)$ into Eq. (28).

Otherwise, set $\lambda_0 = \lambda(t)$ and $\xi = \xi + 1$. Go back to (ii).

## 5 A Search Allocation Game and an Numerical Algorithm for Its Equilibrium

In Section 4, we proposed an algorithm to derive an optimal distribution of searching resource for the maximum expected reward of the searcher given a target strategy $\pi$. Here, we model a search game with the expected payoff of Eq. (14) and discuss a computational algorithm to derive an equilibrium of the game.

We know that there is an equilibrium for our search game because the expected payoff has the strict concavity for the searcher's strategy $\varphi$ and the linearity for the target strategy $\pi$ [24]. At the equilibrium point, the minimax value of the expected payoff equals its maximin value. That is why we focus on the derivation of the maximin value from here. We can transform the maximin optimization as follows.

$$\max_\varphi \min_\pi R(\varphi,\pi) = \max_\varphi \min_\pi \sum_{\omega \in \Omega} \pi(\omega) R(\varphi,\omega) = \max_\varphi \min_{\omega \in \Omega} R(\varphi,\omega)$$

Therefore, the problem is equivalent to the following convex problem, which gives an optimal strategy of the searcher $\varphi^*$.

$$(P_M) \qquad \max_{\varphi,\nu} \ \nu \tag{29}$$

$$s.t. \qquad R(\varphi,\omega) \geq \nu,\ \omega \in \Omega, \tag{30}$$

$$\sum_{i\in K} \varphi(i,t) \leq \Phi(t),\ t \in \widehat{\boldsymbol{T}}, \tag{31}$$

$$\varphi(i,t) \geq 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}}. \tag{32}$$

Setting dual variables $\eta(\omega)$, $\lambda(t)$ and $\mu(i,t)$ corresponding to conditions (30), (31) and (32), respectively, and making a Lagrangean function

$$\begin{aligned} L(\nu,\varphi;\eta,\lambda,\mu) \ \equiv\ & \nu + \sum_{\omega\in\Omega} \eta(\omega)\left(R(\varphi,\omega)-\nu\right) + \sum_{t\in\widehat{\boldsymbol{T}}} \lambda(t)\left(\Phi(t) - \sum_{i\in K}\varphi(i,t)\right) \\ & + \sum_{i\in K, t\in\widehat{\boldsymbol{T}}} \mu(i,t)\varphi(i,t), \end{aligned}$$

we have the following KKT conditions for an optimal solution:

$$R(\varphi,\omega) \geq \nu,\ \omega \in \Omega \tag{33}$$

$$\eta(\omega)\left(R(\varphi,\omega)-\nu\right) = 0,\ \omega\in\Omega \tag{34}$$

$$\eta(\omega) \geq 0,\ \omega \in \Omega \tag{35}$$

$$\frac{\partial L}{\partial \nu} = 1 - \sum_{\omega} \eta(\omega) = 0 \tag{36}$$

$$\frac{\partial L}{\partial \varphi(i,t)} = \sum_{\omega\in\Omega} \eta(\omega) \frac{\partial R(\varphi,\omega)}{\partial\varphi(i,t)} - \lambda(t) + \mu(i,t) = 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}} \tag{37}$$

$$\lambda(t) \geq 0,\ t \in \widehat{\boldsymbol{T}} \tag{38}$$

$$\mu(i,t) \geq 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}} \tag{39}$$

$$\sum_{i\in K} \varphi(i,t) \leq \Phi(t),\ t \in \widehat{\boldsymbol{T}} \tag{40}$$

$$\varphi(i,t) \geq 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}} \tag{41}$$

$$\lambda(t)\left(\Phi(t) - \sum_{i\in K}\varphi(i,t)\right) = 0,\ t \in \widehat{\boldsymbol{T}} \tag{42}$$

$$\mu(i,t)\varphi(i,t) = 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}}. \tag{43}$$

Conditions (37)$\sim$(43) imply the necessary and sufficient conditions of optimal searcher's strategy, which correspond to (16)$\sim$(22). Considering problem $\min_\pi \sum_\omega \pi(\omega)R(\varphi,\omega)$, an optimal target's response $\pi$ to a searcher's strategy $\varphi$ must be $\pi(\omega) = 0$ if $R(\varphi,\omega) > \nu$, where $\nu \equiv \min_\omega R(\varphi,\omega)$. We can replace the condition with $\pi(\omega)\left(R(\varphi,\omega)-\nu\right) = 0$. Therefore, we can see that $\eta$ in conditions (33)$\sim$(36) must be an optimal target strategy $\pi^*$.

¿From the discussion so far, we can propose a computational algorithm to derive an equilibrium by combining two algorithms. We take the algorithm $\Gamma(\pi)$ for an optimal searcher's strategy $\varphi^*$, discussed in Section 4.2. For an optimal target strategy $\pi^*$, we utilize conditions (33)$\sim$(36). In the proposed algorithm, we repeat the derivation of $\varphi^*_\pi$

optimally corresponding to $\pi$ by $\Gamma(\pi)$ while changing $\pi$ and finally make $\pi$ satisfy conditions (33)$\sim$(36). We manipulate $\pi$ using the property about how $R(\varphi^*_\pi, \omega)$ increases/decreases by $\pi(\omega)$, which is stated in the following lemma.

**Lemma 1** (Reference [12])**.** *We set $\pi_1(k) = \pi(k) + \Delta\pi(k)$ for a $k \in \Omega$ and $\pi_1(\omega) = \pi(\omega)$ for any other path $k \in \Omega$ and apply $\pi_1$ to Algorithm $\Gamma(\pi_1)$. The derived optimal solution $\varphi^*_{\pi_1}$ makes payoff $R(\varphi^*_{\pi_1}, k)$ of the path $k$ increase if $\Delta\pi(k) > 0$ and decrease if $\Delta\pi(k) < 0$ from the original value $R(\varphi^*_\pi, k)$.*

**Proof:** Omitted. $\square$

Now we are ready to propose a new algorithm for an equilibrium of our search game from Lemma 1, in which we reach the equilibrium while changing $\pi$.

An algorithm for an equilibrium: $\Lambda$

(E1) Initialize $\pi$ to be $\pi(\omega) = 1/|\Omega|$. Set $l = 0$.

(E2) For given $\pi$, run Algorithm $\Gamma(\pi)$ to obtain an optimal solution $\varphi^*_\pi$.
Normalize $\pi$ such that $\sum_\omega \pi(\omega) = 1$.

(E3) Sort values $\{R(\varphi^*_\pi, \omega), \omega \in \Omega\}$ in the increasing order like $W_1 < W_2 < \cdots < W_M$, where $W_k$ is the $k$-th smallest value of $R(\varphi^*_\pi, \omega)$. We categorize all paths in some sets of paths based on the value such that $\Omega_k \equiv \{\omega \in \Omega | R(\varphi^*_\pi, \omega) = W_k\}$ for $k = 1, \ldots, M$.
If conditions (33)$\sim$(36) are satisfied for $\eta(\omega) = \pi(\omega)$, end.

(E4) Generate a new $\pi$ as follows. If $l$ is even, increase $\pi(k)$ by a little bit $\Delta\pi(k) > 0$ for a $k \in \Omega_1$. If odd, add a tiny negative amount $\Delta\pi(k) < 0$ to $\pi(k)$ for a path $k$ of $k \in \arg\max_{\{\omega|\pi(\omega)>0\}} R(\varphi^*_\pi, \omega)$.
Increase $l$ by one, $l = l + 1$, and go back to Step (E2).

In Algorithm $\Lambda$, we manipulate the expected payoff in such a way that we push down the payoff a little by decreasing the selection probability $\pi(\omega)$ for a path $\omega$ with maximum expected payoff $R(\varphi^*_\pi, \omega)$ and lift it up by increasing the selection probability for a path with minimum payoff. The manipulation leads $\pi$ to the satisfaction of conditions (33)$\sim$(36).

As seen in some numerical algorithms in a general way, the speed of convergence changes depending on tolerance of error and $\Delta\pi(k)$ in this case. We set $\Delta\pi(k)$ in a similar way to Reference [12], as follows. We assume that $\pi(k)$ changes the expect payoff of path $k$, $R(\varphi^*_k, k)$, in a linear way, within the maximum expected reward by $\pi(k) = 1$, $\overline{R_k} \equiv R(\varphi^*_k, k)$, and the minimum payoff by $\pi(k) = 0$, $\underline{R_k} \equiv \min_{k \neq \omega \in \Omega} R(\varphi^*_\omega, k)$. The assumption teaches us a function $(\overline{R_k} - \underline{R_k})\pi(k) + \underline{R_k}$ of variable $\pi(k)$ and the increasing/decreasing by $\Delta\pi(k) = \gamma/(\overline{R_k} - \underline{R_k})$ if we want to lift-up/push-down the expected payoff by $\gamma$. We embed the control of $\pi(k)$ in Algorithm $\Lambda$. Anyway, we adopt the following rule to control $\gamma$, assuming that $k$ belongs to $W_{M'}$ in Step (E4) for odd $l$.

(i) In the case of $M' = 1$, $R(\varphi^*_\pi, \omega)$ coincides for any $\omega$ of $\pi(\omega) > 0$ and the algorithm ends.

(ii) In the case of $M' = 2$, increase the expected payoff of path $k \in \Omega_1$ by $\gamma = (W_2 - W_1)/2$ and decrease that of $k \in \Omega_2$ by the same amount with the intention of coincidence of the expected payoffs for both paths.

(iii) In the case of $M' > 2$, increase the expected payoff of path $k \in \Omega_1$ by $\gamma = (W_2 - W_1)$ and decrease it for path $k \in \Omega'_M$ by $\gamma = (W_{M'} - W_{M'-1})$.

$\pi$ is modified to the following $\pi_1$ in Step (E4) and normalized to $\widehat{\pi}$ in (E2).

$$\pi_1(k) = \pi(k) + \Delta\pi(k), \quad \pi_1(\omega) = \pi(\omega)\ (k \neq \omega \in \Omega).$$

Let us see that the modification makes larger expected payoff, namely, $R(\varphi^*_\pi, \pi) > R(\varphi^*_{\widehat{\pi}}, \widehat{\pi})$, as follows.

$$\begin{aligned}
\Delta R = R(\varphi^*_{\widehat{\pi}}, \widehat{\pi}) - R(\varphi^*_\pi, \pi) &= \sum_\omega \widehat{\pi}(\omega) R(\varphi^*_{\widehat{\pi}}, \omega) - R(\varphi^*_\pi, \pi) \\
&= \sum_\omega \frac{\pi_1(\omega)}{1+\Delta\pi(k)} R(\varphi^*_{\widehat{\pi}}, \omega) - R(\varphi^*_\pi, \pi) \\
&= \sum_\omega \frac{1}{1+\Delta\pi(k)} \left\{ R(\varphi^*_{\widehat{\pi}}, \pi) + \Delta\pi(k) R(\varphi^*_{\widehat{\pi}}, k) - (1+\Delta\pi(k)) R(\varphi^*_\pi, \pi) \right\} \\
&= \sum_\omega \frac{1}{1+\Delta\pi(k)} \left\{ \Delta\pi(k) \left( R(\varphi^*_{\widehat{\pi}}, k) - R(\varphi^*_\pi, \pi) \right) - \left( R(\varphi^*_\pi, \pi) - R(\varphi^*_{\widehat{\pi}}, \pi) \right) \right\}
\end{aligned}$$

In Step (E4), we have $R(\varphi^*_\pi, k) < R(\varphi^*_\pi, \pi)$ because the selected $k \in \Omega_1$ has the minimum payoff among all paths. Therefore, we can set $\Delta\pi(k)$ small enough to keep $R(\varphi^*_{\widehat{\pi}}, k) < R(\varphi^*_\pi, \pi)$. We also have $R(\varphi^*_\pi, \pi) \geq R(\varphi^*_{\widehat{\pi}}, \pi)$ and then $\Delta R < 0$ for even $l$ in (E4). Similarly, we can see $R(\varphi^*_{\widehat{\pi}}, k) > R(\varphi^*_\pi, \pi)$ for a path $k \in \Omega_{M'}$. From $\Delta\pi(k) < 0$, we also have $\Delta R < 0$ for odd $l$. We can verify that, in Step (E4), the changing of $\pi$ controls $R(\varphi^*_\pi, k)$ in the direction of decreasing the expected payoff.

**6 Numerical Examples** Let a search space and a time space be $\boldsymbol{K} = \{1, \ldots, 5\}$ and $\boldsymbol{T} = \widehat{\boldsymbol{T}} = \{1, \ldots, 10\}$, respectively. We set other parameters as follows: $\Phi(t) = 1\ (t \in \widehat{\boldsymbol{T}})$ and $\alpha_i = 0.2\ (i \in \boldsymbol{K})$. Figure 2 shows four target paths, $\Omega$, by illustrating which cell each path runs through time by time in a $|\boldsymbol{K}| \times |\boldsymbol{T}|$ matrix. Path 3 and 4 always stay at Cell 3 and 2, respectively, but Path 1 and 2 run across several cells symmetrically to each other. Crossing points of paths are effective for search operation because the searcher can cover several paths running there at the same time by distributing searching resources there. We can sort all paths into 2, 1, 3 and 4 in the decreasing order in terms of the number of crossing points.

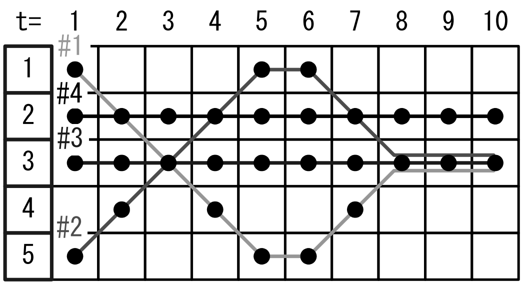


Figure 2: Target paths

We consider the search allocation game with $V(t) = 20\ (t \in \widehat{\boldsymbol{T}})$ as target value and $c_0(i,t) = 1\ (i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}})$ as searching cost.

(1) Case of no false contact (Case 1)

In the case of no false contact, we can analyze optimal strategies of players by setting $Q_t = 0$. The value of the game is 5.8. Table 1a shows an optimal distribution plan of searching resource by the searcher and Table 1b indicates the total amount of resources accumulated on each target path and an optimal selection probability of paths by the target.

As seen from Table 1a, the distribution of resource is concentrated in crossing points from cost-effective point of view and the searcher uses up all available resources, $\Phi(t) = 1$, all time points but $t = 1$. At time $t = 1$, the searcher cannot find any crossing point and use a part of $\Phi(t) = 1$. Because of the concentration strategy on crossing points, the most amount of resources are distributed along Path 2 with the largest number of crossing points and the target never takes the path 2, as seen from Table 1b. The target chooses paths 1, 3 and 4, which have similar amount of distributed resources. The searcher decides his distribution strategy of searching resource, taking account of a tradeoff between effective search and target's preference on paths, which is conversely affected by the distribution strategy. On the target side, he tends to avoid the paths on which effective search is easy to be done because of the possession of many crossing points or others, and to make it difficult for the searcher to have easy anticipation on the target's path and concentrate much searching resources on the path.

Table 1a. Optimal distribution of searching resource (Case 1).

| Cells \ t | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0.137 | 1 | 0 | 0.718 | 0.733 | 1 | 1 | 0 | 0 | 0 |
| 3 | 0.496 | 0 | 1 | 0.282 | 0.267 | 0 | 0 | 1 | 1 | 1 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Total | 0.634 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Table 1b. Accumulated amount of resource and optimal selection of target paths (Case 1)

| Paths | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Amount of accumulated resource | 5.0 | 5.719 | 5.045 | 4.589 |
| Selection probability | 0.26 | 0 | 0.37 | 0.37 |

(2) Effect by false contact (Case 2)

In this case, we change some parameters of Case 1 to $Q_t = 0.5$ ($t \in \widehat{\boldsymbol{T}}$) and $t_f = 3$. The false signal makes the search operation less effective and changes the value of the game to 3.3 from 5.8 of Case 1. Equilibrium is shown in Table 2a and 2b. On happening of false contact, the pre-planned distribution of searching resource cannot be done precisely and comes to no use in a sequent investigation process. The fact lessens the value of crossing point on effective search and makes the searcher have the incentive to transfer some resources from crossing points to other points, which is seen from the comparison between Table 1a and Table 2a. At the same time, the target's avoidance to Path 2 and 1 having many crossing points gets weak, comparing with Case 1, and the selection probabilities of Path 1, 3 and 4 become all equal. Whether or not the distribution of searching resource is planned, real false contacts cancel the distribution plan in its sequent investigation process although the searcher does not expend any searching cost. That is the reason why the searcher performs an active behavior to consume more available resources in this case than Case 1. Practically, the searcher exhausts $\Phi(t) = 1$ at time $t = 1$.

Table 2a. Optimal distribution of searching resource (Case 2)

| Cells \ t | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.05 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0.414 | 1 | 0 | 0.842 | 0.586 | 0.549 | 0.772 | 0 | 0 | 0 |
| 3 | 0.536 | 0 | 1 | 0 | 0.218 | 0.366 | 0.047 | 1 | 1 | 1 |
| 4 | 0 | 0 | 0 | 0.158 | 0 | 0 | 0.181 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 0.196 | 0.085 | 0 | 0 | 0 | 0 |
| Total | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Table 2b. Accumulated amount of resource and optimal selection of target paths (Case 2)

| Paths | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Amount of accumulated resource | 5.67 | 5.614 | 5.168 | 4.162 |
| Selection probability | 0.327 | 0.019 | 0.327 | 0.327 |

(3) Effect of target value (Case 3)

In this case, we increase target value $V(t)$ to 50 from 20 of Case 2. The value of the game is 15.0 although it was 3.3 in Case 2. Table 3a and 3b show an equilibrium in this case. Since the target value increases, the searcher accelerates the distribution of searching resource, expecting higher reward on detection of target and takes the strategy of exhaustion of available resource (EAR strategy). Therefore, the optimal strategy of this case is very similar to Case 2.

Table 3a. Optimal distribution of searching resource (Case 3)

| Cells \ t | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0.573 | 1 | 0 | 0.822 | 0.529 | 0.542 | 0.654 | 0 | 0 | 0 |
| 3 | 0.427 | 0 | 1 | 0 | 0.285 | 0.369 | 0.147 | 1 | 1 | 1 |
| 4 | 0 | 0 | 0 | 0.178 | 0 | 0 | 0.199 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 0.186 | 0.089 | 0 | 0 | 0 | 0 |
| Total | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Table 3b. Accumulated amount of resource and optimal selection of target paths (Case 3)

| Paths | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Amount of accumulated resource | 5.651 | 5.477 | 5.229 | 4.12 |
| Selection probability | 0.327 | 0.016 | 0.329 | 0.329 |

(4) Effect of distribution cost (Case 4)

We increase cost $c_0(i,t)$ to 4 from 1 of Case 2. The value of the game is 2.7, which is smaller than Case 2. The searcher makes much account of the effectiveness of cell for search and avoids distributing searching resource into ineffective cells other than crossing points. He takes the strategy of partially-using available resource (PAR strategy) of $\Phi(t) = 1$ at time $t = 1, 4, 5, 6$. The focus on crossing points by the searcher decreases the selection probabilities of Path 2 and 1, having many crossing points, but increases those of Path 3 and 4, compared with Case 2.

Table 4a. Optimal distribution of searching resource (Case 4)

| Cells \ t | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0.374 | 1 | 0 | 0.697 | 0.379 | 0.465 | 0.435 | 0 | 0 | 0 |
| 3 | 0.139 | 0 | 1 | 0 | 0.256 | 0.222 | 0.223 | 1 | 1 | 1 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Total | 0.514 | 1 | 1 | 0.697 | 0.687 | 0.658 | 1 | 1 | 1 | 1 |

Table 4b. Accumulated amount of resource and optimal selection of target paths (Case 4)

| Paths | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Amount of accumulated resource | 5 | 5.132 | 4.84 | 3.35 |
| Selection probability | 0.117 | 0 | 0.442 | 0.442 |

From studies in Case 1∼4, we enumerate some properties on optimal strategy of player.

1. The distribution plan of searching resource is certainly executed in the case of no false contact. In the false contact model, however, the plan is not necessarily done and the usage of resource has some uncertainty. The fact lessens the effectiveness of search in crossing points, leads the searcher to pay attention to other points for search and mitigates the target's avoidance to paths with many crossing points. The cancellation of the distribution plan means the no-expense of searching cost as well as the no-use of searching resource and the searcher has more activeness to use resource in the presence of false contacts. The value of the game becomes smaller, of course.

2. The increase of target value brings larger value of the game and makes the searcher more active for search. We could enumerate other influences by the increase of target value on the game.

   (1) The searching cost is relatively getting smaller compared with the target value and then the searcher tends to use more resource at the time when he takes the PAR strategy in the case of lower target value. The searcher's tendency causes more resources being distributed in non-crossing points. Because of that, the target is going to increase the selection probability of the paths with many crossing points and to decrease the probability of other paths.

   (2) In the case that the searcher already takes the EAR strategy for available resource every time, the amount of used resource cannot be increased and optimal strategies of players do not change a lot even though the target value gets larger.

3. In the case that the distribution cost of resource increases, the value of the game decreases and the searcher has less incentive to distribute his searching resource. If such a situation causes the decrease of the amount of distributed resources, the decreasing would be mainly adopted to non-crossing points and the target takes paths with many crossing points with less probability and other paths with more probability.

**7 Conclusion** This paper deals with a search game with false contacts in a precise manner. The false contacts are thought to happen independent of detection of true targets and their occurrence probability might be constant in the search operation. The false contact is inevitable to step in the search operation and then it is a most serious phenomenon we should take account of to analyze search operations using any sensor. Nevertheless, there have been just a few researches on the topics because the false contacts are difficult to deal with in terms of modeling and solution.

In this paper, we deal with the false contact as an event with fixed occurrence probability and enumerate all instances with the false contacts as well as the detection of target. At first, we formulate a maximization problem with the objective of the expected reward on the searcher's side into a nonlinear programming problem and propose a numerical algorithm to derive an optimal distribution of searching resource in a search space. Then we analyze the properties of optimal target strategy of taking several paths moving in the space. For the search problem on both sides of the searcher and the target, i.e. the search allocation game, we embed the properties of optimal target strategy in the algorithm proposed for optimal searcher's strategy to construct a repetition algorithm to derive an equilibrium point for the game. We define the payoff of the game as the target value gained by its detection minus the searching cost expended to execute a distribution plan of searching resource, which is a general criterion including the detection probability of target. As long as we take a direction of dealing with the false contact in a precise and practical manner, the dependency of occurrence of the false contacts on cell, time and the amount of distributed searching resource would be future topics to handle.

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